Flashcard Supplement to A First Course in Linear Algebra

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In addition to his teaching at the University of Puget Sound, he has made sabbatical visits to the University of the West Indies (Trinidad campus) and the University of Western Australia. He has also given several courses in the Master's program at the African Institute for Mathematical Sciences, South Africa. He has been a Sage developer since 2008.

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Definition SLE System of Linear Equations

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A system of linear equations is a collection of m equations in the variable quantities $x_1, x_2, x_3, \ldots, x_n$ of the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

where the values of a_{ij} , b_i and x_j , $1 \le i \le m$, $1 \le j \le n$, are from the set of complex numbers, \mathbb{C} .

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Definition SSLE Solution of a System of Linear Equations

 $\mathbf{2}$

A solution of a system of linear equations in n variables, $x_1, x_2, x_3, \ldots, x_n$ (such as the system given in Definition SLE), is an ordered list of n complex numbers, $s_1, s_2, s_3, \ldots, s_n$ such that if we substitute s_1 for x_1, s_2 for x_2, s_3 for x_3, s_n for x_n , then for every equation of the system the left side will equal the right side, i.e. each equation is true simultaneously.

Definition SSSLE Solution Set of a System. The solution set of a linear system of equations system, and nothing more.	
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Definition ESYS Equivalent Systems Two systems of linear equations are equivalent if their solution sets are equal. ©2004—2015 Robert A. Beezer, GFDL License

Definition EO Equation Operations

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Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an equation operation.

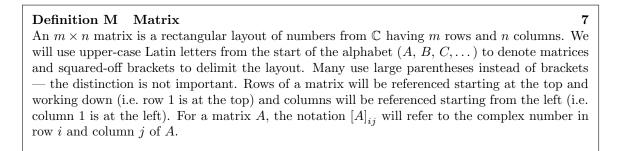
- 1. Swap the locations of two equations in the list of equations.
- 2. Multiply each term of an equation by a nonzero quantity.
- 3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

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Theorem EOPSS Equation Operations Preserve Solution Sets

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If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.



Definition CV Column Vector

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A column vector of size m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a vector. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as \mathbf{u} , \mathbf{v} , \mathbf{w} , \mathbf{x} , \mathbf{y} , \mathbf{z} . Some books like to write vectors with arrows, such as \vec{u} . Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in u. To refer to the entry or component of vector \mathbf{v} in location i of the list, we write $[\mathbf{v}]_i$.

Definition ZCV Zero Column Vector

Q

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The zero vector of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or defined much more compactly, $\left[\mathbf{0}\right]_{i}=0$ for $1\leq i\leq m.$

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Definition CM Coefficient Matrix

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the coefficient matrix is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Definition VOC Vector of Constants

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the vector of constants is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

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Definition SOLV Solution Vector

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the solution vector is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

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Definition MRLS Matrix Representation of a Linear System 13 If A is the coefficient matrix of a system of linear equations and $\mathbf b$ is the vector of constants, then we will write $\mathcal{LS}(A, \mathbf b)$ as a shorthand expression for the system of linear equations, which we will refer to as the matrix representation of the linear system.
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Definition AM Augmented Matrix Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants \mathbf{b} . Then the augmented matrix of the system of equations is the $m \times (n+1)$ matrix whose first n columns are the columns of A and whose last column $(n+1)$ is the column vector \mathbf{b} . This matrix will be written as $[A \mid \mathbf{b}]$.

Definition RO Row Operations

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The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a row operation.

- 1. Swap the locations of two rows.
- 2. Multiply each entry of a single row by a nonzero quantity.
- 3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

- 1. $R_i \leftrightarrow R_j$: Swap the location of rows i and j.
- 2. αR_i : Multiply row i by the nonzero scalar α .
- 3. $\alpha R_i + R_j$: Multiply row i by the scalar α and add to row j.

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Definition REM Row-Equivalent Matrices

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Two matrices, A and B, are row-equivalent if one can be obtained from the other by a sequence of row operations.

Theorem REMES	Row-Equivalent Matric	es represent Equi	ivalent Systems 17
Suppose that A and	B are row-equivalent augm	nented matrices. The	hen the systems of linear
equations that they re	epresent are equivalent syste	ms.	
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Definition RREF Reduced Row-Echelon Form

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A matrix is in reduced row-echelon form if it meets all of the following conditions:

- 1. If there is a row where every entry is zero, then this row lies below any other row that contains a nonzero entry.
- 2. The leftmost nonzero entry of a row is equal to 1.
- 3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
- 4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If s > i, then t > j.

A row of only zero entries is called a zero row and the leftmost nonzero entry of a nonzero row is a leading 1. A column containing a leading 1 will be called a pivot column. The number of nonzero rows will be denoted by r, which is also equal to the number of leading 1's and the number of pivot columns.

The set of column indices for the pivot columns will be denoted by $D = \{d_1, d_2, d_3, \ldots, d_r\}$ where $d_1 < d_2 < d_3 < \cdots < d_r$, while the columns that are not pivot columns will be denoted as $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

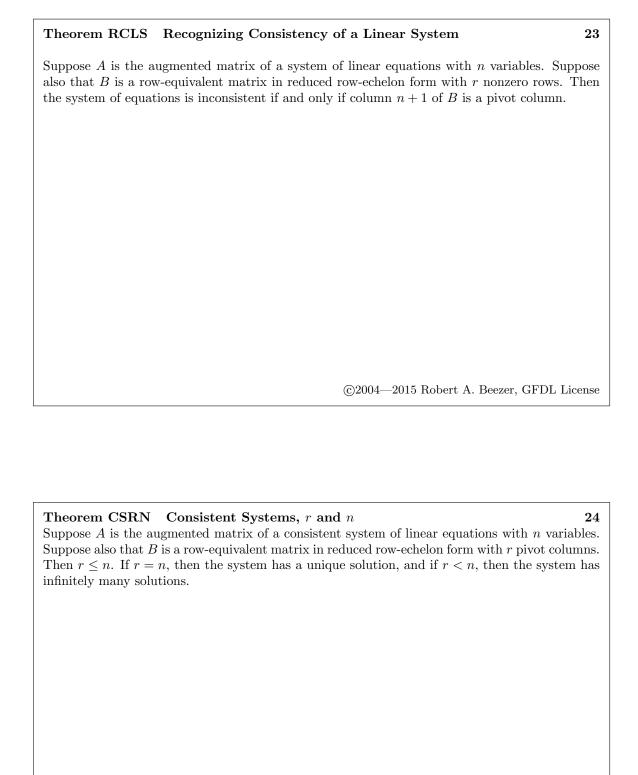
Theorem REMEF Row-Equivalent Matrix in Echelon Form Suppose A is a matrix. Then there is a matrix B so that		
1. A and B are row-equivalent.		
2. B is in reduced row-echelon form.		
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,		
Theorem RREFU Reduced Row-Echelon Form is Unique	20	
Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are roto A and in reduced row-echelon form. Then $B = C$.	ow-equivalent	

Definition CS Consistent System A system of linear equations is consistent if it has is called inconsistent.	21 s at least one solution. Otherwise, the system
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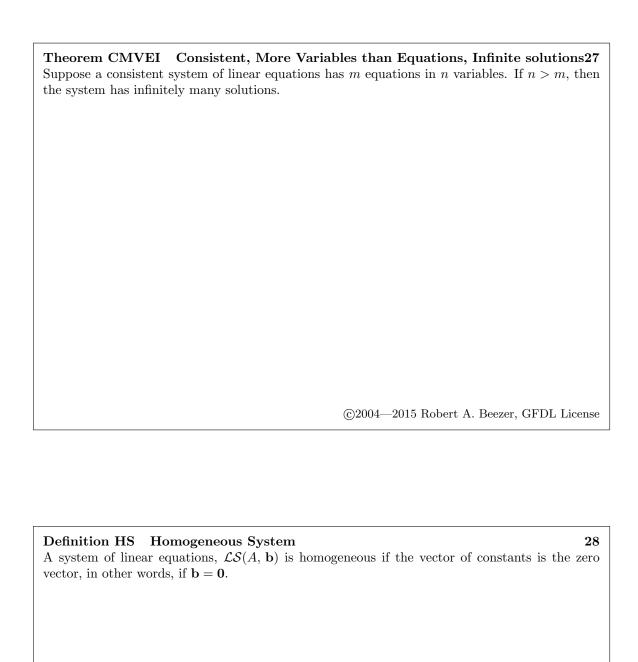
Definition IDV Independent and Dependent Variables

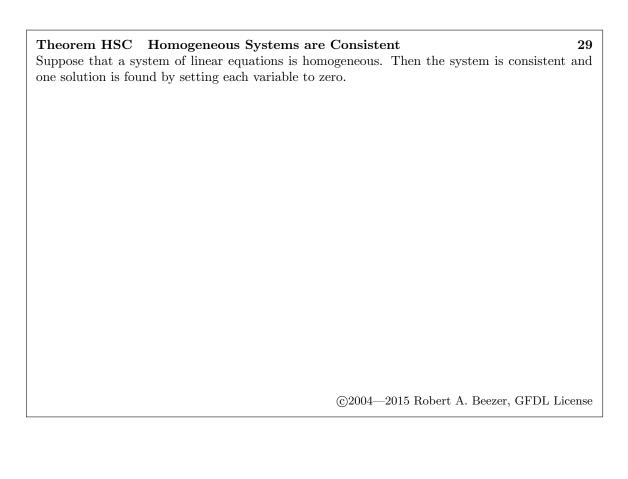
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Suppose A is the augmented matrix of a consistent system of linear equations and B is a row-equivalent matrix in reduced row-echelon form. Suppose j is the index of a pivot column of B. Then the variable x_j is dependent. A variable that is not dependent is called independent or free.



Theorem FVCS Free Variables for Consistent Systems Suppose A is the augmented matrix of a consistent system of linear equations with					
Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with $n-r$ free variables.					
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Theorem PSSLS Possible Solution Sets for Linear Systems A system of linear equations has no solutions, a unique solution or infinitely many	26 v solutions.				

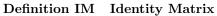




Definition TSHSE Trivial Solution to Homogeneous Systems of Equations 30 Suppose a homogeneous system of linear equations has n variables. The solution $x_1 = 0$, $x_2 = 0$, $x_n = 0$ (i.e. $\mathbf{x} = \mathbf{0}$) is called the trivial solution.

Theorem HMVEI Homogeneous, More Variables than Equations, Infinite solutions 31				
Suppose that a homogeneous system of linear equations has m equations and n variables with				
n > m. Then the system has infinitely many solutions.				
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Definition NSM Null Space of a Matrix 32 The null space of a matrix A , denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions to the homogeneous system $\mathcal{LS}(A, 0)$.				
the homogeneous system $2e(1, 0)$.				

	Square Matrix $m=n$. In this case, we say the matrix has the situation when a matrix is not square, we will call it rectangular.
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Suppose A is a squasystem of equations	Honsingular Matrix 34 re matrix. Suppose further that the solution set to the homogeneous linear $\mathcal{LS}(A, 0)$ is $\{0\}$, in other words, the system has only the trivial solution. is a nonsingular matrix. Otherwise we say A is a singular matrix.



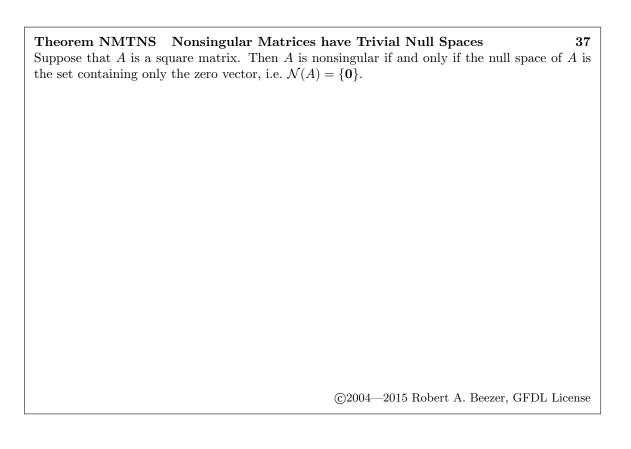
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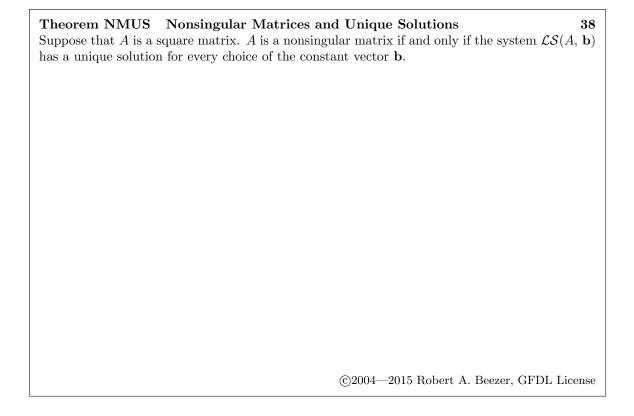
The $m \times m$ identity matrix, I_m , is defined by

$$\left[I_m\right]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad 1 \leq i, j \leq m$$

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Theorem NMRRI Nonsingular Matrices Row Reduce to the Identity matrix 36 Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.





Theorem NME1 Nonsingular Matrix Equivalences, Round 1

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Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- $2.\ A$ row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .

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Definition VSCV Vector Space of Column Vectors

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The vector space \mathbb{C}^m is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers, \mathbb{C} .

Definition CVE Column Vector Equality

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Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$. Then \mathbf{u} and \mathbf{v} are equal, written $\mathbf{u} = \mathbf{v}$ if

$$[\mathbf{u}]_i = [\mathbf{v}]_i$$

$$1 \leq i \leq m$$

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Definition CVA Column Vector Addition

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Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$. The sum of \mathbf{u} and \mathbf{v} is the vector $\mathbf{u} + \mathbf{v}$ defined by

$$[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i$$

$$1 \leq i \leq m$$

Definition CVSM Column Vector Scalar Multiplication

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Suppose $\mathbf{u} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$, then the scalar multiple of \mathbf{u} by α is the vector $\alpha \mathbf{u}$ defined by

$$[\alpha \mathbf{u}]_i = \alpha [\mathbf{u}]_i$$

$$1 \leq i \leq m$$

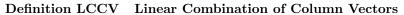
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Theorem VSPCV Vector Space Properties of Column Vectors

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Suppose that \mathbb{C}^m is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- ACC Additive Closure, Column Vectors: If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.
- SCC Scalar Closure, Column Vectors: If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha \mathbf{u} \in \mathbb{C}^m$.
- CC Commutativity, Column Vectors: If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AAC Additive Associativity, Column Vectors: If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- ZC Zero Vector, Column Vectors: There is a vector, $\mathbf{0}$, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.
- AIC Additive Inverses, Column Vectors: If $\mathbf{u} \in \mathbb{C}^m$, then there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMAC Scalar Multiplication Associativity, Column Vectors: If α , $\beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVAC Distributivity across Vector Addition, Column Vectors: If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSAC Distributivity across Scalar Addition, Column Vectors: If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.
- OC One, Column Vectors: If $\mathbf{u} \in \mathbb{C}^m$, then $1\mathbf{u} = \mathbf{u}$.



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Given n vectors \mathbf{u}_1 , \mathbf{u}_2 , \mathbf{u}_3 , ..., \mathbf{u}_n from \mathbb{C}^m and n scalars α_1 , α_2 , α_3 , ..., α_n , their linear combination is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n$$

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Theorem SLSLC Solutions to Linear Systems are Linear Combinations

Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$. Then $\mathbf{x} \in \mathbb{C}^n$ is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if \mathbf{b} equals the linear combination of the columns of A formed with the entries of \mathbf{x} ,

$$\left[\mathbf{x}\right]_{1} \mathbf{A}_{1} + \left[\mathbf{x}\right]_{2} \mathbf{A}_{2} + \left[\mathbf{x}\right]_{3} \mathbf{A}_{3} + \dots + \left[\mathbf{x}\right]_{n} \mathbf{A}_{n} = \mathbf{b}$$

Theorem VFSLS Vector Form of Solutions to Linear Systems

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Suppose that $[A \mid \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A, \mathbf{b})$ of m equations in n variables. Let B be a row-equivalent $m \times (n+1)$ matrix in reduced row-echelon form. Suppose that B has r pivot columns, with indices $D = \{d_1, d_2, d_3, \ldots, d_r\}$, while the n-r non-pivot columns have indices in $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$. Define vectors $\mathbf{c}, \mathbf{u}_j, 1 \le j \le n-r$ of size n by

$$\begin{aligned} \left[\mathbf{c}\right]_i &= \begin{cases} 0 & \text{if } i \in F \\ \left[B\right]_{k,n+1} & \text{if } i \in D, \, i = d_k \end{cases} \\ \left[\mathbf{u}_j\right]_i &= \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -\left[B\right]_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases} . \end{aligned}$$

Then the set of solutions to the system of equations $\mathcal{LS}(A, \mathbf{b})$ is

$$S = \{ \mathbf{c} + \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_{n-r} \mathbf{u}_{n-r} | \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-r} \in \mathbb{C} \}$$

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Theorem PSPHS Particular Solution Plus Homogeneous Solutions

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Suppose that **w** is one solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$. Then **y** is a solution to $\mathcal{LS}(A, \mathbf{b})$ if and only if $\mathbf{y} = \mathbf{w} + \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$.

Definition SSCV Span of a Set of Column Vectors

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Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$, their span, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p$. Symbolically,

$$\langle S \rangle = \left\{ \left. \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_p \mathbf{u}_p \right| \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$
$$= \left\{ \left. \sum_{i=1}^p \alpha_i \mathbf{u}_i \right| \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$

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Theorem SSNS Spanning Sets for Null Spaces

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Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form. Suppose that B has r pivot columns, with indices given by $D = \{d_1, d_2, d_3, \ldots, d_r\}$, while the n-r non-pivot columns have indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$. Construct the n-r vectors \mathbf{z}_j , $1 \le j \le n-r$ of size n,

$$\left[\mathbf{z}_{j}\right]_{i} = \begin{cases} 1 & \text{if } i \in F, \ i = f_{j} \\ 0 & \text{if } i \in F, \ i \neq f_{j} \\ -\left[B\right]_{k,f_{j}} & \text{if } i \in D, \ i = d_{k} \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r}\} \rangle$$

Definition	RLDCV	$R\epsilon$	elation of Linear	Dependence fo	or (Column ${f V}_0$	ectors	
	_							

Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n}$, a true statement of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a relation of linear dependence on S. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \le i \le n$, then we say it is the trivial relation of linear dependence on S.

