A First Course in Linear Algebra
A First Course in Linear Algebra

by
Robert A. Beezer
Department of Mathematics and Computer Science
University of Puget Sound

Version 1.04
Robert A. Beezer is a Professor of Mathematics at the University of Puget Sound, where he has been on the faculty since 1984. He received a B.S. in Mathematics (with an Emphasis in Computer Science) from the University of Santa Clara in 1978, a M.S. in Statistics from the University of Illinois at Urbana-Champaign in 1982 and a Ph.D. in Mathematics from the University of Illinois at Urbana-Champaign in 1984. He teaches calculus, linear algebra and abstract algebra regularly, while his research interests include the applications of linear algebra to graph theory. His professional website is at http://buzzard.ups.edu.
To my wife, Pat.
# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table of Contents</td>
<td>vi</td>
</tr>
<tr>
<td>Contributors</td>
<td>vii</td>
</tr>
<tr>
<td>Definitions</td>
<td>viii</td>
</tr>
<tr>
<td>Theorems</td>
<td>ix</td>
</tr>
<tr>
<td>Notation</td>
<td>x</td>
</tr>
<tr>
<td>Examples</td>
<td>xi</td>
</tr>
<tr>
<td>Preface</td>
<td>xii</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>xvii</td>
</tr>
</tbody>
</table>

## Part C Core

### Chapter SLE Systems of Linear Equations

<table>
<thead>
<tr>
<th>Subsection</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>WILA What is Linear Algebra?</strong></td>
<td>2</td>
</tr>
<tr>
<td><strong>LA “Linear” + “Algebra”</strong></td>
<td>2</td>
</tr>
<tr>
<td><strong>AA An Application</strong></td>
<td>3</td>
</tr>
<tr>
<td><strong>READ Reading Questions</strong></td>
<td>6</td>
</tr>
<tr>
<td><strong>EXC Exercises</strong></td>
<td>7</td>
</tr>
<tr>
<td><strong>SOL Solutions</strong></td>
<td>8</td>
</tr>
<tr>
<td><strong>SSLE Solving Systems of Linear Equations</strong></td>
<td>9</td>
</tr>
<tr>
<td><strong>SLE Systems of Linear Equations</strong></td>
<td>9</td>
</tr>
<tr>
<td><strong>PSS Possibilities for Solution Sets</strong></td>
<td>10</td>
</tr>
<tr>
<td><strong>ESEO Equivalent Systems and Equation Operations</strong></td>
<td>11</td>
</tr>
<tr>
<td><strong>READ Reading Questions</strong></td>
<td>16</td>
</tr>
<tr>
<td><strong>EXC Exercises</strong></td>
<td>17</td>
</tr>
<tr>
<td><strong>SOL Solutions</strong></td>
<td>19</td>
</tr>
<tr>
<td><strong>RREF Reduced Row-Echelon Form</strong></td>
<td>21</td>
</tr>
<tr>
<td><strong>MVNSE Matrix and Vector Notation for Systems of Equations</strong></td>
<td>21</td>
</tr>
<tr>
<td><strong>RO Row Operations</strong></td>
<td>24</td>
</tr>
<tr>
<td><strong>RREF Reduced Row-Echelon Form</strong></td>
<td>26</td>
</tr>
<tr>
<td><strong>READ Reading Questions</strong></td>
<td>32</td>
</tr>
<tr>
<td><strong>EXC Exercises</strong></td>
<td>34</td>
</tr>
<tr>
<td><strong>SOL Solutions</strong></td>
<td>37</td>
</tr>
<tr>
<td><strong>TSS Types of Solution Sets</strong></td>
<td>42</td>
</tr>
<tr>
<td><strong>CS Consistent Systems</strong></td>
<td>42</td>
</tr>
<tr>
<td><strong>FV Free Variables</strong></td>
<td>46</td>
</tr>
<tr>
<td><strong>READ Reading Questions</strong></td>
<td>48</td>
</tr>
<tr>
<td><strong>EXC Exercises</strong></td>
<td>50</td>
</tr>
</tbody>
</table>
Chapter M Matrices

MO Matrix Operations
- MEASM Matrix Equality, Addition, Scalar Multiplication
- VSP Vector Space Properties
- TSM Transposes and Symmetric Matrices
- MCC Matrices and Complex Conjugation
- READ Reading Questions
- EXC Exercises
- SOL Solutions

Chapter VS Vector Spaces

VS Vector Spaces
- EVS Examples of Vector Spaces
| CONTENTS x |
|------------------|------------------|------------------|
| DNMMM Determinants, Nonsingular Matrices, Matrix Multiplication | -------------- 351 |
| READ Reading Questions | -------------- 353 |
| EXC Exercises | -------------- 354 |
| SOL Solutions | -------------- 355 |

### Chapter E Eigenvalues 356

<table>
<thead>
<tr>
<th>EE Eigenvalues and Eigenvectors</th>
<th>-------------- 356</th>
</tr>
</thead>
<tbody>
<tr>
<td>EEM Eigenvalues and Eigenvectors of a Matrix</td>
<td>-------------- 356</td>
</tr>
<tr>
<td>PM Polynomials and Matrices</td>
<td>-------------- 358</td>
</tr>
<tr>
<td>EEE Existence of Eigenvalues and Eigenvectors</td>
<td>-------------- 359</td>
</tr>
<tr>
<td>CEE Computing Eigenvalues and Eigenvectors</td>
<td>-------------- 362</td>
</tr>
<tr>
<td>EC ECEE Examples of Computing Eigenvalues and Eigenvectors</td>
<td>-------------- 365</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>-------------- 372</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>-------------- 373</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>-------------- 374</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>PEE Properties of Eigenvalues and Eigenvectors</th>
<th>-------------- 378</th>
</tr>
</thead>
<tbody>
<tr>
<td>ME Multiplicities of Eigenvalues</td>
<td>-------------- 383</td>
</tr>
<tr>
<td>EHM Eigenvalues of Hermitian Matrices</td>
<td>-------------- 386</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>-------------- 387</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>-------------- 388</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>-------------- 389</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SD Similarity and Diagonalization</th>
<th>-------------- 390</th>
</tr>
</thead>
<tbody>
<tr>
<td>SM Similar Matrices</td>
<td>-------------- 390</td>
</tr>
<tr>
<td>PSM Properties of Similar Matrices</td>
<td>-------------- 391</td>
</tr>
<tr>
<td>D Diagonalization</td>
<td>-------------- 392</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>-------------- 400</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>-------------- 401</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>-------------- 402</td>
</tr>
</tbody>
</table>

### Chapter LT Linear Transformations 405

<table>
<thead>
<tr>
<th>LT Linear Transformations</th>
<th>-------------- 405</th>
</tr>
</thead>
<tbody>
<tr>
<td>LT Linear Transformations</td>
<td>-------------- 405</td>
</tr>
<tr>
<td>MLT Matrices and Linear Transformations</td>
<td>-------------- 409</td>
</tr>
<tr>
<td>LTLC Linear Transformations and Linear Combinations</td>
<td>-------------- 413</td>
</tr>
<tr>
<td>PI Pre-Images</td>
<td>-------------- 415</td>
</tr>
<tr>
<td>NLTO New Linear Transformations From Old</td>
<td>-------------- 417</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>-------------- 420</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>-------------- 422</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>-------------- 424</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ILT Injective Linear Transformations</th>
<th>-------------- 426</th>
</tr>
</thead>
<tbody>
<tr>
<td>EILT Examples of Injective Linear Transformations</td>
<td>-------------- 426</td>
</tr>
<tr>
<td>KLT Kernel of a Linear Transformation</td>
<td>-------------- 429</td>
</tr>
<tr>
<td>ILTLTI Injective Linear Transformations and Linear Independence</td>
<td>-------------- 433</td>
</tr>
<tr>
<td>ILTD Injective Linear Transformations and Dimension</td>
<td>-------------- 434</td>
</tr>
<tr>
<td>CILT Composition of Injective Linear Transformations</td>
<td>-------------- 435</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>-------------- 435</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>-------------- 436</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>-------------- 438</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SLT Surjective Linear Transformations</th>
<th>-------------- 440</th>
</tr>
</thead>
<tbody>
<tr>
<td>ESLT Examples of Surjective Linear Transformations</td>
<td>-------------- 440</td>
</tr>
<tr>
<td>RLT Range of a Linear Transformation</td>
<td>-------------- 443</td>
</tr>
<tr>
<td>SSSLT Spanning Sets and Surjective Linear Transformations</td>
<td>-------------- 448</td>
</tr>
<tr>
<td>SLTD Surjective Linear Transformations and Dimension</td>
<td>-------------- 450</td>
</tr>
<tr>
<td>CSLT Composition of Surjective Linear Transformations</td>
<td>-------------- 450</td>
</tr>
</tbody>
</table>

| Version 1.04 |
# Contents

- **READ Reading Questions** ............................................. 451
- **EXC Exercises** ....................................................... 452
- **SOL Solutions** ......................................................... 454

