# Flash Cards

to accompany

# A First Course in Linear Algebra

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Most recent version can be found at http://linear.ups.edu/.

**Definition SLE** System of Linear Equations

A system of linear equations is a collection of m equations in the variable quantities  $x_1, x_2, x_3, \ldots, x_n$  of the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$
  

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$
  

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$
  

$$\vdots$$
  

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

where the values of  $a_{ij}$ ,  $b_i$  and  $x_j$  are from the set of complex numbers,  $\mathbb{C}$ .

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**Definition ES** Equivalent Systems

Two systems of linear equations are **equivalent** if their solution sets are equal.

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### **Definition EO** Equation Operations

Given a system of linear equations, the following three operations will transform the system into a different one, and each is known as an **equation operation**.

- 1. Swap the locations of two equations in the list.
- 2. Multiply each term of an equation by a nonzero quantity.
- 3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

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Theorem EOPSS Equation Operations Preserve Solution Sets

If we apply one of the three equation operations of Definition EO to the system of linear equations

 $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$   $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$   $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$   $\vdots \qquad \vdots$  $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$ 

then the original system and the transformed system are equivalent.

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#### **Definition M** Matrix

An  $m \times n$  matrix is a rectangular layout of numbers from  $\mathbb{C}$  having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet (A, B, C, ...) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation  $[A]_{ij}$  will refer to the complex number in row i and column j of A.

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#### **Definition AM** Augmented Matrix

Suppose we have a system of m equations in the n variables  $x_1, x_2, x_3, \ldots, x_n$  written as

 $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$   $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$   $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$   $\vdots$  $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$ 

then the **augmented matrix** of the system of equations is the  $m \times (n+1)$  matrix

 $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} & b_3 \\ \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m \end{bmatrix}$ 

## **Definition RO** Row Operations

The following three operations will transform an  $m \times n$  matrix into a different matrix of the same size, and each is known as a **row operation**.

- 1. Swap the locations of two rows.
- 2. Multiply each entry of a single row by a nonzero quantity.
- 3. Multiply each entry of one row by some quantity, and add these values to the entry in the same column of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

- 1.  $R_i \leftrightarrow R_j$ : Swap the location of rows *i* and *j*.
- 2.  $\alpha R_i$ : Multiply row *i* by the nonzero scalar  $\alpha$ .
- 3.  $\alpha R_i + R_j$ : Multiply row *i* by the scalar  $\alpha$  and add to row *j*.

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Definition REM Row-Equivalent Matrices

Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

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Definition RREF Reduced Row-Echelon Form

A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

- 1. A row where every entry is zero lies below any row that contains a nonzero entry.
- 2. The leftmost nonzero entry of a row is equal to 1.
- 3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
- 4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If s > i, then t > j.

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**Definition ZRM** Zero Row of a Matrix

A row of a matrix where every entry is zero is called a **zero row**.

For a matrix in reduced row-echelon form, the leftmost nonzero entry of any row that is not a zero row will be called a **leading 1**.

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**Definition PC** Pivot Columns

For a matrix in reduced row-echelon form, a column containing a leading 1 will be called a **pivot column**.

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**Theorem REMEF** Row-Equivalent Matrix in Echelon Form

Suppose A is a matrix. Then there is a (unique!) matrix B so that

- 1. A and B are row-equivalent.
- 2. B is in reduced row-echelon form.

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**Definition RR** Row-Reducing

To **row-reduce** the matrix A means to apply row operations to A and arrive at a row-equivalent matrix B in reduced row-echelon form.

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A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**.

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**Definition IDV** Independent and Dependent Variables

Suppose A is the augmented matrix of a system of linear equations and B is a row-equivalent matrix in reduced row-echelon form. Suppose j is the index of a column of B that contains the leading 1 for some row (i.e. column j is a pivot column), and this column is not the last column. Then the variable  $x_j$  is **dependent**. A variable that is not dependent is called **independent** or **free**.

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then the system of equations is inconsistent if and only if the leading 1 of row r is located in column n + 1 of B.

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**Theorem ICRN** Inconsistent Systems, r and n

Suppose A is the augmented matrix of a system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. If r = n + 1, then the system of equations is inconsistent.

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Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not zero rows. Then  $r \leq n$ . If r = n, then the system has a unique solution, and if r < n, then the system has infinitely many solutions.

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Theorem FVCS Free Variables for Consistent Systems

# Suppose A is the augmented matrix of a *consistent* system of linear equations with m equations in n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that are not completely zeros. Then the solution set can be described with n - r free variables.

A system of linear equations has no solutions, a unique solution or infinitely many solutions.

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Theorem CMVEI Consistent, More Variables than Equations, Infinite solutions 22

Suppose a consistent system of linear equations has m equations in n variables. If n > m, then the system has infinitely many solutions.

# **Definition HS** Homogeneous System

A system of linear equations is **homogeneous** if each equation has a 0 for its constant term. Such a system then has the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = 0$$
  

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = 0$$
  

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = 0$$
  

$$\vdots$$
  

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = 0$$

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**Theorem HSC** Homogeneous Systems are Consistent

Suppose that a system of linear equations is homogeneous. Then the system is consistent.

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Definition TSHSE Trivial Solution to Homogeneous Systems of Equations

Suppose a homogeneous system of linear equations has n variables. The solution  $x_1 = 0$ ,  $x_2 = 0, \ldots, x_n = 0$  is called the **trivial solution**.

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Theorem HMVEI Homogeneous, More Variables than Equations, Infinite solutions 26

Suppose that a homogeneous system of linear equations has m equations and n variables with n > m. Then the system has infinitely many solutions.

#### Definition CV Column Vector

A column vector of size m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a vector. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as  $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}$ . Some books like to write vectors with arrows, such as  $\vec{u}$ . Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in u. To refer to the entry or component that is number i in the list that is the vector  $\mathbf{v}$  we write  $[\mathbf{v}]_i$ .

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### Definition ZV Zero Vector

The **zero vector** of size *m* is the column vector of size *m* where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0\\0\\0\\\vdots\\0\end{bmatrix}$$

or more compactly,  $[\mathbf{0}]_i = 0$  for  $1 \le i \le m$ .

## **Definition CM** Coefficient Matrix

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$
  

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$
  

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$
  

$$\vdots$$
  

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **coefficient matrix** is the  $m \times n$  matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

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# Definition VOC Vector of Constants 30 For a system of linear equations, $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$ $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$ $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$ : $a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$ the vector of constants is the column vector of size m $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$

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# ${\bf Definition \ SV} \quad {\rm Solution \ Vector}$

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$
  

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$
  

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$
  

$$\vdots$$
  

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the **solution vector** is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

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**Definition NSM** Null Space of a Matrix

The **null space** of a matrix A, denoted  $\mathcal{N}(A)$ , is the set of all the vectors that are solutions to the homogeneous system  $\mathcal{LS}(A, \mathbf{0})$ .

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A matrix with m rows and n columns is **square** if m = n. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular**.

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# **Definition NM** Nonsingular Matrix

Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations  $\mathcal{LS}(A, \mathbf{0})$  is  $\{\mathbf{0}\}$ , i.e. the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

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**Definition IM** Identity Matrix

The  $m \times m$  identity matrix,  $I_m$  is defined by

$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

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**Theorem NSRRI** NonSingular matrices Row Reduce to the Identity matrix

Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Theorem NSTNS NonSingular matrices have Trivial Null Spaces

Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A,  $\mathcal{N}(A)$ , contains only the zero vector, i.e.  $\mathcal{N}(A) = \{\mathbf{0}\}.$ 

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Theorem NSMUS NonSingular Matrices and Unique Solutions

Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every choice of the constant vector  $\mathbf{b}$ .

Theorem NSME1 NonSingular Matrix Equivalences, Round 1

Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .

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Definition VSCV Vector Space of Column Vectors

The vector space  $\mathbb{C}^m$  is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers,  $\mathbb{C}$ .

Definition CVE Column Vector Equality 41 The vectors **u** and **v** are equal, written  $\mathbf{u} = \mathbf{v}$  provided that  $[\mathbf{u}]_i = [\mathbf{v}]_i$   $1 \le i \le m$ 

Given the vectors  ${\bf u}$  and  ${\bf v}$  the  ${\bf sum}$  of  ${\bf u}$  and  ${\bf v}$  is the vector  ${\bf u}+{\bf v}$  defined by

 $[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i \qquad 1 \le i \le m$ 

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Definition CVSM Column Vector Scalar Multiplication

Given the vector **u** and the scalar  $\alpha \in \mathbb{C}$ , the **scalar multiple** of **u** by  $\alpha$ ,  $\alpha$ **u** is defined by

$$\left[\alpha \mathbf{u}\right]_{i} = \alpha \left[\mathbf{u}\right]_{i} \qquad \qquad 1 \le i \le m$$

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**Theorem VSPCV** Vector Space Properties of Column Vectors 44 Suppose that  $\mathbb{C}^m$  is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- ACC Additive Closure, Column Vectors If  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ , then  $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$ .
- SCC Scalar Closure, Column Vectors If  $\alpha \in \mathbb{C}$  and  $\mathbf{u} \in \mathbb{C}^m$ , then  $\alpha \mathbf{u} \in \mathbb{C}^m$ .
- CC Commutativity, Column Vectors If  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ , then  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .
- AAC Additive Associativity, Column Vectors If  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$ , then  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ .
- ZC Zero Vector, Column Vectors There is a vector, 0, called the zero vector, such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$  for all  $\mathbf{u} \in \mathbb{C}^m$ .
- AIC Additive Inverses, Column Vectors If  $\mathbf{u} \in \mathbb{C}^m$ , then there exists a vector  $-\mathbf{u} \in \mathbb{C}^m$  so that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ .
- SMAC Scalar Multiplication Associativity, Column Vectors If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in \mathbb{C}^m$ , then  $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$ .
- DVAC Distributivity across Vector Addition, Column Vectors If  $\alpha \in \mathbb{C}$  and  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ , then  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$ .
- DSAC Distributivity across Scalar Addition, Column Vectors If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in \mathbb{C}^m$ , then  $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$ .