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Definition LICV Linear Independence of Column Vectors

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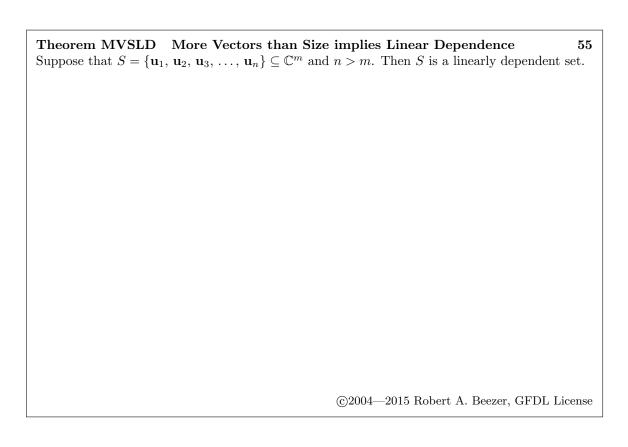
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The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is linearly dependent if there is a relation of linear dependence on S that is not trivial. In the case where the only relation of linear dependence on S is the trivial one, then S is a linearly independent set of vectors.

Theorem LIVRN Linearly Independent Vectors, r and n

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Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\} \subseteq \mathbb{C}^m$ is a set of vectors and A is the $m \times n$ matrix whose columns are the vectors in S. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of pivot columns in B. Then S is linearly independent if and only if n = r.



Theorem NMLIC Nonsingular Matrices have Linearly Independent Columns 56 Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

57

Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of **b**.
- 5. The columns of A form a linearly independent set.

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Theorem BNS Basis for Null Spaces

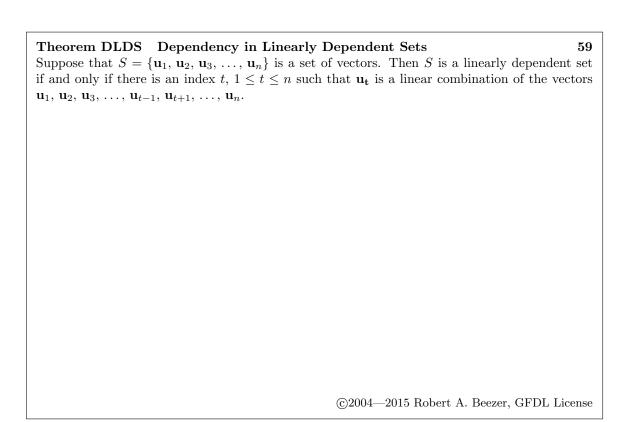
58

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r pivot columns. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have pivot columns. Construct the n-r vectors \mathbf{z}_j , $1 \le j \le n-r$ of size n as

$$\left[\mathbf{z}_{j}\right]_{i} = \begin{cases} 1 & \text{if } i \in F, \ i = f_{j} \\ 0 & \text{if } i \in F, \ i \neq f_{j} \\ -\left[B\right]_{k,f_{j}} & \text{if } i \in D, \ i = d_{k} \end{cases}$$

Define the set $S = {\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_{n-r}}$. Then

- 1. $\mathcal{N}(A) = \langle S \rangle$.
- 2. S is a linearly independent set.



Theorem BS Basis of a Span

60

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ is a set of column vectors. Define $W = \langle S \rangle$ and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with $D = \{d_1, d_2, d_3, \dots, d_r\}$ the set of indices for the pivot columns of B. Then

- 1. $T = {\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \dots \mathbf{v}_{d_r}}$ is a linearly independent set.
- 2. $W = \langle T \rangle$.

$\ \, \textbf{Definition CCCV} \quad \textbf{Complex Conjugate of a Column Vector} \\$

61

Suppose that **u** is a vector from \mathbb{C}^m . Then the conjugate of the vector, $\overline{\mathbf{u}}$, is defined by

$$[\overline{\mathbf{u}}]_i = \overline{[\mathbf{u}]_i}$$

$$1 \leq i \leq m$$

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Theorem CRVA Conjugation Respects Vector Addition

62

Suppose **x** and **y** are two vectors from \mathbb{C}^m . Then

$$\overline{\mathbf{x} + \mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$$



Suppose **x** is a vector from \mathbb{C}^m , and $\alpha \in \mathbb{C}$ is a scalar. Then

$$\overline{\alpha}\overline{\mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$$

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Definition IP Inner Product

64

63

Given the vectors \mathbf{u} , $\mathbf{v} \in \mathbb{C}^m$ the inner product of \mathbf{u} and \mathbf{v} is the scalar quantity in \mathbb{C} ,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \overline{[\mathbf{u}]_1} [\mathbf{v}]_1 + \overline{[\mathbf{u}]_2} [\mathbf{v}]_2 + \overline{[\mathbf{u}]_3} [\mathbf{v}]_3 + \dots + \overline{[\mathbf{u}]_m} [\mathbf{v}]_m = \sum_{i=1}^m \overline{[\mathbf{u}]_i} [\mathbf{v}]_i$$

Theorem IPVA Inner Product and Vector Addition

Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$. Then

- 1. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
- 2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

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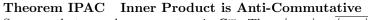
65

66

${\bf Theorem~IPSM~~Inner~Product~and~Scalar~Multiplication}$

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$. Then

- 1. $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \overline{\alpha} \langle \mathbf{u}, \mathbf{v} \rangle$
- 2. $\langle \mathbf{u}, \, \alpha \mathbf{v} \rangle = \alpha \, \langle \mathbf{u}, \, \mathbf{v} \rangle$



Suppose that **u** and **v** are vectors in \mathbb{C}^m . Then $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.

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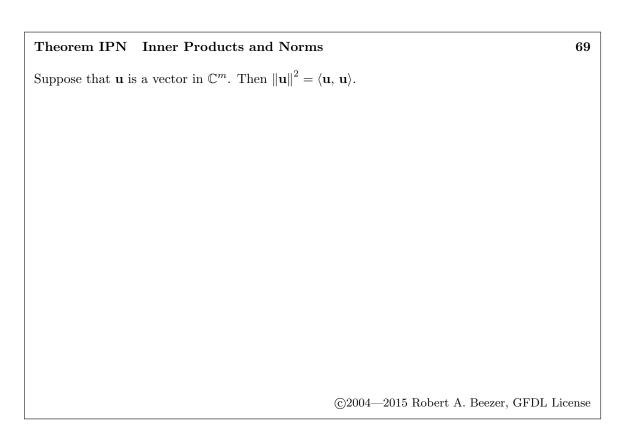
Definition NV Norm of a Vector

68

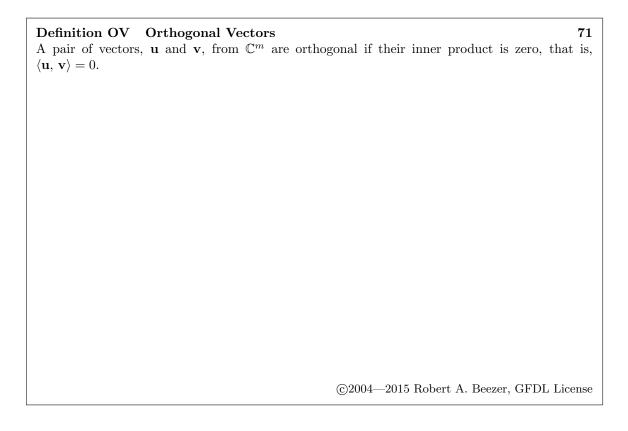
67

The norm of the vector ${\bf u}$ is the scalar quantity in ${\mathbb C}$

$$\|\mathbf{u}\| = \sqrt{|[\mathbf{u}]_1|^2 + |[\mathbf{u}]_2|^2 + |[\mathbf{u}]_3|^2 + \dots + |[\mathbf{u}]_m|^2} = \sqrt{\sum_{i=1}^m |[\mathbf{u}]_i|^2}$$



Theorem PIP Positive Inner Products Suppose that ${\bf u}$ is a vector in \mathbb{C}^m . Then $\langle {\bf u}, {\bf u} \rangle \geq 0$ with equality if and only if ${\bf u}={\bf 0}$.



Definition OSV Orthogonal Set of Vectors

72

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a set of vectors from \mathbb{C}^m . Then S is an orthogonal set if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$.

Definition SUV Standard Unit Vectors

Let $\mathbf{e}_j \in \mathbb{C}^m$, $1 \leq j \leq m$ denote the column vectors defined by

$$\left[\mathbf{e}_{j}\right]_{i} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Then the set

$$\{\mathbf{e}_1, \, \mathbf{e}_2, \, \mathbf{e}_3, \, \dots, \, \mathbf{e}_m\} = \{\, \mathbf{e}_j | \, 1 \le j \le m\}$$

is the set of standard unit vectors in \mathbb{C}^m .

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Theorem OSLI Orthogonal Sets are Linearly Independent

Suppose that S is an orthogonal set of nonzero vectors. Then S is linearly independent.

Theorem GSP Gram-Schmidt Procedure

75

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$ is a linearly independent set of vectors in \mathbb{C}^m . Define the vectors $\mathbf{u}_i, 1 \leq i \leq p$ by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{u}_1, \, \mathbf{v}_i \rangle}{\langle \mathbf{u}_1, \, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{u}_2, \, \mathbf{v}_i \rangle}{\langle \mathbf{u}_2, \, \mathbf{u}_2 \rangle} \mathbf{u}_2 - \frac{\langle \mathbf{u}_3, \, \mathbf{v}_i \rangle}{\langle \mathbf{u}_3, \, \mathbf{u}_3 \rangle} \mathbf{u}_3 - \dots - \frac{\langle \mathbf{u}_{i-1}, \, \mathbf{v}_i \rangle}{\langle \mathbf{u}_{i-1}, \, \mathbf{u}_{i-1} \rangle} \mathbf{u}_{i-1}$$

Let $T = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$. Then T is an orthogonal set of nonzero vectors, and $\langle T \rangle = \langle S \rangle$.

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Definition ONS OrthoNormal Set

76

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \le i \le n$. Then S is an orthonormal set of vectors.

•	77
The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complenumbers.	ex
numbers.	
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Definition ME Matrix Equality 78
The $m \times n$ matrices A and B are equal, written A = B provided $[A]_{ij} = [B]_{ij}$ for all $1 \le i \le m$, $1 \le j \le n$.

Definition MA Matrix Addition

79

Given the $m \times n$ matrices A and B, define the sum of A and B as an $m \times n$ matrix, written A + B, according to

$$[A + B]_{ij} = [A]_{ij} + [B]_{ij}$$

$$1 \leq i \leq m, \, 1 \leq j \leq n$$

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Definition MSM Matrix Scalar Multiplication

80

Given the $m \times n$ matrix A and the scalar $\alpha \in \mathbb{C}$, the scalar multiple of A is an $m \times n$ matrix, written αA and defined according to

$$[\alpha A]_{ij} = \alpha \, [A]_{ij}$$

$$1 \le i \le m, \ 1 \le j \le n$$

Theorem VSPM Vector Space Properties of Matrices

81

Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- ACM Additive Closure, Matrices: If $A, B \in M_{mn}$, then $A + B \in M_{mn}$.
- SCM Scalar Closure, Matrices: If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.
- CM Commutativity, Matrices: If $A, B \in M_{mn}$, then A + B = B + A.
- AAM Additive Associativity, Matrices: If $A, B, C \in M_{mn}$, then A + (B + C) = (A + B) + C.
- ZM Zero Matrix, Matrices: There is a matrix, \mathcal{O} , called the zero matrix, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.
- AIM Additive Inverses, Matrices: If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.
- SMAM Scalar Multiplication Associativity, Matrices: If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha \beta)A$.
- DMAM Distributivity across Matrix Addition, Matrices: If $\alpha \in \mathbb{C}$ and $A, B \in M_{mn}$, then $\alpha(A+B) = \alpha A + \alpha B$.
- DSAM Distributivity across Scalar Addition, Matrices: If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.
- OM One, Matrices: If $A \in M_{mn}$, then 1A = A.

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Definition ZM Zero Matrix

82

The $m \times n$ zero matrix is written as $\mathcal{O} = \mathcal{O}_{m \times n}$ and defined by $[\mathcal{O}]_{ij} = 0$, for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Definition TM Transpose of a Matrix

83

84

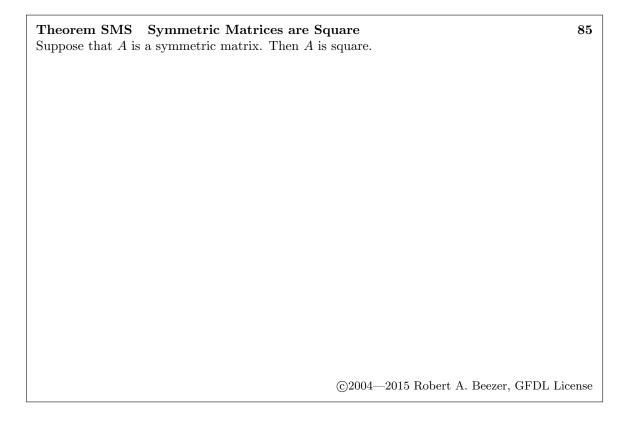
Given an $m \times n$ matrix A, its transpose is the $n \times m$ matrix A^t given by

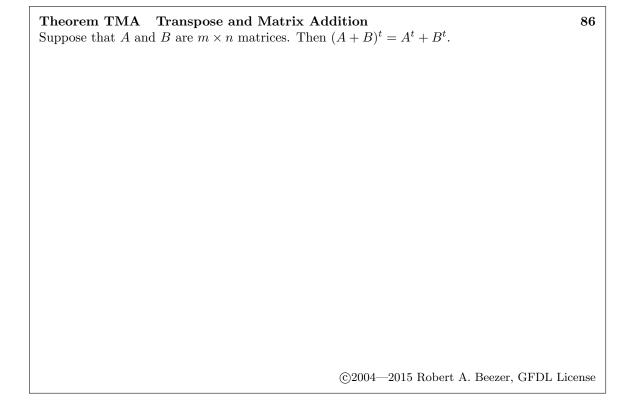
$$\left[A^t\right]_{ij} = [A]_{ji}\,, \quad 1 \leq i \leq n,\, 1 \leq j \leq m.$$

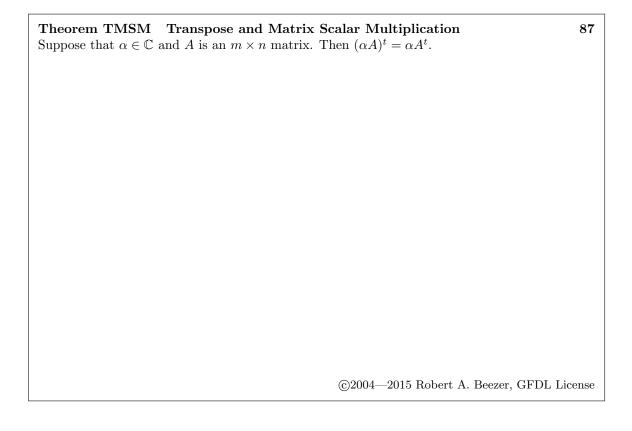
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Definition SYM Symmetric Matrix

The matrix A is symmetric if $A = A^t$.







Suppose that A is an $m \times n$ matrix. Then $(A^t)^t = A$.

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88

Theorem TT Transpose of a Transpose

Definition CCM Complex Conjugate of a Matrix

89

Suppose A is an $m \times n$ matrix. Then the conjugate of A, written \overline{A} is an $m \times n$ matrix defined by

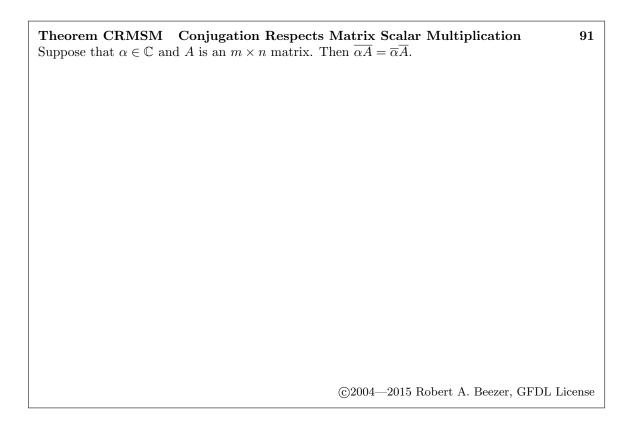
$$\left[\overline{A}\right]_{ij} = \overline{[A]_{ij}}$$

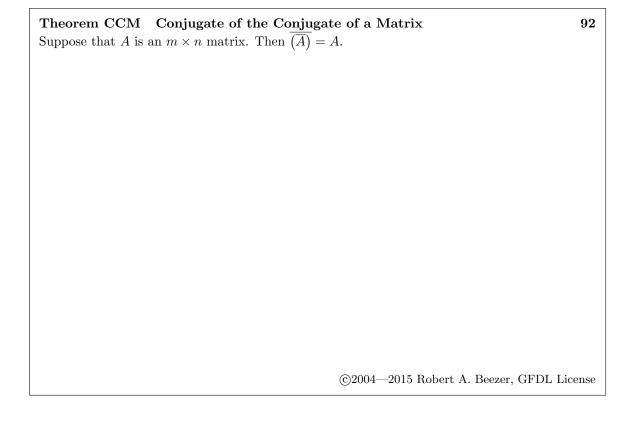
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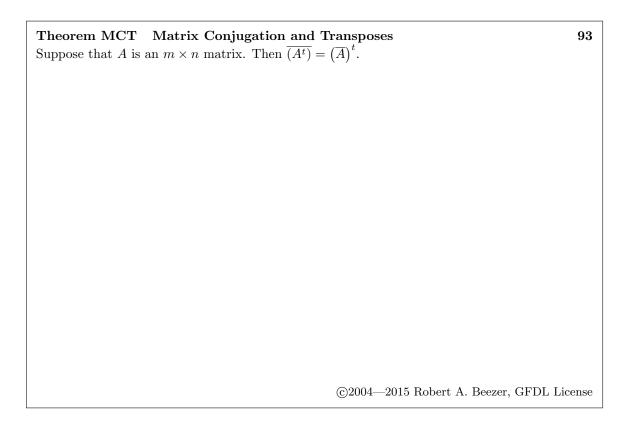
${\bf Theorem~CRMA~~Conjugation~Respects~M\underline{atrix}~A\underline{ddition}}$

Suppose that A and B are $m \times n$ matrices. Then $\overline{A+B} = \overline{A} + \overline{B}$.