## IVLT Invertible Linear Transformations
- **IVLT Invertible Linear Transformations** ................. 456
- **IV Invertibility** ...................................................... 459
- **SI Structure and Isomorphism** ................................. 461
- **RNLT Rank and Nullity of a Linear Transformation** .... 463
- **SLELT Systems of Linear Equations and Linear Transformations** 466
- **READ Reading Questions** ......................................... 467
- **EXC Exercises** ....................................................... 468
- **SOL Solutions** ......................................................... 470

## Chapter R Representations

- **VR Vector Representations** ........................................... 473
  - **CVS Characterization of Vector Spaces** ................. 478
  - **CP Coordinatization Principle** ............................. 479
  - **READ Reading Questions** ......................................... 482
  - **EXC Exercises** ....................................................... 483
  - **SOL Solutions** ......................................................... 484

- **MR Matrix Representations** ....................................... 485
  - **NRFO New Representations from Old** .................... 490
  - **PMR Properties of Matrix Representations** ............. 494
  - **IVLT Invertible Linear Transformations** ................. 499
  - **READ Reading Questions** ......................................... 503
  - **EXC Exercises** ....................................................... 504
  - **SOL Solutions** ......................................................... 507

- **CB Change of Basis** .................................................. 515
  - **EELT Eigenvalues and Eigenvectors of Linear Transformations** 515
  - **CBM Change-of-Basis Matrix** ..................................... 516
  - **MRS Matrix Representations and Similarity** ............. 521
  - **CELT Computing Eigenvectors of Linear Transformations** 527
  - **READ Reading Questions** ......................................... 534
  - **EXC Exercises** ....................................................... 536
  - **SOL Solutions** ......................................................... 537

- **OD Orthonormal Diagonalization** ................................ 540
  - **TM Triangular Matrices** .......................................... 540
  - **UTMR Upper Triangular Matrix Representation** ........ 541
  - **NM Normal Matrices** ............................................... 545
  - **OD Orthonormal Diagonalization** .............................. 545

- **NLT Nilpotent Linear Transformations** ....................... 548
  - **NLT Nilpotent Linear Transformations** .................... 548
  - **PNLT Properties of Nilpotent Linear Transformations** ... 553
  - **CFNLT Canonical Form for Nilpotent Linear Transformations** 557

- **IS Invariant Subspaces** ............................................. 565
  - **IS Invariant Subspaces** .......................................... 565
  - **GEE Generalized Eigenvectors and Eigenspaces** ......... 568
  - **RLT Restrictions of Linear Transformations** ............ 572

- **JCF Jordan Canonical Form** ....................................... 582
  - **UTMR Upper Triangular Matrix Representation** ......... 582
  - **GESD Generalized Eigenspace Decomposition** ........... 584
  - **JCF Jordan Canonical Form** ...................................... 590
  - **CHT Cayley-Hamilton Theorem** ............................... 602

Version 1.04
Appendix GFDL GNU Free Documentation License

1. APPLICABILITY AND DEFINITIONS .............................................. 711
2. VERBATIM COPYING ................................................................ 712
3. COPYING IN QUANTITY .............................................................. 712
4. MODIFICATIONS ....................................................................... 713
5. COMBINING DOCUMENTS ............................................................ 714
6. COLLECTIONS OF DOCUMENTS ................................................. 715
7. AGGREGATION WITH INDEPENDENT WORKS .............................. 715
8. TRANSLATION .......................................................................... 715
9. TERMINATION .......................................................................... 715
10. FUTURE REVISIONS OF THIS LICENSE ..................................... 715
ADDENDUM: How to use this License for your documents ............... 716

Part T Topics

F Fields ....................................................................................... 718
  F Fields ............................................................................... 718
  FF Finite Fields ................................................................. 719
  EXC Exercises ...................................................................... 723
  SOL Solutions ........................................................................ 725

Chapter MD Matrix Decompositions ............................................. 726
  ROD Rank One Decomposition .............................................. 726
  TD Triangular Decomposition .............................................. 731
  TD Triangular Decomposition .............................................. 731
  TDSSE Triangular Decomposition and Solving Systems of Equations ........................................................................ 733
  CTD Computing Triangular Decompositions ........................................ 735
Contributors

Beezer, David. St. Charles Borromeo School
Beezer, Robert. University of Puget Sound  http://buzzard.ups.edu/
Fellez, Sarah. University of Puget Sound
Fickenscher, Eric. University of Puget Sound
Jackson, Martin. University of Puget Sound  http://www.math.ups.edu/~martinj
Linenthal, Jacob. University of Puget Sound
Osborne, Travis. University of Puget Sound
Riegsecker, Joe. Middlebury, Indiana joepy (at) pobox (dot) com
Phelps, Douglas. University of Puget Sound
Shoemaker, Mark. University of Puget Sound
Zimmer, Andy. University of Puget Sound
Definitions

Section WILA
Section SSLE
SLE System of Linear Equations .................................................. 9
ESYS Equivalent Systems .......................................................... 11
EO Equation Operations ........................................................... 11

Section RREF
M Matrix ................................................................. 21
CV Column Vector .......................................................... 21
ZCV Zero Column Vector ...................................................... 22
CM Coefficient Matrix ......................................................... 22
VOC Vector of Constants ....................................................... 22
SV Solution Vector ............................................................ 23
LSMR Matrix Representation of a Linear System ......................... 23
AM Augmented Matrix ......................................................... 24
RO Row Operations .......................................................... 24
REM Row-Equivalent Matrices .............................................. 25
RREF Reduced Row-Echelon Form ........................................... 26
RR Row-Reducing ............................................................. 32

Section TSS
CS Consistent System .......................................................... 42
IDV Independent and Dependent Variables ................................ 44

Section HSE
HS Homogeneous System ..................................................... 52
TSHSE Trivial Solution to Homogeneous Systems of Equations .... 52
NSM Null Space of a Matrix .................................................. 54

Section NM
SQM Square Matrix ........................................................... 61
NM Nonsingular Matrix ....................................................... 61
IM Identity Matrix ............................................................ 62

Section VO
VSCV Vector Space of Column Vectors .................................... 72
CVE Column Vector Equality ................................................ 73
CVA Column Vector Addition .............................................. 73
CVSM Column Vector Scalar Multiplication ............................ 74

Section LC
LCCV Linear Combination of Column Vectors ......................... 79
Section SS
SSCV Span of a Set of Column Vectors 

Section LI
RLDCV Relation of Linear Dependence for Column Vectors
LICV Linear Independence of Column Vectors

Section LDS
Section O
CCCV Complex Conjugate of a Column Vector
IP Inner Product
NV Norm of a Vector
OV Orthogonal Vectors
OSV Orthogonal Set of Vectors
ONS OrthoNormal Set

Section MO
VSM Vector Space of $m \times n$ Matrices
ME Matrix Equality
MA Matrix Addition
MSM Matrix Scalar Multiplication
ZM Zero Matrix
TM Transpose of a Matrix
SYM Symmetric Matrix
CCM Complex Conjugate of a Matrix