**OC** One, Column Vectors II  $\mathbf{u} \in \mathbb{C}^{+}$ , then  $\mathbf{1u} = \mathbf{u}$ .

Given *n* vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$  from  $\mathbb{C}^m$  and *n* scalars  $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ , their **linear** combination is the vector

 $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n.$ 

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**Theorem SLSLC** Solutions to Linear Systems are Linear Combinations

Denote the columns of the  $m \times n$  matrix A as the vectors  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ . Then  $\mathbf{x}$  is a solution to the linear system of equations  $\mathcal{LS}(A, \mathbf{b})$  if and only if

$$\left[\mathbf{x}\right]_{1}\mathbf{A}_{1}+\left[\mathbf{x}\right]_{2}\mathbf{A}_{2}+\left[\mathbf{x}\right]_{3}\mathbf{A}_{3}+\cdots+\left[\mathbf{x}\right]_{n}\mathbf{A}_{n}=\mathbf{b}$$

#### Theorem VFSLS Vector Form of Solutions to Linear Systems

Suppose that  $[A \mid \mathbf{b}]$  is the augmented matrix for a consistent linear system  $\mathcal{LS}(A, \mathbf{b})$  of m equations in n variables. Let B be a row-equivalent  $m \times (n + 1)$  matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices  $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$ , and columns with leading 1's (pivot columns) having indices  $D = \{d_1, d_2, d_3, \ldots, d_r\}$ . Define vectors  $\mathbf{c}, \mathbf{u}_j, 1 \leq j \leq n-r$  of size n by

$$\begin{split} \left[ \mathbf{c} \right]_{i} &= \begin{cases} 0 & \text{if } i \in F \\ \left[ B \right]_{k,n+1} & \text{if } i \in D, \, i = d_{k} \end{cases} \\ \left[ \mathbf{u}_{j} \right]_{i} &= \begin{cases} 1 & \text{if } i \in F, \, i = f_{j} \\ 0 & \text{if } i \in F, \, i \neq f_{j} \\ -\left[ B \right]_{k,f_{j}} & \text{if } i \in D, \, i = d_{k} \end{cases} \end{split}$$

Then the set of solutions to the system of equations  $\mathcal{LS}(A, \mathbf{b})$  is

$$S = \left\{ \mathbf{c} + x_{f_1} \mathbf{u}_1 + x_{f_2} \mathbf{u}_2 + x_{f_3} \mathbf{u}_3 + \dots + x_{f_{n-r}} \mathbf{u}_{n-r} \, \middle| \, x_{f_1}, \, x_{f_2}, \, x_{f_3}, \, \dots, \, x_{f_{n-r}} \in \mathbb{C} \right\}$$

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Theorem PSPHS Particular Solution Plus Homogeneous Solutions

Suppose that **w** is one solution to the linear system of equations  $\mathcal{LS}(A, b)$ . Then **y** is a solution to  $\mathcal{LS}(A, b)$  if and only if  $\mathbf{y} = \mathbf{w} + \mathbf{z}$  for some vector  $\mathbf{z} \in \mathcal{N}(A)$ .

Suppose that A is an  $m \times n$  matrix and that B and C are  $m \times n$  matrices that are row-equivalent to A and in reduced row-echelon form. Then B = C.

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# Definition SSCV Span of a Set of Column Vectors

Given a set of vectors  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p}$ , their **span**,  $\langle S \rangle$ , is the set of all possible linear combinations of  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p$ . Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_p \mathbf{u}_p \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$
$$= \left\{ \sum_{i=1}^p \alpha_i \mathbf{u}_i \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$

Theorem SSNS Spanning Sets for Null Spaces

Suppose that A is an  $m \times n$  matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let  $D = \{d_1, d_2, d_3, \ldots, d_r\}$  be the column indices where B has leading 1's (pivot columns) and  $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$  be the set of column indices where B does not have leading 1's. Construct the n - r vectors  $\mathbf{z}_j$ ,  $1 \le j \le n - r$  of size n as

$$\left[ \mathbf{z}_j \right]_i = \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -\left[B\right]_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r} \} \rangle.$$

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**Definition RLDCV** Relation of Linear Dependence for Column Vectors

Given a set of vectors  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ , a true statement of the form

 $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \dots + \alpha_n\mathbf{u}_n = \mathbf{0}$ 

is a relation of linear dependence on S. If this statement is formed in a trivial fashion, i.e.  $\alpha_i = 0, 1 \le i \le n$ , then we say it is the trivial relation of linear dependence on S.

The set of vectors  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$  is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

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# Theorem LIVHS Linearly Independent Vectors and Homogeneous Systems

Suppose that A is an  $m \times n$  matrix and  $S = {\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n}$  is the set of vectors in  $\mathbb{C}^m$  that are the columns of A. Then S is a linearly independent set if and only if the homogeneous system  $\mathcal{LS}(A, \mathbf{0})$  has a unique solution.

**Theorem LIVRN** Linearly Independent Vectors, r and n

Suppose that A is an  $m \times n$  matrix and  $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$  is the set of vectors in  $\mathbb{C}^m$  that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if n = r.

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**Theorem MVSLD** More Vectors than Size implies Linear Dependence

Suppose that  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$  is the set of vectors in  $\mathbb{C}^m$ , and that n > m. Then S is a linearly dependent set.

Theorem NSLIC NonSingular matrices have Linearly Independent Columns

Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

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# Theorem NSME2 NonSingular Matrix Equivalences, Round 2

Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A form a linearly independent set.

#### Theorem BNS Basis for Null Spaces

Suppose that A is an  $m \times n$  matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let  $D = \{d_1, d_2, d_3, \ldots, d_r\}$  and  $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$  be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the n - r vectors  $\mathbf{z}_j$ ,  $1 \le j \le n - r$  of size n as

$$\left[ \mathbf{z}_j \right]_i = \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -b_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases}$$

Define the set  $S = \{ \mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r} \}$ . Then

1. 
$$\mathcal{N}(A) = \langle S \rangle.$$

2. S is a linearly independent set.

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## Theorem DLDS Dependency in Linearly Dependent Sets

Suppose that  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$  is a set of vectors. Then S is a linearly dependent set if and only if there is an index  $t, 1 \le t \le n$  such that  $\mathbf{u}_t$  is a linear combination of the vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \dots, \mathbf{u}_n$ .

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#### Theorem BS Basis of a Span

Suppose that  $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n}$  is a set of column vectors. Define  $W = \langle S \rangle$  and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with  $D = {d_1, d_2, d_3, \dots, d_r}$  be the set of column indices corresponding to the pivot columns of B. Then

- 1.  $T = {\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \dots, \mathbf{v}_{d_r}}$  is a linearly independent set.
- 2.  $W = \langle T \rangle$ .

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**Definition CCCV** Complex Conjugate of a Column Vector

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Suppose that **u** is a vector from  $\mathbb{C}^m$ . Then the conjugate of the vector,  $\overline{\mathbf{u}}$ , is defined by

 $[\overline{\mathbf{u}}]_i = \overline{[\mathbf{u}]_i} \qquad \qquad 1 \le i \le m$ 

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 Theorem CRVA
 Conjugation Respects Vector Addition
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 Suppose x and y are two vectors from  $\mathbb{C}^m$ . Then
  $\overline{x+y} = \overline{x} + \overline{y}$ 
 $\overline{x+y} = \overline{x} + \overline{y}$   $\overline{c}$  2005 Robert Beezer

Theorem CRSM Conjugation Respects Vector Scalar Multiplication

Suppose **x** is a vector from  $\mathbb{C}^m$ , and  $\alpha \in \mathbb{C}$  is a scalar. Then

 $\overline{\alpha \mathbf{x}} = \overline{\alpha} \, \overline{\mathbf{x}}$ 

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# **Definition IP** Inner Product

Given the vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$  the **inner product** of  $\mathbf{u}$  and  $\mathbf{v}$  is the scalar quantity in  $\mathbb{C}$ ,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \left[ \mathbf{u} \right]_1 \overline{\left[ \mathbf{v} \right]_1} + \left[ \mathbf{u} \right]_2 \overline{\left[ \mathbf{v} \right]_2} + \left[ \mathbf{u} \right]_3 \overline{\left[ \mathbf{v} \right]_3} + \dots + \left[ \mathbf{u} \right]_m \overline{\left[ \mathbf{v} \right]_m} = \sum_{i=1}^m \left[ \mathbf{u} \right]_i \overline{\left[ \mathbf{v} \right]_i}$$

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Theorem IPVA Inner Product and Vector Addition

Suppose  $\mathbf{u}\mathbf{v}, \mathbf{w} \in \mathbb{C}^m$ . Then

1. 
$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$
  
2.  $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ 

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Theorem IPSM Inner Product and Scalar Multiplication

Suppose  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$  and  $\alpha \in \mathbb{C}$ . Then

1.  $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$ 2.  $\langle \mathbf{u}, \alpha \mathbf{v} \rangle = \overline{\alpha} \langle \mathbf{u}, \mathbf{v} \rangle$ 

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**Theorem IPAC** Inner Product is Anti-Commutative

Suppose that **u** and **v** are vectors in  $\mathbb{C}^m$ . Then  $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$ .

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 ${\bf Definition} \ {\bf NV} \quad {\rm Norm} \ {\rm of} \ {\rm a} \ {\rm Vector}$ 

The  $\mathbf{norm}$  of the vector  $\mathbf u$  is the scalar quantity in  $\mathbb C$ 

$$\|\mathbf{u}\| = \sqrt{|[\mathbf{u}]_1|^2 + |[\mathbf{u}]_2|^2 + |[\mathbf{u}]_3|^2 + \dots + |[\mathbf{u}]_m|^2} = \sqrt{\sum_{i=1}^m |[\mathbf{u}]_i|^2}$$

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# Theorem IPN Inner Products and Norms

Suppose that  $\mathbf{u}$  is a vector in  $\mathbb{C}^m$ . Then  $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$ .

Suppose that **u** is a vector in  $\mathbb{C}^m$ . Then  $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$  with equality if and only if  $\mathbf{u} = \mathbf{0}$ .