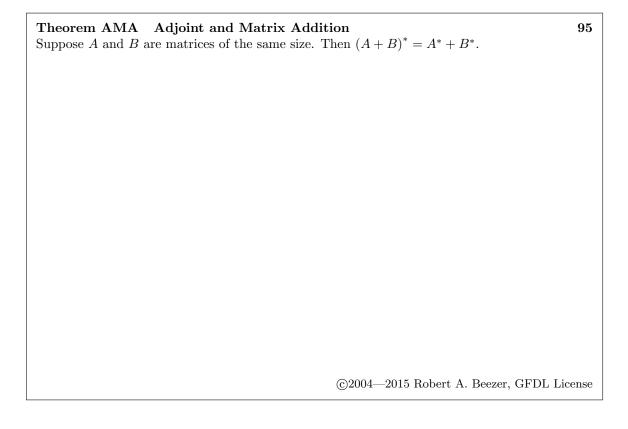
90

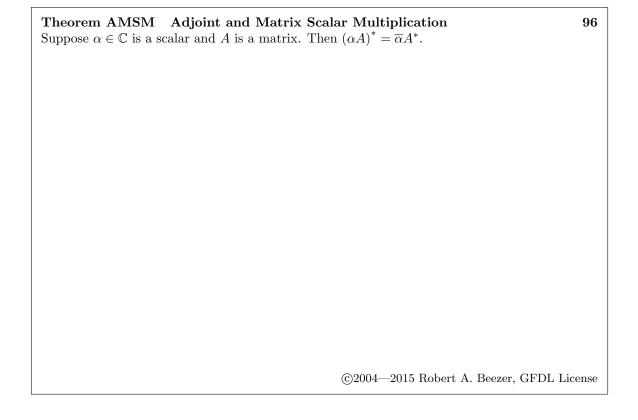


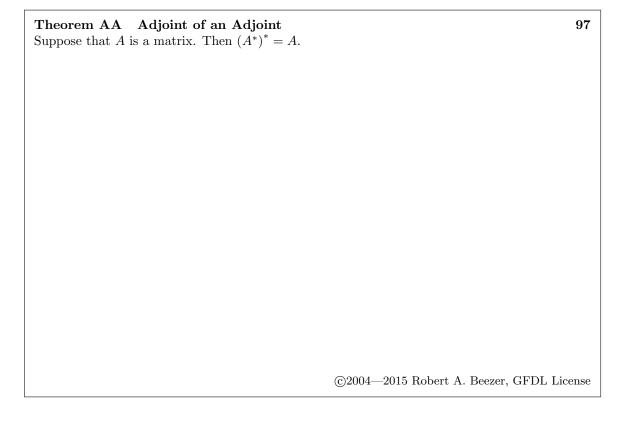




Definition A Adjoint 94 If A is a matrix, then its adjoint is $A^* = \left(\overline{A}\right)^t$.





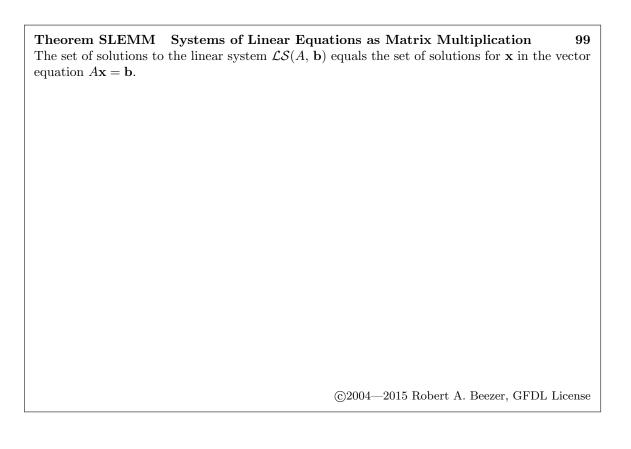


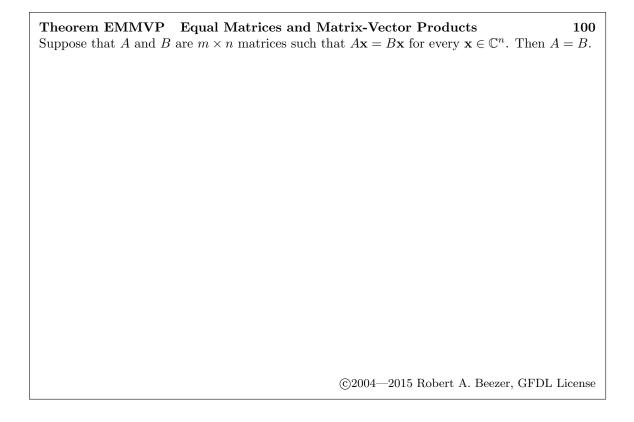
Definition MVP Matrix-Vector Product

98

Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ and \mathbf{u} is a vector of size n. Then the matrix-vector product of A with \mathbf{u} is the linear combination

$$A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \dots + [\mathbf{u}]_n \mathbf{A}_n$$





Definition MM Matrix Multiplication

101

Suppose A is an $m \times n$ matrix and $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$ are the columns of an $n \times p$ matrix B. Then the matrix product of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

$$AB = A \left[\mathbf{B}_1 | \mathbf{B}_2 | \mathbf{B}_3 | \dots | \mathbf{B}_p \right] = \left[A \mathbf{B}_1 | A \mathbf{B}_2 | A \mathbf{B}_3 | \dots | A \mathbf{B}_p \right].$$

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Theorem EMP Entries of Matrix Products

102

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then for $1 \le i \le m$, $1 \le j \le p$, the individual entries of AB are given by

$$[AB]_{ij} = [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \dots + [A]_{in} [B]_{nj}$$
$$= \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$$

Theorem MMZM Matrix Multiplication and the Zero Matrix

103

Suppose A is an $m \times n$ matrix. Then

- 1. $A\mathcal{O}_{n\times p} = \mathcal{O}_{m\times p}$
- $2. \ \mathcal{O}_{p\times m}A = \mathcal{O}_{p\times n}$

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Theorem MMIM Matrix Multiplication and Identity Matrix

104

Suppose A is an $m \times n$ matrix. Then

- 1. $AI_n = A$
- $2. I_m A = A$

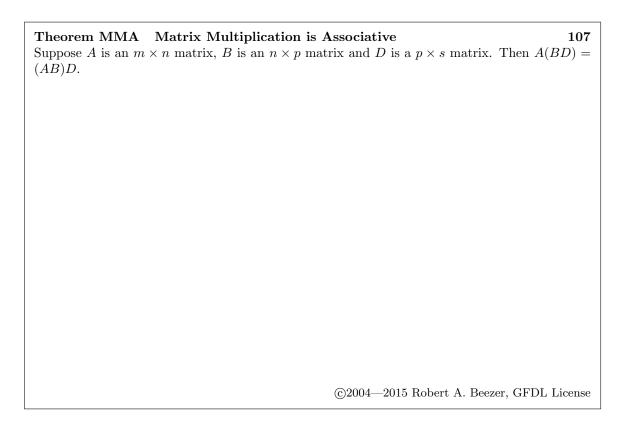
Theorem MMDAA Matrix Multiplication Distributes Across Addition 105 Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then

$$1. \ A(B+C) = AB + AC$$

$$2. \ (B+C)D = BD + CD$$

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Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication 106 Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

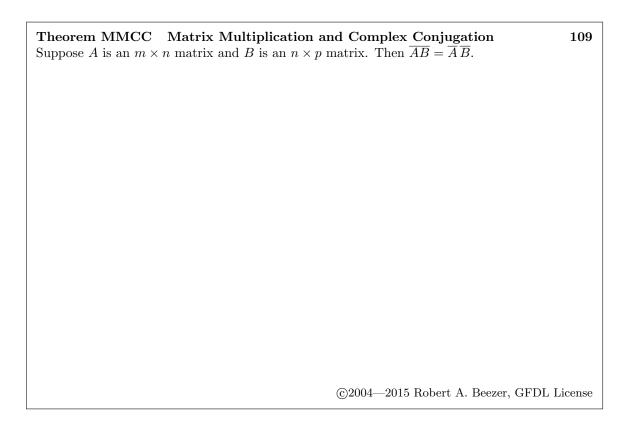


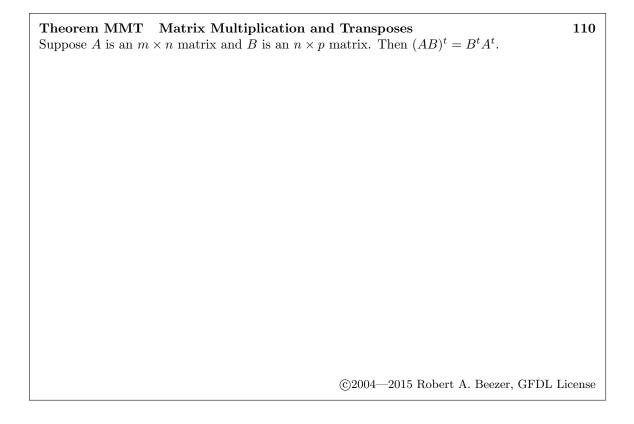
${\bf Theorem~MMIP~~Matrix~Multiplication~and~Inner~Products}$

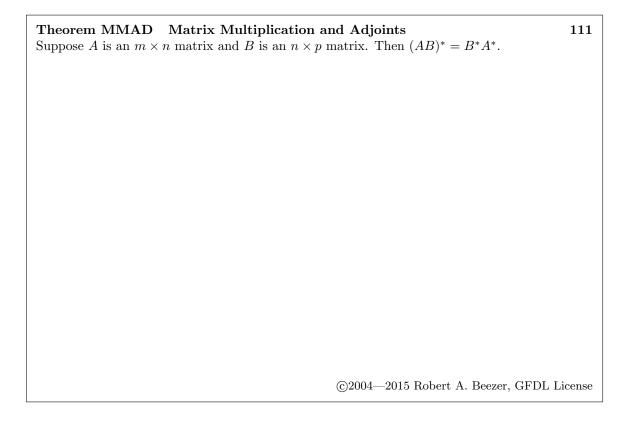
108

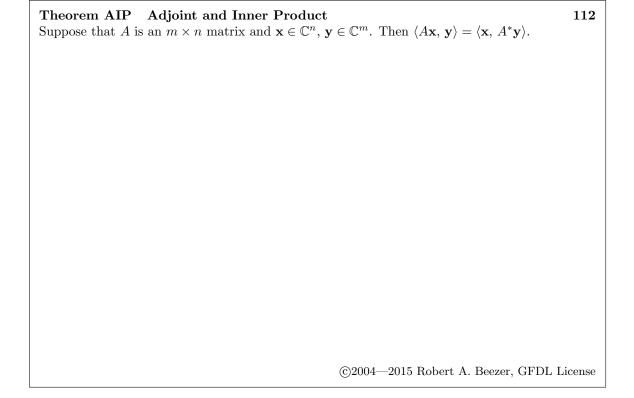
If we consider the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then

$$\langle \mathbf{u},\,\mathbf{v}\rangle = \overline{\mathbf{u}}^t\mathbf{v} = \mathbf{u}^*\mathbf{v}$$











Definition MI Matrix Inverse

115

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is invertible and B is the inverse of A. In this situation, we write $B = A^{-1}$.

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Theorem TTMI Two-by-Two Matrix Inverse

116

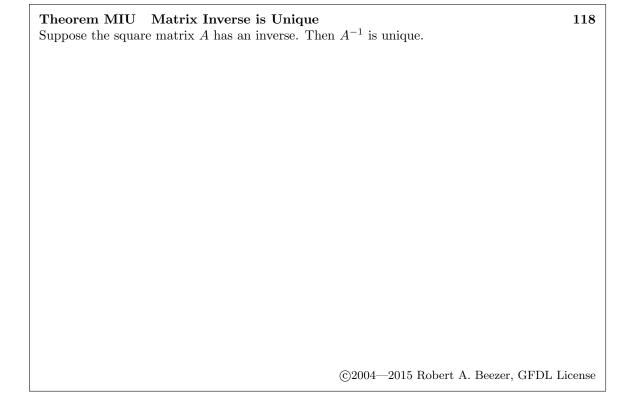
Suppose

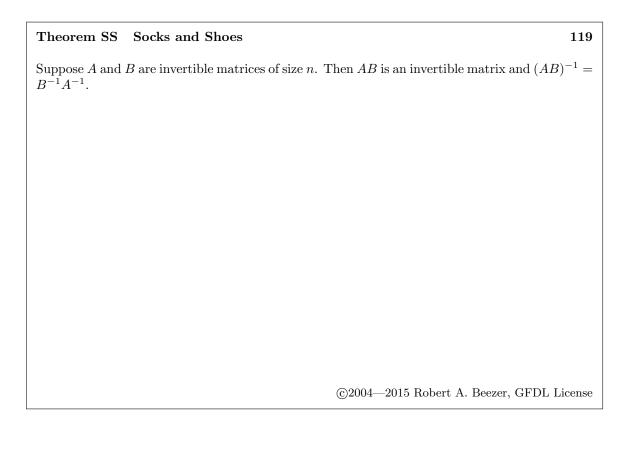
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem CINM Computing the Inverse of a Nonsingular Matrix 117 Suppose A is a nonsingular square matrix of size n . Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A . Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N . Then $AJ = I_n$.
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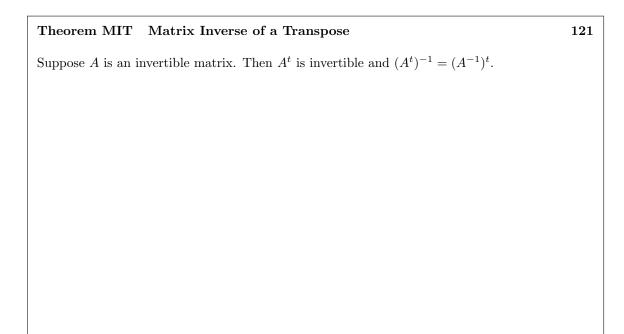




Theorem MIMI Matrix Inverse of a Matrix Inverse

120

Suppose A is an invertible matrix. Then A^{-1} is invertible and $(A^{-1})^{-1} = A$.



Theorem MISM Matrix Inverse of a Scalar Multiple

122

Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ and αA is invertible.

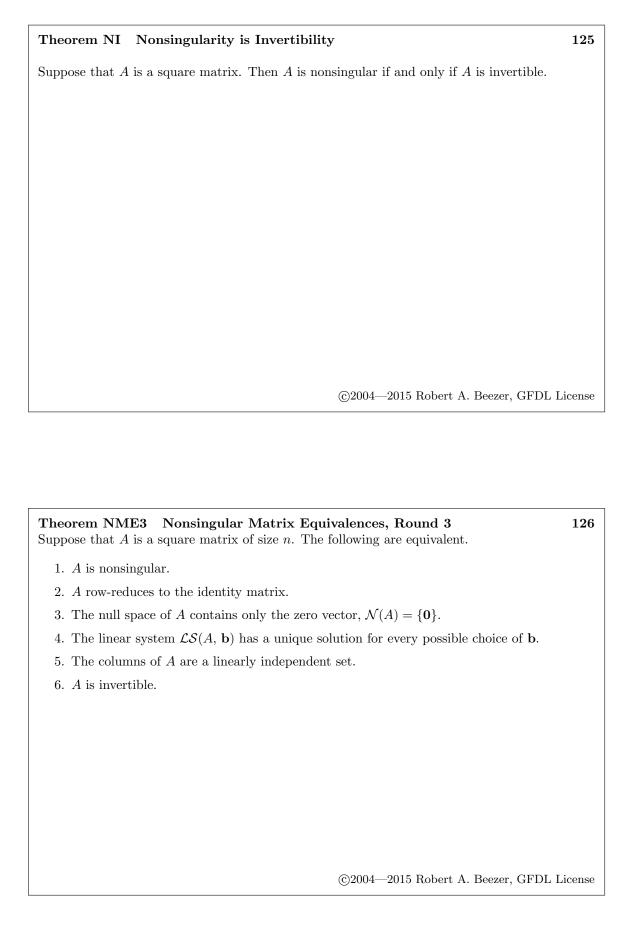
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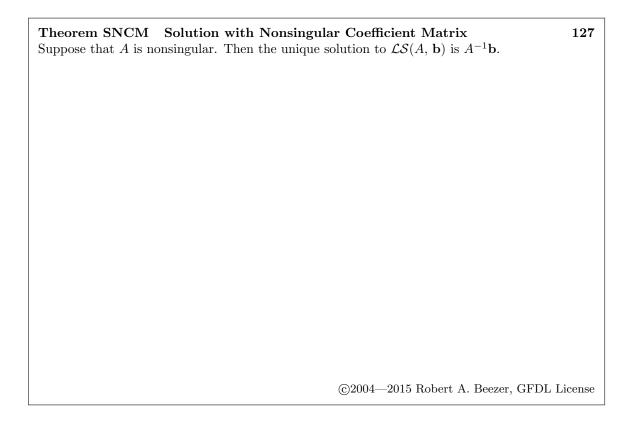
Theorem NPNT Nonsingular Product has Suppose that A and B are square matrices of size a if A and B are both nonsingular.	
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Theorem OSIS One-Sided Inverse is Sufficient

 $\bf 124$

Suppose A and B are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.

