

Flashcard Supplement  
to  
A First Course in Linear Algebra

Robert A. Beezer

University of Puget Sound

Version 3.00-preview(December 28, 2013)

Robert A. Beezer is a Professor of Mathematics at the University of Puget Sound, where he has been on the faculty since 1984. He received a B.S. in Mathematics (with an Emphasis in Computer Science) from the University of Santa Clara in 1978, a M.S. in Statistics from the University of Illinois at Urbana-Champaign in 1982 and a Ph.D. in Mathematics from the University of Illinois at Urbana-Champaign in 1984.

In addition to his teaching at the University of Puget Sound, he has made sabbatical visits to the University of the West Indies (Trinidad campus) and the University of Western Australia. He has also given several courses in the Master's program at the African Institute for Mathematical Sciences, South Africa. He has been a Sage developer since 2008.

He teaches calculus, linear algebra and abstract algebra regularly, while his research interests include the applications of linear algebra to graph theory. His professional website is at <http://buzzard.ups.edu>.

**Edition**

Version 3.20 Flashcard Supplement  
December 28, 2013

**Publisher**

Robert A. Beezer Congruent Press Gig Harbor, Washington, USA

©2004—2013 Robert A. Beezer

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the appendix entitled “GNU Free Documentation License”.

**Definition SLE System of Linear Equations****1**

A system of linear equations is a collection of  $m$  equations in the variable quantities  $x_1, x_2, x_3, \dots, x_n$  of the form,

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

where the values of  $a_{ij}$ ,  $b_i$  and  $x_j$ ,  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ , are from the set of complex numbers,  $\mathbb{C}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition SSLE Solution of a System of Linear Equations****2**

A solution of a system of linear equations in  $n$  variables,  $x_1, x_2, x_3, \dots, x_n$  (such as the system given in Definition SLE), is an ordered list of  $n$  complex numbers,  $s_1, s_2, s_3, \dots, s_n$  such that if we substitute  $s_1$  for  $x_1$ ,  $s_2$  for  $x_2$ ,  $s_3$  for  $x_3$ , ,  $s_n$  for  $x_n$ , then for every equation of the system the left side will equal the right side, i.e. each equation is true simultaneously.

©2004–2013 Robert A. Beezer, GFDL License

**Definition SSSLE Solution Set of a System of Linear Equations**

**3**

The solution set of a linear system of equations is the set which contains every solution to the system, and nothing more.

©2004–2013 Robert A. Beezer, GFDL License

**Definition ESYS Equivalent Systems**

**4**

Two systems of linear equations are equivalent if their solution sets are equal.

©2004–2013 Robert A. Beezer, GFDL License

**Definition EO Equation Operations****5**

Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an equation operation.

1. Swap the locations of two equations in the list of equations.
2. Multiply each term of an equation by a nonzero quantity.
3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

©2004—2013 Robert A. Beezer, GFDL License

**Theorem EOPSS Equation Operations Preserve Solution Sets****6**

If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.

©2004—2013 Robert A. Beezer, GFDL License

**Definition M Matrix****7**

An  $m \times n$  matrix is a rectangular layout of numbers from  $\mathbb{C}$  having  $m$  rows and  $n$  columns. We will use upper-case Latin letters from the start of the alphabet ( $A, B, C, \dots$ ) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix  $A$ , the notation  $[A]_{ij}$  will refer to the complex number in row  $i$  and column  $j$  of  $A$ .

©2004—2013 Robert A. Beezer, GFDL License

**Definition CV Column Vector****8**

A column vector of size  $m$  is an ordered list of  $m$  numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a vector. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as  $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}$ . Some books like to write vectors with arrows, such as  $\vec{u}$ . Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in  $\tilde{u}$ . To refer to the entry or component of vector  $\mathbf{v}$  in location  $i$  of the list, we write  $[\mathbf{v}]_i$ .

©2004—2013 Robert A. Beezer, GFDL License

**Definition ZCV Zero Column Vector****9**

The zero vector of size  $m$  is the column vector of size  $m$  where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or defined much more compactly,  $[\mathbf{0}]_i = 0$  for  $1 \leq i \leq m$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition CM Coefficient Matrix****10**

For a system of linear equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

the coefficient matrix is the  $m \times n$  matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition VOC Vector of Constants**

11

For a system of linear equations,

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

the vector of constants is the column vector of size  $m$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition SOLV Solution Vector**

12

For a system of linear equations,

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

the solution vector is the column vector of size  $n$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

©2004–2013 Robert A. Beezer, GFDL License



**Definition MRLS Matrix Representation of a Linear System****13**

If  $A$  is the coefficient matrix of a system of linear equations and  $\mathbf{b}$  is the vector of constants, then we will write  $\mathcal{LS}(A, \mathbf{b})$  as a shorthand expression for the system of linear equations, which we will refer to as the matrix representation of the linear system.

©2004–2013 Robert A. Beezer, GFDL License

**Definition AM Augmented Matrix****14**

Suppose we have a system of  $m$  equations in  $n$  variables, with coefficient matrix  $A$  and vector of constants  $\mathbf{b}$ . Then the augmented matrix of the system of equations is the  $m \times (n + 1)$  matrix whose first  $n$  columns are the columns of  $A$  and whose last column ( $n + 1$ ) is the column vector  $\mathbf{b}$ . This matrix will be written as  $[A \mid \mathbf{b}]$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition RO Row Operations****15**

The following three operations will transform an  $m \times n$  matrix into a different matrix of the same size, and each is known as a row operation.

1. Swap the locations of two rows.
2. Multiply each entry of a single row by a nonzero quantity.
3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

1.  $R_i \leftrightarrow R_j$ : Swap the location of rows  $i$  and  $j$ .
2.  $\alpha R_i$ : Multiply row  $i$  by the nonzero scalar  $\alpha$ .
3.  $\alpha R_i + R_j$ : Multiply row  $i$  by the scalar  $\alpha$  and add to row  $j$ .

©2004—2013 Robert A. Beezer, GFDL License

**Definition REM Row-Equivalent Matrices****16**

Two matrices,  $A$  and  $B$ , are row-equivalent if one can be obtained from the other by a sequence of row operations.

©2004—2013 Robert A. Beezer, GFDL License

**Theorem REMES Row-Equivalent Matrices represent Equivalent Systems 17**

Suppose that  $A$  and  $B$  are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

©2004–2013 Robert A. Beezer, GFDL License

**Definition RREF Reduced Row-Echelon Form 18**

A matrix is in reduced row-echelon form if it meets all of the following conditions:

1. If there is a row where every entry is zero, then this row lies below any other row that contains a nonzero entry.
2. The leftmost nonzero entry of a row is equal to 1.
3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
4. Consider any two different leftmost nonzero entries, one located in row  $i$ , column  $j$  and the other located in row  $s$ , column  $t$ . If  $s > i$ , then  $t > j$ .

A row of only zero entries is called a zero row and the leftmost nonzero entry of a nonzero row is a leading 1. A column containing a leading 1 will be called a pivot column. The number of nonzero rows will be denoted by  $r$ , which is also equal to the number of leading 1's and the number of pivot columns.

The set of column indices for the pivot columns will be denoted by  $D = \{d_1, d_2, d_3, \dots, d_r\}$  where  $d_1 < d_2 < d_3 < \dots < d_r$ , while the columns that are not pivot columns will be denoted as  $F = \{f_1, f_2, f_3, \dots, f_{n-r}\}$  where  $f_1 < f_2 < f_3 < \dots < f_{n-r}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem REMEF Row-Equivalent Matrix in Echelon Form****19**

Suppose  $A$  is a matrix. Then there is a matrix  $B$  so that

1.  $A$  and  $B$  are row-equivalent.
2.  $B$  is in reduced row-echelon form.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem RREFU Reduced Row-Echelon Form is Unique****20**

Suppose that  $A$  is an  $m \times n$  matrix and that  $B$  and  $C$  are  $m \times n$  matrices that are row-equivalent to  $A$  and in reduced row-echelon form. Then  $B = C$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition CS Consistent System****21**

A system of linear equations is consistent if it has at least one solution. Otherwise, the system is called inconsistent.

©2004–2013 Robert A. Beezer, GFDL License

**Definition IDV Independent and Dependent Variables****22**

Suppose  $A$  is the augmented matrix of a consistent system of linear equations and  $B$  is a row-equivalent matrix in reduced row-echelon form. Suppose  $j$  is the index of a pivot column of  $B$ . Then the variable  $x_j$  is dependent. A variable that is not dependent is called independent or free.

©2004–2013 Robert A. Beezer, GFDL License

Suppose  $A$  is the augmented matrix of a system of linear equations with  $n$  variables. Suppose also that  $B$  is a row-equivalent matrix in reduced row-echelon form with  $r$  nonzero rows. Then the system of equations is inconsistent if and only if column  $n + 1$  of  $B$  is a pivot column.

Suppose  $A$  is the augmented matrix of a consistent system of linear equations with  $n$  variables. Suppose also that  $B$  is a row-equivalent matrix in reduced row-echelon form with  $r$  pivot columns. Then  $r \leq n$ . If  $r = n$ , then the system has a unique solution, and if  $r < n$ , then the system has infinitely many solutions.

**Theorem FVCS Free Variables for Consistent Systems****25**

Suppose  $A$  is the augmented matrix of a consistent system of linear equations with  $n$  variables. Suppose also that  $B$  is a row-equivalent matrix in reduced row-echelon form with  $r$  rows that are not completely zeros. Then the solution set can be described with  $n - r$  free variables.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem PSSLS Possible Solution Sets for Linear Systems****26**

A system of linear equations has no solutions, a unique solution or infinitely many solutions.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CMVEI Consistent, More Variables than Equations, Infinite solutions**<sup>27</sup>

Suppose a consistent system of linear equations has  $m$  equations in  $n$  variables. If  $n > m$ , then the system has infinitely many solutions.

©2004–2013 Robert A. Beezer, GFDL License

**Definition HS Homogeneous System**

**28**

A system of linear equations,  $\mathcal{LS}(A, \mathbf{b})$  is homogeneous if the vector of constants is the zero vector, in other words, if  $\mathbf{b} = \mathbf{0}$ .

©2004–2013 Robert A. Beezer, GFDL License



**Theorem HSC Homogeneous Systems are Consistent****29**

Suppose that a system of linear equations is homogeneous. Then the system is consistent and one solution is found by setting each variable to zero.

©2004–2013 Robert A. Beezer, GFDL License

**Definition TSHSE Trivial Solution to Homogeneous Systems of Equations****30**

Suppose a homogeneous system of linear equations has  $n$  variables. The solution  $x_1 = 0, x_2 = 0, \dots, x_n = 0$  (i.e.  $\mathbf{x} = \mathbf{0}$ ) is called the trivial solution.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem HMVEI Homogeneous, More Variables than Equations, Infinite solutions**  
**31**

Suppose that a homogeneous system of linear equations has  $m$  equations and  $n$  variables with  $n > m$ . Then the system has infinitely many solutions.

©2004–2013 Robert A. Beezer, GFDL License

**Definition NSM Null Space of a Matrix**

**32**

The null space of a matrix  $A$ , denoted  $\mathcal{N}(A)$ , is the set of all the vectors that are solutions to the homogeneous system  $\mathcal{LS}(A, \mathbf{0})$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition SQM Square Matrix****33**

A matrix with  $m$  rows and  $n$  columns is square if  $m = n$ . In this case, we say the matrix has size  $n$ . To emphasize the situation when a matrix is not square, we will call it rectangular.

©2004—2013 Robert A. Beezer, GFDL License

**Definition NM Nonsingular Matrix****34**

Suppose  $A$  is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations  $\mathcal{LS}(A, \mathbf{0})$  is  $\{\mathbf{0}\}$ , in other words, the system has only the trivial solution. Then we say that  $A$  is a nonsingular matrix. Otherwise we say  $A$  is a singular matrix.

©2004—2013 Robert A. Beezer, GFDL License

**Definition IM Identity Matrix****35**

The  $m \times m$  identity matrix,  $I_m$ , is defined by

$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad 1 \leq i, j \leq m$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem NMRRI Nonsingular Matrices Row Reduce to the Identity matrix 36**

Suppose that  $A$  is a square matrix and  $B$  is a row-equivalent matrix in reduced row-echelon form. Then  $A$  is nonsingular if and only if  $B$  is the identity matrix.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem NMTNS Nonsingular Matrices have Trivial Null Spaces****37**

Suppose that  $A$  is a square matrix. Then  $A$  is nonsingular if and only if the null space of  $A$  is the set containing only the zero vector, i.e.  $\mathcal{N}(A) = \{\mathbf{0}\}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem NMUS Nonsingular Matrices and Unique Solutions****38**

Suppose that  $A$  is a square matrix.  $A$  is a nonsingular matrix if and only if the system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every choice of the constant vector  $\mathbf{b}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem NME1 Nonsingular Matrix Equivalences, Round 1****39**

Suppose that  $A$  is a square matrix. The following are equivalent.