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**Definition OV** Orthogonal Vectors

A pair of vectors, **u** and **v**, from  $\mathbb{C}^m$  are **orthogonal** if their inner product is zero, that is,  $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ .

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Suppose that  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$  is a set of vectors from  $\mathbb{C}^m$ . Then the set S is **orthogonal** if every pair of different vectors from S is orthogonal, that is  $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$  whenever  $i \neq j$ .

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#### **Theorem OSLI** Orthogonal Sets are Linearly Independent

Suppose that  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$  is an orthogonal set of nonzero vectors. Then S is linearly independent.

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Theorem GSPCV Gram-Schmidt Procedure, Column Vectors

Suppose that  $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p}$  is a linearly independent set of vectors in  $\mathbb{C}^m$ . Define the vectors  $\mathbf{u}_i, 1 \leq i \leq p$  by

$$\mathbf{u}_{i} = \mathbf{v}_{i} - \frac{\langle \mathbf{v}_{i}, \mathbf{u}_{1} \rangle}{\langle \mathbf{u}_{1}, \mathbf{u}_{1} \rangle} \mathbf{u}_{1} - \frac{\langle \mathbf{v}_{i}, \mathbf{u}_{2} \rangle}{\langle \mathbf{u}_{2}, \mathbf{u}_{2} \rangle} \mathbf{u}_{2} - \frac{\langle \mathbf{v}_{i}, \mathbf{u}_{3} \rangle}{\langle \mathbf{u}_{3}, \mathbf{u}_{3} \rangle} \mathbf{u}_{3} - \dots - \frac{\langle \mathbf{v}_{i}, \mathbf{u}_{i-1} \rangle}{\langle \mathbf{u}_{i-1}, \mathbf{u}_{i-1} \rangle} \mathbf{u}_{i-1}$$

Then if  $T = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p}$ , then T is an orthogonal set of non-zero vectors, and  $\langle T \rangle = \langle S \rangle$ .

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#### Definition ONS OrthoNormal Set

Suppose  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$  is an orthogonal set of vectors such that  $||\mathbf{u}_i|| = 1$  for all  $1 \le i \le n$ . Then S is an **orthonormal** set of vectors.

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The vector space  $M_{mn}$  is the set of all  $m \times n$  matrices with entries from the set of complex numbers.

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**Definition ME** Matrix Equality

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The  $m \times n$  matrices A and B are equal, written A = B provided  $[A]_{ij} = [B]_{ij}$  for all  $1 \le i \le m$ ,  $1 \le j \le n$ .

Definition MA Matrix Addition

Given the  $m \times n$  matrices A and B, define the **sum** of A and B as an  $m \times n$  matrix, written A + B, according to

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij} \qquad 1 \le i \le m, \ 1 \le j \le n$$

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## Definition MSM Matrix Scalar Multiplication

Given the  $m \times n$  matrix A and the scalar  $\alpha \in \mathbb{C}$ , the scalar multiple of A is an  $m \times n$  matrix, written  $\alpha A$  and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij} \qquad 1 \le i \le m, \ 1 \le j \le n$$

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**Theorem VSPM** Vector Space Properties of Matrices

Suppose that  $M_{mn}$  is the set of all  $m \times n$  matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- ACM Additive Closure, Matrices If  $A, B \in M_{mn}$ , then  $A + B \in M_{mn}$ .
- SCM Scalar Closure, Matrices If  $\alpha \in \mathbb{C}$  and  $A \in M_{mn}$ , then  $\alpha A \in M_{mn}$ .
- CM Commutativity, Matrices If  $A, B \in M_{mn}$ , then A + B = B + A.
- AAM Additive Associativity, Matrices If  $A, B, C \in M_{mn}$ , then A + (B + C) = (A + B) + C.
- ZM Zero Vector, Matrices There is a matrix,  $\mathcal{O}$ , called the zero matrix, such that  $A + \mathcal{O} = A$  for all  $A \in M_{mn}$ .
- AIM Additive Inverses, Matrices If  $A \in M_{mn}$ , then there exists a matrix  $-A \in M_{mn}$  so that  $A + (-A) = \mathcal{O}$ .
- SMAM Scalar Multiplication Associativity, Matrices If  $\alpha$ ,  $\beta \in \mathbb{C}$  and  $A \in M_{mn}$ , then  $\alpha(\beta A) = (\alpha \beta)A$ .
- DMAM Distributivity across Matrix Addition, Matrices If  $\alpha \in \mathbb{C}$  and  $A, B \in M_{mn}$ , then  $\alpha(A+B) = \alpha A + \alpha B$ .
- DSAM Distributivity across Scalar Addition, Matrices If  $\alpha, \beta \in \mathbb{C}$  and  $A \in M_{mn}$ , then  $(\alpha + \beta)A = \alpha A + \beta A$ .

**ONI** One, Matrices If  $A \in M_{mn}$ , then 1A = A.

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#### **Definition ZM** Zero Matrix

The  $m \times n$  zero matrix is written as  $\mathcal{O} = \mathcal{O}_{m \times n}$  and defined by  $[\mathcal{O}]_{ij} = 0$ , for all  $1 \le i \le m$ ,  $1 \le j \le n$ .

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## **Definition TM** Transpose of a Matrix

Given an  $m \times n$  matrix A, its **transpose** is the  $n \times m$  matrix  $A^t$  given by

 $\left[A^t\right]_{ij} = [A]_{ji}\,,\quad 1\leq i\leq n,\, 1\leq j\leq m.$ 

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**Definition SYM** Symmetric Matrix

The matrix A is symmetric if  $A = A^t$ .

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<b>Theorem SMS</b> Symmetric Matrices are Square	85
Suppose that $A$ is a symmetric matrix. Then $A$ is square.	

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# Theorem TMA Transpose and Matrix Addition

Suppose that A and B are  $m \times n$  matrices. Then  $(A + B)^t = A^t + B^t$ .

Suppose that  $\alpha \in \mathbb{C}$  and A is an  $m \times n$  matrix. Then  $(\alpha A)^t = \alpha A^t$ .

**Theorem TMSM** Transpose and Matrix Scalar Multiplication

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**Theorem TT** Transpose of a Transpose

Suppose that A is an  $m \times n$  matrix. Then  $(A^t)^t = A$ .

Suppose A is an  $m \times n$  matrix. Then the **conjugate** of A, written  $\overline{A}$  is an  $m \times n$  matrix defined by

$$\left[\overline{A}\right]_{ij} = \overline{\left[A\right]_{ij}}$$

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## Theorem CRMA Conjugation Respects Matrix Addition

Suppose that A and B are  $m \times n$  matrices. Then  $\overline{A + B} = \overline{A} + \overline{B}$ .

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Theorem CRMSM Conjugation Respects Matrix Scalar Multiplication

Suppose that  $\alpha \in \mathbb{C}$  and A is an  $m \times n$  matrix. Then  $\overline{\alpha A} = \overline{\alpha} \overline{A}$ .

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# Theorem MCT Matrix Conjugation and Transposes

Suppose that A is an  $m \times n$  matrix. Then  $\overline{(A^t)} = (\overline{A})^t$ .

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Definition MVP Matrix-Vector Product

Suppose A is an  $m \times n$  matrix with columns  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$  and  $\mathbf{u}$  is a vector of size n. Then the **matrix-vector product** of A with  $\mathbf{u}$  is the linear combination

 $A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \dots + [\mathbf{u}]_n \mathbf{A}_n$ 

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**Theorem SLEMM** Systems of Linear Equations as Matrix Multiplication

Solutions to the linear system  $\mathcal{LS}(A, \mathbf{b})$  are the solutions for  $\mathbf{x}$  in the vector equation  $A\mathbf{x} = \mathbf{b}$ .

Theorem EMMVP Equal Matrices and Matrix-Vector Products

Suppose that A and B are  $m \times n$  matrices such that  $A\mathbf{x} = B\mathbf{x}$  for every  $\mathbf{x} \in \mathbb{C}^n$ . Then A = B.

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## Definition MM Matrix Multiplication

Suppose A is an  $m \times n$  matrix and B is an  $n \times p$  matrix with columns  $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$ . Then the **matrix product** of A with B is the  $m \times p$  matrix where column i is the matrix-vector product  $A\mathbf{B}_i$ . Symbolically,

 $AB = A \left[ \mathbf{B}_1 | \mathbf{B}_2 | \mathbf{B}_3 | \dots | \mathbf{B}_p \right] = \left[ A \mathbf{B}_1 | A \mathbf{B}_2 | A \mathbf{B}_3 | \dots | A \mathbf{B}_p \right].$ 

#### Theorem EMP Entries of Matrix Products

Suppose A is an  $m \times n$  matrix and B = is an  $n \times p$  matrix. Then for  $1 \le i \le m, 1 \le j \le p$ , the individual entries of AB are given by

$$[AB]_{ij} = [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \dots + [A]_{in} [B]_{nj} = \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$$

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## Theorem MMZM Matrix Multiplication and the Zero Matrix

Suppose A is an  $m \times n$  matrix. Then 1.  $A\mathcal{O}_{n \times p} = \mathcal{O}_{m \times p}$ 2.  $\mathcal{O}_{p \times m} A = \mathcal{O}_{p \times n}$ 

**Theorem MMIM** Matrix Multiplication and Identity Matrix

Suppose A is an  $m \times n$  matrix. Then 1.  $AI_n = A$ 2.  $I_m A = A$ 

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## Theorem MMDAA Matrix Multiplication Distributes Across Addition

Suppose A is an  $m \times n$  matrix and B and C are  $n \times p$  matrices and D is a  $p \times s$  matrix. Then 1. A(B+C) = AB + AC2. (B+C)D = BD + CD Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication

Suppose A is an  $m \times n$  matrix and B is an  $n \times p$  matrix. Let  $\alpha$  be a scalar. Then  $\alpha(AB) = (\alpha A)B = A(\alpha B)$ .