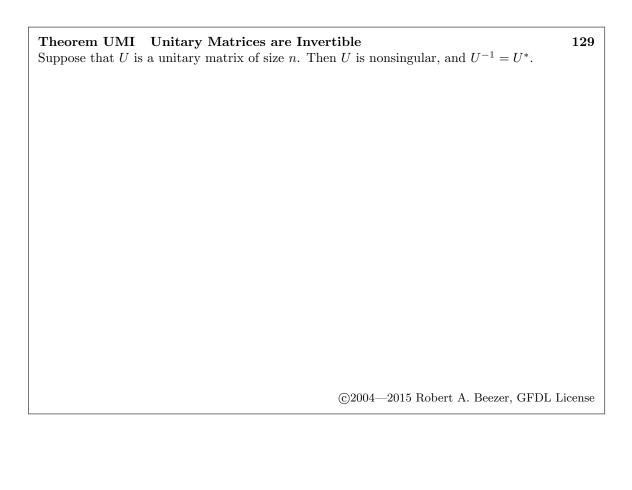




Definition UM Unitary Matrices

128

Suppose that U is a square matrix of size n such that $U^*U = I_n$. Then we say U is unitary.



Theorem CUMOS Columns of Unitary Matrices are Orthonormal Sets 130 Suppose that $S = \{A_1, A_2, A_3, ..., A_n\}$ is the set of columns of a square matrix A of size n. Then A is a unitary matrix if and only if S is an orthonormal set.

Theorem UMPIP Unitary Matrices Preserve Inner Products

131

Suppose that U is a unitary matrix of size n and **u** and **v** are two vectors from \mathbb{C}^n . Then

$$\langle U\mathbf{u}, U\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

and

 $||U\mathbf{v}|| = ||\mathbf{v}||$

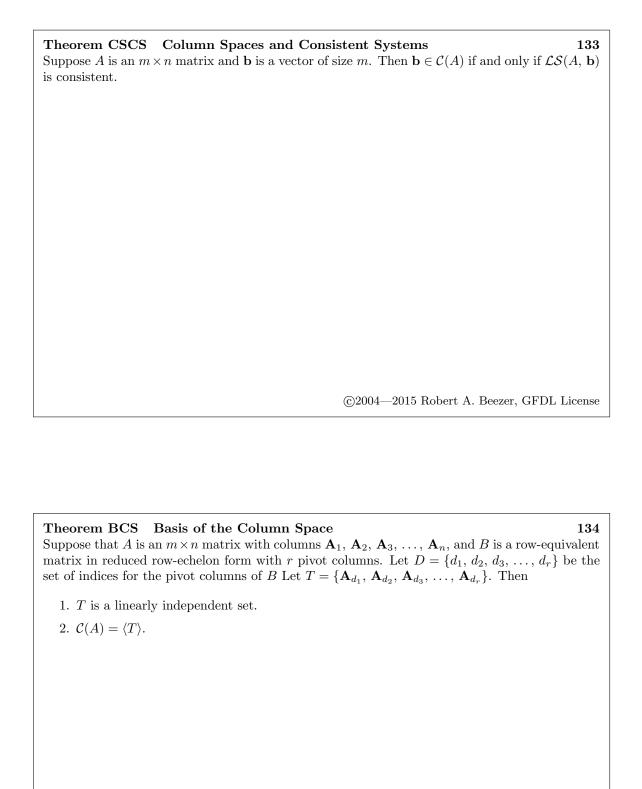
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Definition CSM Column Space of a Matrix

132

Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$. Then the column space of A, written $\mathcal{C}(A)$, is the subset of \mathbb{C}^m containing all linear combinations of the columns of A,

$$C(A) = \langle \{ \mathbf{A}_1, \, \mathbf{A}_2, \, \mathbf{A}_3, \, \dots, \, \mathbf{A}_n \} \rangle$$





135

Suppose A is a square matrix of size n. Then A is nonsingular if and only if $\mathcal{C}(A) = \mathbb{C}^n$.

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Theorem NME4 $\,$ Nonsingular Matrix Equivalences, Round 4

136

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.

Definition RSM Row Space of a Matrix Suppose A is an $m \times n$ matrix. Then the row space of A , $\mathcal{R}(A)$, is the column space of $\mathcal{R}(A) = \mathcal{C}(A^t)$.	137 of A^t , i.e.
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Theorem REMRS Row-Equivalent Matrices have equal Row Spaces Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A)=\mathcal{R}(B)$.	138

Theorem BRS Basis for the Row Space

139

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t . Then

- 1. $\mathcal{R}(A) = \langle S \rangle$.
- 2. S is a linearly independent set.

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Theorem CSRST Column Space, Row Space, Transpose

140

Suppose A is a matrix. Then $C(A) = \mathcal{R}(A^t)$.



141

Suppose A is an $m \times n$ matrix. Then the left null space is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$.

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Definition EEF Extended Echelon Form

142

Suppose A is an $m \times n$ matrix. Extend A on its right side with the addition of an $m \times m$ identity matrix to form an $m \times (n+m)$ matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the extended reduced row-echelon form of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the $m \times n$ matrix formed from the first n columns of N and let J denote the $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the $r \times n$ matrix formed from all of the nonzero rows of B. Let K be the $r \times m$ matrix formed from the first r rows of J, while L will be the $(m-r) \times m$ matrix formed from the bottom m-r rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix} C & K \\ \hline 0 & L \end{bmatrix}$$

Theorem PEEF Properties of Extended Echelon Form

143

Suppose that A is an $m \times n$ matrix and that N is its extended echelon form. Then

- 1. J is nonsingular.
- 2. B = JA.
- 3. If $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$, then $A\mathbf{x} = \mathbf{y}$ if and only if $B\mathbf{x} = J\mathbf{y}$.
- 4. C is in reduced row-echelon form, has no zero rows and has r pivot columns.
- 5. L is in reduced row-echelon form, has no zero rows and has m-r pivot columns.

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Theorem FS Four Subsets

144

Suppose A is an $m \times n$ matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last m-r rows. Then

- 1. The null space of A is the null space of C, $\mathcal{N}(A) = \mathcal{N}(C)$.
- 2. The row space of A is the row space of C, $\mathcal{R}(A) = \mathcal{R}(C)$.
- 3. The column space of A is the null space of L, $C(A) = \mathcal{N}(L)$.
- 4. The left null space of A is the row space of L, $\mathcal{L}(A) = \mathcal{R}(L)$.

Definition VS Vector Space

145

Suppose that V is a set upon which we have defined two operations: (1) vector addition, which combines two elements of V and is denoted by "+", and (2) scalar multiplication, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a vector space over \mathbb{C} if the following ten properties hold.

- AC Additive Closure: If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.
- SC Scalar Closure: If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha \mathbf{u} \in V$.
- C Commutativity: If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AA Additive Associativity: If \mathbf{u} , \mathbf{v} , $\mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- Z Zero Vector: There is a vector, $\mathbf{0}$, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.
- AI Additive Inverses: If $\mathbf{u} \in V$, then there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMA Scalar Multiplication Associativity: If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVA Distributivity across Vector Addition: If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSA Distributivity across Scalar Addition: If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$.
- O One: If $\mathbf{u} \in V$, then $1\mathbf{u} = \mathbf{u}$.

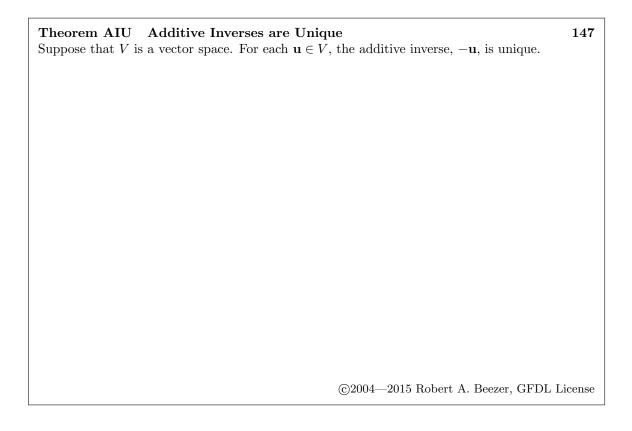
The objects in V are called vectors, no matter what else they might really be, simply by virtue of being elements of a vector space.

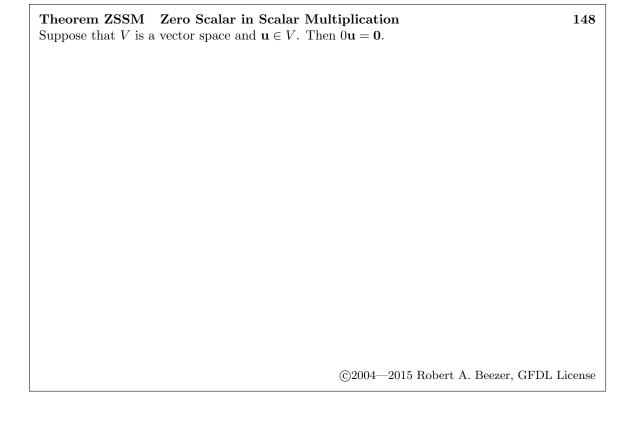
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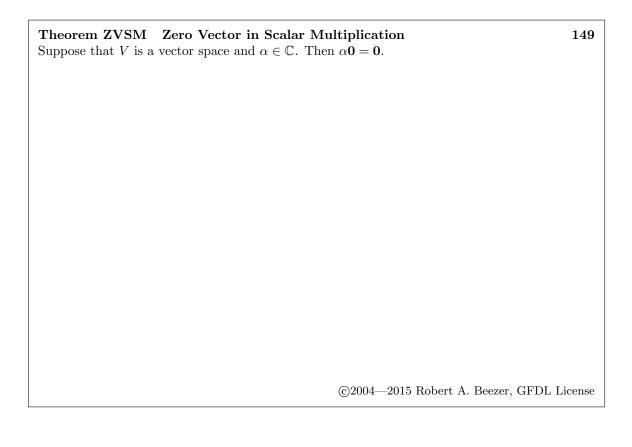
Theorem ZVU Zero Vector is Unique

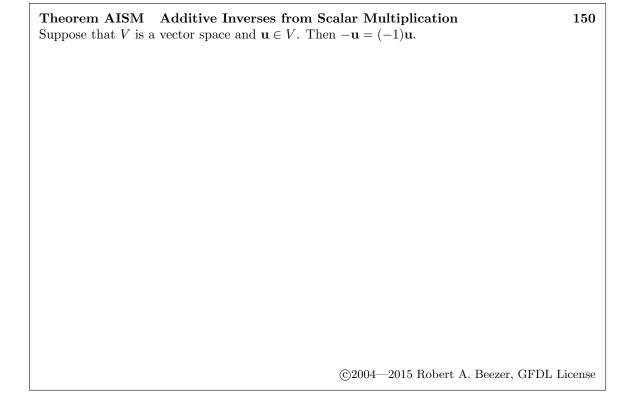
146

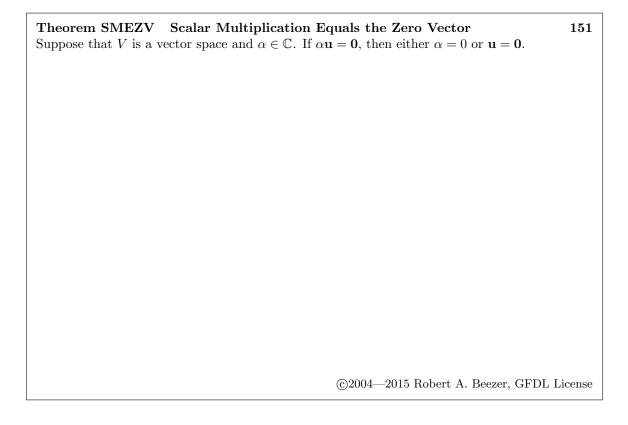
Suppose that V is a vector space. The zero vector, $\mathbf{0}$, is unique.











Definition S Subspace

152

Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of V, $W \subseteq V$. Then W is a subspace of V.

Theorem TSS Testing Subsets for Subspaces

153

Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

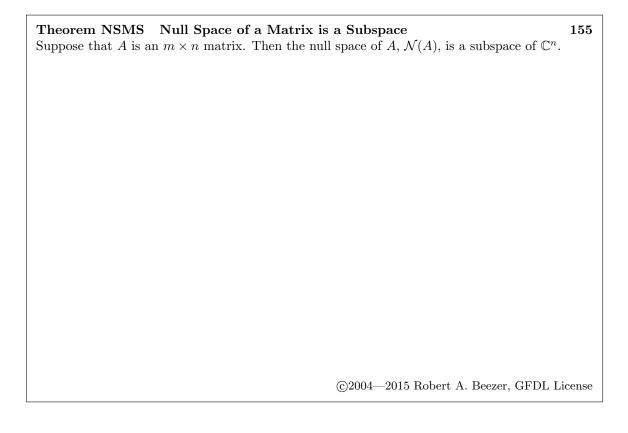
- 1. W is nonempty, $W \neq \emptyset$.
- 2. If $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.
- 3. If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha \mathbf{x} \in W$.

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Definition TS Trivial Subspaces

154

Given the vector space V, the subspaces V and $\{0\}$ are each called a trivial subspace.



Definition LC Linear Combination

156

Suppose that V is a vector space. Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their linear combination is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n.$$

Definition SS Span of a Set

157

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$, their span, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$. Symbolically,

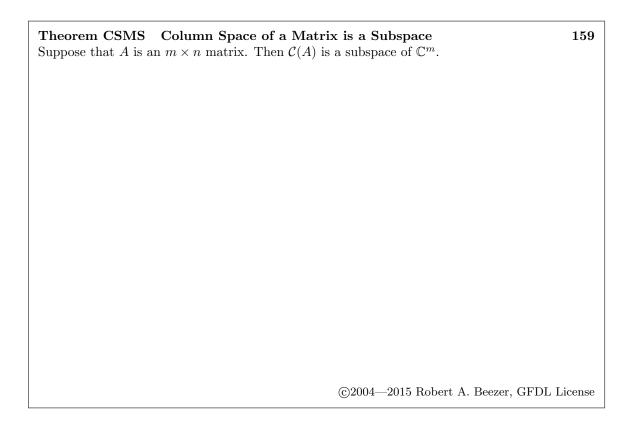
$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_t \mathbf{u}_t \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$
$$= \left\{ \sum_{i=1}^t \alpha_i \mathbf{u}_i \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$

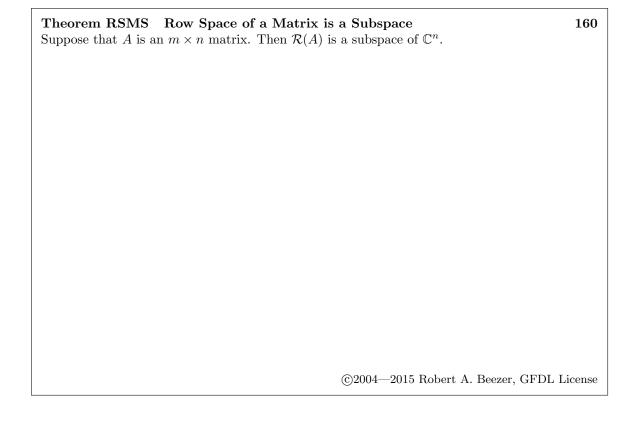
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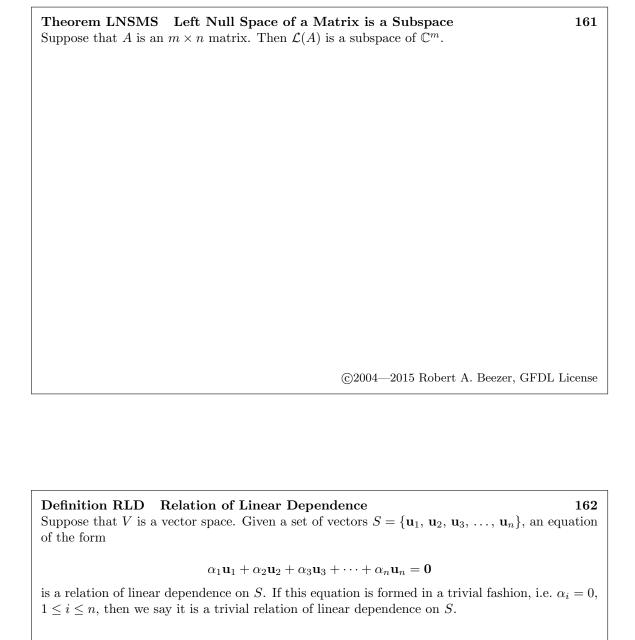
Theorem SSS Span of a Set is a Subspace

158

Suppose V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\} \subseteq V$, their span, $\langle S \rangle$, is a subspace.







Definition LI Linear Independence Suppose that V is a vector space. The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ from V is linearly dependent if there is a relation of linear dependence on S that is not trivial. In the case where the only relation of linear dependence on S is the trivial one, then S is a linearly independent set of vectors.
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Definition SSVS Spanning Set of a Vector Space Suppose V is a vector space. A subset S of V is a spanning set of V if $\langle S \rangle = V$. In this case, we also frequently say S spans V .

${\bf Theorem~VRRB~~Vector~Representation~Relative~to~a~Basis}$

165

Suppose that V is a vector space and $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$ is a linearly independent set that spans V. Let \mathbf{w} be any vector in V. Then there exist unique scalars $a_1, a_2, a_3, \dots, a_m$ such that

$$\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_m \mathbf{v}_m.$$

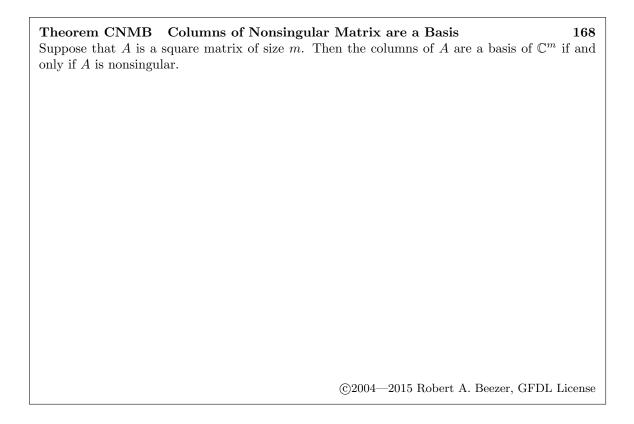
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Definition B Basis

166

Suppose V is a vector space. Then a subset $S \subseteq V$ is a basis of V if it is linearly independent and spans V.