Section MM
MVP Matrix-Vector Product
MM Matrix Multiplication

Section MISLE
MI Matrix Inverse
SUV Standard Unit Vectors

Section MINM
UM Unitary Matrices
A Adjoint
HM Hermitian Matrix

Section CRS
CSM Column Space of a Matrix
RSM Row Space of a Matrix

Section FS
LNS Left Null Space
EEF Extended Echelon Form

Section VS
VS Vector Space

Section S
S Subspace
TS Trivial Subspaces
LC Linear Combination
<table>
<thead>
<tr>
<th>SS</th>
<th>Span of a Set</th>
<th>270</th>
</tr>
</thead>
<tbody>
<tr>
<td>LI</td>
<td>Linear Independence</td>
<td>280</td>
</tr>
<tr>
<td>RLD</td>
<td>Relation of Linear Dependence</td>
<td>280</td>
</tr>
<tr>
<td>TSVS</td>
<td>To Span a Vector Space</td>
<td>284</td>
</tr>
<tr>
<td>B</td>
<td>Basis</td>
<td>294</td>
</tr>
<tr>
<td>D</td>
<td>Dimension</td>
<td>307</td>
</tr>
<tr>
<td>NOM</td>
<td>Nullity Of a Matrix</td>
<td>312</td>
</tr>
<tr>
<td>ROM</td>
<td>Rank Of a Matrix</td>
<td>313</td>
</tr>
<tr>
<td>DS</td>
<td>Direct Sum</td>
<td>326</td>
</tr>
<tr>
<td>ELEM</td>
<td>Elementary Matrices</td>
<td>333</td>
</tr>
<tr>
<td>SM</td>
<td>SubMatrix</td>
<td>337</td>
</tr>
<tr>
<td>DM</td>
<td>Determinant of a Matrix</td>
<td>337</td>
</tr>
<tr>
<td>EEM</td>
<td>Eigenvalues and Eigenvectors of a Matrix</td>
<td>356</td>
</tr>
<tr>
<td>CP</td>
<td>Characteristic Polynomial</td>
<td>363</td>
</tr>
<tr>
<td>EM</td>
<td>Eigenspace of a Matrix</td>
<td>364</td>
</tr>
<tr>
<td>AME</td>
<td>Algebraic Multiplicity of an Eigenvalue</td>
<td>366</td>
</tr>
<tr>
<td>GME</td>
<td>Geometric Multiplicity of an Eigenvalue</td>
<td>366</td>
</tr>
<tr>
<td>SIM</td>
<td>Similar Matrices</td>
<td>390</td>
</tr>
<tr>
<td>DIM</td>
<td>Diagonal Matrix</td>
<td>393</td>
</tr>
<tr>
<td>DZM</td>
<td>Diagonalizable Matrix</td>
<td>393</td>
</tr>
<tr>
<td>LT</td>
<td>Linear Transformation</td>
<td>405</td>
</tr>
<tr>
<td>PI</td>
<td>Pre-Image</td>
<td>415</td>
</tr>
<tr>
<td>LTA</td>
<td>Linear Transformation Addition</td>
<td>417</td>
</tr>
<tr>
<td>LTSM</td>
<td>Linear Transformation Scalar Multiplication</td>
<td>418</td>
</tr>
<tr>
<td>LTC</td>
<td>Linear Transformation Composition</td>
<td>419</td>
</tr>
<tr>
<td>ILT</td>
<td>Injective Linear Transformation</td>
<td>426</td>
</tr>
<tr>
<td>KLT</td>
<td>Kernel of a Linear Transformation</td>
<td>429</td>
</tr>
<tr>
<td>SLT</td>
<td>Surjective Linear Transformation</td>
<td>440</td>
</tr>
<tr>
<td>RLT</td>
<td>Range of a Linear Transformation</td>
<td>444</td>
</tr>
<tr>
<td>IDLT</td>
<td>Identity Linear Transformation</td>
<td>456</td>
</tr>
<tr>
<td>Acronym</td>
<td>Definition</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>---------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>IVLT</td>
<td>Invertible Linear Transformations</td>
<td>456</td>
</tr>
<tr>
<td>IVS</td>
<td>Isomorphic Vector Spaces</td>
<td>461</td>
</tr>
<tr>
<td>ROLT</td>
<td>Rank Of a Linear Transformation</td>
<td>463</td>
</tr>
<tr>
<td>NOLT</td>
<td>Nullity Of a Linear Transformation</td>
<td>463</td>
</tr>
<tr>
<td>VR</td>
<td>Vector Representation</td>
<td>473</td>
</tr>
<tr>
<td>MR</td>
<td>Matrix Representation</td>
<td>485</td>
</tr>
<tr>
<td>CB</td>
<td>Eigenvalue and Eigenvector of a Linear Transformation</td>
<td>515</td>
</tr>
<tr>
<td>CBM</td>
<td>Change-of-Basis Matrix</td>
<td>516</td>
</tr>
<tr>
<td>UTM</td>
<td>Upper Triangular Matrix</td>
<td>540</td>
</tr>
<tr>
<td>LTM</td>
<td>Lower Triangular Matrix</td>
<td>540</td>
</tr>
<tr>
<td>NRML</td>
<td>Normal Matrix</td>
<td>545</td>
</tr>
<tr>
<td>NLT</td>
<td>Nilpotent Linear Transformation</td>
<td>548</td>
</tr>
<tr>
<td>JB</td>
<td>Jordan Block</td>
<td>550</td>
</tr>
<tr>
<td>IS</td>
<td>Invariant Subspace</td>
<td>565</td>
</tr>
<tr>
<td>GEV</td>
<td>Generalized Eigenvector</td>
<td>568</td>
</tr>
<tr>
<td>GES</td>
<td>Generalized Eigenspace</td>
<td>568</td>
</tr>
<tr>
<td>LTR</td>
<td>Linear Transformation Restriction</td>
<td>572</td>
</tr>
<tr>
<td>IE</td>
<td>Index of an Eigenvalue</td>
<td>578</td>
</tr>
<tr>
<td>JCF</td>
<td>Jordan Canonical Form</td>
<td>590</td>
</tr>
<tr>
<td>CNE</td>
<td>Complex Number Equality</td>
<td>612</td>
</tr>
<tr>
<td>CNA</td>
<td>Complex Number Addition</td>
<td>612</td>
</tr>
<tr>
<td>CNM</td>
<td>Complex Number Multiplication</td>
<td>612</td>
</tr>
<tr>
<td>CCN</td>
<td>Conjugate of a Complex Number</td>
<td>613</td>
</tr>
<tr>
<td>MCN</td>
<td>Modulus of a Complex Number</td>
<td>614</td>
</tr>
<tr>
<td>SET</td>
<td>Set</td>
<td>615</td>
</tr>
<tr>
<td>SSET</td>
<td>Subset</td>
<td>615</td>
</tr>
<tr>
<td>ES</td>
<td>Empty Set</td>
<td>615</td>
</tr>
<tr>
<td>SE</td>
<td>Set Equality</td>
<td>616</td>
</tr>
<tr>
<td>C</td>
<td>Cardinality</td>
<td>616</td>
</tr>
<tr>
<td>SU</td>
<td>Set Union</td>
<td>617</td>
</tr>
<tr>
<td>SI</td>
<td>Set Intersection</td>
<td>617</td>
</tr>
<tr>
<td>SC</td>
<td>Set Complement</td>
<td>617</td>
</tr>
<tr>
<td>F</td>
<td>Field</td>
<td>718</td>
</tr>
<tr>
<td>Theorem</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>NME2</td>
<td>Nonsingular Matrix Equivalences, Round 2</td>
<td>127</td>
</tr>
<tr>
<td>BNS</td>
<td>Basis for Null Spaces</td>
<td>128</td>
</tr>
<tr>
<td></td>
<td><strong>Section LDS</strong></td>
<td></td>
</tr>
<tr>
<td>DLDS</td>
<td>Dependency in Linearly Dependent Sets</td>
<td>139</td>
</tr>
<tr>
<td>BS</td>
<td>Basis of a Span</td>
<td>143</td>
</tr>
<tr>
<td></td>
<td><strong>Section O</strong></td>
<td></td>
</tr>
<tr>
<td>CRVA</td>
<td>Conjugation Respects Vector Addition</td>
<td>151</td>
</tr>
<tr>
<td>CRSM</td>
<td>Conjugation Respects Vector Scalar Multiplication</td>
<td>151</td>
</tr>
<tr>
<td>IPVA</td>
<td>Inner Product and Vector Addition</td>
<td>153</td>
</tr>
<tr>
<td>IPSM</td>
<td>Inner Product and Scalar Multiplication</td>
<td>153</td>
</tr>
<tr>
<td>IPAC</td>
<td>Inner Product is Anti-Commutative</td>
<td>154</td>
</tr>
<tr>
<td>IPN</td>
<td>Inner Products and Norms</td>
<td>155</td>
</tr>
<tr>
<td>PIP</td>
<td>Positive Inner Products</td>
<td>156</td>
</tr>
<tr>
<td>OSLI</td>
<td>Orthogonal Sets are Linearly Independent</td>
<td>157</td>
</tr>
<tr>
<td>GSPCV</td>
<td>Gram-Schmidt Procedure, Column Vectors</td>
<td>158</td>
</tr>
<tr>
<td></td>
<td><strong>Section MO</strong></td>
<td></td>
</tr>
<tr>
<td>VSPM</td>
<td>Vector Space Properties of Matrices</td>
<td>164</td>
</tr>
<tr>
<td>SMS</td>
<td>Symmetric Matrices are Square</td>
<td>167</td>
</tr>
<tr>
<td>TMA</td>
<td>Transpose and Matrix Addition</td>
<td>167</td>
</tr>
<tr>
<td>TMSM</td>
<td>Transpose and Matrix Scalar Multiplication</td>
<td>167</td>
</tr>
<tr>
<td>IT</td>
<td>Transpose of a Transpose</td>
<td>167</td>
</tr>
<tr>
<td>CRMA</td>
<td>Conjugation Respects Matrix Addition</td>
<td>168</td>
</tr>
<tr>
<td>CRMSM</td>
<td>Conjugation Respects Matrix Scalar Multiplication</td>
<td>169</td>
</tr>
<tr>
<td>MCT</td>
<td>Matrix Conjugation and Transposes</td>
<td>169</td>
</tr>
<tr>
<td></td>
<td><strong>Section MM</strong></td>
<td></td>
</tr>
<tr>
<td>SLEMM</td>
<td>Systems of Linear Equations as Matrix Multiplication</td>
<td>173</td>
</tr>
<tr>
<td>EMMVP</td>
<td>Equal Matrices and Matrix-Vector Products</td>
<td>175</td>
</tr>
<tr>
<td>EMP</td>
<td>Entries of Matrix Products</td>
<td>177</td>
</tr>
<tr>
<td>MMZM</td>
<td>Matrix Multiplication and the Zero Matrix</td>
<td>178</td>
</tr>
<tr>
<td>MMIM</td>
<td>Matrix Multiplication and Identity Matrix</td>
<td>179</td>
</tr>
<tr>
<td>MMDAA</td>
<td>Matrix Multiplication Distributes Across Addition</td>
<td>179</td>