1.  $A$  is nonsingular.
2.  $A$  row-reduces to the identity matrix.
3. The null space of  $A$  contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}$ .
4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition VSCV Vector Space of Column Vectors****40**

The vector space  $\mathbb{C}^m$  is the set of all column vectors (Definition CV) of size  $m$  with entries from the set of complex numbers,  $\mathbb{C}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition CVE Column Vector Equality**

41

Suppose that  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ . Then  $\mathbf{u}$  and  $\mathbf{v}$  are equal, written  $\mathbf{u} = \mathbf{v}$  if

$$[\mathbf{u}]_i = [\mathbf{v}]_i \qquad 1 \leq i \leq m$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition CVA Column Vector Addition**

42

Suppose that  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ . The sum of  $\mathbf{u}$  and  $\mathbf{v}$  is the vector  $\mathbf{u} + \mathbf{v}$  defined by

$$[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i \qquad 1 \leq i \leq m$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition CVSM Column Vector Scalar Multiplication**

43

Suppose  $\mathbf{u} \in \mathbb{C}^m$  and  $\alpha \in \mathbb{C}$ , then the scalar multiple of  $\mathbf{u}$  by  $\alpha$  is the vector  $\alpha\mathbf{u}$  defined by

$$[\alpha\mathbf{u}]_i = \alpha [\mathbf{u}]_i \quad 1 \leq i \leq m$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem VSPCV Vector Space Properties of Column Vectors**

44

Suppose that  $\mathbb{C}^m$  is the set of column vectors of size  $m$  (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- ACC Additive Closure, Column Vectors: If  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ , then  $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$ .
- SCC Scalar Closure, Column Vectors: If  $\alpha \in \mathbb{C}$  and  $\mathbf{u} \in \mathbb{C}^m$ , then  $\alpha\mathbf{u} \in \mathbb{C}^m$ .
- CC Commutativity, Column Vectors: If  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ , then  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .
- AAC Additive Associativity, Column Vectors: If  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$ , then  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ .
- ZC Zero Vector, Column Vectors: There is a vector,  $\mathbf{0}$ , called the zero vector, such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$  for all  $\mathbf{u} \in \mathbb{C}^m$ .
- AIC Additive Inverses, Column Vectors: If  $\mathbf{u} \in \mathbb{C}^m$ , then there exists a vector  $-\mathbf{u} \in \mathbb{C}^m$  so that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ .
- SMAC Scalar Multiplication Associativity, Column Vectors: If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in \mathbb{C}^m$ , then  $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$ .
- DVAC Distributivity across Vector Addition, Column Vectors: If  $\alpha \in \mathbb{C}$  and  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ , then  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ .
- DSAC Distributivity across Scalar Addition, Column Vectors: If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in \mathbb{C}^m$ , then  $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ .
- OC One, Column Vectors: If  $\mathbf{u} \in \mathbb{C}^m$ , then  $1\mathbf{u} = \mathbf{u}$ .

©2004–2013 Robert A. Beezer, GFDL License



**Definition LCCV Linear Combination of Column Vectors**

45

Given  $n$  vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n$  from  $\mathbb{C}^m$  and  $n$  scalars  $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ , their linear combination is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem SLSLC Solutions to Linear Systems are Linear Combinations**

46

Denote the columns of the  $m \times n$  matrix  $A$  as the vectors  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n$ . Then  $\mathbf{x} \in \mathbb{C}^n$  is a solution to the linear system of equations  $\mathcal{LS}(A, \mathbf{b})$  if and only if  $\mathbf{b}$  equals the linear combination of the columns of  $A$  formed with the entries of  $\mathbf{x}$ ,

$$[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \cdots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem VFSL S Vector Form of Solutions to Linear Systems**

47

Suppose that  $[A | \mathbf{b}]$  is the augmented matrix for a consistent linear system  $\mathcal{LS}(A, \mathbf{b})$  of  $m$  equations in  $n$  variables. Let  $B$  be a row-equivalent  $m \times (n + 1)$  matrix in reduced row-echelon form. Suppose that  $B$  has  $r$  pivot columns, with indices  $D = \{d_1, d_2, d_3, \dots, d_r\}$ , while the  $n - r$  non-pivot columns have indices in  $F = \{f_1, f_2, f_3, \dots, f_{n-r}, n + 1\}$ . Define vectors  $\mathbf{c}, \mathbf{u}_j$ ,  $1 \leq j \leq n - r$  of size  $n$  by

$$[\mathbf{c}]_i = \begin{cases} 0 & \text{if } i \in F \\ [B]_{k,n+1} & \text{if } i \in D, i = d_k \end{cases}$$
$$[\mathbf{u}_j]_i = \begin{cases} 1 & \text{if } i \in F, i = f_j \\ 0 & \text{if } i \in F, i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D, i = d_k \end{cases}.$$

Then the set of solutions to the system of equations  $\mathcal{LS}(A, \mathbf{b})$  is

$$S = \{ \mathbf{c} + \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_{n-r} \mathbf{u}_{n-r} \mid \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-r} \in \mathbb{C} \}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem PSPHS Particular Solution Plus Homogeneous Solutions**

48

Suppose that  $\mathbf{w}$  is one solution to the linear system of equations  $\mathcal{LS}(A, \mathbf{b})$ . Then  $\mathbf{y}$  is a solution to  $\mathcal{LS}(A, \mathbf{b})$  if and only if  $\mathbf{y} = \mathbf{w} + \mathbf{z}$  for some vector  $\mathbf{z} \in \mathcal{N}(A)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition SSCV Span of a Set of Column Vectors**

49

Given a set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$ , their span,  $\langle S \rangle$ , is the set of all possible linear combinations of  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p$ . Symbolically,

$$\begin{aligned}\langle S \rangle &= \{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_p \mathbf{u}_p \mid \alpha_i \in \mathbb{C}, 1 \leq i \leq p \} \\ &= \left\{ \sum_{i=1}^p \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, 1 \leq i \leq p \right\}\end{aligned}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem SSNS Spanning Sets for Null Spaces**

50

Suppose that  $A$  is an  $m \times n$  matrix, and  $B$  is a row-equivalent matrix in reduced row-echelon form. Suppose that  $B$  has  $r$  pivot columns, with indices given by  $D = \{d_1, d_2, d_3, \dots, d_r\}$ , while the  $n - r$  non-pivot columns have indices  $F = \{f_1, f_2, f_3, \dots, f_{n-r}, n + 1\}$ . Construct the  $n - r$  vectors  $\mathbf{z}_j$ ,  $1 \leq j \leq n - r$  of size  $n$ ,

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F, i = f_j \\ 0 & \text{if } i \in F, i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D, i = d_k \end{cases}$$

Then the null space of  $A$  is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_{n-r}\} \rangle$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition RLDCV Relation of Linear Dependence for Column Vectors****51**

Given a set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ , a true statement of the form

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

is a relation of linear dependence on  $S$ . If this statement is formed in a trivial fashion, i.e.  $\alpha_i = 0$ ,  $1 \leq i \leq n$ , then we say it is the trivial relation of linear dependence on  $S$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition LICV Linear Independence of Column Vectors****52**

The set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  is linearly independent if there is a relation of linear dependence on  $S$  that is not trivial. In the case where the only relation of linear dependence on  $S$  is the trivial one, then  $S$  is a linearly independent set of vectors.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem LIVHS Linearly Independent Vectors and Homogeneous Systems 53**

Suppose that  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\} \subseteq \mathbb{C}^m$  is a set of vectors and  $A$  is the  $m \times n$  matrix whose columns are the vectors in  $S$ . Then  $S$  is a linearly independent set if and only if the homogeneous system  $\mathcal{LS}(A, \mathbf{0})$  has a unique solution.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem LIVRN Linearly Independent Vectors,  $r$  and  $n$  54**

Suppose that  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\} \subseteq \mathbb{C}^m$  is a set of vectors and  $A$  is the  $m \times n$  matrix whose columns are the vectors in  $S$ . Let  $B$  be a matrix in reduced row-echelon form that is row-equivalent to  $A$  and let  $r$  denote the number of pivot columns in  $B$ . Then  $S$  is linearly independent if and only if  $n = r$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MVSLD More Vectors than Size implies Linear Dependence 55**

Suppose that  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\} \subseteq \mathbb{C}^m$  and  $n > m$ . Then  $S$  is a linearly dependent set.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem NMLIC Nonsingular Matrices have Linearly Independent Columns 56**

Suppose that  $A$  is a square matrix. Then  $A$  is nonsingular if and only if the columns of  $A$  form a linearly independent set.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem NME2 Nonsingular Matrix Equivalences, Round 2**

57

Suppose that  $A$  is a square matrix. The following are equivalent.

1.  $A$  is nonsingular.
2.  $A$  row-reduces to the identity matrix.
3. The null space of  $A$  contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}$ .
4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
5. The columns of  $A$  form a linearly independent set.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem BNS Basis for Null Spaces**

58

Suppose that  $A$  is an  $m \times n$  matrix, and  $B$  is a row-equivalent matrix in reduced row-echelon form with  $r$  pivot columns. Let  $D = \{d_1, d_2, d_3, \dots, d_r\}$  and  $F = \{f_1, f_2, f_3, \dots, f_{n-r}\}$  be the sets of column indices where  $B$  does and does not (respectively) have pivot columns. Construct the  $n - r$  vectors  $\mathbf{z}_j$ ,  $1 \leq j \leq n - r$  of size  $n$  as

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F, i = f_j \\ 0 & \text{if } i \in F, i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D, i = d_k \end{cases}$$

Define the set  $S = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_{n-r}\}$ . Then

1.  $\mathcal{N}(A) = \langle S \rangle$ .
2.  $S$  is a linearly independent set.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DLDS Dependency in Linearly Dependent Sets****59**

Suppose that  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  is a set of vectors. Then  $S$  is a linearly dependent set if and only if there is an index  $t$ ,  $1 \leq t \leq n$  such that  $\mathbf{u}_t$  is a linear combination of the vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \dots, \mathbf{u}_n$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem BS Basis of a Span****60**

Suppose that  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$  is a set of column vectors. Define  $W = \langle S \rangle$  and let  $A$  be the matrix whose columns are the vectors from  $S$ . Let  $B$  be the reduced row-echelon form of  $A$ , with  $D = \{d_1, d_2, d_3, \dots, d_r\}$  the set of indices for the pivot columns of  $B$ . Then

1.  $T = \{\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \dots, \mathbf{v}_{d_r}\}$  is a linearly independent set.
2.  $W = \langle T \rangle$ .

©2004–2013 Robert A. Beezer, GFDL License



**Definition CCCV Complex Conjugate of a Column Vector****61**

Suppose that  $\mathbf{u}$  is a vector from  $\mathbb{C}^m$ . Then the conjugate of the vector,  $\bar{\mathbf{u}}$ , is defined by

$$[\bar{\mathbf{u}}]_i = \overline{[\mathbf{u}]_i} \quad 1 \leq i \leq m$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CRVA Conjugation Respects Vector Addition****62**

Suppose  $\mathbf{x}$  and  $\mathbf{y}$  are two vectors from  $\mathbb{C}^m$ . Then

$$\overline{\mathbf{x} + \mathbf{y}} = \bar{\mathbf{x}} + \bar{\mathbf{y}}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CRSM Conjugation Respects Vector Scalar Multiplication****63**

Suppose  $\mathbf{x}$  is a vector from  $\mathbb{C}^m$ , and  $\alpha \in \mathbb{C}$  is a scalar. Then

$$\overline{\alpha \mathbf{x}} = \bar{\alpha} \bar{\mathbf{x}}$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition IP Inner Product****64**

Given the vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$  the inner product of  $\mathbf{u}$  and  $\mathbf{v}$  is the scalar quantity in  $\mathbb{C}$ ,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \overline{[\mathbf{u}]_1} [\mathbf{v}]_1 + \overline{[\mathbf{u}]_2} [\mathbf{v}]_2 + \overline{[\mathbf{u}]_3} [\mathbf{v}]_3 + \cdots + \overline{[\mathbf{u}]_m} [\mathbf{v}]_m = \sum_{i=1}^m \overline{[\mathbf{u}]_i} [\mathbf{v}]_i$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem IPVA Inner Product and Vector Addition****65**Suppose  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$ . Then

1.  $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
2.  $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem IPSM Inner Product and Scalar Multiplication****66**Suppose  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$  and  $\alpha \in \mathbb{C}$ . Then

1.  $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \bar{\alpha} \langle \mathbf{u}, \mathbf{v} \rangle$
2.  $\langle \mathbf{u}, \alpha \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem IPAC Inner Product is Anti-Commutative**

67

Suppose that  $\mathbf{u}$  and  $\mathbf{v}$  are vectors in  $\mathbb{C}^m$ . Then  $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition NV Norm of a Vector**

68

The norm of the vector  $\mathbf{u}$  is the scalar quantity in  $\mathbb{C}$

$$\|\mathbf{u}\| = \sqrt{|\mathbf{u}_1|^2 + |\mathbf{u}_2|^2 + |\mathbf{u}_3|^2 + \cdots + |\mathbf{u}_m|^2} = \sqrt{\sum_{i=1}^m |\mathbf{u}_i|^2}$$

©2004–2013 Robert A. Beezer, GFDL License

Suppose that  $\mathbf{u}$  is a vector in  $\mathbb{C}^m$ . Then  $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$ .

Suppose that  $\mathbf{u}$  is a vector in  $\mathbb{C}^m$ . Then  $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$  with equality if and only if  $\mathbf{u} = \mathbf{0}$ .

**Definition OV Orthogonal Vectors****71**

A pair of vectors,  $\mathbf{u}$  and  $\mathbf{v}$ , from  $\mathbb{C}^m$  are orthogonal if their inner product is zero, that is,  $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition OSV Orthogonal Set of Vectors****72**

Suppose that  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  is a set of vectors from  $\mathbb{C}^m$ . Then  $S$  is an orthogonal set if every pair of different vectors from  $S$  is orthogonal, that is  $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$  whenever  $i \neq j$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition SUV Standard Unit Vectors**

73

Let  $\mathbf{e}_j \in \mathbb{C}^m$ ,  $1 \leq j \leq m$  denote the column vectors defined by

$$[\mathbf{e}_j]_i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Then the set

$$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_m\} = \{\mathbf{e}_j \mid 1 \leq j \leq m\}$$

is the set of standard unit vectors in  $\mathbb{C}^m$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem OSLI Orthogonal Sets are Linearly Independent**

74

Suppose that  $S$  is an orthogonal set of nonzero vectors. Then  $S$  is linearly independent.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem GSP Gram-Schmidt Procedure****75**

Suppose that  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$  is a linearly independent set of vectors in  $\mathbb{C}^m$ . Define the vectors  $\mathbf{u}_i$ ,  $1 \leq i \leq p$  by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{u}_1, \mathbf{v}_i \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{u}_2, \mathbf{v}_i \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 - \frac{\langle \mathbf{u}_3, \mathbf{v}_i \rangle}{\langle \mathbf{u}_3, \mathbf{u}_3 \rangle} \mathbf{u}_3 - \dots - \frac{\langle \mathbf{u}_{i-1}, \mathbf{v}_i \rangle}{\langle \mathbf{u}_{i-1}, \mathbf{u}_{i-1} \rangle} \mathbf{u}_{i-1}$$

Let  $T = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$ . Then  $T$  is an orthogonal set of nonzero vectors, and  $\langle T \rangle = \langle S \rangle$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition ONS OrthoNormal Set****76**

Suppose  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  is an orthogonal set of vectors such that  $\|\mathbf{u}_i\| = 1$  for all  $1 \leq i \leq n$ . Then  $S$  is an orthonormal set of vectors.