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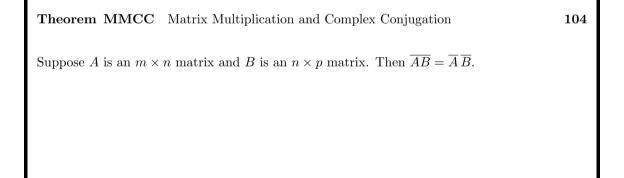
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Theorem MMA Matrix Multiplication is Associative

Suppose A is an  $m \times n$  matrix, B is an  $n \times p$  matrix and D is a  $p \times s$  matrix. Then A(BD) = (AB)D.

**Theorem MMIP** Matrix Multiplication and Inner Products If we consider the vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$  as  $m \times 1$  matrices then  $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^t \overline{\mathbf{v}}$ 

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Suppose A is an  $m \times n$  matrix and B is an  $n \times p$  matrix. Then  $(AB)^t = B^t A^t$ .

Theorem MMT Matrix Multiplication and Transposes

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**Definition MI** Matrix Inverse

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Suppose A and B are square matrices of size n such that  $AB = I_n$  and  $BA = I_n$ . Then A is **invertible** and B is the **inverse** of A. In this situation, we write  $B = A^{-1}$ .

#### Definition SUV Standard Unit Vectors

Let  $\mathbf{e}_j \in \mathbb{C}^m$  denote the column vector that is column j of the  $m \times m$  identity matrix  $I_m$ . Then the set

$$\{\mathbf{e}_1, \, \mathbf{e}_2, \, \mathbf{e}_3, \, \dots, \, \mathbf{e}_m\} = \{\, \mathbf{e}_j \mid 1 \le j \le m\}$$

is the set of standard unit vectors in  $\mathbb{C}^m$ .

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Theorem TTMI Two-by-Two Matrix Inverse

Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if  $ad - bc \neq 0$ . When A is invertible, we have

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

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**Theorem CINSM** Computing the Inverse of a NonSingular Matrix

Suppose A is a nonsingular square matrix of size n. Create the  $n \times 2n$  matrix M by placing the  $n \times n$  identity matrix  $I_n$  to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then  $AJ = I_n$ .

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Theorem MIU Matrix Inverse is Unique

Suppose the square matrix A has an inverse. Then  $A^{-1}$  is unique.

Suppose A and B are invertible matrices of size n. Then  $(AB)^{-1} = B^{-1}A^{-1}$  and AB is an invertible matrix.

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Theorem MIMI Matrix Inverse of a Matrix Inverse

Suppose A is an invertible matrix. Then  $(A^{-1})^{-1} = A$  and  $A^{-1}$  is invertible.

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Suppose A is an invertible matrix. Then  $(A^t)^{-1} = (A^{-1})^t$  and  $A^t$  is invertible.

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**Theorem MISM** Matrix Inverse of a Scalar Multiple

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Suppose A is an invertible matrix and  $\alpha$  is a nonzero scalar. Then  $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$  and  $\alpha A$  is invertible.

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Suppose that A and B are matrices of size n, and one, or both, is singular. Then their product, AB, is singular.

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**Theorem OSIS** One-Sided Inverse is Sufficient

Suppose A and B are square matrices of size n such that  $AB = I_n$ . Then  $BA = I_n$ .

Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.

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 Theorem NSME3
 NonSingular Matrix Equivalences, Round 3
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Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.

# Suppose that A is nonsingular. Then the unique solution to $\mathcal{LS}(A, \mathbf{b})$ is $A^{-1}\mathbf{b}$ .

Theorem SNSCM Solution with NonSingular Coefficient Matrix

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**Definition OM** Orthogonal Matrices

Suppose that Q is a square matrix of size n such that  $(\overline{Q})^t Q = I_n$ . Then we say Q is **orthogonal**.

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<b>Theorem OMI</b> Orthogonal Matrices are Invertible	121
Suppose that Q is an orthogonal matrix of size n. Then Q is nonsingular, and $Q^{-1} = (\overline{Q})$	<sup>t</sup> .

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Theorem COMOS Columns of Orthogonal Matrices are Orthonormal Sets 122

Suppose that A is a square matrix of size n with columns  $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ . Then A is an orthogonal matrix if and only if S is an orthonormal set.

Theorem OMPIP Orthogonal Matrices Preserve Inner Products Suppose that Q is an orthogonal matrix of size n and **u** and **v** are two vectors from  $\mathbb{C}^n$ . Then

$$\langle Q\mathbf{u}, Q\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$
 and  $\|Q\mathbf{v}\| = \|\mathbf{v}\|$ 

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Definition A Adjoint

If A is a square matrix, then its **adjoint** is  $A^{H} = (\overline{A})^{t}$ .

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**Definition HM** Hermitian Matrix125The square matrix A is Hermitian (or self-adjoint) if  $A = (\overline{A})^t$ 

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## **Definition CSM** Column Space of a Matrix

Suppose that A is an  $m \times n$  matrix with columns  $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n\}$ . Then the **column space** of A, written  $\mathcal{C}(A)$ , is the subset of  $\mathbb{C}^m$  containing all linear combinations of the columns of A,

 $\mathcal{C}(A) = \langle \{\mathbf{A}_1, \, \mathbf{A}_2, \, \mathbf{A}_3, \, \dots, \, \mathbf{A}_n \} \rangle$ 

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Suppose A is an  $m \times n$  matrix and **b** is a vector of size m. Then  $\mathbf{b} \in \mathcal{C}(A)$  if and only if  $\mathcal{LS}(A, \mathbf{b})$  is consistent.

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#### **Theorem BCS** Basis of the Column Space

Suppose that A is an  $m \times n$  matrix with columns  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ , and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let  $D = \{d_1, d_2, d_3, \ldots, d_r\}$  be the set of column indices where B has leading 1's. Let  $T = \{\mathbf{A}_{d_1}, \mathbf{A}_{d_2}, \mathbf{A}_{d_3}, \ldots, \mathbf{A}_{d_r}\}$ . Then

- 1. T is a linearly independent set.
- 2.  $\mathcal{C}(A) = \langle T \rangle$ .

Suppose A is a square matrix of size n. Then A is nonsingular if and only if  $\mathcal{C}(A) = \mathbb{C}^n$ .

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 Theorem NSME4
 NonSingular Matrix Equivalences, Round 4
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 Suppose that A is a square matrix of size n. The following are equivalent.
 1.

 1. A is nonsingular.
 2.

 2. A row-reduces to the identity matrix.

- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}$ .
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .

Suppose A is an  $m \times n$  matrix. Then the **row space** of A,  $\mathcal{R}(A)$ , is the column space of  $A^t$ , i.e.  $\mathcal{R}(A) = \mathcal{C}(A^t)$ .

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Theorem REMRSRow-Equivalent Matrices have equal Row Spaces132

Suppose A and B are row-equivalent matrices. Then  $\mathcal{R}(A) = \mathcal{R}(B)$ .

## Theorem BRS Basis for the Row Space

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of  $B^t$ . Then

- 1.  $\mathcal{R}(A) = \langle S \rangle$ .
- 2. S is a linearly independent set.

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Theorem CSRST Column Space, Row Space, Transpose

Suppose A is a matrix. Then  $\mathcal{C}(A) = \mathcal{R}(A^t)$ .

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Suppose A is an  $m \times n$  matrix. Then the **left null space** is defined as  $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$ .

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## Definition EEF Extended Echelon Form

Suppose A is an  $m \times n$  matrix. Add m new columns to A that together equal an  $m \times m$  identity matrix to form an  $m \times (n+m)$  matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the **extended reduced row-echelon form** of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the  $m \times n$  matrix formed from the first n columns of N and let J denote the  $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the  $r \times n$  matrix formed from all of the non-zero rows of B. Let K be the  $r \times m$  matrix formed from the first r rows of J, while L will be the  $(m - r) \times m$ matrix formed from the bottom m - r rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix} C & K \\ \hline 0 & L \end{bmatrix}$$

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#### **Theorem PEEF** Properties of Extended Echelon Form

Suppose that A is an  $m \times n$  matrix and that N is its extended echelon form. Then

- 1. J is nonsingular.
- 2. B = JA.
- 3. If  $\mathbf{x} \in \mathbb{C}^n$  and  $\mathbf{y} \in \mathbb{C}^m$ , then  $A\mathbf{x} = \mathbf{y}$  if and only if  $B\mathbf{x} = J\mathbf{y}$ .
- 4. C is in reduced row-echelon form, has no zero rows and has r pivot columns.
- 5. L is in reduced row-echelon form, has no zero rows and has m r pivot columns.

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## Theorem FS Four Subsets

Suppose A is an  $m \times n$  matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last m - r rows. Then

- 1. The null space of A is the null space of C,  $\mathcal{N}(A) = \mathcal{N}(C)$ .
- 2. The row space of A is the row space of C,  $\mathcal{R}(A) = \mathcal{R}(C)$ .
- 3. The column space of A is the null space of L,  $C(A) = \mathcal{N}(L)$ .
- 4. The left null space of A is the row space of L,  $\mathcal{L}(A) = \mathcal{R}(L)$ .

## **Definition VS** Vector Space

Suppose that V is a set upon which we have defined two operations: (1) vector addition, which combines two elements of V and is denoted by "+", and (2) scalar multiplication, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a vector space if the following ten properties hold.

- AC Additive Closure If  $\mathbf{u}, \mathbf{v} \in V$ , then  $\mathbf{u} + \mathbf{v} \in V$ .
- SC Scalar Closure If  $\alpha \in \mathbb{C}$  and  $\mathbf{u} \in V$ , then  $\alpha \mathbf{u} \in V$ .
- C Commutativity If  $\mathbf{u}, \mathbf{v} \in V$ , then  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .
- AA Additive Associativity If  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ , then  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ .
- Z Zero Vector There is a vector, 0, called the zero vector, such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$  for all  $\mathbf{u} \in V$ .
- AI Additive Inverses If  $\mathbf{u} \in V$ , then there exists a vector  $-\mathbf{u} \in V$  so that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ .
- SMA Scalar Multiplication Associativity If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in V$ , then  $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$ .
- DVA Distributivity across Vector Addition If  $\alpha \in \mathbb{C}$  and  $\mathbf{u}, \mathbf{v} \in V$ , then  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$ .
- DSA Distributivity across Scalar Addition If  $\alpha, \beta \in \mathbb{C}$  and  $\mathbf{u} \in V$ , then  $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$ .