Theorem SUVB Standard Unit Vectors are a Basis 167 The set of standard unit vectors for \mathbb{C}^m (Definition SUV), $B = \{\mathbf{e}_i 1 \le i \le m\}$ is a basis for the vector space \mathbb{C}^m .		
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Theorem NME5 Nonsingular Matrix Equivalences, Round 5

169

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- $2.\ A$ row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .

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Theorem COB Coordinates and Orthonormal Bases

170

Suppose that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$ is an orthonormal basis of the subspace W of \mathbb{C}^m . For any $\mathbf{w} \in W$,

$$\mathbf{w} = \langle \mathbf{v}_1, \, \mathbf{w} \rangle \, \mathbf{v}_1 + \langle \mathbf{v}_2, \, \mathbf{w} \rangle \, \mathbf{v}_2 + \langle \mathbf{v}_3, \, \mathbf{w} \rangle \, \mathbf{v}_3 + \cdots + \langle \mathbf{v}_p, \, \mathbf{w} \rangle \, \mathbf{v}_p$$



Let A be an $n \times n$ matrix and $B = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n\}$ be an orthonormal basis of \mathbb{C}^n . Define

$$C = \{A\mathbf{x}_1, A\mathbf{x}_2, A\mathbf{x}_3, \dots, A\mathbf{x}_n\}$$

Then A is a unitary matrix if and only if C is an orthonormal basis of \mathbb{C}^n .

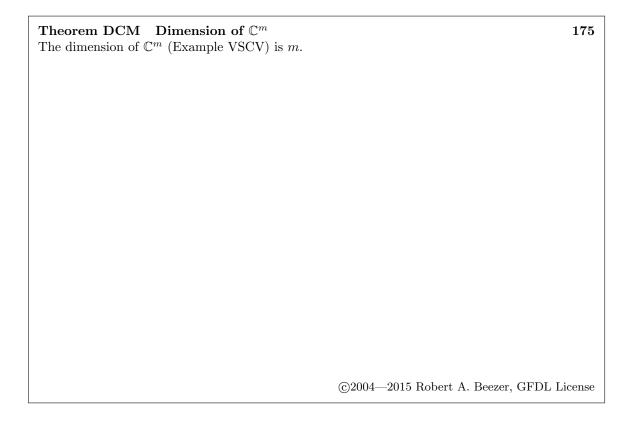
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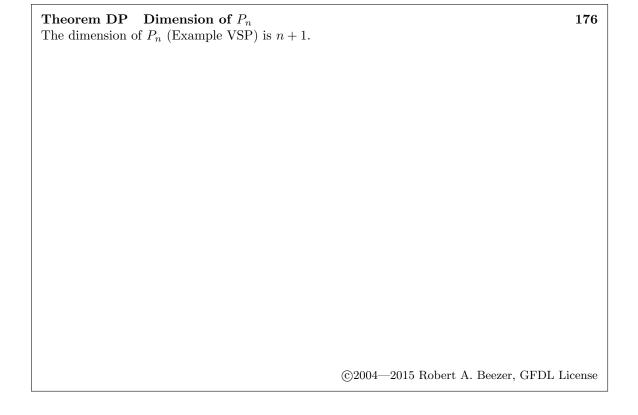
Definition D Dimension

172

Suppose that V is a vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a basis of V. Then the dimension of V is defined by dim (V) = t. If V has no finite bases, we say V has infinite dimension.

Theorem SSLD Spanning Sets and Linear Dependence 173 Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V . Then any set of $t+1$ or more vectors from V is linearly dependent.	
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Theorem BIS Bases have Identical Sizes Suppose that V is a vector space with a finite basis B and a second basis C . Then B and C have the same size.	





Theorem DM Dimension of M_{mn}	177
The dimension of M_{mn} (Example VSM) is mn .	
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Definition NOM Nullity Of a Matrix

178

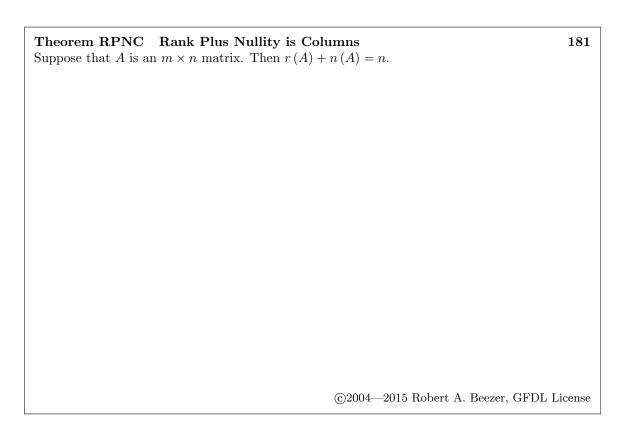
Suppose that A is an $m \times n$ matrix. Then the nullity of A is the dimension of the null space of A, $n(A) = \dim(\mathcal{N}(A))$.

Definition ROM Rank Of a Matrix Suppose that A is an $m \times n$ matrix. Then the rank A , $r(A) = \dim(\mathcal{C}(A))$.	${\bf 179}$ of A is the dimension of the column space of
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Theorem CRN Computing Rank and Nullity

180

Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form. Let r denote the number of pivot columns (or the number of nonzero rows). Then r(A) = r and n(A) = n - r.



Theorem RNNM Rank and Nullity of a Nonsingular Matrix

182

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. The rank of A is n, r(A) = n.
- 3. The nullity of A is zero, n(A) = 0.

Theorem NME6 Nonsingular Matrix Equivalences, Round 6

183

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- $2.\ A$ row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.

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Theorem ELIS Extending Linearly Independent Sets

184

Suppose V is a vector space and S is a linearly independent set of vectors from V. Suppose w is a vector such that $\mathbf{w} \notin \langle S \rangle$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

Theorem G Goldilocks

185

Suppose that V is a vector space of dimension t. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$ be a set of vectors from V. Then

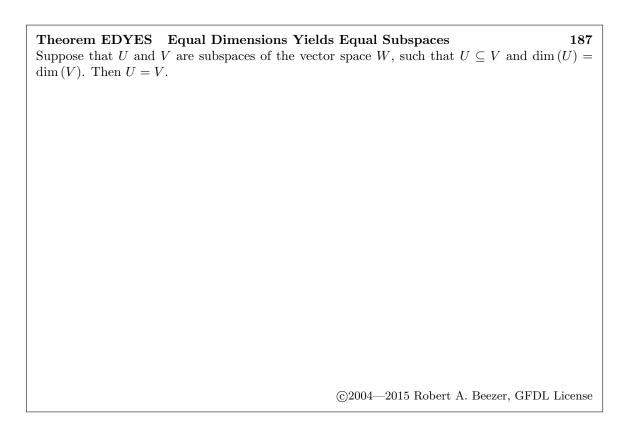
- 1. If m > t, then S is linearly dependent.
- 2. If m < t, then S does not span V.
- 3. If m = t and S is linearly independent, then S spans V.
- 4. If m = t and S spans V, then S is linearly independent.

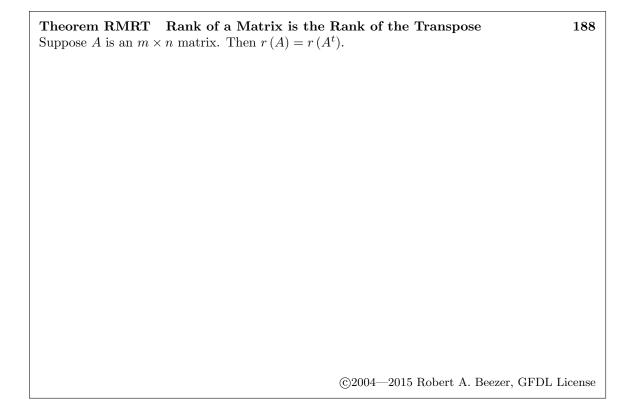
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Theorem PSSD Proper Subspaces have Smaller Dimension

186

Suppose that U and V are subspaces of the vector space W, such that $U \subsetneq V$. Then $\dim(U) < \dim(V)$.





Theorem DFS Dimensions of Four Subspaces

189

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

- 1. dim $(\mathcal{N}(A)) = n r$
- 2. dim $(\mathcal{C}(A)) = r$
- 3. dim $(\mathcal{R}(A)) = r$
- 4. dim $(\mathcal{L}(A)) = m r$

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Definition ELEM Elementary Matrices

190

1. For $i \neq j$, $E_{i,j}$ is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

2. For $\alpha \neq 0$, $E_i(\alpha)$ is the square matrix of size n with

$$[E_{i}(\alpha)]_{k\ell} = \begin{cases} 0 & \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. For $i \neq j$, $E_{i,j}(\alpha)$ is the square matrix of size n with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \end{cases}$$
$$1 & k = j, \ell = j \\ \alpha & k = j, \ell = i \end{cases}$$

Theorem EMDRO Elementary Matrices Do Row Operations

191

Suppose that A is an $m \times n$ matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO). Then there is an elementary matrix of size m that will convert A to B via matrix multiplication on the left. More precisely,

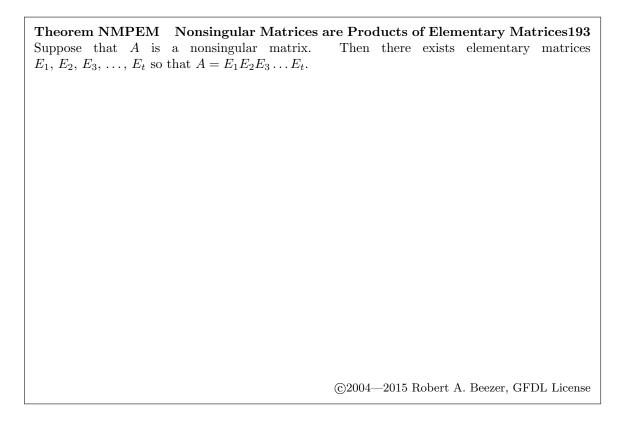
- 1. If the row operation swaps rows i and j, then $B = E_{i,j}A$.
- 2. If the row operation multiplies row i by α , then $B = E_i(\alpha) A$.
- 3. If the row operation multiplies row i by α and adds the result to row j, then $B=E_{i,j}\left(\alpha\right)A$.

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Theorem EMN Elementary Matrices are Nonsingular

192

If E is an elementary matrix, then E is nonsingular.



Definition SM SubMatrix

194

Suppose that A is an $m \times n$ matrix. Then the submatrix A(i|j) is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

Definition DM Determinant of a Matrix

195

Suppose A is a square matrix. Then its determinant, $\det(A) = |A|$, is an element of \mathbb{C} defined recursively by:

- 1. If A is a 1×1 matrix, then $\det(A) = [A]_{11}$.
- 2. If A is a matrix of size n with $n \geq 2$, then

$$\begin{split} \det{(A)} &= [A]_{11} \det{(A\,(1|1))} - [A]_{12} \det{(A\,(1|2))} + [A]_{13} \det{(A\,(1|3))} - \\ & [A]_{14} \det{(A\,(1|4))} + \dots + (-1)^{n+1} \, [A]_{1n} \det{(A\,(1|n))} \end{split}$$

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Theorem DMST Determinant of Matrices of Size Two Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det{(A)} = ad - bc$.

Theorem DER Determinant Expansion about Rows

197

Suppose that A is a square matrix of size n. Then for $1 \le i \le n$

$$\begin{split} \det{(A)} &= (-1)^{i+1} \left[A \right]_{i1} \det{(A\left(i|1\right))} + (-1)^{i+2} \left[A \right]_{i2} \det{(A\left(i|2\right))} \\ &+ (-1)^{i+3} \left[A \right]_{i3} \det{(A\left(i|3\right))} + \dots + (-1)^{i+n} \left[A \right]_{in} \det{(A\left(i|n\right))} \end{split}$$

which is known as expansion about row i.

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${\bf Theorem~DT~~Determinant~of~the~Transpose}$

198

Suppose that A is a square matrix. Then $\det(A^t) = \det(A)$.

Theorem DEC Determinant Expansion about Columns

199

Suppose that A is a square matrix of size n. Then for $1 \leq j \leq n$

$$\begin{split} \det{(A)} &= (-1)^{1+j} \left[A \right]_{1j} \det{(A \, (1|j))} + (-1)^{2+j} \left[A \right]_{2j} \det{(A \, (2|j))} \\ &+ (-1)^{3+j} \left[A \right]_{3j} \det{(A \, (3|j))} + \dots + (-1)^{n+j} \left[A \right]_{nj} \det{(A \, (n|j))} \end{split}$$

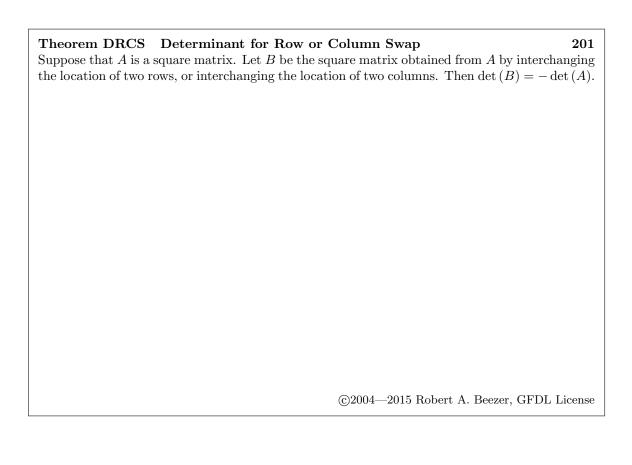
which is known as expansion about column j.

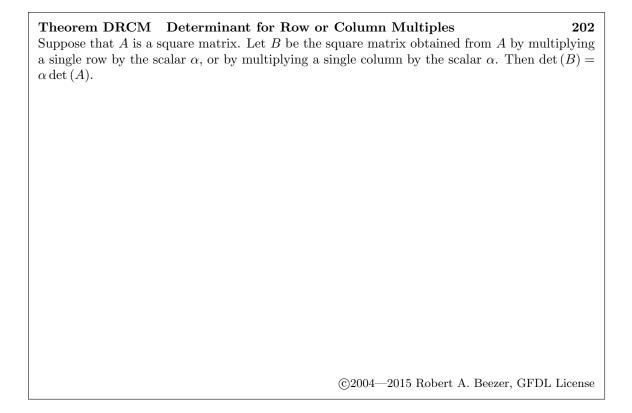
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Theorem DZRC Determinant with Zero Row or Column

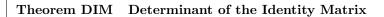
200

Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then $\det(A) = 0$.









205

For every $n \geq 1$, $\det(I_n) = 1$.

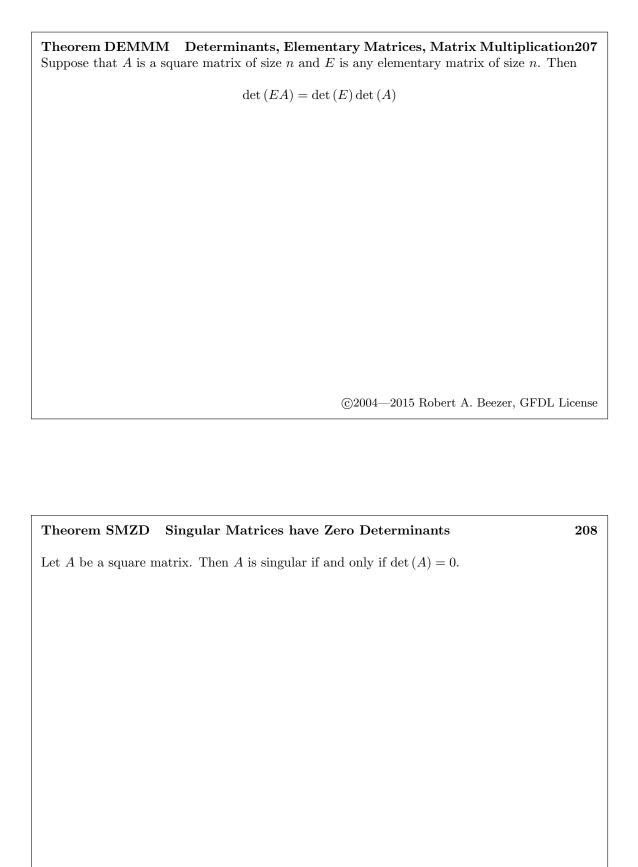
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Theorem DEM Determinants of Elementary Matrices

206

For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

- 1. $\det(E_{i,j}) = -1$
- 2. $\det (E_i(\alpha)) = \alpha$
- 3. $\det (E_{i,j}(\alpha)) = 1$



Theorem NME7 Nonsingular Matrix Equivalences, Round 7

209

Suppose that A is a square matrix of size n. The following are equivalent.

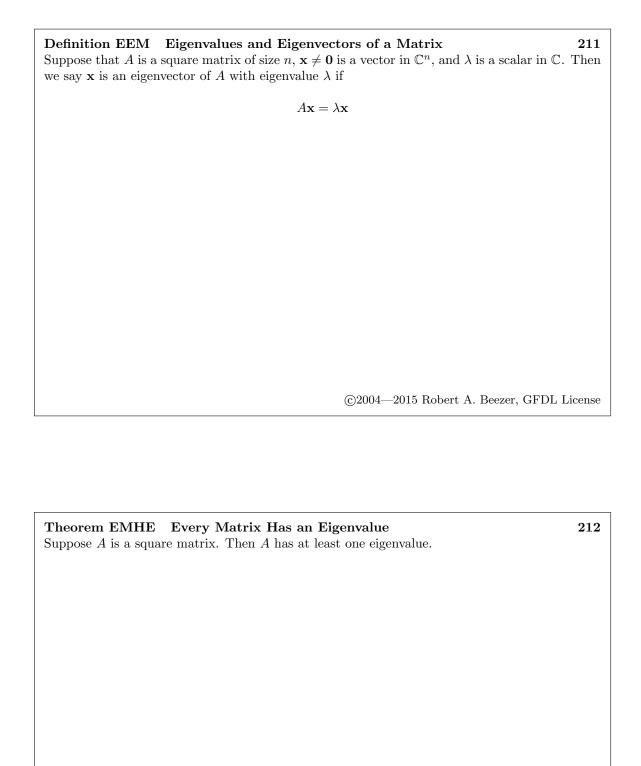
- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $det(A) \neq 0$.