©2004–2013 Robert A. Beezer, GFDL License



**Definition VSM Vector Space of  $m \times n$  Matrices****77**

The vector space  $M_{mn}$  is the set of all  $m \times n$  matrices with entries from the set of complex numbers.

©2004–2013 Robert A. Beezer, GFDL License

**Definition ME Matrix Equality****78**

The  $m \times n$  matrices  $A$  and  $B$  are equal, written  $A = B$  provided  $[A]_{ij} = [B]_{ij}$  for all  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition MA Matrix Addition****79**

Given the  $m \times n$  matrices  $A$  and  $B$ , define the sum of  $A$  and  $B$  as an  $m \times n$  matrix, written  $A + B$ , according to

$$[A + B]_{ij} = [A]_{ij} + [B]_{ij} \qquad 1 \leq i \leq m, 1 \leq j \leq n$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition MSM Matrix Scalar Multiplication****80**

Given the  $m \times n$  matrix  $A$  and the scalar  $\alpha \in \mathbb{C}$ , the scalar multiple of  $A$  is an  $m \times n$  matrix, written  $\alpha A$  and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij} \qquad 1 \leq i \leq m, 1 \leq j \leq n$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem VSPM Vector Space Properties of Matrices****81**

Suppose that  $M_{mn}$  is the set of all  $m \times n$  matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- ACM Additive Closure, Matrices: If  $A, B \in M_{mn}$ , then  $A + B \in M_{mn}$ .
- SCM Scalar Closure, Matrices: If  $\alpha \in \mathbb{C}$  and  $A \in M_{mn}$ , then  $\alpha A \in M_{mn}$ .
- CM Commutativity, Matrices: If  $A, B \in M_{mn}$ , then  $A + B = B + A$ .
- AAM Additive Associativity, Matrices: If  $A, B, C \in M_{mn}$ , then  $A + (B + C) = (A + B) + C$ .
- ZM Zero Matrix, Matrices: There is a matrix,  $\mathcal{O}$ , called the zero matrix, such that  $A + \mathcal{O} = A$  for all  $A \in M_{mn}$ .
- AIM Additive Inverses, Matrices: If  $A \in M_{mn}$ , then there exists a matrix  $-A \in M_{mn}$  so that  $A + (-A) = \mathcal{O}$ .
- SMAM Scalar Multiplication Associativity, Matrices: If  $\alpha, \beta \in \mathbb{C}$  and  $A \in M_{mn}$ , then  $\alpha(\beta A) = (\alpha\beta)A$ .
- DMAM Distributivity across Matrix Addition, Matrices: If  $\alpha \in \mathbb{C}$  and  $A, B \in M_{mn}$ , then  $\alpha(A + B) = \alpha A + \alpha B$ .
- DSAM Distributivity across Scalar Addition, Matrices: If  $\alpha, \beta \in \mathbb{C}$  and  $A \in M_{mn}$ , then  $(\alpha + \beta)A = \alpha A + \beta A$ .
- OM One, Matrices: If  $A \in M_{mn}$ , then  $1A = A$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition ZM Zero Matrix****82**

The  $m \times n$  zero matrix is written as  $\mathcal{O} = \mathcal{O}_{m \times n}$  and defined by  $[\mathcal{O}]_{ij} = 0$ , for all  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition TM Transpose of a Matrix****83**

Given an  $m \times n$  matrix  $A$ , its transpose is the  $n \times m$  matrix  $A^t$  given by

$$[A^t]_{ij} = [A]_{ji}, \quad 1 \leq i \leq n, 1 \leq j \leq m.$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition SYM Symmetric Matrix****84**

The matrix  $A$  is symmetric if  $A = A^t$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem SMS Symmetric Matrices are Square**

85

Suppose that  $A$  is a symmetric matrix. Then  $A$  is square.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem TMA Transpose and Matrix Addition**

86

Suppose that  $A$  and  $B$  are  $m \times n$  matrices. Then  $(A + B)^t = A^t + B^t$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem TMSM Transpose and Matrix Scalar Multiplication****87**

Suppose that  $\alpha \in \mathbb{C}$  and  $A$  is an  $m \times n$  matrix. Then  $(\alpha A)^t = \alpha A^t$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem TT Transpose of a Transpose****88**

Suppose that  $A$  is an  $m \times n$  matrix. Then  $(A^t)^t = A$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition CCM Complex Conjugate of a Matrix****89**

Suppose  $A$  is an  $m \times n$  matrix. Then the conjugate of  $A$ , written  $\overline{A}$  is an  $m \times n$  matrix defined by

$$[\overline{A}]_{ij} = \overline{[A]_{ij}}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CRMA Conjugation Respects Matrix Addition****90**

Suppose that  $A$  and  $B$  are  $m \times n$  matrices. Then  $\overline{A + B} = \overline{A} + \overline{B}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CRMSM Conjugation Respects Matrix Scalar Multiplication****91**

Suppose that  $\alpha \in \mathbb{C}$  and  $A$  is an  $m \times n$  matrix. Then  $\overline{\alpha A} = \overline{\alpha} \overline{A}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CCM Conjugate of the Conjugate of a Matrix****92**

Suppose that  $A$  is an  $m \times n$  matrix. Then  $\overline{\overline{A}} = A$ .

©2004–2013 Robert A. Beezer, GFDL License



**Theorem MCT Matrix Conjugation and Transposes****93**

Suppose that  $A$  is an  $m \times n$  matrix. Then  $\overline{(A^t)} = (\overline{A})^t$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition A Adjoint****94**

If  $A$  is a matrix, then its adjoint is  $A^* = (\overline{A})^t$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem AMA Adjoint and Matrix Addition****95**

Suppose  $A$  and  $B$  are matrices of the same size. Then  $(A + B)^* = A^* + B^*$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem AMSM Adjoint and Matrix Scalar Multiplication****96**

Suppose  $\alpha \in \mathbb{C}$  is a scalar and  $A$  is a matrix. Then  $(\alpha A)^* = \bar{\alpha}A^*$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem AA Adjoint of an Adjoint**

97

Suppose that  $A$  is a matrix. Then  $(A^*)^* = A$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition MVP Matrix-Vector Product**

98

Suppose  $A$  is an  $m \times n$  matrix with columns  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n$  and  $\mathbf{u}$  is a vector of size  $n$ . Then the matrix-vector product of  $A$  with  $\mathbf{u}$  is the linear combination

$$A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \cdots + [\mathbf{u}]_n \mathbf{A}_n$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem SLEMM Systems of Linear Equations as Matrix Multiplication 99**

The set of solutions to the linear system  $\mathcal{LS}(A, \mathbf{b})$  equals the set of solutions for  $\mathbf{x}$  in the vector equation  $A\mathbf{x} = \mathbf{b}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem EMMVP Equal Matrices and Matrix-Vector Products 100**

Suppose that  $A$  and  $B$  are  $m \times n$  matrices such that  $A\mathbf{x} = B\mathbf{x}$  for every  $\mathbf{x} \in \mathbb{C}^n$ . Then  $A = B$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition MM Matrix Multiplication****101**

Suppose  $A$  is an  $m \times n$  matrix and  $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \dots, \mathbf{B}_p$  are the columns of an  $n \times p$  matrix  $B$ . Then the matrix product of  $A$  with  $B$  is the  $m \times p$  matrix where column  $i$  is the matrix-vector product  $A\mathbf{B}_i$ . Symbolically,

$$AB = A[\mathbf{B}_1|\mathbf{B}_2|\mathbf{B}_3|\dots|\mathbf{B}_p] = [A\mathbf{B}_1|A\mathbf{B}_2|A\mathbf{B}_3|\dots|A\mathbf{B}_p].$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem EMP Entries of Matrix Products****102**

Suppose  $A$  is an  $m \times n$  matrix and  $B$  is an  $n \times p$  matrix. Then for  $1 \leq i \leq m$ ,  $1 \leq j \leq p$ , the individual entries of  $AB$  are given by

$$\begin{aligned} [AB]_{ij} &= [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \dots + [A]_{in} [B]_{nj} \\ &= \sum_{k=1}^n [A]_{ik} [B]_{kj} \end{aligned}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MMZM Matrix Multiplication and the Zero Matrix****103**

Suppose  $A$  is an  $m \times n$  matrix. Then

1.  $A\mathcal{O}_{n \times p} = \mathcal{O}_{m \times p}$
2.  $\mathcal{O}_{p \times m}A = \mathcal{O}_{p \times n}$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MMIM Matrix Multiplication and Identity Matrix****104**

Suppose  $A$  is an  $m \times n$  matrix. Then

1.  $AI_n = A$
2.  $I_mA = A$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MMDAA Matrix Multiplication Distributes Across Addition 105**

Suppose  $A$  is an  $m \times n$  matrix and  $B$  and  $C$  are  $n \times p$  matrices and  $D$  is a  $p \times s$  matrix. Then

1.  $A(B + C) = AB + AC$
2.  $(B + C)D = BD + CD$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication 106**

Suppose  $A$  is an  $m \times n$  matrix and  $B$  is an  $n \times p$  matrix. Let  $\alpha$  be a scalar. Then  $\alpha(AB) = (\alpha A)B = A(\alpha B)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MMA Matrix Multiplication is Associative****107**

Suppose  $A$  is an  $m \times n$  matrix,  $B$  is an  $n \times p$  matrix and  $D$  is a  $p \times s$  matrix. Then  $A(BD) = (AB)D$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MMIP Matrix Multiplication and Inner Products****108**

If we consider the vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$  as  $m \times 1$  matrices then

$$\langle \mathbf{u}, \mathbf{v} \rangle = \bar{\mathbf{u}}^t \mathbf{v} = \mathbf{u}^* \mathbf{v}$$

©2004–2013 Robert A. Beezer, GFDL License



**Theorem MMCC Matrix Multiplication and Complex Conjugation**

109

Suppose  $A$  is an  $m \times n$  matrix and  $B$  is an  $n \times p$  matrix. Then  $\overline{AB} = \overline{A} \overline{B}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MMT Matrix Multiplication and Transposes**

110

Suppose  $A$  is an  $m \times n$  matrix and  $B$  is an  $n \times p$  matrix. Then  $(AB)^t = B^t A^t$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MMAD Matrix Multiplication and Adjoints**

111

Suppose  $A$  is an  $m \times n$  matrix and  $B$  is an  $n \times p$  matrix. Then  $(AB)^* = B^*A^*$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem AIP Adjoint and Inner Product**

112

Suppose that  $A$  is an  $m \times n$  matrix and  $\mathbf{x} \in \mathbb{C}^n$ ,  $\mathbf{y} \in \mathbb{C}^m$ . Then  $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A^*\mathbf{y} \rangle$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition HM Hermitian Matrix****113**

The square matrix  $A$  is Hermitian (or self-adjoint) if  $A = A^*$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem HMIP Hermitian Matrices and Inner Products****114**

Suppose that  $A$  is a square matrix of size  $n$ . Then  $A$  is Hermitian if and only if  $\langle A\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, A\mathbf{y} \rangle$  for all  $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition MI Matrix Inverse****115**

Suppose  $A$  and  $B$  are square matrices of size  $n$  such that  $AB = I_n$  and  $BA = I_n$ . Then  $A$  is invertible and  $B$  is the inverse of  $A$ . In this situation, we write  $B = A^{-1}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem TTMI Two-by-Two Matrix Inverse****116**

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then  $A$  is invertible if and only if  $ad - bc \neq 0$ . When  $A$  is invertible, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CINM Computing the Inverse of a Nonsingular Matrix****117**

Suppose  $A$  is a nonsingular square matrix of size  $n$ . Create the  $n \times 2n$  matrix  $M$  by placing the  $n \times n$  identity matrix  $I_n$  to the right of the matrix  $A$ . Let  $N$  be a matrix that is row-equivalent to  $M$  and in reduced row-echelon form. Finally, let  $J$  be the matrix formed from the final  $n$  columns of  $N$ . Then  $AJ = I_n$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MIU Matrix Inverse is Unique****118**

Suppose the square matrix  $A$  has an inverse. Then  $A^{-1}$  is unique.

©2004–2013 Robert A. Beezer, GFDL License

Suppose  $A$  and  $B$  are invertible matrices of size  $n$ . Then  $AB$  is an invertible matrix and  $(AB)^{-1} = B^{-1}A^{-1}$ .

Suppose  $A$  is an invertible matrix. Then  $A^{-1}$  is invertible and  $(A^{-1})^{-1} = A$ .

**Theorem MIT Matrix Inverse of a Transpose****121**

Suppose  $A$  is an invertible matrix. Then  $A^t$  is invertible and  $(A^t)^{-1} = (A^{-1})^t$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MISM Matrix Inverse of a Scalar Multiple****122**

Suppose  $A$  is an invertible matrix and  $\alpha$  is a nonzero scalar. Then  $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$  and  $\alpha A$  is invertible.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem NPNT Nonsingular Product has Nonsingular Terms****123**

Suppose that  $A$  and  $B$  are square matrices of size  $n$ . The product  $AB$  is nonsingular if and only if  $A$  and  $B$  are both nonsingular.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem OSIS One-Sided Inverse is Sufficient****124**

Suppose  $A$  and  $B$  are square matrices of size  $n$  such that  $AB = I_n$ . Then  $BA = I_n$ .