 $\hat{\mathbf{O}}$  One if  $\mathbf{u} \in V$ , then  $\mathbf{u} = \mathbf{u}$ .

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.

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Theorem **ZVU** Zero Vector is Unique

Suppose that V is a vector space. The zero vector,  $\mathbf{0}$ , is unique.

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Suppose that V is a vector space. For each  $\mathbf{u} \in V$ , the additive inverse,  $-\mathbf{u}$ , is unique.

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**Theorem ZSSM** Zero Scalar in Scalar Multiplication

Suppose that V is a vector space and  $\mathbf{u} \in V$ . Then  $0\mathbf{u} = \mathbf{0}$ .

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 Theorem ZVSM
 Zero Vector in Scalar Multiplication
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 Suppose that V is a vector space and  $\alpha \in \mathbb{C}$ . Then  $\alpha \mathbf{0} = \mathbf{0}$ .
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# **Theorem AISM** Additive Inverses from Scalar Multiplication

Suppose that V is a vector space and  $\mathbf{u} \in V$ . Then  $-\mathbf{u} = (-1)\mathbf{u}$ .

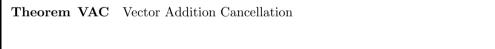
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Suppose that V is a vector space and  $\alpha \in \mathbb{C}$ . If  $\alpha \mathbf{u} = \mathbf{0}$ , then either  $\alpha = 0$  or  $\mathbf{u} = \mathbf{0}$ .

**Theorem SMEZV** Scalar Multiplication Equals the Zero Vector

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Suppose that V is a vector space, and  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ . If  $\mathbf{w} + \mathbf{u} = \mathbf{w} + \mathbf{v}$ , then  $\mathbf{u} = \mathbf{v}$ .

Suppose V is a vector space,  $\mathbf{u}, \mathbf{v} \in V$  and  $\alpha$  is a nonzero scalar from  $\mathbb{C}$ . If  $\alpha \mathbf{u} = \alpha \mathbf{v}$ , then  $\mathbf{u} = \mathbf{v}$ .

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**Theorem CVSM** Canceling Vectors in Scalar Multiplication

Suppose V is a vector space,  $\mathbf{u} \neq \mathbf{0}$  is a vector in V and  $\alpha, \beta \in \mathbb{C}$ . If  $\alpha \mathbf{u} = \beta \mathbf{u}$ , then  $\alpha = \beta$ .

#### **Definition S** Subspace

Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of V,  $W \subseteq V$ . Then W is a subspace of V.

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### Theorem TSS Testing Subsets for Subspaces

Suppose that V is a vector space and W is a subset of V,  $W \subseteq V$ . Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

- 1. W is non-empty,  $W \neq \emptyset$ .
- 2. If  $\mathbf{x} \in W$  and  $\mathbf{y} \in W$ , then  $\mathbf{x} + \mathbf{y} \in W$ .
- 3. If  $\alpha \in \mathbb{C}$  and  $\mathbf{x} \in W$ , then  $\alpha \mathbf{x} \in W$ .

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Given the vector space V, the subspaces V and  $\{0\}$  are each called a **trivial subspace**.

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Theorem NSMS Null Space of a Matrix is a Subspace

Suppose that A is an  $m \times n$  matrix. Then the null space of A,  $\mathcal{N}(A)$ , is a subspace of  $\mathbb{C}^n$ .

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Suppose that V is a vector space. Given n vectors  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$  and n scalars  $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ , their linear combination is the vector

 $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \dots + \alpha_n\mathbf{u}_n.$ 

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# ${\bf Definition \ SS} \quad {\rm Span \ of \ a \ Set}$

Suppose that V is a vector space. Given a set of vectors  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t}$ , their **span**,  $\langle S \rangle$ , is the set of all possible linear combinations of  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$ . Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_t \mathbf{u}_t \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$
$$= \left\{ \sum_{i=1}^t \alpha_i \mathbf{u}_i \mid \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$

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# **Theorem SSS** Span of a Set is a Subspace

Suppose V is a vector space. Given a set of vectors  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t} \subseteq V$ , their span,  $\langle S \rangle$ , is a subspace.

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**Theorem CSMS** Column Space of a Matrix is a Subspace

Suppose that A is an  $m \times n$  matrix. Then  $\mathcal{C}(A)$  is a subspace of  $\mathbb{C}^m$ .

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<b>Theorem RSMS</b> Row Space of a Matrix is a Subspace	157
Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of $\mathbb{C}^n$ .	

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**Theorem LNSMS** Left Null Space of a Matrix is a Subspace

Suppose that A is an  $m \times n$  matrix. Then  $\mathcal{L}(A)$  is a subspace of  $\mathbb{C}^m$ .

### Definition RLD Relation of Linear Dependence

Suppose that V is a vector space. Given a set of vectors  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$ , an equation of the form

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \dots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

is a relation of linear dependence on S. If this equation is formed in a trivial fashion, i.e.  $\alpha_i = 0, 1 \le i \le n$ , then we say it is a trivial relation of linear dependence on S.

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**Definition LI** Linear Independence

Suppose that V is a vector space. The set of vectors  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$  from V is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

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Suppose V is a vector space. A subset S of V is a **spanning set** for V if  $\langle S \rangle = V$ . In this case, we also say S **spans** V.

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### **Definition B** Basis

Suppose V is a vector space. Then a subset  $S \subseteq V$  is a **basis** of V if it is linearly independent and spans V.

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**Theorem SUVB** Standard Unit Vectors are a Basis 163 The set of standard unit vectors for  $\mathbb{C}^m$ ,  $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_m\} = \{\mathbf{e}_i \mid 1 \le i \le m\}$  is a basis for the vector space  $\mathbb{C}^m$ .

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Theorem CNSMB Columns of NonSingular Matrix are a Basis

Suppose that A is a square matrix of size m. Then the columns of A are a basis of  $\mathbb{C}^m$  if and only if A is nonsingular.

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Suppose that A is a square matrix of size n. The following are equivalent.

1. A is nonsingular.

Theorem NSME5

- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .

NonSingular Matrix Equivalences, Round 5

- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
- 8. The columns of A are a basis for  $\mathbb{C}^n$ .

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Theorem VRRB Vector Representation Relative to a Basis

Suppose that V is a vector space with basis  $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m}$  and that **w** is a vector in V. Then there exist *unique* scalars  $a_1, a_2, a_3, \dots, a_m$  such that

$$\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_m \mathbf{v}_m.$$

#### **Definition D** Dimension

Suppose that V is a vector space and  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$  is a basis of V. Then the **dimension** of V is defined by dim (V) = t. If V has no finite bases, we say V has infinite dimension.

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# Theorem SSLD Spanning Sets and Linear Dependence

Suppose that  $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t}$  is a finite set of vectors which spans the vector space V. Then any set of t + 1 or more vectors from V is linearly dependent.

Suppose that V is a vector space with a finite basis B and a second basis C. Then B and C have the same size.

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**Theorem DCM** Dimension of  $\mathbb{C}^m$ 

The dimension of  $\mathbb{C}^m$  (Example VSCV) is m.

The dimension of  $P_n$  (Example VSP) is n + 1.

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**Theorem DM** Dimension of  $M_{mn}$ 

The dimension of  $M_{mn}$  (Example VSM) is mn.

Suppose that A is an  $m \times n$  matrix. Then the **nullity** of A is the dimension of the null space of A,  $n(A) = \dim(\mathcal{N}(A))$ .

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### **Definition ROM** Rank Of a Matrix

Suppose that A is an  $m \times n$  matrix. Then the **rank** of A is the dimension of the column space of A,  $r(A) = \dim (\mathcal{C}(A))$ .

Suppose that A is an  $m \times n$  matrix and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then r(A) = r and n(A) = n - r.

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# **Theorem RPNC** Rank Plus Nullity is Columns

Suppose that A is an  $m \times n$  matrix. Then r(A) + n(A) = n.

**Theorem RNNSM** Rank and Nullity of a NonSingular Matrix

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. The rank of A is n, r(A) = n.
- 3. The nullity of A is zero, n(A) = 0.

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**Theorem NSME6** NonSingular Matrix Equivalences, Round 6 Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}$ .
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
- 8. The columns of A are a basis for  $\mathbb{C}^n$ .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.

Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose w is a vector such that  $\mathbf{w} \notin \langle S \rangle$ . Then the set  $S' = S \cup \{\mathbf{w}\}$  is linearly independent.

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# **Theorem G** Goldilocks

Suppose that V is a vector space of dimension t. Let  $S = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m}$  be a set of vectors from V. Then

- 1. If m > t, then S is linearly dependent.
- 2. If m < t, then S does not span V.
- 3. If m = t and S is linearly independent, then S spans V.
- 4. If m = t and S spans V, then S is linearly independent.

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Suppose that U and V are subspaces of the vector space W, such that  $U \subseteq V$  and dim $(U) = \dim(V)$ . Then U = V.

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**Theorem RMRT** Rank of a Matrix is the Rank of the Transpose

Suppose A is an  $m \times n$  matrix. Then  $r(A) = r(A^t)$ .

Suppose that  $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p}$  is an orthonormal basis of the subspace W of  $\mathbb{C}^m$ . For any  $\mathbf{w} \in W$ ,

 $\mathbf{w} = \langle \mathbf{w}, \, \mathbf{v}_1 \rangle \, \mathbf{v}_1 + \langle \mathbf{w}, \, \mathbf{v}_2 \rangle \, \mathbf{v}_2 + \langle \mathbf{w}, \, \mathbf{v}_3 \rangle \, \mathbf{v}_3 + \dots + \langle \mathbf{w}, \, \mathbf{v}_p \rangle \, \mathbf{v}_p$ 

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# **Definition SM** SubMatrix

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Suppose that A is an  $m \times n$  matrix. Then the **submatrix**  $A_{ij}$  is the  $(m-1) \times (n-1)$  matrix obtained from A by removing row i and column j.