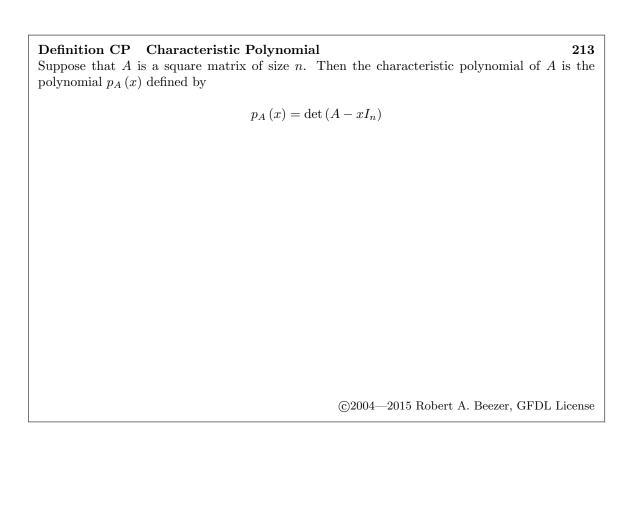
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Theorem DRMM Determinant Respects Matrix Multiplication

210

Suppose that A and B are square matrices of the same size. Then $\det(AB) = \det(A) \det(B)$.





$\begin{array}{ll} {\bf Theorem~EMRCP} & {\bf Eigenvalues~of~a~Matrix~are~Roots~of~Characteristic~Polynomials} \\ {\bf 214} & \\ \end{array}$

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$.

Definition EM Eigenspace of a Matrix Suppose that A is a square matrix and λ is an eigenvalue of A . Then the eigenspace of A for $\mathcal{E}_A(\lambda)$, is the set of all the eigenvectors of A for λ , together with the inclusion of the zero vectors	λ ,
$\mathcal{C}_A(\lambda)$, is the set of all the eigenvectors of λ 1 for λ , together with the inclusion of the zero vector.	1.
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Theorem EMS Eigenspace for a Matrix is a Subspace 21 Suppose A is a square matrix of size n and λ is an eigenvalue of A . Then the eigenspace \mathcal{E}_A (λ is a subspace of the vector space \mathbb{C}^n .	

Theorem EMNS Eigenspace of a Matrix is a Null Space Suppose A is a square matrix of size n and λ is an eigenvalue of A . Then	217
$\mathcal{E}_{A}\left(\lambda ight)=\mathcal{N}(A-\lambda I_{n})$	
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Definition AME Algebraic Multiplicity of an Eigenvalue	218
Suppose that A is a square matrix and λ is an eigenvalue of A. Then the alge of λ , $\alpha_A(\lambda)$, is the highest power of $(x - \lambda)$ that divides the characteristic poly	

Definition GME Geometric Multiplicity Suppose that A is a square matrix and λ is an error of λ , $\gamma_A(\lambda)$, is the dimension of the eigenspace \mathcal{E}	genvalue of A . Then the geometric multiplicity
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Theorem EDELI Eigenvectors with Disti 220 Suppose that A is an $n \times n$ square matrix and S with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p$ such that λ independent set.	nct Eigenvalues are Linearly Independent $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_p\}$ is a set of eigenvectors $i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly

) is an eigenvalue of A .	r Matrices have Zero Eigenvalues . Then A is singular if and only if $\lambda =$	

Theorem NME8 Nonsingular Matrix Equivalences, Round 8

222

Suppose that A is a square matrix of size n. The following are equivalent.

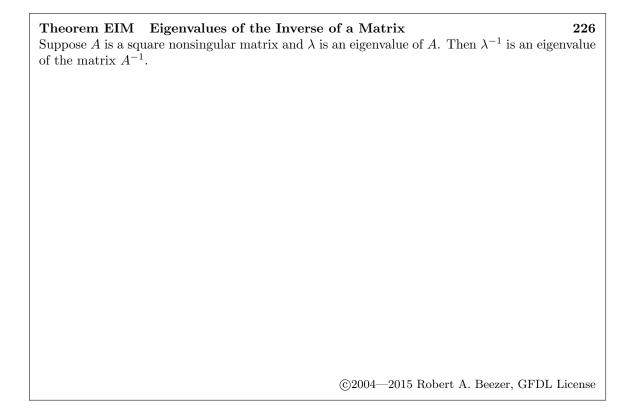
- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $det(A) \neq 0$.
- 12. $\lambda = 0$ is not an eigenvalue of A.

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Theorem ESMM Eigenvalues of a Scalar Multiple of a Matrix Suppose A is a square matrix and λ is an eigenvalue of A . Then $\alpha\lambda$ is an eigenvalue of A .	223 envalue of αA .
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Theorem EOMP Eigenvalues Of Matrix Powers 224 Suppose A is a square matrix, λ is an eigenvalue of A, and $s \geq 0$ is an integer. Then λ^s is an eigenvalue of A^s .

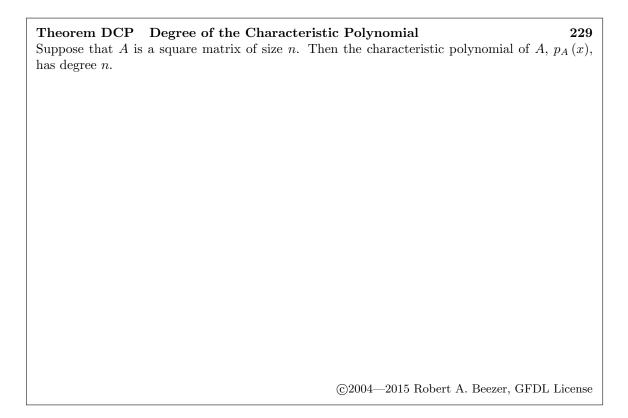
Theorem EPM Eigenvalues of the Polynomial of a Matrix 225 Suppose A is a square matrix and λ is an eigenvalue of A . Let $q(x)$ be a polynomial in the variable x . Then $q(\lambda)$ is an eigenvalue of the matrix $q(A)$.
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Theorem ETM Eigenvalues of the Transpose Suppose A is a square matrix and λ is an eigenvalue	
A^t .	or in Thom will all eigenvalue of the matrix
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Theorem ERMCP Eigenvalues of Real Matrices come in Conjugate Pairs 228

Suppose A is a square matrix with real entries and \mathbf{x} is an eigenvector of A for the eigenvalue λ . Then $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$.



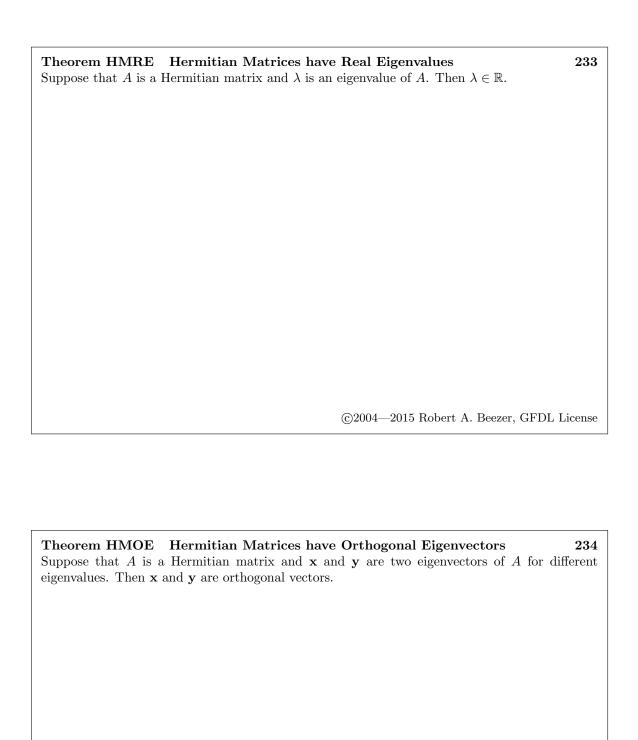
Theorem NEM Number of Eigenvalues of a Matrix

230

Suppose that $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$ are the distinct eigenvalues of a square matrix A of size n. Then

$$\sum_{i=1}^{k} \alpha_A \left(\lambda_i \right) = n$$

Theorem ME Multiplicities of an Eigenvalue	231
Suppose that A is a square matrix of size n and λ is an eigenvalue. Then	
$1 \le \gamma_A(\lambda) \le \alpha_A(\lambda) \le n$	
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Theorem MNEM Maximum Number of Eigenvalues of a Matrix 232 Suppose that A is a square matrix of size n . Then A cannot have more than n distinct eigenvalues.	



Definition SIM Similar Matrices Suppose A and B are two square matrices of size n . Then A and B are similar if there ex nonsingular matrix of size n , S , such that $A = S^{-1}BS$.	235 cists a
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Theorem SER Similarity is an Equivalence Relation Suppose A, B and C are square matrices of size n . Then	236
1. A is similar to A. (Reflexive)	
2. If A is similar to B , then B is similar to A . (Symmetric)	
3. If A is similar to B and B is similar to C , then A is similar to C . (Transitive)	

Theorem SMEE Similar Matrices Suppose A and B are similar matrices. equal, that is, $p_A(x) = p_B(x)$.	have Equal Eigenvalues ${\bf 237}$ Then the characteristic polynomials of A and B are
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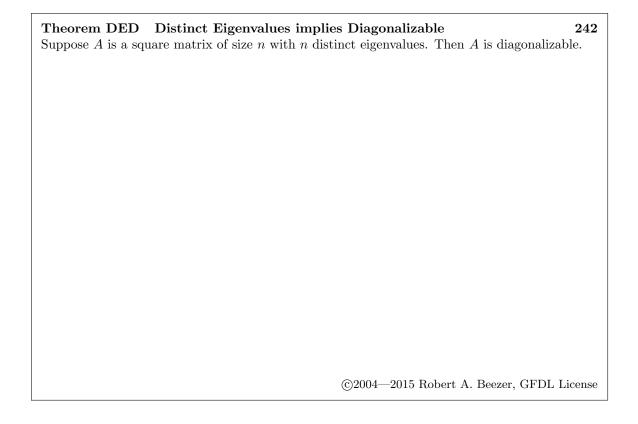
238

Definition DIM Diagonal Matrix Suppose that A is a square matrix. Then A is a diagonal matrix if $[A]_{ij} = 0$ whenever $i \neq j$.

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Theorem DC Diagonalization Characterization 240 Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a

Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a linearly independent set S that contains n eigenvectors of A. ©2004—2015 Robert A. Beezer, GFDL License

Theorem DMFE Diagonalizable Matrices h Suppose A is a square matrix. Then A is diagonalize eigenvalue λ of A .	
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Definition LT Linear Transformation

243

A linear transformation, $T \colon U \to V$, is a function that carries elements of the vector space U (called the domain) to the vector space V (called the codomain), and which has two additional properties

- 1. $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$ for all $\mathbf{u}_1, \mathbf{u}_2 \in U$
- 2. $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

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${\bf Theorem~LTTZZ~~Linear~Transformations~Take~Zero~to~Zero}$

244

Suppose $T \colon U \to V$ is a linear transformation. Then $T\left(\mathbf{0}\right) = \mathbf{0}$.

Theorem MBLT Matrices Build Linear Transformations 245 Suppose that A is an $m \times n$ matrix. Define a function $T: \mathbb{C}^n \to \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation.	
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Theorem MLTCV Matrix of a Linear Transformation, Column Vectors 246

Suppose that $T: \mathbb{C}^n \to \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Theorem LTLC Linear Transformations and Linear Combinations

247

Suppose that $T: U \to V$ is a linear transformation, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$ are vectors from U and $a_1, a_2, a_3, \ldots, a_t$ are scalars from \mathbb{C} . Then

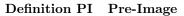
$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \dots + a_t\mathbf{u}_t) = a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \dots + a_tT(\mathbf{u}_t)$$

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Theorem LTDB Linear Transformation Defined on a Basis

248

Suppose U is a vector space with basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ and the vector space V contains the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n$ (which may not be distinct). Then there is a unique linear transformation, $T: U \to V$, such that $T(\mathbf{u}_i) = \mathbf{v}_i$, $1 \le i \le n$.



249

Suppose that $T: U \to V$ is a linear transformation. For each \mathbf{v} , define the pre-image of \mathbf{v} to be the subset of U given by

$$T^{-1}(\mathbf{v}) = \{ \mathbf{u} \in U | T(\mathbf{u}) = \mathbf{v} \}$$

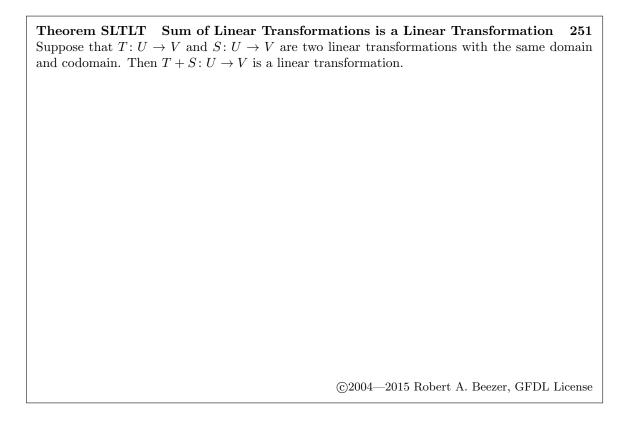
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Definition LTA Linear Transformation Addition

250

Suppose that $T\colon U\to V$ and $S\colon U\to V$ are two linear transformations with the same domain and codomain. Then their sum is the function $T+S\colon U\to V$ whose outputs are defined by

$$(T+S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

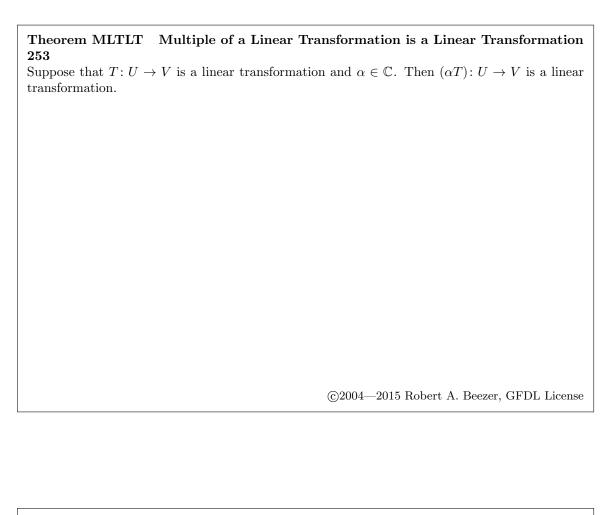


Definition LTSM Linear Transformation Scalar Multiplication

252

Suppose that $T:U\to V$ is a linear transformation and $\alpha\in\mathbb{C}$. Then the scalar multiple is the function $\alpha T\colon U\to V$ whose outputs are defined by

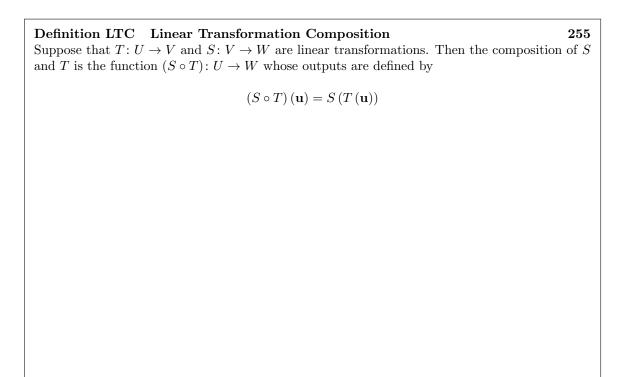
$$(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u})$$



Theorem VSLT Vector Space of Linear Transformations

254

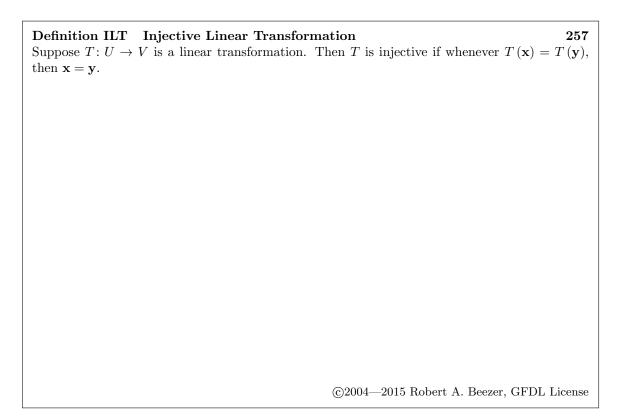
Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, $\mathcal{L}T(U,V)$, is a vector space when the operations are those given in Definition LTA and Definition LTSM.



Theorem CLTLT Composition of Linear Transformations is a Linear Transformation 256

Suppose that $T\colon U\to V$ and $S\colon V\to W$ are linear transformations. Then $(S\circ T)\colon U\to W$ is a linear transformation.