©2004–2013 Robert A. Beezer, GFDL License



Suppose that  $A$  is a square matrix. Then  $A$  is nonsingular if and only if  $A$  is invertible.

Suppose that  $A$  is a square matrix of size  $n$ . The following are equivalent.

1.  $A$  is nonsingular.
2.  $A$  row-reduces to the identity matrix.
3. The null space of  $A$  contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}$ .
4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
5. The columns of  $A$  are a linearly independent set.
6.  $A$  is invertible.

**Theorem SNCM Solution with Nonsingular Coefficient Matrix****127**

Suppose that  $A$  is nonsingular. Then the unique solution to  $\mathcal{LS}(A, \mathbf{b})$  is  $A^{-1}\mathbf{b}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition UM Unitary Matrices****128**

Suppose that  $U$  is a square matrix of size  $n$  such that  $U^*U = I_n$ . Then we say  $U$  is unitary.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem UMI Unitary Matrices are Invertible****129**

Suppose that  $U$  is a unitary matrix of size  $n$ . Then  $U$  is nonsingular, and  $U^{-1} = U^*$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CUMOS Columns of Unitary Matrices are Orthonormal Sets****130**

Suppose that  $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$  is the set of columns of a square matrix  $A$  of size  $n$ . Then  $A$  is a unitary matrix if and only if  $S$  is an orthonormal set.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem UMPIP Unitary Matrices Preserve Inner Products****131**

Suppose that  $U$  is a unitary matrix of size  $n$  and  $\mathbf{u}$  and  $\mathbf{v}$  are two vectors from  $\mathbb{C}^n$ . Then

$$\langle U\mathbf{u}, U\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle \quad \text{and} \quad \|U\mathbf{v}\| = \|\mathbf{v}\|$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition CSM Column Space of a Matrix****132**

Suppose that  $A$  is an  $m \times n$  matrix with columns  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n$ . Then the column space of  $A$ , written  $\mathcal{C}(A)$ , is the subset of  $\mathbb{C}^m$  containing all linear combinations of the columns of  $A$ ,

$$\mathcal{C}(A) = \langle \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\} \rangle$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CSCS Column Spaces and Consistent Systems****133**

Suppose  $A$  is an  $m \times n$  matrix and  $\mathbf{b}$  is a vector of size  $m$ . Then  $\mathbf{b} \in \mathcal{C}(A)$  if and only if  $\mathcal{LS}(A, \mathbf{b})$  is consistent.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem BCS Basis of the Column Space****134**

Suppose that  $A$  is an  $m \times n$  matrix with columns  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n$ , and  $B$  is a row-equivalent matrix in reduced row-echelon form with  $r$  nonzero rows. Let  $D = \{d_1, d_2, d_3, \dots, d_r\}$  be the set of indices for the pivot columns of  $B$ . Let  $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \dots, \mathbf{A}_{d_r}\}$ . Then

1.  $T$  is a linearly independent set.
2.  $\mathcal{C}(A) = \langle T \rangle$ .

©2004–2013 Robert A. Beezer, GFDL License

Suppose  $A$  is a square matrix of size  $n$ . Then  $A$  is nonsingular if and only if  $\mathcal{C}(A) = \mathbb{C}^n$ .

Suppose that  $A$  is a square matrix of size  $n$ . The following are equivalent.

1.  $A$  is nonsingular.
2.  $A$  row-reduces to the identity matrix.
3. The null space of  $A$  contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}$ .
4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
5. The columns of  $A$  are a linearly independent set.
6.  $A$  is invertible.
7. The column space of  $A$  is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .

**Definition RSM Row Space of a Matrix****137**

Suppose  $A$  is an  $m \times n$  matrix. Then the row space of  $A$ ,  $\mathcal{R}(A)$ , is the column space of  $A^t$ , i.e.  $\mathcal{R}(A) = \mathcal{C}(A^t)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem REMRS Row-Equivalent Matrices have equal Row Spaces****138**

Suppose  $A$  and  $B$  are row-equivalent matrices. Then  $\mathcal{R}(A) = \mathcal{R}(B)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem BRS Basis for the Row Space****139**

Suppose that  $A$  is a matrix and  $B$  is a row-equivalent matrix in reduced row-echelon form. Let  $S$  be the set of nonzero columns of  $B^t$ . Then

1.  $\mathcal{R}(A) = \langle S \rangle$ .
2.  $S$  is a linearly independent set.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CSRST Column Space, Row Space, Transpose****140**

Suppose  $A$  is a matrix. Then  $\mathcal{C}(A) = \mathcal{R}(A^t)$ .

©2004–2013 Robert A. Beezer, GFDL License



**Definition LNS Left Null Space**

141

Suppose  $A$  is an  $m \times n$  matrix. Then the left null space is defined as  $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition EEF Extended Echelon Form**

142

Suppose  $A$  is an  $m \times n$  matrix. Extend  $A$  on its right side with the addition of an  $m \times m$  identity matrix to form an  $m \times (n + m)$  matrix  $M$ . Use row operations to bring  $M$  to reduced row-echelon form and call the result  $N$ .  $N$  is the extended reduced row-echelon form of  $A$ , and we will standardize on names for five submatrices ( $B, C, J, K, L$ ) of  $N$ .

Let  $B$  denote the  $m \times n$  matrix formed from the first  $n$  columns of  $N$  and let  $J$  denote the  $m \times m$  matrix formed from the last  $m$  columns of  $N$ . Suppose that  $B$  has  $r$  nonzero rows. Further partition  $N$  by letting  $C$  denote the  $r \times n$  matrix formed from all of the nonzero rows of  $B$ . Let  $K$  be the  $r \times m$  matrix formed from the first  $r$  rows of  $J$ , while  $L$  will be the  $(m - r) \times m$  matrix formed from the bottom  $m - r$  rows of  $J$ . Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \left[ \begin{array}{c|c} C & K \\ \hline 0 & L \end{array} \right]$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem PEEF Properties of Extended Echelon Form**

143

Suppose that  $A$  is an  $m \times n$  matrix and that  $N$  is its extended echelon form. Then

1.  $J$  is nonsingular.
2.  $B = JA$ .
3. If  $\mathbf{x} \in \mathbb{C}^n$  and  $\mathbf{y} \in \mathbb{C}^m$ , then  $A\mathbf{x} = \mathbf{y}$  if and only if  $B\mathbf{x} = J\mathbf{y}$ .
4.  $C$  is in reduced row-echelon form, has no zero rows and has  $r$  pivot columns.
5.  $L$  is in reduced row-echelon form, has no zero rows and has  $m - r$  pivot columns.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem FS Four Subsets**

144

Suppose  $A$  is an  $m \times n$  matrix with extended echelon form  $N$ . Suppose the reduced row-echelon form of  $A$  has  $r$  nonzero rows. Then  $C$  is the submatrix of  $N$  formed from the first  $r$  rows and the first  $n$  columns and  $L$  is the submatrix of  $N$  formed from the last  $m$  columns and the last  $m - r$  rows. Then

1. The null space of  $A$  is the null space of  $C$ ,  $\mathcal{N}(A) = \mathcal{N}(C)$ .
2. The row space of  $A$  is the row space of  $C$ ,  $\mathcal{R}(A) = \mathcal{R}(C)$ .
3. The column space of  $A$  is the null space of  $L$ ,  $\mathcal{C}(A) = \mathcal{N}(L)$ .
4. The left null space of  $A$  is the row space of  $L$ ,  $\mathcal{L}(A) = \mathcal{R}(L)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition VS Vector Space**

145

Suppose that  $V$  is a set upon which we have defined two operations: (1) vector addition, which combines two elements of  $V$  and is denoted by “+”, and (2) scalar multiplication, which combines a complex number with an element of  $V$  and is denoted by juxtaposition. Then  $V$ , along with the two operations, is a vector space over  $\mathbb{C}$  if the following ten properties hold.

- AC Additive Closure: If  $\mathbf{u}, \mathbf{v} \in V$ , then  $\mathbf{u} + \mathbf{v} \in V$ .
- SC Scalar Closure: If  $\alpha \in \mathbb{C}$  and  $\mathbf{u} \in V$ , then  $\alpha\mathbf{u} \in V$ .
- C Commutativity: If  $\mathbf{u}, \mathbf{v} \in V$ , then  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .
- AA Additive Associativity: If  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ , then  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ .
- Z Zero Vector: There is a vector,  $\mathbf{0}$ , called the zero vector, such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$  for all  $\mathbf{u} \in V$ .
- AI Additive Inverses: If  $\mathbf{u} \in V$ , then there exists a vector  $-\mathbf{u} \in V$  so that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ .
- SMA Scalar Multiplication Associativity: If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in V$ , then  $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$ .
- DVA Distributivity across Vector Addition: If  $\alpha \in \mathbb{C}$  and  $\mathbf{u}, \mathbf{v} \in V$ , then  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ .
- DSA Distributivity across Scalar Addition: If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in V$ , then  $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ .
- O One: If  $\mathbf{u} \in V$ , then  $1\mathbf{u} = \mathbf{u}$ .

The objects in  $V$  are called vectors, no matter what else they might really be, simply by virtue of being elements of a vector space.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ZVU Zero Vector is Unique**

146

Suppose that  $V$  is a vector space. The zero vector,  $\mathbf{0}$ , is unique.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem AIU Additive Inverses are Unique**

147

Suppose that  $V$  is a vector space. For each  $\mathbf{u} \in V$ , the additive inverse,  $-\mathbf{u}$ , is unique.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ZSSM Zero Scalar in Scalar Multiplication**

148

Suppose that  $V$  is a vector space and  $\mathbf{u} \in V$ . Then  $0\mathbf{u} = \mathbf{0}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ZVSM Zero Vector in Scalar Multiplication**

149

Suppose that  $V$  is a vector space and  $\alpha \in \mathbb{C}$ . Then  $\alpha\mathbf{0} = \mathbf{0}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem AISM Additive Inverses from Scalar Multiplication**

150

Suppose that  $V$  is a vector space and  $\mathbf{u} \in V$ . Then  $-\mathbf{u} = (-1)\mathbf{u}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem SMEZV Scalar Multiplication Equals the Zero Vector****151**

Suppose that  $V$  is a vector space and  $\alpha \in \mathbb{C}$ . If  $\alpha \mathbf{u} = \mathbf{0}$ , then either  $\alpha = 0$  or  $\mathbf{u} = \mathbf{0}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition S Subspace****152**

Suppose that  $V$  and  $W$  are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that  $W$  is a subset of  $V$ ,  $W \subseteq V$ . Then  $W$  is a subspace of  $V$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem TSS Testing Subsets for Subspaces****153**

Suppose that  $V$  is a vector space and  $W$  is a subset of  $V$ ,  $W \subseteq V$ . Endow  $W$  with the same operations as  $V$ . Then  $W$  is a subspace if and only if three conditions are met

1.  $W$  is non-empty,  $W \neq \emptyset$ .
2. If  $\mathbf{x} \in W$  and  $\mathbf{y} \in W$ , then  $\mathbf{x} + \mathbf{y} \in W$ .
3. If  $\alpha \in \mathbb{C}$  and  $\mathbf{x} \in W$ , then  $\alpha\mathbf{x} \in W$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition TS Trivial Subspaces****154**

Given the vector space  $V$ , the subspaces  $V$  and  $\{\mathbf{0}\}$  are each called a trivial subspace.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem NSMS Null Space of a Matrix is a Subspace****155**

Suppose that  $A$  is an  $m \times n$  matrix. Then the null space of  $A$ ,  $\mathcal{N}(A)$ , is a subspace of  $\mathbb{C}^n$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition LC Linear Combination****156**

Suppose that  $V$  is a vector space. Given  $n$  vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n$  and  $n$  scalars  $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ , their linear combination is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n.$$

©2004–2013 Robert A. Beezer, GFDL License



**Definition SS Span of a Set**

157

Suppose that  $V$  is a vector space. Given a set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$ , their span,  $\langle S \rangle$ , is the set of all possible linear combinations of  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$ . Symbolically,

$$\begin{aligned}\langle S \rangle &= \{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_t \mathbf{u}_t \mid \alpha_i \in \mathbb{C}, 1 \leq i \leq t \} \\ &= \left\{ \sum_{i=1}^t \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, 1 \leq i \leq t \right\}\end{aligned}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem SSS Span of a Set is a Subspace**

158

Suppose  $V$  is a vector space. Given a set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\} \subseteq V$ , their span,  $\langle S \rangle$ , is a subspace.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CSMS Column Space of a Matrix is a Subspace**

159

Suppose that  $A$  is an  $m \times n$  matrix. Then  $\mathcal{C}(A)$  is a subspace of  $\mathbb{C}^m$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem RSMS Row Space of a Matrix is a Subspace**

160

Suppose that  $A$  is an  $m \times n$  matrix. Then  $\mathcal{R}(A)$  is a subspace of  $\mathbb{C}^n$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem LNSMS Left Null Space of a Matrix is a Subspace**

161

Suppose that  $A$  is an  $m \times n$  matrix. Then  $\mathcal{L}(A)$  is a subspace of  $\mathbb{C}^m$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition RLD Relation of Linear Dependence**

162

Suppose that  $V$  is a vector space. Given a set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ , an equation of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a relation of linear dependence on  $S$ . If this equation is formed in a trivial fashion, i.e.  $\alpha_i = 0$ ,  $1 \leq i \leq n$ , then we say it is a trivial relation of linear dependence on  $S$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition LI Linear Independence****163**

Suppose that  $V$  is a vector space. The set of vectors  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  from  $V$  is linearly dependent if there is a relation of linear dependence on  $S$  that is not trivial. In the case where the only relation of linear dependence on  $S$  is the trivial one, then  $S$  is a linearly independent set of vectors.