#### **Definition DM** Determinant of a Matrix

Suppose A is a square matrix. Then its **determinant**, det (A) = |A|, is an element of  $\mathbb{C}$  defined recursively by:

- If A = [a] is a 1 × 1 matrix, then det (A) = a.
- If A is a matrix of size n with  $n \ge 2$ , then

 $\det (A) = [A]_{11} \det (A_{11}) - [A]_{12} \det (A_{12}) + [A]_{13} \det (A_{13}) - \dots + (-1)^{n+1} [A]_{1n} \det (A_{1n})$ 

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Theorem DMST Determinant of Matrices of Size Two

Suppose that  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Then det (A) = ad - bc

Suppose A is an  $n \times n$  matrix and  $A_{ij}$  is the  $(n-1) \times (n-1)$  submatrix formed by removing row i and column j. Then the **minor** for A at location ij is the determinant of the submatrix,  $M_{A,ij} = \det(A_{ij})$ .

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### **Definition CIM** Cofactor In a Matrix

Suppose A is an  $n \times n$  matrix and  $A_{ij}$  is the  $(n-1) \times (n-1)$  submatrix formed by removing row i and column j. Then the **cofactor** for A at location ij is the signed determinant of the submatrix,  $C_{A,ij} = (-1)^{i+j} \det(A_{ij})$ .

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**Theorem DERC** Determinant Expansion about Rows and Columns Suppose that A is a square matrix of size n. Then  $\det (A) = a_{i1}C_{A,i1} + a_{i2}C_{A,i2} + a_{i3}C_{A,i3} + \dots + a_{in}C_{A,in} \qquad 1 \le i \le n$ 

which is known as **expansion** about row i, and

$$\det (A) = [A]_{1j} C_{A,1j} + [A]_{2j} C_{A,2j} + [A]_{3j} C_{A,3j} + \dots + [A]_{nj} C_{A,nj} \qquad 1 \le j \le n$$

which is known as **expansion** about column j.

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**Theorem DT** Determinant of the Transpose

Suppose that A is a square matrix. Then  $\det(A^t) = \det(A)$ .

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# Theorem DRMM Determinant Respects Matrix Multiplication

Suppose that A and B are square matrices of the same size. Then  $\det(AB) = \det(A) \det(B)$ .

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**Theorem SMZD** Singular Matrices have Zero Determinants

Let A be a square matrix. Then A is singular if and only if det (A) = 0.

**Theorem NSME7** NonSingular Matrix Equivalences, Round 7 Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
- 8. The columns of A are a basis for  $\mathbb{C}^n$ .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero,  $\det(A) \neq 0$ .

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Definition EEM Eigenvalues and Eigenvectors of a Matrix

Suppose that A is a square matrix of size  $n, \mathbf{x} \neq \mathbf{0}$  is a vector in  $\mathbb{C}^n$ , and  $\lambda$  is a scalar in  $\mathbb{C}$ . Then we say  $\mathbf{x}$  is an **eigenvector** of A with **eigenvalue**  $\lambda$  if

 $A\mathbf{x} = \lambda \mathbf{x}$ 

Suppose A is a square matrix. Then A has at least one eigenvalue.

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# Definition CP Characteristic Polynomial

Suppose that A is a square matrix of size n. Then the **characteristic polynomial** of A is the polynomial  $p_A(x)$  defined by

 $p_A(x) = \det\left(A - xI_n\right)$ 

**Theorem EMRCP** Eigenvalues of a Matrix are Roots of Characteristic Polynomials 197

Suppose A is a square matrix. Then  $\lambda$  is an eigenvalue of A if and only if  $p_A(\lambda) = 0$ .

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# **Definition EM** Eigenspace of a Matrix

Suppose that A is a square matrix and  $\lambda$  is an eigenvalue of A. Then the **eigenspace** of A for  $\lambda$ ,  $E_A(\lambda)$ , is the set of all the eigenvectors of A for  $\lambda$ , together with the inclusion of the zero vector.

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Suppose A is a square matrix of size n and  $\lambda$  is an eigenvalue of A. Then the eigenspace  $E_A(\lambda)$  is a subspace of the vector space  $\mathbb{C}^n$ .

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**Theorem EMNS** Eigenspace of a Matrix is a Null Space

Suppose A is a square matrix of size n and  $\lambda$  is an eigenvalue of A. Then

 $E_A(\lambda) = \mathcal{N}(A - \lambda I_n)$ 

Suppose that A is a square matrix and  $\lambda$  is an eigenvalue of A. Then the **algebraic multiplicity** of  $\lambda$ ,  $\alpha_A(\lambda)$ , is the highest power of  $(x - \lambda)$  that divides the characteristic polynomial,  $p_A(x)$ .

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Definition GME Geometric Multiplicity of an Eigenvalue

Suppose that A is a square matrix and  $\lambda$  is an eigenvalue of A. Then the **geometric multiplicity** of  $\lambda$ ,  $\gamma_A(\lambda)$ , is the dimension of the eigenspace  $E_A(\lambda)$ .

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Theorem EDELI Eigenvectors with Distinct Eigenvalues are Linearly Independent 203

Suppose that A is a square matrix and  $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_p\}$  is a set of eigenvectors with eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_p$  such that  $\lambda_i \neq \lambda_j$  whenever  $i \neq j$ . Then S is a linearly independent set.

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Theorem SMZE Singular Matrices have Zero Eigenvalues

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Suppose A is a square matrix. Then A is singular if and only if  $\lambda = 0$  is an eigenvalue of A.

**Theorem NSME8** NonSingular Matrix Equivalences, Round 8 Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2.  ${\cal A}$  row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of  $\mathbf{b}$ .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ .
- 8. The columns of A are a basis for  $\mathbb{C}^n$ .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero,  $\det(A) \neq 0$ .
- 12.  $\lambda = 0$  is not an eigenvalue of A.

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**Theorem ESMM** Eigenvalues of a Scalar Multiple of a Matrix

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Suppose A is a square matrix and  $\lambda$  is an eigenvalue of A. Then  $\alpha\lambda$  is an eigenvalue of  $\alpha A$ .

Suppose A is a square matrix,  $\lambda$  is an eigenvalue of A, and  $s \ge 0$  is an integer. Then  $\lambda^s$  is an eigenvalue of  $A^s$ .

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**Theorem EPM** Eigenvalues of the Polynomial of a Matrix

Suppose A is a square matrix and  $\lambda$  is an eigenvalue of A. Let q(x) be a polynomial in the variable x. Then  $q(\lambda)$  is an eigenvalue of the matrix q(A).

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Suppose A is a square nonsingular matrix and  $\lambda$  is an eigenvalue of A. Then  $\frac{1}{\lambda}$  is an eigenvalue of the matrix  $A^{-1}$ .

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**Theorem ETM** Eigenvalues of the Transpose of a Matrix

 $A^t$ .

Suppose A is a square matrix and  $\lambda$  is an eigenvalue of A. Then  $\lambda$  is an eigenvalue of the matrix

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Suppose A is a square matrix with real entries and  $\mathbf{x}$  is an eigenvector of A for the eigenvalue  $\lambda$ . Then  $\overline{\mathbf{x}}$  is an eigenvector of A for the eigenvalue  $\overline{\lambda}$ .

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**Theorem DCP** Degree of the Characteristic Polynomial

Suppose that A is a square matrix of size n. Then the characteristic polynomial of A,  $p_A(x)$ , has degree n.

**Theorem NEM** Number of Eigenvalues of a Matrix

Suppose that A is a square matrix of size n with distinct eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$ . Then

$$\sum_{i=1}^{k} \alpha_A\left(\lambda_i\right) = n$$

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**Theorem ME** Multiplicities of an Eigenvalue

Suppose that A is a square matrix of size n and  $\lambda$  is an eigenvalue. Then

 $1 \le \gamma_A(\lambda) \le \alpha_A(\lambda) \le n$ 

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Suppose that A is a square matrix of size n. Then A cannot have more than n distinct eigenvalues.

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**Theorem HMRE** Hermitian Matrices have Real Eigenvalues

Suppose that A is a Hermitian matrix and  $\lambda$  is an eigenvalue of A. Then  $\lambda \in \mathbb{R}$ .

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Suppose that A is a Hermitian matrix and  $\mathbf{x}$  and  $\mathbf{y}$  are two eigenvectors of A for different eigenvalues. Then  $\mathbf{x}$  and  $\mathbf{y}$  are orthogonal vectors.

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**Definition SIM** Similar Matrices

Suppose A and B are two square matrices of size n. Then A and B are similar if there exists a nonsingular matrix of size n, S, such that  $A = S^{-1}BS$ .

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- 1. A is similar to A. (Reflexive)
- 2. If A is similar to B, then B is similar to A. (Symmetric)
- 3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive)

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Theorem SMEE Similar Matrices have Equal Eigenvalues

Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is  $p_A(x) = p_B(x)$ .

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Suppose that A is a square matrix. Then A is a **diagonal matrix** if  $[A]_{ij} = 0$  whenever  $i \neq j$ .

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 ${\bf Definition} \ {\bf DZM} \quad {\rm Diagonalizable} \ {\rm Matrix}$ 

Suppose A is a square matrix. Then A is **diagonalizable** if A is similar to a diagonal matrix.

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Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a linearly independent set S that contains n eigenvectors of A.

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**Theorem DMLE** Diagonalizable Matrices have Large Eigenspaces

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Suppose A is a square matrix. Then A is diagonalizable if and only if  $\gamma_A(\lambda) = \alpha_A(\lambda)$  for every eigenvalue  $\lambda$  of A.

Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

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**Definition LT** Linear Transformation

A linear transformation,  $T: U \mapsto V$ , is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

- 1.  $T(\mathbf{u}_{1} + \mathbf{u}_{2}) = T(\mathbf{u}_{1}) + T(\mathbf{u}_{2})$  for all  $\mathbf{u}_{1}, \mathbf{u}_{2} \in U$
- 2.  $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$  for all  $\mathbf{u} \in U$  and all  $\alpha \in \mathbb{C}$

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**Theorem LTTZZ** Linear Transformations Take Zero to Zero

Suppose  $T: U \mapsto V$  is a linear transformation. Then  $T(\mathbf{0}) = \mathbf{0}$ .