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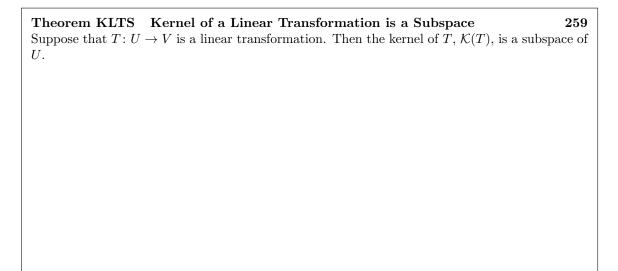


Definition KLT Kernel of a Linear Transformation

258

Suppose $T\colon U\to V$ is a linear transformation. Then the kernel of T is the set

$$\mathcal{K}(T) = \{ \mathbf{u} \in U | T(\mathbf{u}) = \mathbf{0} \}$$

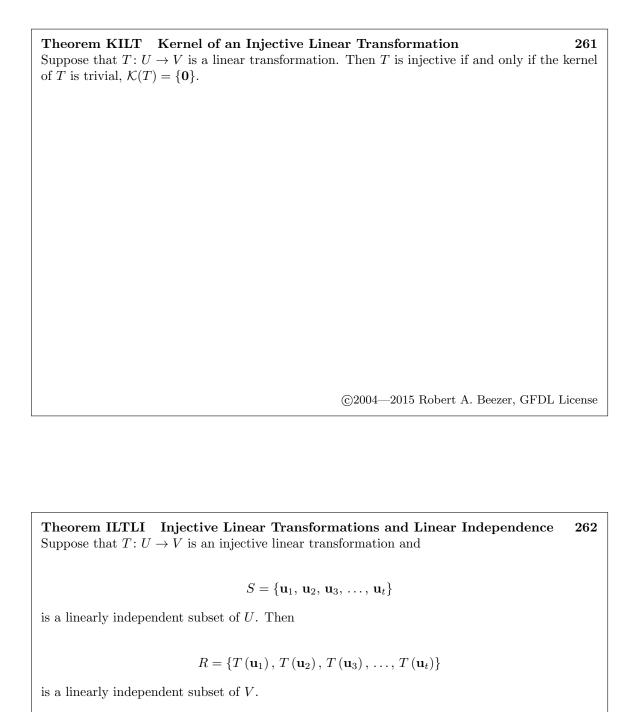


Theorem KPI Kernel and Pre-Image

260

Suppose $T: U \to V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is nonempty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then

$$T^{-1}(\mathbf{v}) = \{ \mathbf{u} + \mathbf{z} | \mathbf{z} \in \mathcal{K}(T) \} = \mathbf{u} + \mathcal{K}(T)$$



Suppose that $T \colon U \to V$ is a linear transformation and

$$B = {\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_m}$$

is a basis of U. Then T is injective if and only if

$$C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}\$$

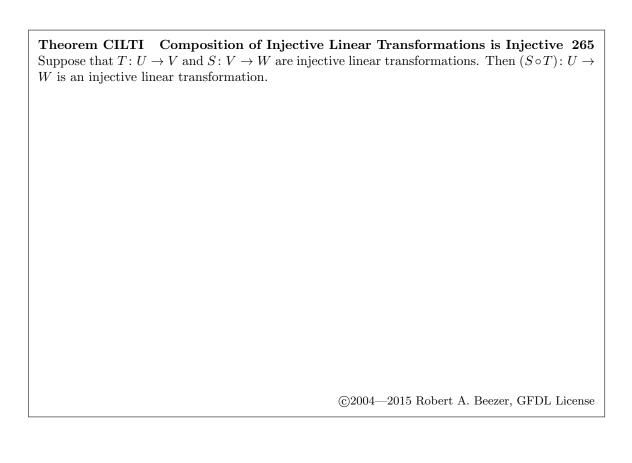
is a linearly independent subset of V.

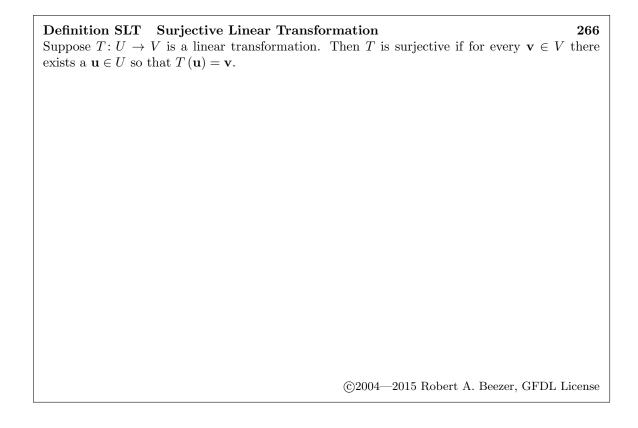
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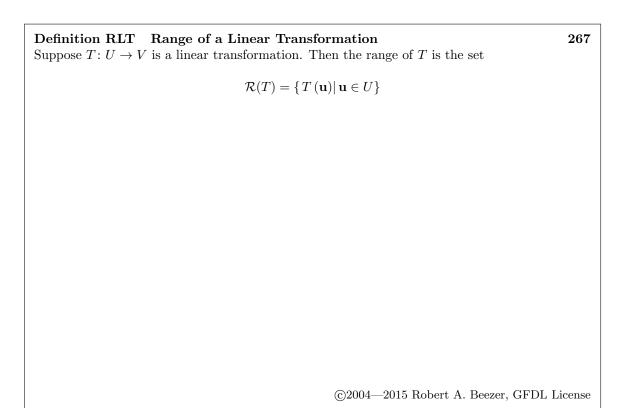
263

264

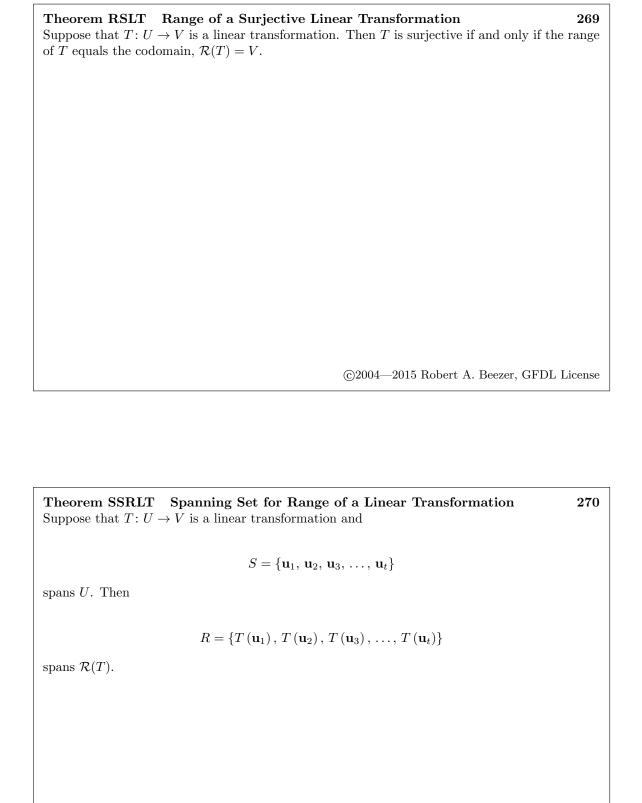
Suppose that $T: U \to V$ is an injective linear transformation. Then $\dim(U) \leq \dim(V)$.

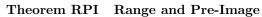






Theorem RLTS Range of a Linear Transformation is a Subspace 268 Suppose that $T: U \to V$ is a linear transformation. Then the range of T, $\mathcal{R}(T)$, is a subspace of V.





271

Suppose that $T \colon U \to V$ is a linear transformation. Then

$$\mathbf{v} \in \mathcal{R}(T)$$
 if and only if $T^{-1}(\mathbf{v}) \neq \emptyset$

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Theorem SLTB Surjective Linear Transformations and Bases

272

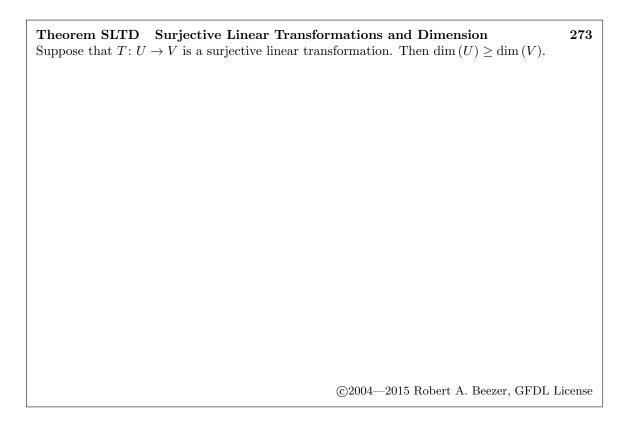
Suppose that $T \colon U \to V$ is a linear transformation and

$$B = \{\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_m\}$$

is a basis of U. Then T is surjective if and only if

$$C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}\$$

is a spanning set for V.



Theorem CSLTS Composition of Surjective Linear Transformations is Surjective 274

Suppose that $T\colon U\to V$ and $S\colon V\to W$ are surjective linear transformations. Then $(S\circ T)\colon U\to W$ is a surjective linear transformation.

Definition IDLT Identity Linear Transformation

275

The identity linear transformation on the vector space W is defined as

$$I_W \colon W \to W, \qquad I_W (\mathbf{w}) = \mathbf{w}$$

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Definition IVLT Invertible Linear Transformations

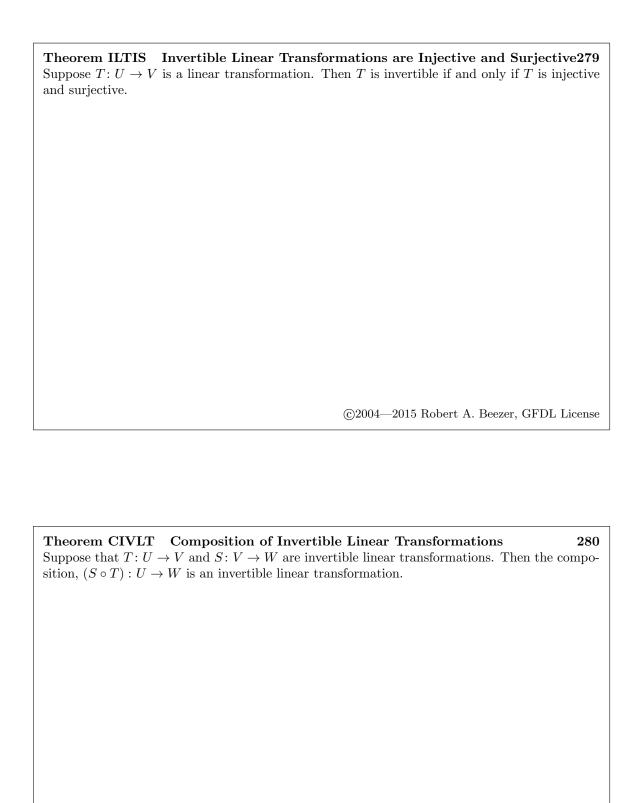
276

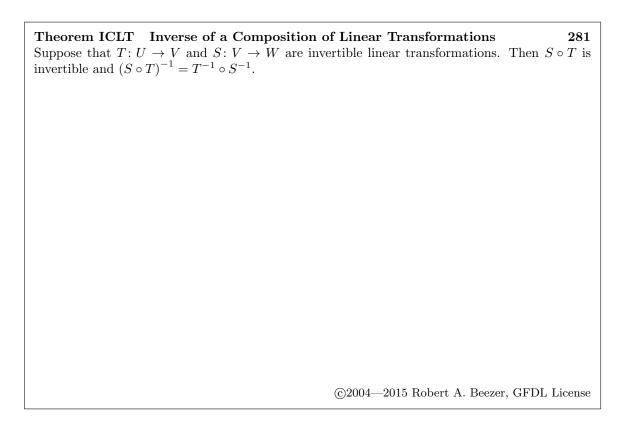
Suppose that $T: U \to V$ is a linear transformation. If there is a function $S: V \to U$ such that

$$S \circ T = I_U \qquad \qquad T \circ S = I_V$$

then T is invertible. In this case, we call S the inverse of T and write $S = T^{-1}$.



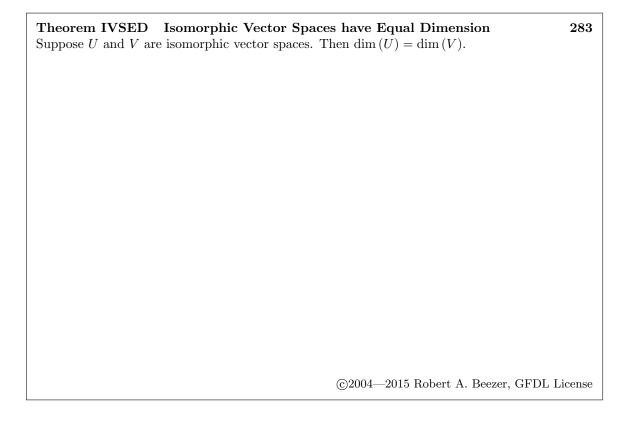




Definition IVS Isomorphic Vector Spaces

282

Two vector spaces U and V are isomorphic if there exists an invertible linear transformation T with domain U and codomain V, $T:U\to V$. In this case, we write $U\cong V$, and the linear transformation T is known as an isomorphism between U and V.

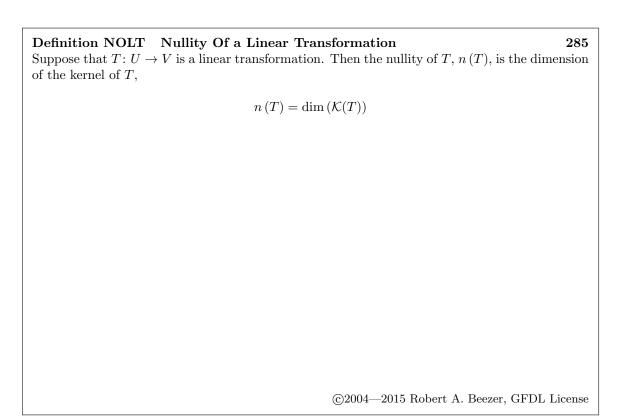


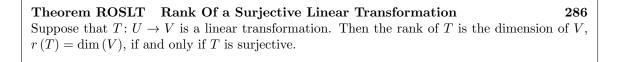
Definition ROLT Rank Of a Linear Transformation

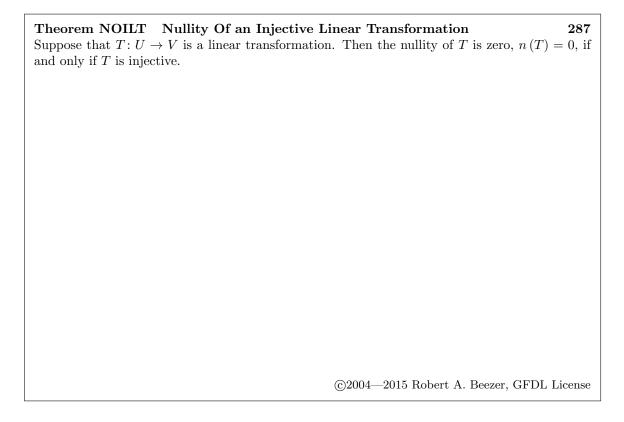
284

Suppose that $T:U\to V$ is a linear transformation. Then the rank of $T,\,r\left(T\right)$, is the dimension of the range of T,

$$r(T) = \dim (\mathcal{R}(T))$$







Theorem RPNDD Rank Plus Nullity is Domain Dimension

288

Suppose that $T \colon U \to V$ is a linear transformation. Then

$$r(T) + n(T) = \dim(U)$$

Definition VR Vector Representation

289

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$. Define a function $\rho_B \colon V \to \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$ define the column vector $\rho_B(\mathbf{w}) \in \mathbb{C}^n$ by

$$\mathbf{w} = [\rho_B(\mathbf{w})]_1 \mathbf{v}_1 + [\rho_B(\mathbf{w})]_2 \mathbf{v}_2 + [\rho_B(\mathbf{w})]_3 \mathbf{v}_3 + \dots + [\rho_B(\mathbf{w})]_n \mathbf{v}_n$$

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Theorem VRLT Vector Representation is a Linear Transformation

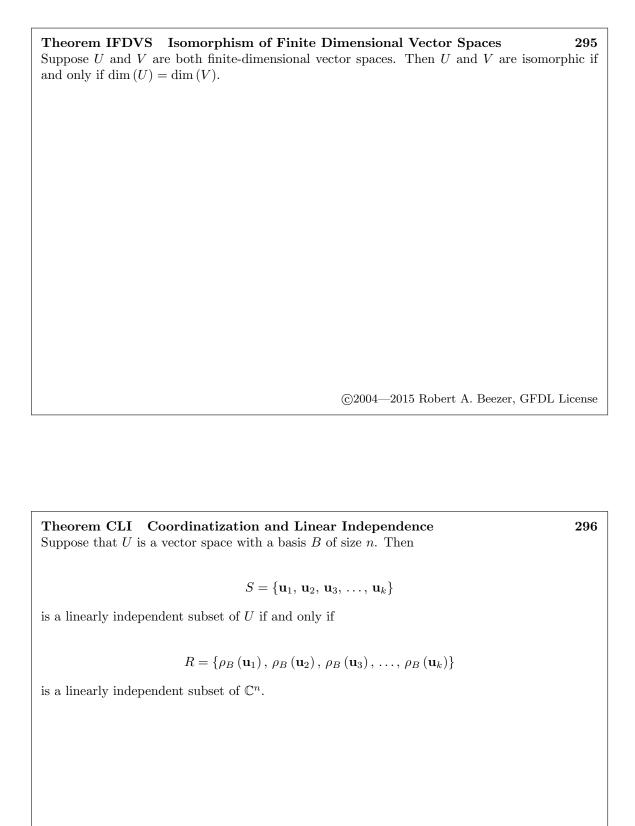
290

The function ρ_B (Definition VR) is a linear transformation.

Theorem VRI Vector Representation is Injective lines of the function ρ_B (Definition VR) is an injective lines		291
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Theorem VRS Vector Representation is Surjective 292 The function ρ_B (Definition VR) is a surjective linear transformation.