©2004–2013 Robert A. Beezer, GFDL License

**Definition SSVS Spanning Set of a Vector Space****164**

Suppose  $V$  is a vector space. A subset  $S$  of  $V$  is a spanning set of  $V$  if  $\langle S \rangle = V$ . In this case, we also frequently say  $S$  spans  $V$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem VRRB Vector Representation Relative to a Basis****165**

Suppose that  $V$  is a vector space and  $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$  is a linearly independent set that spans  $V$ . Let  $\mathbf{w}$  be any vector in  $V$ . Then there exist unique scalars  $a_1, a_2, a_3, \dots, a_m$  such that

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_m\mathbf{v}_m.$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition B Basis****166**

Suppose  $V$  is a vector space. Then a subset  $S \subseteq V$  is a basis of  $V$  if it is linearly independent and spans  $V$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem SUVB Standard Unit Vectors are a Basis****167**

The set of standard unit vectors for  $\mathbb{C}^m$  (Definition SUV),  $B = \{\mathbf{e}_i \mid 1 \leq i \leq m\}$  is a basis for the vector space  $\mathbb{C}^m$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CNMB Columns of Nonsingular Matrix are a Basis****168**

Suppose that  $A$  is a square matrix of size  $m$ . Then the columns of  $A$  are a basis of  $\mathbb{C}^m$  if and only if  $A$  is nonsingular.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem NME5 Nonsingular Matrix Equivalences, Round 5**

169

Suppose that  $A$  is a square matrix of size  $n$ . The following are equivalent.

1.  $A$  is nonsingular.
2.  $A$  row-reduces to the identity matrix.
3. The null space of  $A$  contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}$ .
4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
5. The columns of  $A$  are a linearly independent set.
6.  $A$  is invertible.
7. The column space of  $A$  is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
8. The columns of  $A$  are a basis for  $\mathbb{C}^n$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem COB Coordinates and Orthonormal Bases**

170

Suppose that  $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$  is an orthonormal basis of the subspace  $W$  of  $\mathbb{C}^m$ . For any  $\mathbf{w} \in W$ ,

$$\mathbf{w} = \langle \mathbf{v}_1, \mathbf{w} \rangle \mathbf{v}_1 + \langle \mathbf{v}_2, \mathbf{w} \rangle \mathbf{v}_2 + \langle \mathbf{v}_3, \mathbf{w} \rangle \mathbf{v}_3 + \cdots + \langle \mathbf{v}_p, \mathbf{w} \rangle \mathbf{v}_p$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem UMCOB Unitary Matrices Convert Orthonormal Bases****171**

Let  $A$  be an  $n \times n$  matrix and  $B = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n\}$  be an orthonormal basis of  $\mathbb{C}^n$ . Define

$$C = \{A\mathbf{x}_1, A\mathbf{x}_2, A\mathbf{x}_3, \dots, A\mathbf{x}_n\}$$

Then  $A$  is a unitary matrix if and only if  $C$  is an orthonormal basis of  $\mathbb{C}^n$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition D Dimension****172**

Suppose that  $V$  is a vector space and  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$  is a basis of  $V$ . Then the dimension of  $V$  is defined by  $\dim(V) = t$ . If  $V$  has no finite bases, we say  $V$  has infinite dimension.

©2004–2013 Robert A. Beezer, GFDL License



**Theorem SSLD Spanning Sets and Linear Dependence****173**

Suppose that  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$  is a finite set of vectors which spans the vector space  $V$ . Then any set of  $t + 1$  or more vectors from  $V$  is linearly dependent.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem BIS Bases have Identical Sizes****174**

Suppose that  $V$  is a vector space with a finite basis  $B$  and a second basis  $C$ . Then  $B$  and  $C$  have the same size.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DCM Dimension of  $\mathbb{C}^m$**

175

The dimension of  $\mathbb{C}^m$  (Example VSCV) is  $m$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DP Dimension of  $P_n$**

176

The dimension of  $P_n$  (Example VSP) is  $n + 1$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DM Dimension of  $M_{mn}$** 

177

The dimension of  $M_{mn}$  (Example VSM) is  $mn$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition NOM Nullity Of a Matrix**

178

Suppose that  $A$  is an  $m \times n$  matrix. Then the nullity of  $A$  is the dimension of the null space of  $A$ ,  $n(A) = \dim(\mathcal{N}(A))$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition ROM Rank Of a Matrix****179**

Suppose that  $A$  is an  $m \times n$  matrix. Then the rank of  $A$  is the dimension of the column space of  $A$ ,  $r(A) = \dim(\mathcal{C}(A))$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CRN Computing Rank and Nullity****180**

Suppose that  $A$  is an  $m \times n$  matrix and  $B$  is a row-equivalent matrix in reduced row-echelon form. Let  $r$  denote the number of pivot columns (or the number of nonzero rows). Then  $r(A) = r$  and  $n(A) = n - r$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem RPNC Rank Plus Nullity is Columns****181**

Suppose that  $A$  is an  $m \times n$  matrix. Then  $r(A) + n(A) = n$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem RNNM Rank and Nullity of a Nonsingular Matrix****182**

Suppose that  $A$  is a square matrix of size  $n$ . The following are equivalent.

1.  $A$  is nonsingular.
2. The rank of  $A$  is  $n$ ,  $r(A) = n$ .
3. The nullity of  $A$  is zero,  $n(A) = 0$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem NME6 Nonsingular Matrix Equivalences, Round 6**

183

Suppose that  $A$  is a square matrix of size  $n$ . The following are equivalent.

1.  $A$  is nonsingular.
2.  $A$  row-reduces to the identity matrix.
3. The null space of  $A$  contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}$ .
4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
5. The columns of  $A$  are a linearly independent set.
6.  $A$  is invertible.
7. The column space of  $A$  is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
8. The columns of  $A$  are a basis for  $\mathbb{C}^n$ .
9. The rank of  $A$  is  $n$ ,  $r(A) = n$ .
10. The nullity of  $A$  is zero,  $n(A) = 0$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ELIS Extending Linearly Independent Sets**

184

Suppose  $V$  is a vector space and  $S$  is a linearly independent set of vectors from  $V$ . Suppose  $\mathbf{w}$  is a vector such that  $\mathbf{w} \notin \langle S \rangle$ . Then the set  $S' = S \cup \{\mathbf{w}\}$  is linearly independent.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem G Goldilocks****185**

Suppose that  $V$  is a vector space of dimension  $t$ . Let  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$  be a set of vectors from  $V$ . Then

1. If  $m > t$ , then  $S$  is linearly dependent.
2. If  $m < t$ , then  $S$  does not span  $V$ .
3. If  $m = t$  and  $S$  is linearly independent, then  $S$  spans  $V$ .
4. If  $m = t$  and  $S$  spans  $V$ , then  $S$  is linearly independent.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem PSSD Proper Subspaces have Smaller Dimension****186**

Suppose that  $U$  and  $V$  are subspaces of the vector space  $W$ , such that  $U \subsetneq V$ . Then  $\dim(U) < \dim(V)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem EDYES Equal Dimensions Yields Equal Subspaces**

187

Suppose that  $U$  and  $V$  are subspaces of the vector space  $W$ , such that  $U \subseteq V$  and  $\dim(U) = \dim(V)$ . Then  $U = V$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem RMRT Rank of a Matrix is the Rank of the Transpose**

188

Suppose  $A$  is an  $m \times n$  matrix. Then  $r(A) = r(A^t)$ .

©2004–2013 Robert A. Beezer, GFDL License



Suppose that  $A$  is an  $m \times n$  matrix, and  $B$  is a row-equivalent matrix in reduced row-echelon form with  $r$  nonzero rows. Then

1.  $\dim(\mathcal{N}(A)) = n - r$
2.  $\dim(\mathcal{C}(A)) = r$
3.  $\dim(\mathcal{R}(A)) = r$
4.  $\dim(\mathcal{L}(A)) = m - r$

1. For  $i \neq j$ ,  $E_{i,j}$  is the square matrix of size  $n$  with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

2. For  $\alpha \neq 0$ ,  $E_i(\alpha)$  is the square matrix of size  $n$  with

$$[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. For  $i \neq j$ ,  $E_{i,j}(\alpha)$  is the square matrix of size  $n$  with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = i \end{cases}$$

Suppose that  $A$  is an  $m \times n$  matrix, and  $B$  is a matrix of the same size that is obtained from  $A$  by a single row operation (Definition RO). Then there is an elementary matrix of size  $m$  that will convert  $A$  to  $B$  via matrix multiplication on the left. More precisely,

1. If the row operation swaps rows  $i$  and  $j$ , then  $B = E_{i,j}A$ .
2. If the row operation multiplies row  $i$  by  $\alpha$ , then  $B = E_i(\alpha)A$ .
3. If the row operation multiplies row  $i$  by  $\alpha$  and adds the result to row  $j$ , then  $B = E_{i,j}(\alpha)A$ .

If  $E$  is an elementary matrix, then  $E$  is nonsingular.

**Theorem NMPER** Nonsingular Matrices are Products of Elementary Matrices **193**

Suppose that  $A$  is a nonsingular matrix. Then there exists elementary matrices  $E_1, E_2, E_3, \dots, E_t$  so that  $A = E_1 E_2 E_3 \dots E_t$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition SM** SubMatrix

**194**

Suppose that  $A$  is an  $m \times n$  matrix. Then the submatrix  $A(i|j)$  is the  $(m - 1) \times (n - 1)$  matrix obtained from  $A$  by removing row  $i$  and column  $j$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition DM Determinant of a Matrix****195**

Suppose  $A$  is a square matrix. Then its determinant,  $\det(A) = |A|$ , is an element of  $\mathbb{C}$  defined recursively by:

1. If  $A$  is a  $1 \times 1$  matrix, then  $\det(A) = [A]_{11}$ .
2. If  $A$  is a matrix of size  $n$  with  $n \geq 2$ , then

$$\det(A) = [A]_{11} \det(A(1|1)) - [A]_{12} \det(A(1|2)) + [A]_{13} \det(A(1|3)) - [A]_{14} \det(A(1|4)) + \cdots + (-1)^{n+1} [A]_{1n} \det(A(1|n))$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DMST Determinant of Matrices of Size Two****196**

Suppose that  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Then  $\det(A) = ad - bc$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DER Determinant Expansion about Rows**

197

Suppose that  $A$  is a square matrix of size  $n$ . Then for  $1 \leq i \leq n$

$$\det(A) = (-1)^{i+1} [A]_{i1} \det(A(i|1)) + (-1)^{i+2} [A]_{i2} \det(A(i|2)) \\ + (-1)^{i+3} [A]_{i3} \det(A(i|3)) + \cdots + (-1)^{i+n} [A]_{in} \det(A(i|n))$$

which is known as expansion about row  $i$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DT Determinant of the Transpose**

198

Suppose that  $A$  is a square matrix. Then  $\det(A^t) = \det(A)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DEC Determinant Expansion about Columns****199**

Suppose that  $A$  is a square matrix of size  $n$ . Then for  $1 \leq j \leq n$

$$\det(A) = (-1)^{1+j} [A]_{1j} \det(A(1|j)) + (-1)^{2+j} [A]_{2j} \det(A(2|j)) \\ + (-1)^{3+j} [A]_{3j} \det(A(3|j)) + \cdots + (-1)^{n+j} [A]_{nj} \det(A(n|j))$$

which is known as expansion about column  $j$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DZRC Determinant with Zero Row or Column****200**

Suppose that  $A$  is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then  $\det(A) = 0$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DRCS Determinant for Row or Column Swap****201**

Suppose that  $A$  is a square matrix. Let  $B$  be the square matrix obtained from  $A$  by interchanging the location of two rows, or interchanging the location of two columns. Then  $\det(B) = -\det(A)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DRCM Determinant for Row or Column Multiples****202**

Suppose that  $A$  is a square matrix. Let  $B$  be the square matrix obtained from  $A$  by multiplying a single row by the scalar  $\alpha$ , or by multiplying a single column by the scalar  $\alpha$ . Then  $\det(B) = \alpha \det(A)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DERC Determinant with Equal Rows or Columns****203**

Suppose that  $A$  is a square matrix with two equal rows, or two equal columns. Then  $\det(A) = 0$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DRCMA Determinant for Row or Column Multiples and Addition****204**

Suppose that  $A$  is a square matrix. Let  $B$  be the square matrix obtained from  $A$  by multiplying a row by the scalar  $\alpha$  and then adding it to another row, or by multiplying a column by the scalar  $\alpha$  and then adding it to another column. Then  $\det(B) = \det(A)$ .

©2004–2013 Robert A. Beezer, GFDL License



For every  $n \geq 1$ ,  $\det(I_n) = 1$ .

For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

1.  $\det(E_{i,j}) = -1$
2.  $\det(E_i(\alpha)) = \alpha$
3.  $\det(E_{i,j}(\alpha)) = 1$

**Theorem DEMMM** Determinants, Elementary Matrices, Matrix Multiplication 207

Suppose that  $A$  is a square matrix of size  $n$  and  $E$  is any elementary matrix of size  $n$ . Then

$$\det(EA) = \det(E) \det(A)$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem SMZD** Singular Matrices have Zero Determinants

208

Let  $A$  be a square matrix. Then  $A$  is singular if and only if  $\det(A) = 0$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem NME7 Nonsingular Matrix Equivalences, Round 7****209**

Suppose that  $A$  is a square matrix of size  $n$ . The following are equivalent.