</tr>
<tr>
<td>MMSMM</td>
<td>Matrix Multiplication and Scalar Matrix Multiplication</td>
<td>180</td>
</tr>
<tr>
<td>MMA</td>
<td>Matrix Multiplication is Associative</td>
<td>180</td>
</tr>
<tr>
<td>MMIP</td>
<td>Matrix Multiplication and Inner Products</td>
<td>181</td>
</tr>
<tr>
<td>MMCC</td>
<td>Matrix Multiplication and Complex Conjugation</td>
<td>181</td>
</tr>
<tr>
<td>MMT</td>
<td>Matrix Multiplication and Transposes</td>
<td>182</td>
</tr>
<tr>
<td></td>
<td><strong>Section MISLE</strong></td>
<td></td>
</tr>
<tr>
<td>TTMI</td>
<td>Two-by-Two Matrix Inverse</td>
<td>191</td>
</tr>
<tr>
<td>CINM</td>
<td>Computing the Inverse of a Nonsingular Matrix</td>
<td>193</td>
</tr>
<tr>
<td>MIU</td>
<td>Matrix Inverse is Unique</td>
<td>195</td>
</tr>
<tr>
<td>SS</td>
<td>Socks and Shoes</td>
<td>195</td>
</tr>
<tr>
<td>MIMI</td>
<td>Matrix Inverse of a Matrix Inverse</td>
<td>196</td>
</tr>
<tr>
<td>MIT</td>
<td>Matrix Inverse of a Transpose</td>
<td>196</td>
</tr>
<tr>
<td>MISM</td>
<td>Matrix Inverse of a Scalar Multiple</td>
<td>196</td>
</tr>
<tr>
<td></td>
<td><strong>Section MINM</strong></td>
<td></td>
</tr>
<tr>
<td>NPNT</td>
<td>Nonsingular Product has Nonsingular Terms</td>
<td>202</td>
</tr>
<tr>
<td>OSIS</td>
<td>One-Sided Inverse is Sufficient</td>
<td>203</td>
</tr>
<tr>
<td>NI</td>
<td>Nonsingularity is Invertibility</td>
<td>204</td>
</tr>
<tr>
<td>Section</td>
<td>Theorem</td>
<td></td>
</tr>
<tr>
<td>---------</td>
<td>---------</td>
<td></td>
</tr>
<tr>
<td>CRS</td>
<td>NME3: Nonsingular Matrix Equivalences, Round 3</td>
<td></td>
</tr>
<tr>
<td></td>
<td>SNCM: Solution with Nonsingular Coefficient Matrix</td>
<td></td>
</tr>
<tr>
<td></td>
<td>UMI: Unitary Matrices are Invertible</td>
<td></td>
</tr>
<tr>
<td></td>
<td>CUMOS: Columns of Unitary Matrices are Orthonormal Sets</td>
<td></td>
</tr>
<tr>
<td></td>
<td>UMPIP: Unitary Matrices Preserve Inner Products</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Section CRS</td>
<td></td>
</tr>
<tr>
<td></td>
<td>CSCS: Column Spaces and Consistent Systems</td>
<td></td>
</tr>
<tr>
<td></td>
<td>BCS: Basis of the Column Space</td>
<td></td>
</tr>
<tr>
<td></td>
<td>CSNM: Column Space of a Nonsingular Matrix</td>
<td></td>
</tr>
<tr>
<td></td>
<td>NME4: Nonsingular Matrix Equivalences, Round 4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>REMRS: Row-Equivalent Matrices have equal Row Spaces</td>
<td></td>
</tr>
<tr>
<td></td>
<td>BRS: Basis for the Row Space</td>
<td></td>
</tr>
<tr>
<td></td>
<td>CSRST: Column Space, Row Space, Transpose</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Section FS</td>
<td></td>
</tr>
<tr>
<td></td>
<td>PEEF: Properties of Extended Echelon Form</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FS: Four Subsets</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Section VS</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ZVU: Zero Vector is Unique</td>
<td></td>
</tr>
<tr>
<td></td>
<td>AIU: Additive Inverses are Unique</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ZSSM: Zero Scalar in Scalar Multiplication</td>
<td></td>
</tr>
<tr>
<td></td>
<td>ZVSM: Zero Vector in Scalar Multiplication</td>
<td></td>
</tr>
<tr>
<td></td>
<td>AISM: Additive Inverses from Scalar Multiplication</td>
<td></td>
</tr>
<tr>
<td></td>
<td>SMEZV: Scalar Multiplication Equals the Zero Vector</td>
<td></td>
</tr>
<tr>
<td></td>
<td>VAC: Vector Addition Cancellation</td>
<td></td>
</tr>
<tr>
<td></td>
<td>CSSM: Canceling Scalars in Scalar Multiplication</td>
<td></td>
</tr>
<tr>
<td></td>
<td>CVSM: Canceling Vectors in Scalar Multiplication</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Section S</td>
<td></td>
</tr>
<tr>
<td></td>
<td>TSS: Testing Subsets for Subspaces</td>
<td></td>
</tr>
<tr>
<td></td>
<td>NSMS: Null Space of a Matrix is a Subspace</td>
<td></td>
</tr>
<tr>
<td></td>
<td>SSS: Span of a Set is a Subspace</td>
<td></td>
</tr>
<tr>
<td></td>
<td>CSMS: Column Space of a Matrix is a Subspace</td>
<td></td>
</tr>
<tr>
<td></td>
<td>RSMS: Row Space of a Matrix is a Subspace</td>
<td></td>
</tr>
<tr>
<td></td>
<td>LNSMS: Left Null Space of a Matrix is a Subspace</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Section LISS</td>
<td></td>
</tr>
<tr>
<td></td>
<td>VRRB: Vector Representation Relative to a Basis</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Section B</td>
<td></td>
</tr>
<tr>
<td></td>
<td>SUVB: Standard Unit Vectors are a Basis</td>
<td></td>
</tr>
<tr>
<td></td>
<td>CNMB: Columns of Nonsingular Matrix are a Basis</td>
<td></td>
</tr>
<tr>
<td></td>
<td>NME5: Nonsingular Matrix Equivalences, Round 5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>COB: Coordinates and Orthonormal Bases</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Section D</td>
<td></td>
</tr>
<tr>
<td></td>
<td>SSLD: Spanning Sets and Linear Dependence</td>
<td></td>
</tr>
<tr>
<td></td>
<td>BIS: Bases have Identical Sizes</td>
<td></td>
</tr>
<tr>
<td></td>
<td>DCM: Dimension of $\mathbb{C}^m$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>DP: Dimension of $P_m$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>DM: Dimension of $M_{mn}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>CRN: Computing Rank and Nullity</td>
<td></td>
</tr>
<tr>
<td>Section</td>
<td>Theorem</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>---------</td>
<td>------</td>
</tr>
<tr>
<td>PD</td>
<td>RPNC</td>
<td>314</td>
</tr>
<tr>
<td></td>
<td>RNNM</td>
<td>314</td>
</tr>
<tr>
<td></td>
<td>NME6</td>
<td>315</td>
</tr>
<tr>
<td></td>
<td>ELIS</td>
<td>320</td>
</tr>
<tr>
<td></td>
<td>G</td>
<td>320</td>
</tr>
<tr>
<td></td>
<td>PSSD</td>
<td>323</td>
</tr>
<tr>
<td></td>
<td>EDYSES</td>
<td>323</td>
</tr>
<tr>
<td></td>
<td>RMRT</td>
<td>324</td>
</tr>
<tr>
<td></td>
<td>DFS</td>
<td>325</td>
</tr>
<tr>
<td></td>
<td>DSFB</td>
<td>326</td>
</tr>
<tr>
<td></td>
<td>DSFOS</td>
<td>327</td>
</tr>
<tr>
<td></td>
<td>DSZV</td>
<td>327</td>
</tr>
<tr>
<td></td>
<td>DSLI</td>
<td>328</td>
</tr>
<tr>
<td></td>
<td>DSD</td>
<td>329</td>
</tr>
<tr>
<td></td>
<td>RDS</td>
<td>330</td>
</tr>
<tr>
<td>DM</td>
<td>EMDRO</td>
<td>334</td>
</tr>
<tr>
<td></td>
<td>EMN</td>
<td>336</td>
</tr>
<tr>
<td></td>
<td>NMPEM</td>
<td>337</td>
</tr>
<tr>
<td></td>
<td>DMST</td>
<td>338</td>
</tr>
<tr>
<td></td>
<td>DER</td>
<td>339</td>
</tr>
<tr>
<td></td>
<td>DT</td>
<td>340</td>
</tr>
<tr>
<td></td>
<td>DEC</td>
<td>340</td>
</tr>
<tr>
<td>PDM</td>
<td>DZRC</td>
<td>345</td>
</tr>
<tr>
<td></td>
<td>DRCS</td>
<td>345</td>
</tr>
<tr>
<td></td>
<td>DRCM</td>
<td>346</td>
</tr>
<tr>
<td></td>
<td>DERC</td>
<td>347</td>
</tr>
<tr>
<td></td>
<td>DRCMA</td>
<td>347</td>
</tr>
<tr>
<td></td>
<td>DIM</td>
<td>349</td>
</tr>
<tr>
<td></td>
<td>DEM</td>
<td>350</td>
</tr>
<tr>
<td></td>
<td>DEMM</td>
<td>350</td>
</tr>
<tr>
<td></td>
<td>SMZD</td>
<td>351</td>
</tr>
<tr>
<td></td>
<td>NME7</td>
<td>352</td>
</tr>
<tr>
<td></td>
<td>DRMM</td>
<td>353</td>
</tr>
<tr>
<td>EE</td>
<td>EMHE</td>
<td>359</td>
</tr>
<tr>
<td></td>
<td>EMRCP</td>
<td>363</td>
</tr>
<tr>
<td></td>
<td>EMS</td>
<td>364</td>
</tr>
<tr>
<td></td>
<td>EMNS</td>
<td>364</td>
</tr>
<tr>
<td>PEE</td>
<td>EDELI</td>
<td>378</td>
</tr>
<tr>
<td></td>
<td>SMZE</td>
<td>379</td>
</tr>
<tr>
<td></td>
<td>NME8</td>
<td>379</td>
</tr>
<tr>
<td></td>
<td>ESMM</td>
<td>379</td>
</tr>
<tr>
<td></td>
<td>EOMP</td>
<td>380</td>
</tr>
<tr>
<td></td>
<td>EPM</td>
<td>380</td>
</tr>
</tbody>
</table>