**Theorem MBLT** Matrices Build Linear Transformations **229** Suppose that *A* is an  $m \times n$  matrix. Define a function  $T: \mathbb{C}^n \mapsto \mathbb{C}^m$  by  $T(\mathbf{x}) = A\mathbf{x}$ . Then *T* is a linear transformation.

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**Theorem MLTCV** Matrix of a Linear Transformation, Column Vectors

Suppose that  $T: \mathbb{C}^n \to \mathbb{C}^m$  is a linear transformation. Then there is an  $m \times n$  matrix A such that  $T(\mathbf{x}) = A\mathbf{x}$ .

Theorem LTLC Linear Transformations and Linear Combinations

Suppose that  $T: U \mapsto V$  is a linear transformation,  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$  are vectors from U and  $a_1, a_2, a_3, \ldots, a_t$  are scalars from  $\mathbb{C}$ . Then

 $T (a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + a_3 \mathbf{u}_3 + \dots + a_t \mathbf{u}_t) = a_1 T (\mathbf{u}_1) + a_2 T (\mathbf{u}_2) + a_3 T (\mathbf{u}_3) + \dots + a_t T (\mathbf{u}_t)$ 

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Theorem LTDB Linear Transformation Defined on a Basis

Suppose that  $T: U \mapsto V$  is a linear transformation,  $B = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n}$  is a basis for U and  $\mathbf{w}$  is a vector from U. Let  $a_1, a_2, a_3, \ldots, a_n$  be the scalars from  $\mathbb{C}$  such that

 $\mathbf{w} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \dots + a_n\mathbf{u}_n$ 

Then

$$T(\mathbf{w}) = a_1 T(\mathbf{u}_1) + a_2 T(\mathbf{u}_2) + a_3 T(\mathbf{u}_3) + \dots + a_n T(\mathbf{u}_n)$$

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**Definition PI** Pre-Image

Suppose that  $T: U \mapsto V$  is a linear transformation. For each **v**, define the **pre-image** of **v** to be the subset of U given by

 $T^{-1}(\mathbf{v}) = \{ \mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{v} \}$ 

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### Definition LTA Linear Transformation Addition

Suppose that  $T: U \mapsto V$  and  $S: U \mapsto V$  are two linear transformations with the same domain and codomain. Then their **sum** is the function  $T + S: U \mapsto V$  whose outputs are defined by

 $(T+S)\left(\mathbf{u}\right) = T\left(\mathbf{u}\right) + S\left(\mathbf{u}\right)$ 

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**Theorem SLTLT** Sum of Linear Transformations is a Linear Transformation

Suppose that  $T: U \mapsto V$  and  $S: U \mapsto V$  are two linear transformations with the same domain and codomain. Then  $T + S: U \mapsto V$  is a linear transformation.

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Definition LTSM Linear Transformation Scalar Multiplication

Suppose that  $T: U \mapsto V$  is a linear transformation and  $\alpha \in \mathbb{C}$ . Then the scalar multiple is the function  $\alpha T: U \mapsto V$  whose outputs are defined by

 $\left(\alpha T\right)\left(\mathbf{u}\right)=\alpha T\left(\mathbf{u}\right)$ 

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Theorem MLTLT Multiple of a Linear Transformation is a Linear Transformation 237

Suppose that  $T: U \mapsto V$  is a linear transformation and  $\alpha \in \mathbb{C}$ . Then  $(\alpha T): U \mapsto V$  is a linear transformation.

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Theorem VSLT Vector Space of Linear Transformations

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Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, LT (U, V) is a vector space when the operations are those given in Definition LTA and Definition LTSM.

Definition LTC Linear Transformation Composition

Suppose that  $T: U \mapsto V$  and  $S: V \mapsto W$  are linear transformations. Then the **composition** of S and T is the function  $(S \circ T): U \mapsto W$  whose outputs are defined by

 $(S \circ T) (\mathbf{u}) = S (T (\mathbf{u}))$ 

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Theorem CLTLT Composition of Linear Transformations is a Linear Transformation 240

Suppose that  $T: U \mapsto V$  and  $S: V \mapsto W$  are linear transformations. Then  $(S \circ T): U \mapsto W$  is a linear transformation.

Suppose  $T: U \mapsto V$  is a linear transformation. Then T is **injective** if whenever  $T(\mathbf{x}) = T(\mathbf{y})$ , then  $\mathbf{x} = \mathbf{y}$ .

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Definition KLT Kernel of a Linear Transformation

Suppose  $T\colon U\mapsto V$  is a linear transformation. Then the  ${\bf kernel}$  of T is the set

 $\mathcal{K}(T) = \{ \mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{0} \}$ 

Suppose that  $T: U \mapsto V$  is a linear transformation. Then the kernel of T,  $\mathcal{K}(T)$ , is a subspace of U.

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### Theorem KPI Kernel and Pre-Image

Suppose  $T: U \mapsto V$  is a linear transformation and  $\mathbf{v} \in V$ . If the preimage  $T^{-1}(\mathbf{v})$  is non-empty, and  $\mathbf{u} \in T^{-1}(\mathbf{v})$  then

 $T^{-1}\left(\mathbf{v}\right) = \left\{ \mathbf{u} + \mathbf{z} \mid \mathbf{z} \in \mathcal{K}(T) \right\} = \mathbf{u} + \mathcal{K}(T)$ 

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Suppose that  $T: U \mapsto V$  is a linear transformation. Then T is injective if and only if the kernel of T is trivial,  $\mathcal{K}(T) = \{\mathbf{0}\}.$ 

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**Theorem ILTLI** Injective Linear Transformations and Linear Independence

Suppose that  $T: U \mapsto V$  is an injective linear transformation and  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t\}$  is a linearly independent subset of U. Then  $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \ldots, T(\mathbf{u}_t)\}$  is a linearly independent subset of V.

Theorem ILTB Injective Linear Transformations and Bases

Suppose that  $T: U \mapsto V$  is a linear transformation and  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$  is a basis of U. Then T is injective if and only if  $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$  is a linearly independent subset of V.

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**Theorem ILTD** Injective Linear Transformations and Dimension

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Suppose that  $T: U \mapsto V$  is an injective linear transformation. Then dim  $(U) \leq \dim(V)$ .

### **Theorem CILTI** Composition of Injective Linear Transformations is Injective

Suppose that  $T: U \mapsto V$  and  $S: V \mapsto W$  are injective linear transformations. Then  $(S \circ T): U \mapsto W$  is an injective linear transformation.

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Definition SLT Surjective Linear Transformation

Suppose  $T: U \mapsto V$  is a linear transformation. Then T is **surjective** if for every  $\mathbf{v} \in V$  there exists a  $\mathbf{u} \in U$  so that  $T(\mathbf{u}) = \mathbf{v}$ .

**Definition RLT** Range of a Linear Transformation

Suppose  $T: U \mapsto V$  is a linear transformation. Then the **range** of T is the set

 $\mathcal{R}(T) = \{ T(\mathbf{u}) \mid \mathbf{u} \in U \}$ 

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**Theorem RLTS** Range of a Linear Transformation is a Subspace

Suppose that  $T: U \mapsto V$  is a linear transformation. Then the range of T,  $\mathcal{R}(T)$ , is a subspace of V.

Theorem RSLT Range of a Surjective Linear Transformation

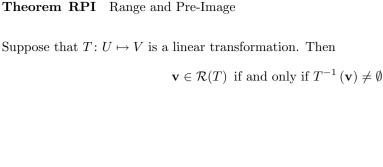
Suppose that  $T: U \mapsto V$  is a linear transformation. Then T is surjective if and only if the range of T equals the codomain,  $\mathcal{R}(T) = V$ .

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**Theorem SSRLT** Spanning Set for Range of a Linear Transformation

Suppose that  $T: U \mapsto V$  is a linear transformation and  $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t}$  spans U. Then  $R = {T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)}$  spans  $\mathcal{R}(T)$ .



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Theorem SLTB Surjective Linear Transformations and Bases

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Suppose that  $T: U \mapsto V$  is a linear transformation and  $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$  is a basis of U. Then T is surjective if and only if  $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$  is a spanning set for V.

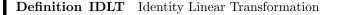
#### Theorem SLTD Surjective Linear Transformations and Dimension

Suppose that  $T: U \mapsto V$  is a surjective linear transformation. Then dim  $(U) \ge \dim(V)$ .

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Theorem CSLTS Composition of Surjective Linear Transformations is Surjective 258

Suppose that  $T: U \mapsto V$  and  $S: V \mapsto W$  are surjective linear transformations. Then  $(S \circ T): U \mapsto W$  is a surjective linear transformation.



The identity linear transformation on the vector space W is defined as

 $I_W \colon W \mapsto W, \qquad I_W(\mathbf{w}) = \mathbf{w}$ 

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**Definition IVLT** Invertible Linear Transformations

Suppose that  $T\colon U\mapsto V$  is a linear transformation. If there is a function  $S\colon V\mapsto U$  such that

 $S \circ T = I_U \qquad \qquad T \circ S = I_V$ 

then T is **invertible**. In this case, we call S the **inverse** of T and write  $S = T^{-1}$ .

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Theorem ILTLT Inverse of a Linear Transformation is a Linear Transformation 261

Suppose that  $T: U \mapsto V$  is an invertible linear transformation. Then the function  $T^{-1}: V \mapsto U$  is a linear transformation.

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**Theorem IILT** Inverse of an Invertible Linear Transformation

Suppose that  $T: U \mapsto V$  is an invertible linear transformation. Then  $T^{-1}$  is an invertible linear transformation and  $(T^{-1})^{-1} = T$ .

**Theorem ILTIS** Invertible Linear Transformations are Injective and Surjective

Suppose  $T: U \mapsto V$  is a linear transformation. Then T is invertible if and only if T is injective and surjective.

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Suppose that  $T: U \mapsto V$  and  $S: V \mapsto W$  are invertible linear transformations. Then the composition,  $(S \circ T): U \mapsto W$  is an invertible linear transformation.