1.  $A$  is nonsingular.
2.  $A$  row-reduces to the identity matrix.
3. The null space of  $A$  contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}$ .
4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
5. The columns of  $A$  are a linearly independent set.
6.  $A$  is invertible.
7. The column space of  $A$  is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
8. The columns of  $A$  are a basis for  $\mathbb{C}^n$ .
9. The rank of  $A$  is  $n$ ,  $r(A) = n$ .
10. The nullity of  $A$  is zero,  $n(A) = 0$ .
11. The determinant of  $A$  is nonzero,  $\det(A) \neq 0$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DRMM Determinant Respects Matrix Multiplication****210**

Suppose that  $A$  and  $B$  are square matrices of the same size. Then  $\det(AB) = \det(A)\det(B)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition EEM Eigenvalues and Eigenvectors of a Matrix****211**

Suppose that  $A$  is a square matrix of size  $n$ ,  $\mathbf{x} \neq \mathbf{0}$  is a vector in  $\mathbb{C}^n$ , and  $\lambda$  is a scalar in  $\mathbb{C}$ . Then we say  $\mathbf{x}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$  if

$$A\mathbf{x} = \lambda\mathbf{x}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem EMHE Every Matrix Has an Eigenvalue****212**

Suppose  $A$  is a square matrix. Then  $A$  has at least one eigenvalue.

©2004–2013 Robert A. Beezer, GFDL License

**Definition CP Characteristic Polynomial****213**

Suppose that  $A$  is a square matrix of size  $n$ . Then the characteristic polynomial of  $A$  is the polynomial  $p_A(x)$  defined by

$$p_A(x) = \det(A - xI_n)$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem EMRCP Eigenvalues of a Matrix are Roots of Characteristic Polynomials**  
**214**

Suppose  $A$  is a square matrix. Then  $\lambda$  is an eigenvalue of  $A$  if and only if  $p_A(\lambda) = 0$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition EM Eigenspace of a Matrix****215**

Suppose that  $A$  is a square matrix and  $\lambda$  is an eigenvalue of  $A$ . Then the eigenspace of  $A$  for  $\lambda$ ,  $\mathcal{E}_A(\lambda)$ , is the set of all the eigenvectors of  $A$  for  $\lambda$ , together with the inclusion of the zero vector.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem EMS Eigenspace for a Matrix is a Subspace****216**

Suppose  $A$  is a square matrix of size  $n$  and  $\lambda$  is an eigenvalue of  $A$ . Then the eigenspace  $\mathcal{E}_A(\lambda)$  is a subspace of the vector space  $\mathbb{C}^n$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem EMNS Eigenspace of a Matrix is a Null Space****217**

Suppose  $A$  is a square matrix of size  $n$  and  $\lambda$  is an eigenvalue of  $A$ . Then

$$\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition AME Algebraic Multiplicity of an Eigenvalue****218**

Suppose that  $A$  is a square matrix and  $\lambda$  is an eigenvalue of  $A$ . Then the algebraic multiplicity of  $\lambda$ ,  $\alpha_A(\lambda)$ , is the highest power of  $(x - \lambda)$  that divides the characteristic polynomial,  $p_A(x)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition GME Geometric Multiplicity of an Eigenvalue****219**

Suppose that  $A$  is a square matrix and  $\lambda$  is an eigenvalue of  $A$ . Then the geometric multiplicity of  $\lambda$ ,  $\gamma_A(\lambda)$ , is the dimension of the eigenspace  $\mathcal{E}_A(\lambda)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem EDELI Eigenvectors with Distinct Eigenvalues are Linearly Independent**  
**220**

Suppose that  $A$  is an  $n \times n$  square matrix and  $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_p\}$  is a set of eigenvectors with eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_p$  such that  $\lambda_i \neq \lambda_j$  whenever  $i \neq j$ . Then  $S$  is a linearly independent set.

©2004–2013 Robert A. Beezer, GFDL License



**Theorem SMZE Singular Matrices have Zero Eigenvalues****221**

Suppose  $A$  is a square matrix. Then  $A$  is singular if and only if  $\lambda = 0$  is an eigenvalue of  $A$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem NME8 Nonsingular Matrix Equivalences, Round 8****222**

Suppose that  $A$  is a square matrix of size  $n$ . The following are equivalent.

1.  $A$  is nonsingular.
2.  $A$  row-reduces to the identity matrix.
3. The null space of  $A$  contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}$ .
4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
5. The columns of  $A$  are a linearly independent set.
6.  $A$  is invertible.
7. The column space of  $A$  is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
8. The columns of  $A$  are a basis for  $\mathbb{C}^n$ .
9. The rank of  $A$  is  $n$ ,  $r(A) = n$ .
10. The nullity of  $A$  is zero,  $n(A) = 0$ .
11. The determinant of  $A$  is nonzero,  $\det(A) \neq 0$ .
12.  $\lambda = 0$  is not an eigenvalue of  $A$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ESMM Eigenvalues of a Scalar Multiple of a Matrix**

**223**

Suppose  $A$  is a square matrix and  $\lambda$  is an eigenvalue of  $A$ . Then  $\alpha\lambda$  is an eigenvalue of  $\alpha A$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem EOMP Eigenvalues Of Matrix Powers**

**224**

Suppose  $A$  is a square matrix,  $\lambda$  is an eigenvalue of  $A$ , and  $s \geq 0$  is an integer. Then  $\lambda^s$  is an eigenvalue of  $A^s$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem EPM Eigenvalues of the Polynomial of a Matrix****225**

Suppose  $A$  is a square matrix and  $\lambda$  is an eigenvalue of  $A$ . Let  $q(x)$  be a polynomial in the variable  $x$ . Then  $q(\lambda)$  is an eigenvalue of the matrix  $q(A)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem EIM Eigenvalues of the Inverse of a Matrix****226**

Suppose  $A$  is a square nonsingular matrix and  $\lambda$  is an eigenvalue of  $A$ . Then  $\lambda^{-1}$  is an eigenvalue of the matrix  $A^{-1}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ETM Eigenvalues of the Transpose of a Matrix****227**

Suppose  $A$  is a square matrix and  $\lambda$  is an eigenvalue of  $A$ . Then  $\lambda$  is an eigenvalue of the matrix  $A^t$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ERMCP Eigenvalues of Real Matrices come in Conjugate Pairs****228**

Suppose  $A$  is a square matrix with real entries and  $\mathbf{x}$  is an eigenvector of  $A$  for the eigenvalue  $\lambda$ . Then  $\bar{\mathbf{x}}$  is an eigenvector of  $A$  for the eigenvalue  $\bar{\lambda}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DCP Degree of the Characteristic Polynomial****229**

Suppose that  $A$  is a square matrix of size  $n$ . Then the characteristic polynomial of  $A$ ,  $p_A(x)$ , has degree  $n$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem NEM Number of Eigenvalues of a Matrix****230**

Suppose that  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$  are the distinct eigenvalues of a square matrix  $A$  of size  $n$ . Then

$$\sum_{i=1}^k \alpha_A(\lambda_i) = n$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ME Multiplicities of an Eigenvalue****231**

Suppose that  $A$  is a square matrix of size  $n$  and  $\lambda$  is an eigenvalue. Then

$$1 \leq \gamma_A(\lambda) \leq \alpha_A(\lambda) \leq n$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MNEM Maximum Number of Eigenvalues of a Matrix****232**

Suppose that  $A$  is a square matrix of size  $n$ . Then  $A$  cannot have more than  $n$  distinct eigenvalues.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem HMRE Hermitian Matrices have Real Eigenvalues**

**233**

Suppose that  $A$  is a Hermitian matrix and  $\lambda$  is an eigenvalue of  $A$ . Then  $\lambda \in \mathbb{R}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem HMOE Hermitian Matrices have Orthogonal Eigenvectors**

**234**

Suppose that  $A$  is a Hermitian matrix and  $\mathbf{x}$  and  $\mathbf{y}$  are two eigenvectors of  $A$  for different eigenvalues. Then  $\mathbf{x}$  and  $\mathbf{y}$  are orthogonal vectors.

©2004–2013 Robert A. Beezer, GFDL License

**Definition SIM Similar Matrices****235**

Suppose  $A$  and  $B$  are two square matrices of size  $n$ . Then  $A$  and  $B$  are similar if there exists a nonsingular matrix of size  $n$ ,  $S$ , such that  $A = S^{-1}BS$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem SER Similarity is an Equivalence Relation****236**

Suppose  $A$ ,  $B$  and  $C$  are square matrices of size  $n$ . Then

1.  $A$  is similar to  $A$ . (Reflexive)
2. If  $A$  is similar to  $B$ , then  $B$  is similar to  $A$ . (Symmetric)
3. If  $A$  is similar to  $B$  and  $B$  is similar to  $C$ , then  $A$  is similar to  $C$ . (Transitive)

©2004–2013 Robert A. Beezer, GFDL License



**Theorem SMEE Similar Matrices have Equal Eigenvalues****237**

Suppose  $A$  and  $B$  are similar matrices. Then the characteristic polynomials of  $A$  and  $B$  are equal, that is,  $p_A(x) = p_B(x)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition DIM Diagonal Matrix****238**

Suppose that  $A$  is a square matrix. Then  $A$  is a diagonal matrix if  $[A]_{ij} = 0$  whenever  $i \neq j$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition DZM Diagonalizable Matrix****239**

Suppose  $A$  is a square matrix. Then  $A$  is diagonalizable if  $A$  is similar to a diagonal matrix.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DC Diagonalization Characterization****240**

Suppose  $A$  is a square matrix of size  $n$ . Then  $A$  is diagonalizable if and only if there exists a linearly independent set  $S$  that contains  $n$  eigenvectors of  $A$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DMFE Diagonalizable Matrices have Full Eigenspaces**

**241**

Suppose  $A$  is a square matrix. Then  $A$  is diagonalizable if and only if  $\gamma_A(\lambda) = \alpha_A(\lambda)$  for every eigenvalue  $\lambda$  of  $A$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem DED Distinct Eigenvalues implies Diagonalizable**

**242**

Suppose  $A$  is a square matrix of size  $n$  with  $n$  distinct eigenvalues. Then  $A$  is diagonalizable.

©2004–2013 Robert A. Beezer, GFDL License

**Definition LT Linear Transformation****243**

A linear transformation,  $T: U \rightarrow V$ , is a function that carries elements of the vector space  $U$  (called the domain) to the vector space  $V$  (called the codomain), and which has two additional properties

1.  $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$  for all  $\mathbf{u}_1, \mathbf{u}_2 \in U$
2.  $T(\alpha\mathbf{u}) = \alpha T(\mathbf{u})$  for all  $\mathbf{u} \in U$  and all  $\alpha \in \mathbb{C}$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem LTTZZ Linear Transformations Take Zero to Zero****244**

Suppose  $T: U \rightarrow V$  is a linear transformation. Then  $T(\mathbf{0}) = \mathbf{0}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MBLT Matrices Build Linear Transformations****245**

Suppose that  $A$  is an  $m \times n$  matrix. Define a function  $T: \mathbb{C}^n \rightarrow \mathbb{C}^m$  by  $T(\mathbf{x}) = A\mathbf{x}$ . Then  $T$  is a linear transformation.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MLTCV Matrix of a Linear Transformation, Column Vectors****246**

Suppose that  $T: \mathbb{C}^n \rightarrow \mathbb{C}^m$  is a linear transformation. Then there is an  $m \times n$  matrix  $A$  such that  $T(\mathbf{x}) = A\mathbf{x}$ .

©2004–2013 Robert A. Beezer, GFDL License

Suppose that  $T: U \rightarrow V$  is a linear transformation,  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$  are vectors from  $U$  and  $a_1, a_2, a_3, \dots, a_t$  are scalars from  $\mathbb{C}$ . Then

$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t) = a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \cdots + a_tT(\mathbf{u}_t)$$

Suppose  $U$  is a vector space with basis  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  and the vector space  $V$  contains the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n$  (which may not be distinct). Then there is a unique linear transformation,  $T: U \rightarrow V$ , such that  $T(\mathbf{u}_i) = \mathbf{v}_i$ ,  $1 \leq i \leq n$ .

**Definition PI Pre-Image****249**

Suppose that  $T: U \rightarrow V$  is a linear transformation. For each  $\mathbf{v}$ , define the pre-image of  $\mathbf{v}$  to be the subset of  $U$  given by

$$T^{-1}(\mathbf{v}) = \{\mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{v}\}$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition LTA Linear Transformation Addition****250**

Suppose that  $T: U \rightarrow V$  and  $S: U \rightarrow V$  are two linear transformations with the same domain and codomain. Then their sum is the function  $T + S: U \rightarrow V$  whose outputs are defined by

$$(T + S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem SLTLT Sum of Linear Transformations is a Linear Transformation 251**

Suppose that  $T: U \rightarrow V$  and  $S: U \rightarrow V$  are two linear transformations with the same domain and codomain. Then  $T + S: U \rightarrow V$  is a linear transformation.

©2004–2013 Robert A. Beezer, GFDL License

**Definition LTSM Linear Transformation Scalar Multiplication 252**

Suppose that  $T: U \rightarrow V$  is a linear transformation and  $\alpha \in \mathbb{C}$ . Then the scalar multiple is the function  $\alpha T: U \rightarrow V$  whose outputs are defined by

$$(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u})$$

©2004–2013 Robert A. Beezer, GFDL License



**Theorem MLTLT Multiple of a Linear Transformation is a Linear Transformation 253**

Suppose that  $T: U \rightarrow V$  is a linear transformation and  $\alpha \in \mathbb{C}$ . Then  $(\alpha T): U \rightarrow V$  is a linear transformation.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem VSLT Vector Space of Linear Transformations**

**254**

Suppose that  $U$  and  $V$  are vector spaces. Then the set of all linear transformations from  $U$  to  $V$ ,  $\mathcal{LT}(U, V)$ , is a vector space when the operations are those given in Definition LTA and Definition LTSM.