Version 1.04
Section SD
SER Similarity is an Equivalence Relation
SMEE Similar Matrices have Equal Eigenvalues
DC Diagonalization Characterization
DMFE Diagonalizable Matrices have Full Eigenspaces
DED Distinct Eigenvalues implies Diagonalizable

Section LT
LTTZZ Linear Transformations Take Zero to Zero
MBLT Matrices Build Linear Transformations
MLTCV Matrix of a Linear Transformation, Column Vectors
LTLC Linear Transformations and Linear Combinations
LTDB Linear Transformation Defined on a Basis
SLTLT Sum of Linear Transformations is a Linear Transformation
MLTLT Multiple of a Linear Transformation is a Linear Transformation
VSLT Vector Space of Linear Transformations
CLTLT Composition of Linear Transformations is a Linear Transformation

Section ILT
KLTS Kernel of a Linear Transformation is a Subspace
KPI Kernel and Pre-Image
KILT Kernel of an Injective Linear Transformation
ILTLI Injective Linear Transformations and Linear Independence
ILTB Injective Linear Transformations and Bases
ILTD Injective Linear Transformations and Dimension
CILTI Composition of Injective Linear Transformations is Injective

Section SLT
RLTS Range of a Linear Transformation is a Subspace
RSLT Range of a Surjective Linear Transformation
SSRLT Spanning Set for Range of a Linear Transformation
RPI Range and Pre-Image
SLTB Surjective Linear Transformations and Bases
SLTD Surjective Linear Transformations and Dimension
CSLTS Composition of Surjective Linear Transformations is Surjective