Suppose that  $T: U \mapsto V$  and  $S: V \mapsto W$  are invertible linear transformations. Then  $S \circ T$  is invertible and  $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$ .

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#### **Definition IVS** Isomorphic Vector Spaces

Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain  $V, T: U \mapsto V$ . In this case, we write  $U \cong V$ , and the linear transformation T is known as an **isomorphism** between U and V.

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Suppose U and V are isomorphic vector spaces. Then  $\dim(U) = \dim(V)$ .

Theorem IVSED Isomorphic Vector Spaces have Equal Dimension

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Definition ROLT Rank Of a Linear Transformation

Suppose that  $T: U \mapsto V$  is a linear transformation. Then the **rank** of T, r(T), is the dimension of the range of T,

 $r(T) = \dim\left(\mathcal{R}(T)\right)$ 

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Definition NOLT Nullity Of a Linear Transformation

Suppose that  $T: U \mapsto V$  is a linear transformation. Then the **nullity** of T, n(T), is the dimension of the kernel of T,

 $n(T) = \dim\left(\mathcal{K}(T)\right)$ 

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**Theorem ROSLT** Rank Of a Surjective Linear Transformation

Suppose that  $T: U \mapsto V$  is a linear transformation. Then the rank of T is the dimension of V,  $r(T) = \dim(V)$ , if and only if T is surjective.

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Suppose that  $T: U \mapsto V$  is an injective linear transformation. Then the nullity of T is zero, n(T) = 0, if and only if T is injective.

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**Theorem RPNDD** Rank Plus Nullity is Domain Dimension

Suppose that  $T\colon U\mapsto V$  is a linear transformation. Then

 $r(T) + n(T) = \dim(U)$ 

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#### **Definition VR** Vector Representation

Suppose that V is a vector space with a basis  $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n}$ . Define a function  $\rho_B: V \mapsto \mathbb{C}^n$  as follows. For  $\mathbf{w} \in V$ , find scalars  $a_1, a_2, a_3, \dots, a_n$  so that

$$\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_n \mathbf{v}_n$$

then

$$\left[\rho_B\left(\mathbf{w}\right)\right]_i = a_i \qquad \qquad 1 \le i \le n$$

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**Theorem VRLT** Vector Representation is a Linear Transformation

The function  $\rho_B$  (Definition VR) is a linear transformation.

. .

<b>Theorem VRI</b> Vector Representation is Injective	275
The function $\rho_B$ (Definition VR) is an injective linear transformation.	

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 ${\bf Theorem} \ {\bf VRS} \quad {\rm Vector} \ {\rm Representation} \ {\rm is} \ {\rm Surjective}$ 

The function  $\rho_B$  (Definition VR) is a surjective linear transformation.

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The function  $\rho_B$  (Definition VR) is an invertible linear transformation.

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Theorem CFDVS Characterization of Finite Dimensional Vector Spaces 278

Suppose that V is a vector space with dimension n. Then V is isomorphic to  $\mathbb{C}^n$ .

Theorem IFDVS Isomorphism of Finite Dimensional Vector Spaces

Suppose U and V are both finite-dimensional vector spaces. Then U and V are isomorphic if and only if dim  $(U) = \dim(V)$ .

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Theorem CLI Coordinatization and Linear Independence

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Suppose that U is a vector space with a basis B of size n. Then  $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\}$  is a linearly independent subset of U if and only if  $R = \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \ldots, \rho_B(\mathbf{u}_k)\}$  is a linearly independent subset of  $\mathbb{C}^n$ .

Suppose that U is a vector space with a basis B of size n. Then  $\mathbf{u} \in \langle \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\} \rangle$  if and only if  $\rho_B(\mathbf{u}) \in \langle \{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \ldots, \rho_B(\mathbf{u}_k)\} \rangle$ .

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#### Definition MR Matrix Representation

Suppose that  $T: U \mapsto V$  is a linear transformation,  $B = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n}$  is a basis for U of size n, and C is a basis for V of size m. Then the **matrix representation** of T relative to B and C is the  $m \times n$  matrix,

 $M_{B,C}^{T} = \left[\rho_{C}\left(T\left(\mathbf{u}_{1}\right)\right) \middle| \rho_{C}\left(T\left(\mathbf{u}_{2}\right)\right) \middle| \rho_{C}\left(T\left(\mathbf{u}_{3}\right)\right) \middle| \dots \left|\rho_{C}\left(T\left(\mathbf{u}_{n}\right)\right)\right]$ 

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**Theorem FTMR** Fundamental Theorem of Matrix Representation

Suppose that  $T: U \mapsto V$  is a linear transformation, B is a basis for U, C is a basis for V and  $M_{B,C}^T$  is the matrix representation of T relative to B and C. Then, for any  $\mathbf{u} \in U$ ,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_{C}^{-1}\left(M_{B,C}^{T}\left(\rho_{B}\left(\mathbf{u}\right)\right)\right)$$

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Theorem MRSLT Matrix Representation of a Sum of Linear Transformations 284

Suppose that  $T: U \mapsto V$  and  $S: U \mapsto V$  are linear transformations, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$$

Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation 285

Suppose that  $T \colon U \mapsto V$  is a linear transformation,  $\alpha \in \mathbb{C}$ , B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$

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Theorem MRCLT Matrix Representation of a Composition of Linear Transformations 286

Suppose that  $T: U \mapsto V$  and  $S: V \mapsto W$  are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

 $M_{B,D}^{S \circ T} = M_{C,D}^S M_{B,C}^T$ 

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Theorem KNSI Kernel and Null Space Isomorphism

Suppose that  $T: U \mapsto V$  is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of  $M_{B,C}^T$ ,

$$\mathcal{K}(T) \cong \mathcal{N}(M_{B,C}^T)$$

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Theorem RCSI Range and Column Space Isomorphism

Suppose that  $T: U \mapsto V$  is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of  $M_{B,C}^T$ ,

 $\mathcal{R}(T) \cong \mathcal{C}\big(M_{B,C}^T\big)$ 

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Theorem IMR Invertible Matrix Representations

Suppose that  $T: U \mapsto V$  is an invertible linear transformation, B is a basis for U and C is a basis for V. Then the matrix representation of T relative to B and C,  $M_{B,C}^T$  is an invertible matrix, and

$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^{T}\right)^{-1}$$

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Theorem IMILT Invertible Matrices, Invertible Linear Transformation

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Suppose that A is a square matrix of size n and  $T: \mathbb{C}^n \to \mathbb{C}^n$  is the linear transformation defined by  $T(\mathbf{x}) = A\mathbf{x}$ . Then A is invertible matrix if and only if T is an invertible linear transformation.

291 Theorem NSME9 NonSingular Matrix Equivalences, Round 9 Suppose that A is a square matrix of size n. The following are equivalent. 1. A is nonsingular. 2. A row-reduces to the identity matrix. 3. The null space of A contains only the zero vector,  $\mathcal{N}(A) = \{\mathbf{0}\}$ . 4. The linear system  $\mathcal{LS}(A, \mathbf{b})$  has a unique solution for every possible choice of **b**. 5. The columns of A are a linearly independent set. 6. A is invertible. 7. The column space of A is  $\mathbb{C}^n$ ,  $\mathcal{C}(A) = \mathbb{C}^n$ . 8. The columns of A are a basis for  $\mathbb{C}^n$ . 9. The rank of A is n, r(A) = n. 10. The nullity of A is zero, n(A) = 0. 11. The determinant of A is nonzero,  $\det(A) \neq 0$ . 12.  $\lambda = 0$  is not an eigenvalue of A. 13. The linear transformation  $T: \mathbb{C}^n \mapsto \mathbb{C}^n$  defined by  $T(\mathbf{x}) = A\mathbf{x}$  is invertible. ©2005 Robert Beezer

**Definition EELT** Eigenvalue and Eigenvector of a Linear Transformation

Suppose that  $T: V \mapsto V$  is a linear transformation. Then a nonzero vector  $\mathbf{v} \in V$  is an **eigenvector** of T for the **eigenvalue**  $\lambda$  if  $T(\mathbf{v}) = \lambda \mathbf{v}$ .

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#### Definition CBM Change-of-Basis Matrix

Suppose that V is a vector space, and  $I_V: V \mapsto V$  is the identity linear transformation on V. Let  $B = {\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n}$  and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of  $I_V$  relative to B and C,

$$C_{B,C} = M_{B,C}^{I_V}$$
  
=  $[\rho_C (I_V (\mathbf{v}_1)) | \rho_C (I_V (\mathbf{v}_2)) | \rho_C (I_V (\mathbf{v}_3)) | \dots | \rho_C (I_V (\mathbf{v}_n))]$   
=  $[\rho_C (\mathbf{v}_1) | \rho_C (\mathbf{v}_2) | \rho_C (\mathbf{v}_3) | \dots | \rho_C (\mathbf{v}_n)]$ 

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Theorem CB Change-of-Basis

Suppose that  $\mathbf{v}$  is a vector in the vector space V and B and C are bases of V. Then

 $\rho_{C}\left(\mathbf{v}\right) = C_{B,C}\rho_{B}\left(\mathbf{v}\right)$ 

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Theorem ICBM Inverse of Change-of-Basis Matrix

Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix  $C_{B,C}$  is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

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Theorem MRCB Matrix Representation and Change of Basis

Suppose that  $T: U \mapsto V$  is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

 $M_{B,D}^T = C_{E,D} M_{C,E}^T C_{B,C}$ 

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Suppose that  $T \colon V \mapsto V$  is a linear transformation and B and C are bases of V. Then

$$M_{B,B}^{T} = C_{B,C}^{-1} M_{C,C}^{T} C_{B,C}$$

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Suppose that  $T: V \mapsto V$  is a linear transformation and B is a basis of V. Then  $\mathbf{v} \in V$  is an eigenvector of T for the eigenvalue  $\lambda$  if and only if  $\rho_B(\mathbf{v})$  is an eigenvector of  $M_{B,B}^T$  for the eigenvalue  $\lambda$ .