©2004–2013 Robert A. Beezer, GFDL License

**Definition LTC Linear Transformation Composition****255**

Suppose that  $T: U \rightarrow V$  and  $S: V \rightarrow W$  are linear transformations. Then the composition of  $S$  and  $T$  is the function  $(S \circ T): U \rightarrow W$  whose outputs are defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CLTLT Composition of Linear Transformations is a Linear Transformation****256**

Suppose that  $T: U \rightarrow V$  and  $S: V \rightarrow W$  are linear transformations. Then  $(S \circ T): U \rightarrow W$  is a linear transformation.

©2004–2013 Robert A. Beezer, GFDL License

**Definition ILT** **Injective Linear Transformation****257**

Suppose  $T: U \rightarrow V$  is a linear transformation. Then  $T$  is injective if whenever  $T(\mathbf{x}) = T(\mathbf{y})$ , then  $\mathbf{x} = \mathbf{y}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition KLT** **Kernel of a Linear Transformation****258**

Suppose  $T: U \rightarrow V$  is a linear transformation. Then the kernel of  $T$  is the set

$$\mathcal{K}(T) = \{\mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{0}\}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem KLTS Kernel of a Linear Transformation is a Subspace****259**

Suppose that  $T: U \rightarrow V$  is a linear transformation. Then the kernel of  $T$ ,  $\mathcal{K}(T)$ , is a subspace of  $U$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem KPI Kernel and Pre-Image****260**

Suppose  $T: U \rightarrow V$  is a linear transformation and  $\mathbf{v} \in V$ . If the preimage  $T^{-1}(\mathbf{v})$  is non-empty, and  $\mathbf{u} \in T^{-1}(\mathbf{v})$  then

$$T^{-1}(\mathbf{v}) = \{\mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{K}(T)\} = \mathbf{u} + \mathcal{K}(T)$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem KILT Kernel of an Injective Linear Transformation****261**

Suppose that  $T: U \rightarrow V$  is a linear transformation. Then  $T$  is injective if and only if the kernel of  $T$  is trivial,  $\mathcal{K}(T) = \{\mathbf{0}\}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ILTLI Injective Linear Transformations and Linear Independence****262**

Suppose that  $T: U \rightarrow V$  is an injective linear transformation and

$$S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$$

is a linearly independent subset of  $U$ . Then

$$R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}$$

is a linearly independent subset of  $V$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ILTB Injective Linear Transformations and Bases****263**

Suppose that  $T: U \rightarrow V$  is a linear transformation and

$$B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$$

is a basis of  $U$ . Then  $T$  is injective if and only if

$$C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$$

is a linearly independent subset of  $V$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ILTD Injective Linear Transformations and Dimension****264**

Suppose that  $T: U \rightarrow V$  is an injective linear transformation. Then  $\dim(U) \leq \dim(V)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CILTI Composition of Injective Linear Transformations is Injective 265**

Suppose that  $T: U \rightarrow V$  and  $S: V \rightarrow W$  are injective linear transformations. Then  $(S \circ T): U \rightarrow W$  is an injective linear transformation.

©2004–2013 Robert A. Beezer, GFDL License

**Definition SLT Surjective Linear Transformation**

**266**

Suppose  $T: U \rightarrow V$  is a linear transformation. Then  $T$  is surjective if for every  $\mathbf{v} \in V$  there exists a  $\mathbf{u} \in U$  so that  $T(\mathbf{u}) = \mathbf{v}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition RLT Range of a Linear Transformation****267**

Suppose  $T: U \rightarrow V$  is a linear transformation. Then the range of  $T$  is the set

$$\mathcal{R}(T) = \{T(\mathbf{u}) \mid \mathbf{u} \in U\}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem RLTS Range of a Linear Transformation is a Subspace****268**

Suppose that  $T: U \rightarrow V$  is a linear transformation. Then the range of  $T$ ,  $\mathcal{R}(T)$ , is a subspace of  $V$ .

©2004–2013 Robert A. Beezer, GFDL License



**Theorem RSLT Range of a Surjective Linear Transformation****269**

Suppose that  $T: U \rightarrow V$  is a linear transformation. Then  $T$  is surjective if and only if the range of  $T$  equals the codomain,  $\mathcal{R}(T) = V$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem SSRLT Spanning Set for Range of a Linear Transformation****270**

Suppose that  $T: U \rightarrow V$  is a linear transformation and

$$S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$$

spans  $U$ . Then

$$R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}$$

spans  $\mathcal{R}(T)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem RPI Range and Pre-Image**

271

Suppose that  $T: U \rightarrow V$  is a linear transformation. Then

$$\mathbf{v} \in \mathcal{R}(T) \text{ if and only if } T^{-1}(\mathbf{v}) \neq \emptyset$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem SLTB Surjective Linear Transformations and Bases**

272

Suppose that  $T: U \rightarrow V$  is a linear transformation and

$$B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$$

is a basis of  $U$ . Then  $T$  is surjective if and only if

$$C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$$

is a spanning set for  $V$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem SLTD Surjective Linear Transformations and Dimension****273**

Suppose that  $T: U \rightarrow V$  is a surjective linear transformation. Then  $\dim(U) \geq \dim(V)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CSLTS Composition of Surjective Linear Transformations is Surjective**  
**274**

Suppose that  $T: U \rightarrow V$  and  $S: V \rightarrow W$  are surjective linear transformations. Then  $(S \circ T): U \rightarrow W$  is a surjective linear transformation.

©2004–2013 Robert A. Beezer, GFDL License

**Definition IDLT Identity Linear Transformation****275**

The identity linear transformation on the vector space  $W$  is defined as

$$I_W: W \rightarrow W, \quad I_W(\mathbf{w}) = \mathbf{w}$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition IVLT Invertible Linear Transformations****276**

Suppose that  $T: U \rightarrow V$  is a linear transformation. If there is a function  $S: V \rightarrow U$  such that

$$S \circ T = I_U \qquad T \circ S = I_V$$

then  $T$  is invertible. In this case, we call  $S$  the inverse of  $T$  and write  $S = T^{-1}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem IL/TLT Inverse of a Linear Transformation is a Linear Transformation 277**

Suppose that  $T: U \rightarrow V$  is an invertible linear transformation. Then the function  $T^{-1}: V \rightarrow U$  is a linear transformation.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem IILT Inverse of an Invertible Linear Transformation 278**

Suppose that  $T: U \rightarrow V$  is an invertible linear transformation. Then  $T^{-1}$  is an invertible linear transformation and  $(T^{-1})^{-1} = T$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ILTIS Invertible Linear Transformations are Injective and Surjective** 279

Suppose  $T: U \rightarrow V$  is a linear transformation. Then  $T$  is invertible if and only if  $T$  is injective and surjective.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CIVLT Composition of Invertible Linear Transformations** 280

Suppose that  $T: U \rightarrow V$  and  $S: V \rightarrow W$  are invertible linear transformations. Then the composition,  $(S \circ T): U \rightarrow W$  is an invertible linear transformation.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ICLT Inverse of a Composition of Linear Transformations****281**

Suppose that  $T: U \rightarrow V$  and  $S: V \rightarrow W$  are invertible linear transformations. Then  $S \circ T$  is invertible and  $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition IVS Isomorphic Vector Spaces****282**

Two vector spaces  $U$  and  $V$  are isomorphic if there exists an invertible linear transformation  $T$  with domain  $U$  and codomain  $V$ ,  $T: U \rightarrow V$ . In this case, we write  $U \cong V$ , and the linear transformation  $T$  is known as an isomorphism between  $U$  and  $V$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem IVSED Isomorphic Vector Spaces have Equal Dimension****283**

Suppose  $U$  and  $V$  are isomorphic vector spaces. Then  $\dim(U) = \dim(V)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition ROLT Rank Of a Linear Transformation****284**

Suppose that  $T: U \rightarrow V$  is a linear transformation. Then the rank of  $T$ ,  $r(T)$ , is the dimension of the range of  $T$ ,

$$r(T) = \dim(\mathcal{R}(T))$$

©2004–2013 Robert A. Beezer, GFDL License



**Definition NOLT Nullity Of a Linear Transformation****285**

Suppose that  $T: U \rightarrow V$  is a linear transformation. Then the nullity of  $T$ ,  $n(T)$ , is the dimension of the kernel of  $T$ ,

$$n(T) = \dim(\mathcal{K}(T))$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ROSLT Rank Of a Surjective Linear Transformation****286**

Suppose that  $T: U \rightarrow V$  is a linear transformation. Then the rank of  $T$  is the dimension of  $V$ ,  $r(T) = \dim(V)$ , if and only if  $T$  is surjective.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem NOILT Nullity Of an Injective Linear Transformation****287**

Suppose that  $T: U \rightarrow V$  is a linear transformation. Then the nullity of  $T$  is zero,  $n(T) = 0$ , if and only if  $T$  is injective.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem RPNDT Rank Plus Nullity is Domain Dimension****288**

Suppose that  $T: U \rightarrow V$  is a linear transformation. Then

$$r(T) + n(T) = \dim(U)$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition VR Vector Representation****289**

Suppose that  $V$  is a vector space with a basis  $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ . Define a function  $\rho_B: V \rightarrow \mathbb{C}^n$  as follows. For  $\mathbf{w} \in V$  define the column vector  $\rho_B(\mathbf{w}) \in \mathbb{C}^n$  by

$$\mathbf{w} = [\rho_B(\mathbf{w})]_1 \mathbf{v}_1 + [\rho_B(\mathbf{w})]_2 \mathbf{v}_2 + [\rho_B(\mathbf{w})]_3 \mathbf{v}_3 + \cdots + [\rho_B(\mathbf{w})]_n \mathbf{v}_n$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem VRLT Vector Representation is a Linear Transformation****290**

The function  $\rho_B$  (Definition VR) is a linear transformation.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem VRI Vector Representation is Injective****291**

The function  $\rho_B$  (Definition VR) is an injective linear transformation.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem VRS Vector Representation is Surjective****292**

The function  $\rho_B$  (Definition VR) is a surjective linear transformation.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem VRILT Vector Representation is an Invertible Linear Transformation** 293

The function  $\rho_B$  (Definition VR) is an invertible linear transformation.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CFDVS Characterization of Finite Dimensional Vector Spaces** 294

Suppose that  $V$  is a vector space with dimension  $n$ . Then  $V$  is isomorphic to  $\mathbb{C}^n$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem IFDVS Isomorphism of Finite Dimensional Vector Spaces****295**

Suppose  $U$  and  $V$  are both finite-dimensional vector spaces. Then  $U$  and  $V$  are isomorphic if and only if  $\dim(U) = \dim(V)$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CLI Coordinatization and Linear Independence****296**

Suppose that  $U$  is a vector space with a basis  $B$  of size  $n$ . Then

$$S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\}$$

is a linearly independent subset of  $U$  if and only if

$$R = \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\}$$

is a linearly independent subset of  $\mathbb{C}^n$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CSS Coordinatization and Spanning Sets****297**

Suppose that  $U$  is a vector space with a basis  $B$  of size  $n$ . Then

$$\mathbf{u} \in \langle \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_k\} \rangle$$

if and only if

$$\rho_B(\mathbf{u}) \in \langle \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\} \rangle$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition MR Matrix Representation****298**

Suppose that  $T: U \rightarrow V$  is a linear transformation,  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$  is a basis for  $U$  of size  $n$ , and  $C$  is a basis for  $V$  of size  $m$ . Then the matrix representation of  $T$  relative to  $B$  and  $C$  is the  $m \times n$  matrix,

$$M_{B,C}^T = [\rho_C(T(\mathbf{u}_1)) \mid \rho_C(T(\mathbf{u}_2)) \mid \rho_C(T(\mathbf{u}_3)) \mid \dots \mid \rho_C(T(\mathbf{u}_n))]$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem FTMR Fundamental Theorem of Matrix Representation****299**

Suppose that  $T: U \rightarrow V$  is a linear transformation,  $B$  is a basis for  $U$ ,  $C$  is a basis for  $V$  and  $M_{B,C}^T$  is the matrix representation of  $T$  relative to  $B$  and  $C$ . Then, for any  $\mathbf{u} \in U$ ,

$$\rho_C(T(\mathbf{u})) = M_{B,C}^T(\rho_B(\mathbf{u}))$$

or equivalently

$$T(\mathbf{u}) = \rho_C^{-1}(M_{B,C}^T(\rho_B(\mathbf{u})))$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MRSLT Matrix Representation of a Sum of Linear Transformations****300**

Suppose that  $T: U \rightarrow V$  and  $S: U \rightarrow V$  are linear transformations,  $B$  is a basis of  $U$  and  $C$  is a basis of  $V$ . Then

$$M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$$

©2004–2013 Robert A. Beezer, GFDL License



**Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation 301**

Suppose that  $T: U \rightarrow V$  is a linear transformation,  $\alpha \in \mathbb{C}$ ,  $B$  is a basis of  $U$  and  $C$  is a basis of  $V$ . Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MRCLT Matrix Representation of a Composition of Linear Transformations 302**

Suppose that  $T: U \rightarrow V$  and  $S: V \rightarrow W$  are linear transformations,  $B$  is a basis of  $U$ ,  $C$  is a basis of  $V$ , and  $D$  is a basis of  $W$ . Then

$$M_{B,D}^{S \circ T} = M_{C,D}^S M_{B,C}^T$$

©2004–2013 Robert A. Beezer, GFDL License

Suppose that  $T: U \rightarrow V$  is a linear transformation,  $B$  is a basis for  $U$  of size  $n$ , and  $C$  is a basis for  $V$ . Then the kernel of  $T$  is isomorphic to the null space of  $M_{B,C}^T$ ,

$$\mathcal{K}(T) \cong \mathcal{N}(M_{B,C}^T)$$

Suppose that  $T: U \rightarrow V$  is a linear transformation,  $B$  is a basis for  $U$  of size  $n$ , and  $C$  is a basis for  $V$  of size  $m$ . Then the range of  $T$  is isomorphic to the column space of  $M_{B,C}^T$ ,

$$\mathcal{R}(T) \cong \mathcal{C}(M_{B,C}^T)$$

**Theorem IMR Invertible Matrix Representations****305**

Suppose that  $T: U \rightarrow V$  is a linear transformation,  $B$  is a basis for  $U$  and  $C$  is a basis for  $V$ . Then  $T$  is an invertible linear transformation if and only if the matrix representation of  $T$  relative to  $B$  and  $C$ ,  $M_{B,C}^T$  is an invertible matrix. When  $T$  is invertible,

$$M_{C,B}^{T^{-1}} = (M_{B,C}^T)^{-1}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem IMILT Invertible Matrices, Invertible Linear Transformation****306**

Suppose that  $A$  is a square matrix of size  $n$  and  $T: \mathbb{C}^n \rightarrow \mathbb{C}^n$  is the linear transformation defined by  $T(\mathbf{x}) = A\mathbf{x}$ . Then  $A$  is an invertible matrix if and only if  $T$  is an invertible linear transformation.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem NME9 Nonsingular Matrix Equivalences, Round 9**

307

Suppose that  $A$  is a square matrix of size  $n$ . The following are equivalent.