Section IVLT
ILTLT Inverse of a Linear Transformation is a Linear Transformation
IILT Inverse of an Invertible Linear Transformation
ILTI Composition of Invertible Linear Transformations are Injective and Surjective
CIVLT Composition of Invertible Linear Transformations
ICLT Inverse of a Composition of Linear Transformations
IVSED Isomorphic Vector Spaces have Equal Dimension
ROS LT Rank Of a Surjective Linear Transformation
<table>
<thead>
<tr>
<th>Theorem</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOILT Nullity Of an Injective Linear Transformation</td>
<td>463</td>
</tr>
<tr>
<td>RPNDD Rank Plus Nullity is Domain Dimension</td>
<td>464</td>
</tr>
</tbody>
</table>

| Section VR     |
|------------------|------|
| VRLT Vector Representation is a Linear Transformation | 473  |
| VRI Vector Representation isInjective                   | 477  |
| VRS Vector Representation is Surjective                  | 478  |
| VRILT Vector Representation is an Invertible Linear Transformation | 478  |

| Section MR     |
|------------------|------|
| FTMR Fundamental Theorem of Matrix Representation       | 487  |
| MRSLT Matrix Representation of a Sum of Linear Transformations | 490  |
| MRMLT Matrix Representation of a Multiple of a Linear Transformation | 491  |
| MRCLT Matrix Representation of a Composition of Linear Transformations | 491  |
| KNSI Kernel and Null Space Isomorphism                      | 495  |
| RCSI Range and Column Space Isomorphism                      | 497  |
| IMR Invertible Matrix Representations                        | 499  |
| IMILT Invertible Matrices, Invertible Linear Transformation | 501  |
| NME9 Nonsingular Matrix Equivalences, Round 9             | 502  |

| Section CB     |
|------------------|------|
| CB Change-of-Basis | 516  |
| ICBM Inverse of Change-of-Basis Matrix                     | 517  |
| MRCB Matrix Representation and Change of Basis              | 521  |
| SCB Similarity and Change of Basis                          | 524  |
| EER Eigenvalues, Eigenvectors, Representations              | 527  |

| Section OD     |
|------------------|------|
| PTMT Product of Triangular Matrices is Triangular         | 540  |
| ITMT Inverse of a Triangular Matrix is Triangular          | 541  |
| UTMR Upper Triangular Matrix Representation                | 541  |
| OBUTR Orthonormal Basis for Upper Triangular Representation | 544  |
| OD Orthonormal Diagonalization                             | 546  |

| Section NLT    |
|------------------|------|
| NJB Nilpotent Jordan Blocks                                | 552  |
| ENLT Eigenvalues of Nilpotent Linear Transformations       | 553  |
| DNLT Diagonalizable Nilpotent Linear Transformations       | 553  |
| KPLT Kernels of Powers of Linear Transformations           | 554  |
| KPNLT Kernels of Powers of Nilpotent Linear Transformations| 555  |
| CFNLT Canonical Form for Nilpotent Linear Transformations | 557  |

<p>| Section IS     |
|------------------|------|
| EIS Eigenspaces are Invariant Subspaces                     | 566  |
| KPIIS Kernels of Powers are Invariant Subspaces             | 567  |
| GESIS Generalized Eigenspace is an Invariant Subspace       | 569  |
| GEK Generalized Eigenspace as a Kernel                       | 569  |
| RGEN Restriction to Generalized Eigenspace is Nilpotent     | 578  |
| MRRGE Matrix Representation of a Restriction to a Generalized Eigenspace | 580  |</p>
<table>
<thead>
<tr>
<th>Section</th>
<th>Topic</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>JCF</td>
<td>Upper Triangular Matrix Representation</td>
<td>582</td>
</tr>
<tr>
<td></td>
<td>Generalized Eigenspace Decomposition</td>
<td>584</td>
</tr>
<tr>
<td></td>
<td>Dimension of Generalized Eigenspaces</td>
<td>590</td>
</tr>
<tr>
<td></td>
<td>Jordan Canonical Form for a Linear Transformation</td>
<td>590</td>
</tr>
<tr>
<td></td>
<td>Cayley-Hamilton Theorem</td>
<td>603</td>
</tr>
<tr>
<td>CNO</td>
<td>Properties of Complex Number Arithmetic</td>
<td>612</td>
</tr>
<tr>
<td></td>
<td>Complex Conjugation Respects Addition</td>
<td>613</td>
</tr>
<tr>
<td></td>
<td>Complex Conjugation Respects Multiplication</td>
<td>613</td>
</tr>
<tr>
<td></td>
<td>Complex Conjugation Twice</td>
<td>613</td>
</tr>
<tr>
<td>SET</td>
<td>Field of Integers Modulo a Prime</td>
<td>719</td>
</tr>
<tr>
<td>PT</td>
<td>Rank One Decomposition</td>
<td>726</td>
</tr>
<tr>
<td>F</td>
<td>Triangular Decomposition</td>
<td>731</td>
</tr>
<tr>
<td></td>
<td>Triangular Decomposition, Entry by Entry</td>
<td>735</td>
</tr>
</tbody>
</table>
### Notation