Theorem CSS Coordinatization and Spanning Sets

297

Suppose that U is a vector space with a basis B of size n. Then

$$\mathbf{u} \in \langle \{\mathbf{u}_1, \, \mathbf{u}_2, \, \mathbf{u}_3, \, \dots, \, \mathbf{u}_k\} \rangle$$

if and only if

$$\rho_B(\mathbf{u}) \in \langle \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \dots, \rho_B(\mathbf{u}_k)\} \rangle$$

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Definition MR Matrix Representation

298

Suppose that $T: U \to V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a basis for U of size n, and C is a basis for V of size m. Then the matrix representation of T relative to B and C is the $m \times n$ matrix,

$$M_{B,C}^{T} = \left[\rho_{C}\left(T\left(\mathbf{u}_{1}\right)\right) \middle| \rho_{C}\left(T\left(\mathbf{u}_{2}\right)\right) \middle| \rho_{C}\left(T\left(\mathbf{u}_{3}\right)\right) \middle| \dots \middle| \rho_{C}\left(T\left(\mathbf{u}_{n}\right)\right) \right]$$

Theorem FTMR Fundamental Theorem of Matrix Representation

299

Suppose that $T: U \to V$ is a linear transformation, B is a basis for U, C is a basis for V and $M_{B,C}^T$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u} \in U$,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_C^{-1}\left(M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)\right)$$

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Theorem MRSLT Matrix Representation of a Sum of Linear Transformations 300 Suppose that $T: U \to V$ and $S: U \to V$ are linear transformations, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$$

$\begin{array}{ll} {\bf Theorem~MRMLT} & {\bf Matrix~Representation~of~a~Multiple~of~a~Linear~Transformation} \\ {\bf 301} & \\ \end{array}$

Suppose that $T:U\to V$ is a linear transformation, $\alpha\in\mathbb{C},\,B$ is a basis of U and C is a basis of V. Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$

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Theorem MRCLT Matrix Representation of a Composition of Linear Transformations 302

Suppose that $T: U \to V$ and $S: V \to W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^S M_{B,C}^T$$

Theorem KNSI Kernel and Null Space Isomorphism

303

Suppose that $T: U \to V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M_{B,C}^T$,

$$\mathcal{K}(T) \cong \mathcal{N}(M_{B,C}^T)$$

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Theorem RCSI Range and Column Space Isomorphism

304

Suppose that $T: U \to V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of $M_{B,C}^T$,

$$\mathcal{R}(T) \cong \mathcal{C}(M_{B,C}^T)$$

Theorem IMR Invertible Matrix Representations

305

Suppose that $T: U \to V$ is a linear transformation, B is a basis for U and C is a basis for V. Then T is an invertible linear transformation if and only if the matrix representation of T relative to B and C, $M_{B,C}^T$ is an invertible matrix. When T is invertible,

$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^T\right)^{-1}$$

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Theorem IMILT Invertible Matrices, Invertible Linear Transformation 306 Suppose that A is a square matrix of size n and $T: \mathbb{C}^n \to \mathbb{C}^n$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Then A is an invertible matrix if and only if T is an invertible linear transformation.

Theorem NME9 Nonsingular Matrix Equivalences, Round 9

307

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $\det(A) \neq 0$.
- 12. $\lambda = 0$ is not an eigenvalue of A.
- 13. The linear transformation $T: \mathbb{C}^n \to \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible.

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Definition EELT Eigenvalue and Eigenvector of a Linear Transformation 308 Suppose that $T: V \to V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if $T(\mathbf{v}) = \lambda \mathbf{v}$.

Definition CBM Change-of-Basis Matrix

309

Suppose that V is a vector space, and $I_V: V \to V$ is the identity linear transformation on V. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ and C be two bases of V. Then the change-of-basis matrix from B to C is the matrix representation of I_V relative to B and C,

$$C_{B,C} = M_{B,C}^{I_V}$$

$$= \left[\rho_C \left(I_V \left(\mathbf{v}_1 \right) \right) \middle| \rho_C \left(I_V \left(\mathbf{v}_2 \right) \right) \middle| \rho_C \left(I_V \left(\mathbf{v}_3 \right) \right) \middle| \dots \middle| \rho_C \left(I_V \left(\mathbf{v}_n \right) \right) \right]$$

$$= \left[\rho_C \left(\mathbf{v}_1 \right) \middle| \rho_C \left(\mathbf{v}_2 \right) \middle| \rho_C \left(\mathbf{v}_3 \right) \middle| \dots \middle| \rho_C \left(\mathbf{v}_n \right) \right]$$

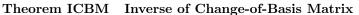
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Theorem CB Change-of-Basis

310

Suppose that \mathbf{v} is a vector in the vector space V and B and C are bases of V. Then

$$\rho_C\left(\mathbf{v}\right) = C_{B,C}\rho_B\left(\mathbf{v}\right)$$



311

Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

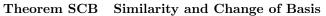
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Theorem MRCB Matrix Representation and Change of Basis

312

Suppose that $T\colon U\to V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

$$M_{B,D}^T = C_{E,D} M_{C,E}^T C_{B,C}$$



313

Suppose that $T: V \to V$ is a linear transformation and B and C are bases of V. Then

$$M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$$

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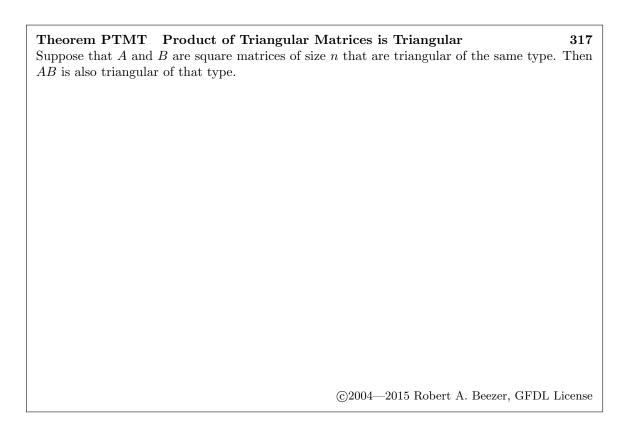
Theorem EER Eigenvalues, Eigenvectors, Representations

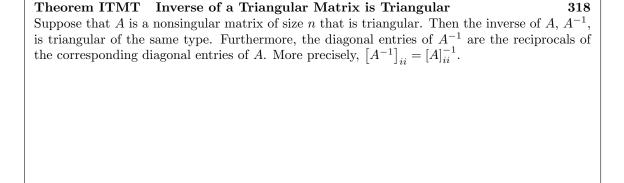
314

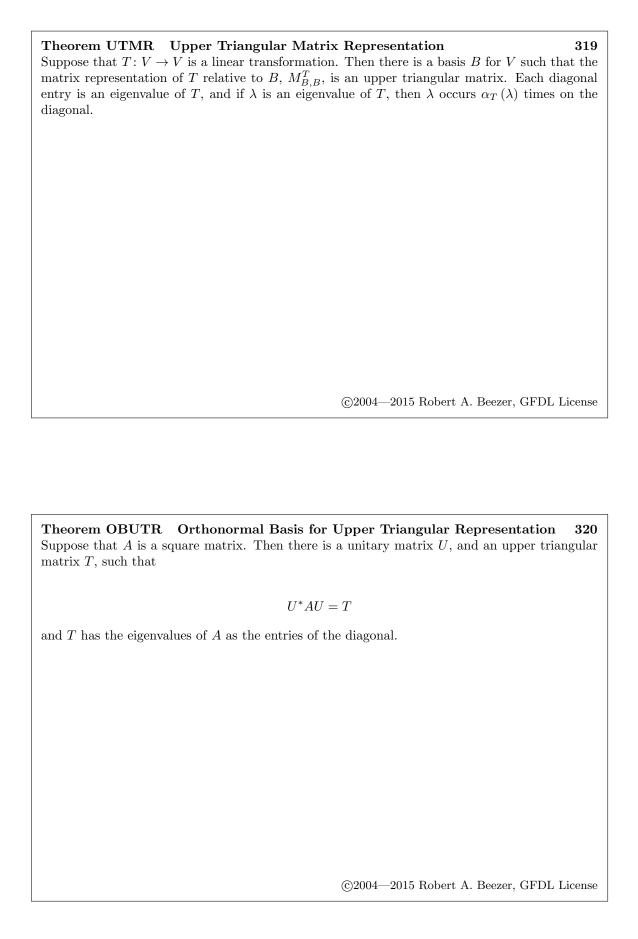
Suppose that $T: V \to V$ is a linear transformation and B is a basis of V. Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M_{B,B}^T$ for the eigenvalue λ .

	Upper Triangular Matrix		315
The $n \times n$ square m	atrix A is upper triangular if $ $	$[A]_{ij} = 0$ whenever $i > j$.	
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Definition LTM Lower Triangular Matrix 316 The $n \times n$ square matrix A is lower triangular if $[A]_{ij} = 0$ whenever i < j.







Definition NRML Normal Matrix The square matrix A is normal if $A^*A = AA^*$.	321
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Theorem OD Orthonormal Diagonalization

 $\bf 322$

Suppose that A is a square matrix. Then there is a unitary matrix U and a diagonal matrix D, with diagonal entries equal to the eigenvalues of A, such that $U^*AU = D$ if and only if A is a normal matrix.

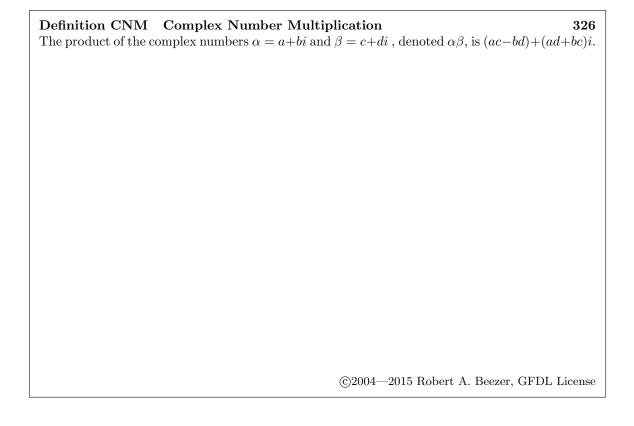
Theorem OBNM Orthonormal Bases and Suppose that A is a normal matrix of size n . There of eigenvectors of A .	
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 $\bf 324$

Definition CNE Complex Number Equality

The complex numbers $\alpha = a + bi$ and $\beta = c + di$ are equal, denoted $\alpha = \beta$, if a = c and b = d.

	Complex Number Additi		325
The sum of the com	piex numbers $\alpha = a + bi$ and β	$\beta = c + di$, denoted $\alpha + \beta$, is $(a + c) + (b + \beta)$	a)i.
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Theorem PCNA Properties of Complex Number Arithmetic

327

The operations of addition and multiplication of complex numbers have the following properties.

- ACCN Additive Closure, Complex Numbers: If $\alpha, \beta \in \mathbb{C}$, then $\alpha + \beta \in \mathbb{C}$.
- MCCN Multiplicative Closure, Complex Numbers: If $\alpha, \beta \in \mathbb{C}$, then $\alpha\beta \in \mathbb{C}$.
- CACN Commutativity of Addition, Complex Numbers: For any α , $\beta \in \mathbb{C}$, $\alpha + \beta = \beta + \alpha$.
- CMCN Commutativity of Multiplication, Complex Numbers: For any $\alpha, \beta \in \mathbb{C}, \alpha\beta = \beta\alpha$.
- AACN Additive Associativity, Complex Numbers: For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
- MACN Multiplicative Associativity, Complex Numbers: For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.
- DCN Distributivity, Complex Numbers: For any α , β , $\gamma \in \mathbb{C}$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
- ZCN Zero, Complex Numbers: There is a complex number 0 = 0 + 0i so that for any $\alpha \in \mathbb{C}$, $0 + \alpha = \alpha$.
- OCN One, Complex Numbers: There is a complex number 1 = 1 + 0i so that for any $\alpha \in \mathbb{C}$,
- AICN Additive Inverse, Complex Numbers: For every $\alpha \in \mathbb{C}$ there exists $-\alpha \in \mathbb{C}$ so that $\alpha + (-\alpha) = 0$.
- MICN Multiplicative Inverse, Complex Numbers: For every $\alpha \in \mathbb{C}$, $\alpha \neq 0$ there exists $\frac{1}{\alpha} \in \mathbb{C}$ so that $\alpha\left(\frac{1}{\alpha}\right) = 1$.

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Theorem ZPCN Zero Product, Complex Numbers Suppose $\alpha \in \mathbb{C}$. Then $0\alpha = 0$.

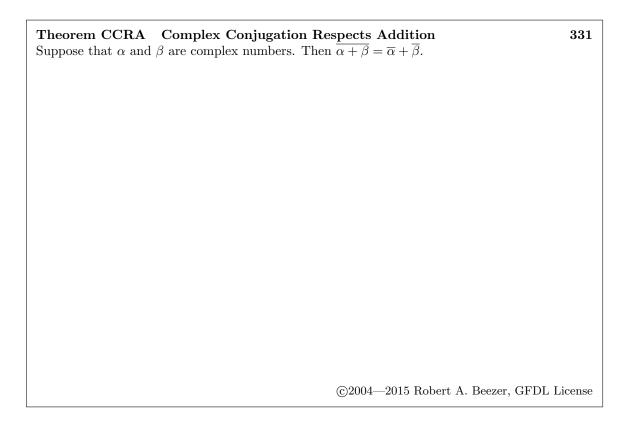
328

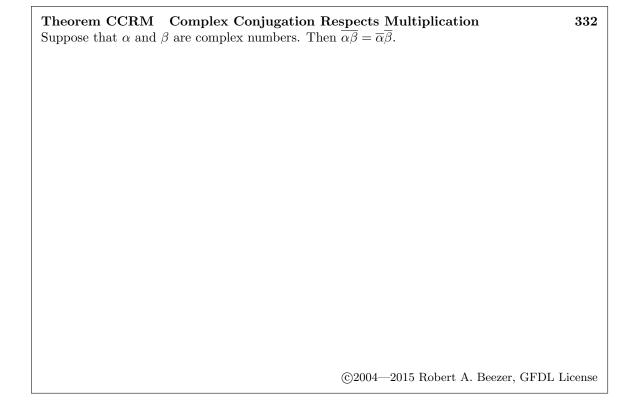
Theorem ZPZT Zero Product, Zero Terms	329
Suppose $\alpha, \beta \in \mathbb{C}$. Then $\alpha\beta = 0$ if and only if at least on	e of $\alpha = 0$ or $\beta = 0$.
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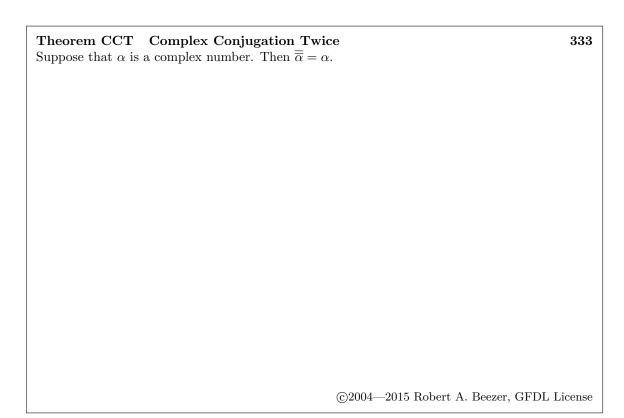
Definition CCN Conjugate of a Complex Number

330

The conjugate of the complex number $\alpha = a + bi \in \mathbb{C}$ is the complex number $\overline{\alpha} = a - bi$.







Definition MCN Modulus of a Complex Number

334

The modulus of the complex number $\alpha = a + bi \in \mathbb{C}$, is the nonnegative real number

$$|\alpha| = \sqrt{\overline{\alpha}\alpha} = \sqrt{a^2 + b^2}.$$

Definition SET Set	335
A set is an unordered collection of objects. If S is	a set and x is an object that is in the set S ,
we write $x \in S$. If x is not in S, then we write x	$\not\in S$. We refer to the objects in a set as its
elements.	
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Definition SSET Subset

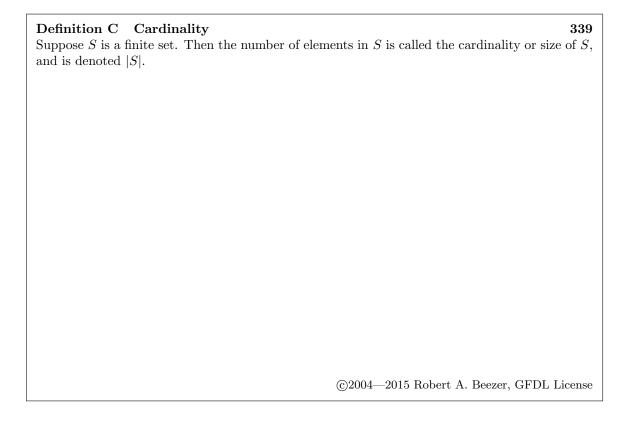
336

If S and T are two sets, then S is a subset of T, written $S \subseteq T$ if whenever $x \in S$ then $x \in T$.

Definition ES Empty Set	337
The empty set is the set with no elements. It is de	enoted by \emptyset .
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Definition SE Set Equality Two sets, S and T, are equal, if $S \subseteq T$ and $T \subseteq S$. In this case, we write S = T.

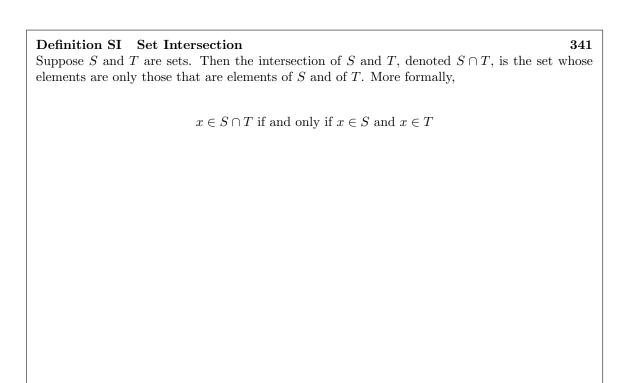


Definition SU Set Union

340

Suppose S and T are sets. Then the union of S and T, denoted $S \cup T$, is the set whose elements are those that are elements of S or of T, or both. More formally,

 $x \in S \cup T$ if and only if $x \in S$ or $x \in T$



Definition SC Set Complement

342

Suppose S is a set that is a subset of a universal set U. Then the complement of S, denoted \overline{S} , is the set whose elements are those that are elements of U and not elements of S. More formally,

 $x \in \overline{S}$ if and only if $x \in U$ and $x \notin S$

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