1.  $A$  is nonsingular.
2.  $A$  row-reduces to the identity matrix.
3. The null space of  $A$  contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}$ .
4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
5. The columns of  $A$  are a linearly independent set.
6.  $A$  is invertible.
7. The column space of  $A$  is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
8. The columns of  $A$  are a basis for  $\mathbb{C}^n$ .
9. The rank of  $A$  is  $n$ ,  $r(A) = n$ .
10. The nullity of  $A$  is zero,  $n(A) = 0$ .
11. The determinant of  $A$  is nonzero,  $\det(A) \neq 0$ .
12.  $\lambda = 0$  is not an eigenvalue of  $A$ .
13. The linear transformation  $T: \mathbb{C}^n \rightarrow \mathbb{C}^n$  defined by  $T(\mathbf{x}) = A\mathbf{x}$  is invertible.

©2004–2013 Robert A. Beezer, GFDL License

**Definition EELT Eigenvalue and Eigenvector of a Linear Transformation**

308

Suppose that  $T: V \rightarrow V$  is a linear transformation. Then a nonzero vector  $\mathbf{v} \in V$  is an eigenvector of  $T$  for the eigenvalue  $\lambda$  if  $T(\mathbf{v}) = \lambda\mathbf{v}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition CBM Change-of-Basis Matrix****309**

Suppose that  $V$  is a vector space, and  $I_V: V \rightarrow V$  is the identity linear transformation on  $V$ . Let  $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$  and  $C$  be two bases of  $V$ . Then the change-of-basis matrix from  $B$  to  $C$  is the matrix representation of  $I_V$  relative to  $B$  and  $C$ ,

$$\begin{aligned} C_{B,C} &= M_{B,C}^{I_V} \\ &= [\rho_C(I_V(\mathbf{v}_1)) \mid \rho_C(I_V(\mathbf{v}_2)) \mid \rho_C(I_V(\mathbf{v}_3)) \mid \dots \mid \rho_C(I_V(\mathbf{v}_n))] \\ &= [\rho_C(\mathbf{v}_1) \mid \rho_C(\mathbf{v}_2) \mid \rho_C(\mathbf{v}_3) \mid \dots \mid \rho_C(\mathbf{v}_n)] \end{aligned}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CB Change-of-Basis****310**

Suppose that  $\mathbf{v}$  is a vector in the vector space  $V$  and  $B$  and  $C$  are bases of  $V$ . Then

$$\rho_C(\mathbf{v}) = C_{B,C} \rho_B(\mathbf{v})$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ICBM Inverse of Change-of-Basis Matrix****311**

Suppose that  $V$  is a vector space, and  $B$  and  $C$  are bases of  $V$ . Then the change-of-basis matrix  $C_{B,C}$  is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem MRCB Matrix Representation and Change of Basis****312**

Suppose that  $T: U \rightarrow V$  is a linear transformation,  $B$  and  $C$  are bases for  $U$ , and  $D$  and  $E$  are bases for  $V$ . Then

$$M_{B,D}^T = C_{E,D} M_{C,E}^T C_{B,C}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem SCB Similarity and Change of Basis****313**

Suppose that  $T: V \rightarrow V$  is a linear transformation and  $B$  and  $C$  are bases of  $V$ . Then

$$M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$$

©2004–2013 Robert A. Beezer, GFDL License

**Theorem EER Eigenvalues, Eigenvectors, Representations****314**

Suppose that  $T: V \rightarrow V$  is a linear transformation and  $B$  is a basis of  $V$ . Then  $\mathbf{v} \in V$  is an eigenvector of  $T$  for the eigenvalue  $\lambda$  if and only if  $\rho_B(\mathbf{v})$  is an eigenvector of  $M_{B,B}^T$  for the eigenvalue  $\lambda$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition UTM Upper Triangular Matrix****315**

The  $n \times n$  square matrix  $A$  is upper triangular if  $[A]_{ij} = 0$  whenever  $i > j$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition LTM Lower Triangular Matrix****316**

The  $n \times n$  square matrix  $A$  is lower triangular if  $[A]_{ij} = 0$  whenever  $i < j$ .

©2004–2013 Robert A. Beezer, GFDL License



**Theorem PTMT Product of Triangular Matrices is Triangular****317**

Suppose that  $A$  and  $B$  are square matrices of size  $n$  that are triangular of the same type. Then  $AB$  is also triangular of that type.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ITMT Inverse of a Triangular Matrix is Triangular****318**

Suppose that  $A$  is a nonsingular matrix of size  $n$  that is triangular. Then the inverse of  $A$ ,  $A^{-1}$ , is triangular of the same type. Furthermore, the diagonal entries of  $A^{-1}$  are the reciprocals of the corresponding diagonal entries of  $A$ . More precisely,  $[A^{-1}]_{ii} = [A]_{ii}^{-1}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem UTMR Upper Triangular Matrix Representation****319**

Suppose that  $T: V \rightarrow V$  is a linear transformation. Then there is a basis  $B$  for  $V$  such that the matrix representation of  $T$  relative to  $B$ ,  $M_{B,B}^T$ , is an upper triangular matrix. Each diagonal entry is an eigenvalue of  $T$ , and if  $\lambda$  is an eigenvalue of  $T$ , then  $\lambda$  occurs  $\alpha_T(\lambda)$  times on the diagonal.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem OBUTR Orthonormal Basis for Upper Triangular Representation****320**

Suppose that  $A$  is a square matrix. Then there is a unitary matrix  $U$ , and an upper triangular matrix  $T$ , such that

$$U^*AU = T$$

and  $T$  has the eigenvalues of  $A$  as the entries of the diagonal.

©2004–2013 Robert A. Beezer, GFDL License

**Definition NRML Normal Matrix****321**

The square matrix  $A$  is normal if  $A^*A = AA^*$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem OD Orthonormal Diagonalization****322**

Suppose that  $A$  is a square matrix. Then there is a unitary matrix  $U$  and a diagonal matrix  $D$ , with diagonal entries equal to the eigenvalues of  $A$ , such that  $U^*AU = D$  if and only if  $A$  is a normal matrix.

©2004–2013 Robert A. Beezer, GFDL License

**Theorem OBNM Orthonormal Bases and Normal Matrices****323**

Suppose that  $A$  is a normal matrix of size  $n$ . Then there is an orthonormal basis of  $\mathbb{C}^n$  composed of eigenvectors of  $A$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition CNE Complex Number Equality****324**

The complex numbers  $\alpha = a + bi$  and  $\beta = c + di$  are equal, denoted  $\alpha = \beta$ , if  $a = c$  and  $b = d$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition CNA Complex Number Addition****325**

The sum of the complex numbers  $\alpha = a + bi$  and  $\beta = c + di$ , denoted  $\alpha + \beta$ , is  $(a + c) + (b + d)i$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition CNM Complex Number Multiplication****326**

The product of the complex numbers  $\alpha = a + bi$  and  $\beta = c + di$ , denoted  $\alpha\beta$ , is  $(ac - bd) + (ad + bc)i$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem PCNA Properties of Complex Number Arithmetic****327**

The operations of addition and multiplication of complex numbers have the following properties.

- ACCN Additive Closure, Complex Numbers: If  $\alpha, \beta \in \mathbb{C}$ , then  $\alpha + \beta \in \mathbb{C}$ .
- MCCN Multiplicative Closure, Complex Numbers: If  $\alpha, \beta \in \mathbb{C}$ , then  $\alpha\beta \in \mathbb{C}$ .
- CACN Commutativity of Addition, Complex Numbers: For any  $\alpha, \beta \in \mathbb{C}$ ,  $\alpha + \beta = \beta + \alpha$ .
- CMCN Commutativity of Multiplication, Complex Numbers: For any  $\alpha, \beta \in \mathbb{C}$ ,  $\alpha\beta = \beta\alpha$ .
- AACN Additive Associativity, Complex Numbers: For any  $\alpha, \beta, \gamma \in \mathbb{C}$ ,  $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ .
- MACN Multiplicative Associativity, Complex Numbers: For any  $\alpha, \beta, \gamma \in \mathbb{C}$ ,  $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ .
- DCN Distributivity, Complex Numbers: For any  $\alpha, \beta, \gamma \in \mathbb{C}$ ,  $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ .
- ZCN Zero, Complex Numbers: There is a complex number  $0 = 0+0i$  so that for any  $\alpha \in \mathbb{C}$ ,  $0 + \alpha = \alpha$ .
- OCN One, Complex Numbers: There is a complex number  $1 = 1+0i$  so that for any  $\alpha \in \mathbb{C}$ ,  $1\alpha = \alpha$ .
- AICN Additive Inverse, Complex Numbers: For every  $\alpha \in \mathbb{C}$  there exists  $-\alpha \in \mathbb{C}$  so that  $\alpha + (-\alpha) = 0$ .
- MICN Multiplicative Inverse, Complex Numbers: For every  $\alpha \in \mathbb{C}$ ,  $\alpha \neq 0$  there exists  $\frac{1}{\alpha} \in \mathbb{C}$  so that  $\alpha \left(\frac{1}{\alpha}\right) = 1$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ZPCN Zero Product, Complex Numbers****328**

Suppose  $\alpha \in \mathbb{C}$ . Then  $0\alpha = 0$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem ZPZT Zero Product, Zero Terms****329**

Suppose  $\alpha, \beta \in \mathbb{C}$ . Then  $\alpha\beta = 0$  if and only if at least one of  $\alpha = 0$  or  $\beta = 0$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition CCN Conjugate of a Complex Number****330**

The conjugate of the complex number  $\alpha = a + bi \in \mathbb{C}$  is the complex number  $\bar{\alpha} = a - bi$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CCRA Complex Conjugation Respects Addition****331**

Suppose that  $\alpha$  and  $\beta$  are complex numbers. Then  $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$ .

©2004–2013 Robert A. Beezer, GFDL License

**Theorem CCRM Complex Conjugation Respects Multiplication****332**

Suppose that  $\alpha$  and  $\beta$  are complex numbers. Then  $\overline{\alpha\beta} = \overline{\alpha}\overline{\beta}$ .

©2004–2013 Robert A. Beezer, GFDL License



**Theorem CCT Complex Conjugation Twice****333**

Suppose that  $\alpha$  is a complex number. Then  $\overline{\overline{\alpha}} = \alpha$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition MCN Modulus of a Complex Number****334**

The modulus of the complex number  $\alpha = a + bi \in \mathbb{C}$ , is the nonnegative real number

$$|\alpha| = \sqrt{\overline{\alpha}\alpha} = \sqrt{a^2 + b^2}.$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition SET Set****335**

A set is an unordered collection of objects. If  $S$  is a set and  $x$  is an object that is in the set  $S$ , we write  $x \in S$ . If  $x$  is not in  $S$ , then we write  $x \notin S$ . We refer to the objects in a set as its elements.

©2004–2013 Robert A. Beezer, GFDL License

**Definition SSET Subset****336**

If  $S$  and  $T$  are two sets, then  $S$  is a subset of  $T$ , written  $S \subseteq T$  if whenever  $x \in S$  then  $x \in T$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition ES Empty Set****337**

The empty set is the set with no elements. It is denoted by  $\emptyset$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition SE Set Equality****338**

Two sets,  $S$  and  $T$ , are equal, if  $S \subseteq T$  and  $T \subseteq S$ . In this case, we write  $S = T$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition C Cardinality****339**

Suppose  $S$  is a finite set. Then the number of elements in  $S$  is called the cardinality or size of  $S$ , and is denoted  $|S|$ .

©2004–2013 Robert A. Beezer, GFDL License

**Definition SU Set Union****340**

Suppose  $S$  and  $T$  are sets. Then the union of  $S$  and  $T$ , denoted  $S \cup T$ , is the set whose elements are those that are elements of  $S$  or of  $T$ , or both. More formally,

$$x \in S \cup T \text{ if and only if } x \in S \text{ or } x \in T$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition SI Set Intersection****341**

Suppose  $S$  and  $T$  are sets. Then the intersection of  $S$  and  $T$ , denoted  $S \cap T$ , is the set whose elements are only those that are elements of  $S$  and of  $T$ . More formally,

$$x \in S \cap T \text{ if and only if } x \in S \text{ and } x \in T$$

©2004–2013 Robert A. Beezer, GFDL License

**Definition SC Set Complement****342**

Suppose  $S$  is a set that is a subset of a universal set  $U$ . Then the complement of  $S$ , denoted  $\bar{S}$ , is the set whose elements are those that are elements of  $U$  and not elements of  $S$ . More formally,

$$x \in \bar{S} \text{ if and only if } x \in U \text{ and } x \notin S$$

©2004–2013 Robert A. Beezer, GFDL License