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>Matrix</td>
<td>21</td>
</tr>
<tr>
<td>MC</td>
<td>$[A]_{ij}$: Matrix Components</td>
<td>21</td>
</tr>
<tr>
<td>CV</td>
<td>$\mathbf{v}$: Column Vector</td>
<td>22</td>
</tr>
<tr>
<td>CVC</td>
<td>$[\mathbf{v}]_i$: Column Vector Components</td>
<td>22</td>
</tr>
<tr>
<td>ZCV</td>
<td>$\mathbf{0}$: Zero Column Vector</td>
<td>22</td>
</tr>
<tr>
<td>LSMR</td>
<td>$\mathcal{L}(A, \mathbf{b})$: Matrix Representation of a Linear System</td>
<td>23</td>
</tr>
<tr>
<td>AM</td>
<td>$[A]</td>
<td>\mathbf{b}]$: Augmented Matrix</td>
</tr>
<tr>
<td>RO</td>
<td>$R_i \leftrightarrow R_j$, $\alpha R_i + R_j$: Row Operations</td>
<td>25</td>
</tr>
<tr>
<td>RREFA</td>
<td>$r$, $D$, $F$: Reduced Row-Echelon Form Analysis</td>
<td>27</td>
</tr>
<tr>
<td>NSM</td>
<td>$\mathcal{N}(A)$: Null Space of a Matrix</td>
<td>54</td>
</tr>
<tr>
<td>IM</td>
<td>$I_m$: Identity Matrix</td>
<td>62</td>
</tr>
<tr>
<td>VSCV</td>
<td>$\mathbb{C}^m$: Vector Space of Column Vectors</td>
<td>72</td>
</tr>
<tr>
<td>CVE</td>
<td>$\mathbf{u} = \mathbf{v}$: Column Vector Equality</td>
<td>73</td>
</tr>
<tr>
<td>CVA</td>
<td>$\mathbf{u} + \mathbf{v}$: Column Vector Addition</td>
<td>73</td>
</tr>
<tr>
<td>CVSM</td>
<td>$\alpha \mathbf{u}$: Column Vector Scalar Multiplication</td>
<td>74</td>
</tr>
<tr>
<td>SSV</td>
<td>$(S)$: Span of a Set of Vectors</td>
<td>102</td>
</tr>
<tr>
<td>CCCV</td>
<td>$\mathbf{u}$: Complex Conjugate of a Column Vector</td>
<td>151</td>
</tr>
<tr>
<td>IP</td>
<td>$(\mathbf{u}, \mathbf{v})$: Inner Product</td>
<td>152</td>
</tr>
<tr>
<td>NV</td>
<td>$|\mathbf{v}|$: Norm of a Vector</td>
<td>155</td>
</tr>
<tr>
<td>VSM</td>
<td>$M_m$: Vector Space of Matrices</td>
<td>163</td>
</tr>
<tr>
<td>ME</td>
<td>$A = B$: Matrix Equality</td>
<td>163</td>
</tr>
<tr>
<td>MA</td>
<td>$A + B$: Matrix Addition</td>
<td>164</td>
</tr>
<tr>
<td>MSM</td>
<td>$\alpha A$: Matrix Scalar Multiplication</td>
<td>164</td>
</tr>
<tr>
<td>ZM</td>
<td>$O$: Zero Matrix</td>
<td>166</td>
</tr>
<tr>
<td>TM</td>
<td>$A^*$: Transpose of a Matrix</td>
<td>166</td>
</tr>
<tr>
<td>CCM</td>
<td>$A$: Complex Conjugate of a Matrix</td>
<td>168</td>
</tr>
<tr>
<td>MVP</td>
<td>$A \mathbf{u}$: Matrix-Vector Product</td>
<td>173</td>
</tr>
<tr>
<td>MI</td>
<td>$A^{-1}$: Matrix Inverse</td>
<td>189</td>
</tr>
<tr>
<td>A</td>
<td>$A^*$: Adjoint</td>
<td>207</td>
</tr>
<tr>
<td>CSM</td>
<td>$\mathcal{C}(A)$: Column Space of a Matrix</td>
<td>211</td>
</tr>
<tr>
<td>RSM</td>
<td>$\mathcal{R}(A)$: Row Space of a Matrix</td>
<td>217</td>
</tr>
<tr>
<td>LNS</td>
<td>$\mathcal{L}(A)$: Left Null Space</td>
<td>231</td>
</tr>
<tr>
<td>D</td>
<td>$\dim(V)$: Dimension</td>
<td>307</td>
</tr>
<tr>
<td>NOM</td>
<td>$n(A)$: Nullity of a Matrix</td>
<td>312</td>
</tr>
<tr>
<td>ROM</td>
<td>$r(A)$: Rank of a Matrix</td>
<td>313</td>
</tr>
<tr>
<td>DS</td>
<td>$V = U \oplus W$: Direct Sum</td>
<td>326</td>
</tr>
<tr>
<td>ELEM</td>
<td>$E_{ij}$, $E_i(\alpha)$, $E_{i,j}(\alpha)$: Elementary Matrix</td>
<td>334</td>
</tr>
<tr>
<td>SM</td>
<td>$A(i,j)$: SubMatrix</td>
<td>337</td>
</tr>
<tr>
<td>DM</td>
<td>$\det(A)$, $</td>
<td>A</td>
</tr>
<tr>
<td>LT</td>
<td>$T: U \rightarrow V$: Linear Transformation</td>
<td>405</td>
</tr>
<tr>
<td>KLT</td>
<td>$\mathcal{K}(T)$: Kernel of a Linear Transformation</td>
<td>429</td>
</tr>
<tr>
<td>RLT</td>
<td>$\mathcal{R}(T)$: Range of a Linear Transformation</td>
<td>444</td>
</tr>
<tr>
<td>ROLT</td>
<td>$r(T)$: Rank of a Linear Transformation</td>
<td>463</td>
</tr>
<tr>
<td>NOTATION</td>
<td>Definition</td>
<td>Page</td>
</tr>
<tr>
<td>----------</td>
<td>------------</td>
<td>------</td>
</tr>
<tr>
<td>NOLT</td>
<td>$n(T)$: Nullity of a Linear Transformation</td>
<td>463</td>
</tr>
<tr>
<td>JB</td>
<td>$J_n(\lambda)$: Jordan Block</td>
<td>550</td>
</tr>
<tr>
<td>GES</td>
<td>$G_T(\lambda)$: Generalized Eigenspace</td>
<td>569</td>
</tr>
<tr>
<td>LTR</td>
<td>$T</td>
<td>_U$: Linear Transformation Restriction</td>
</tr>
<tr>
<td>IE</td>
<td>$\nu_T(\lambda)$: Index of an Eigenvalue</td>
<td>578</td>
</tr>
<tr>
<td>CNE</td>
<td>$\alpha = \beta$: Complex Number Equality</td>
<td>612</td>
</tr>
<tr>
<td>CNA</td>
<td>$\alpha + \beta$: Complex Number Addition</td>
<td>612</td>
</tr>
<tr>
<td>CNM</td>
<td>$\alpha \beta$: Complex Number Multiplication</td>
<td>612</td>
</tr>
<tr>
<td>CCN</td>
<td>$\overline{\alpha}$: Conjugate of a Complex Number</td>
<td>613</td>
</tr>
<tr>
<td>SETM</td>
<td>$x \in S$: Set Membership</td>
<td>615</td>
</tr>
<tr>
<td>SSET</td>
<td>$S \subseteq T$: Subset</td>
<td>615</td>
</tr>
<tr>
<td>ES</td>
<td>$\emptyset$: Empty Set</td>
<td>615</td>
</tr>
<tr>
<td>SE</td>
<td>$S = T$: Set Equality</td>
<td>616</td>
</tr>
<tr>
<td>C</td>
<td>$</td>
<td>S</td>
</tr>
<tr>
<td>SU</td>
<td>$S \cup T$: Set Union</td>
<td>617</td>
</tr>
<tr>
<td>SI</td>
<td>$S \cap T$: Set Intersection</td>
<td>617</td>
</tr>
<tr>
<td>SC</td>
<td>$\mathcal{S}$: Set Complement</td>
<td>617</td>
</tr>
</tbody>
</table>
Examples

Section WILA
- TMP Trail Mix Packaging ........................................... 3

Section SSLE
- STNE Solving two (nonlinear) equations .................................. 9
- NSE Notation for a system of equations .................................. 10
- TTS Three typical systems ............................................. 10
- US Three equations, one solution ....................................... 14
- IS Three equations, infinitely many solutions ......................... 15

Section RREF
- AM A matrix ................................................................. 21
- NSLE Notation for systems of linear equations ......................... 23
- AMAA Augmented matrix for Archetype A ................................ 24
- TREM Two row-equivalent matrices .................................... 25
- USR Three equations, one solution, reprised ......................... 25
- RREF A matrix in reduced row-echelon form .......................... 27
- NRREF A matrix not in reduced row-echelon form ..................... 27
- SAB Solutions for Archetype B .......................................... 29
- SAA Solutions for Archetype A ........................................... 30
- SAE Solutions for Archetype E ........................................... 31

Section TSS
- RREFN Reduced row-echelon form notation ............................ 42
- ISSI Describing infinite solution sets, Archetype I .................... 43
- FDV Free and dependent variables ...................................... 44
- CFV Counting free variables ............................................. 47
- OSGMD One solution gives many, Archetype D ......................... 48

Section HSE
- AHSAC Archetype C as a homogeneous system ......................... 52
- HUSAB Homogeneous, unique solution, Archetype B ................... 52
- HISAA Homogeneous, infinite solutions, Archetype A ............... 53
- HISAD Homogeneous, infinite solutions, Archetype D ............... 53
- NSEAI Null space elements of Archetype I ............................... 54
- CNS1 Computing a null space, #1 ........................................ 55
- CNS2 Computing a null space, #2 ........................................ 56

Section NM
- S A singular matrix, Archetype A ....................................... 61
- NM A nonsingular matrix, Archetype B .................................. 62
- IM An identity matrix ..................................................... 62
- SRR Singular matrix, row-reduced ...................................... 62
### Section MO

| MA | Addition of two matrices in $M_{23}$ | 164 |
| MSM | Scalar multiplication in $M_{32}$ | 164 |
| TM | Transpose of a $3 \times 4$ matrix | 166 |
| SYM | A symmetric $5 \times 5$ matrix | 166 |
| CCM | Complex conjugate of a matrix | 168 |

### Section MM

| MTV | A matrix times a vector | 173 |
| MNSLE | Matrix notation for systems of linear equations | 174 |
| MBC | Money’s best cities | 174 |
| PTM | Product of two matrices | 176 |
| MMNC | Matrix multiplication is not commutative | 177 |
| PTMEE | Product of two matrices, entry-by-entry | 178 |

### Section MISLE

| SABMI | Solutions to Archetype B with a matrix inverse | 188 |
| MWIAA | A matrix without an inverse, Archetype A | 189 |
| MI | Matrix inverse | 190 |
| CMI | Computing a matrix inverse | 192 |
| CMIAB | Computing a matrix inverse, Archetype B | 194 |

### Section MINM

| UM3 | Unitary matrix of size 3 | 205 |
| UPM | Unitary permutation matrix | 205 |
| OSMC | Orthonormal set from matrix columns | 206 |

### Section CRS

| CSMCS | Column space of a matrix and consistent systems | 211 |
| MCSM | Membership in the column space of a matrix | 212 |
| CSTW | Column space, two ways | 213 |
| CSOCD | Column space, original columns, Archetype D | 214 |
| CSAA | Column space of Archetype A | 215 |
| CSAB | Column space of Archetype B | 216 |
| RSAI | Row space of Archetype I | 217 |
| RSREM | Row spaces of two row-equivalent matrices | 219 |
| IAS | Improving a span | 220 |
| CSROI | Column space from row operations, Archetype I | 221 |

### Section FS

| LNS | Left null space | 231 |
| CSANS | Column space as null space | 232 |
| SEEF | Submatrices of extended echelon form | 235 |
| FS1 | Four subsets, #1 | 241 |
| FS2 | Four subsets, #2 | 241 |
| FSAG | Four subsets, Archetype G | 242 |

### Section VS

<p>| VSCV | The vector space $C^n$ | 253 |
| VSM | The vector space of matrices, $M_{mn}$ | 253 |
| VSP | The vector space of polynomials, $P_n$ | 253 |
| VSIS | The vector space of infinite sequences | 254 |
| VSF | The vector space of functions | 254 |</p>
<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>SC3 A subspace of $\mathbb{C}^3$</td>
<td>264</td>
</tr>
<tr>
<td></td>
<td>SP4 A subspace of $P_4$</td>
<td>264</td>
</tr>
<tr>
<td></td>
<td>NSC2Z A non-subspace in $\mathbb{C}^2$, zero vector</td>
<td>267</td>
</tr>
<tr>
<td></td>
<td>NSC2A A non-subspace in $\mathbb{C}^2$, additive closure</td>
<td>267</td>
</tr>
<tr>
<td></td>
<td>NSC2S A non-subspace in $\mathbb{C}^2$, scalar multiplication closure</td>
<td>267</td>
</tr>
<tr>
<td></td>
<td>RSNS Recasting a subspace as a null space</td>
<td>268</td>
</tr>
<tr>
<td></td>
<td>LCM A linear combination of matrices</td>
<td>269</td>
</tr>
<tr>
<td></td>
<td>SSP Span of a set of polynomials</td>
<td>271</td>
</tr>
<tr>
<td></td>
<td>SM32 A subspace of $M_{32}$</td>
<td>272</td>
</tr>
<tr>
<td>LISS</td>
<td>LIP4 Linear independence in $P_4$</td>
<td>280</td>
</tr>
<tr>
<td></td>
<td>LIM32 Linear independence in $M_{32}$</td>
<td>282</td>
</tr>
<tr>
<td></td>
<td>LIC Linearly independent set in the crazy vector space</td>
<td>283</td>
</tr>
<tr>
<td></td>
<td>SSP4 Spanning set in $P_4$</td>
<td>284</td>
</tr>
<tr>
<td></td>
<td>SSM22 Spanning set in $M_{32}$</td>
<td>286</td>
</tr>
<tr>
<td></td>
<td>SSC Spanning set in the crazy vector space</td>
<td>287</td>
</tr>
<tr>
<td></td>
<td>AVR A vector representation</td>
<td>288</td>
</tr>
<tr>
<td>B</td>
<td>BP Bases for $P_n$</td>
<td>295</td>
</tr>
<tr>
<td></td>
<td>BM A basis for the vector space of matrices</td>
<td>295</td>
</tr>
<tr>
<td></td>
<td>BSP4 A basis for a subspace of $P_4$</td>
<td>295</td>
</tr>
<tr>
<td></td>
<td>BSM22 A basis for a subspace of $M_{22}$</td>
<td>296</td>
</tr>
<tr>
<td></td>
<td>BC Basis for the crazy vector space</td>
<td>297</td>
</tr>
<tr>
<td></td>
<td>RSB Row space basis</td>
<td>297</td>
</tr>
<tr>
<td></td>
<td>RS Reducing a span</td>
<td>298</td>
</tr>
<tr>
<td></td>
<td>CABAK Columns as Basis, Archetype K</td>
<td>299</td>
</tr>
<tr>
<td></td>
<td>CROB4 Coordinatization relative to an orthonormal basis, $\mathbb{C}^4$</td>
<td>301</td>
</tr>
<tr>
<td></td>
<td>CROB3 Coordinatization relative to an orthonormal basis, $\mathbb{C}^3$</td>
<td>302</td>
</tr>
<tr>
<td>D</td>
<td>LDP4 Linearly dependent set in $P_4$</td>
<td>310</td>
</tr>
<tr>
<td></td>
<td>DSM22 Dimension of a subspace of $M_{22}$</td>
<td>311</td>
</tr>
<tr>
<td></td>
<td>DSP4 Dimension of a subspace of $P_4$</td>
<td>312</td>
</tr>
<tr>
<td></td>
<td>DC Dimension of the crazy vector space</td>
<td>312</td>
</tr>
<tr>
<td></td>
<td>VSPUD Vector space of polynomials with unbounded degree</td>
<td>312</td>
</tr>
<tr>
<td></td>
<td>RNM Rank and nullity of a matrix</td>
<td>313</td>
</tr>
<tr>
<td></td>
<td>RNSM Rank and nullity of a square matrix</td>
<td>314</td>
</tr>
<tr>
<td>PD</td>
<td>BPR Bases for $P_n$, reprised</td>
<td>321</td>
</tr>
<tr>
<td></td>
<td>BDM22 Basis by dimension in $M_{22}$</td>
<td>322</td>
</tr>
<tr>
<td></td>
<td>SVP4 Sets of vectors in $P_4$</td>
<td>322</td>
</tr>
<tr>
<td></td>
<td>RRTI Rank, rank of transpose, Archetype I</td>
<td>324</td>
</tr>
<tr>
<td></td>
<td>SDS Simple direct sum</td>
<td>326</td>
</tr>
<tr>
<td>DM</td>
<td>EMRO Elementary matrices and row operations</td>
<td>334</td>
</tr>
<tr>
<td>Example</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>----------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>SS</td>
<td>Some submatrices</td>
<td>337</td>
</tr>
<tr>
<td>D33M</td>
<td>Determinant of a $3 \times 3$ matrix</td>
<td>338</td>
</tr>
<tr>
<td>TCSD</td>
<td>Two computations, same determinant</td>
<td>340</td>
</tr>
<tr>
<td>DUTM</td>
<td>Determinant of an upper triangular matrix</td>
<td>341</td>
</tr>
<tr>
<td>DRO</td>
<td>Determinant by row operations</td>
<td>348</td>
</tr>
<tr>
<td>ZNDAB</td>
<td>Zero and nonzero determinant, Archetypes A and B</td>
<td>352</td>
</tr>
<tr>
<td>SEE</td>
<td>Some eigenvalues and eigenvectors</td>
<td>356</td>
</tr>
<tr>
<td>PM</td>
<td>Polynomial of a matrix</td>
<td>358</td>
</tr>
<tr>
<td>CAEHW</td>
<td>Computing an eigenvalue the hard way</td>
<td>361</td>
</tr>
<tr>
<td>CPMS3</td>
<td>Characteristic polynomial of a matrix, size 3</td>
<td>363</td>
</tr>
<tr>
<td>EMS3</td>
<td>Eigenvalues of a matrix, size 3</td>
<td>363</td>
</tr>
<tr>
<td>ESMS3</td>
<td>Eigenspaces of a matrix, size 3</td>
<td>365</td>
</tr>
<tr>
<td>EMMS4</td>
<td>Eigenvalue multiplicities, matrix of size 4</td>
<td>366</td>
</tr>
<tr>
<td>ESMS4</td>
<td>Eigenvalues, symmetric matrix of size 4</td>
<td>367</td>
</tr>
<tr>
<td>HMEM5</td>
<td>High multiplicity eigenvalues, matrix of size 5</td>
<td>367</td>
</tr>
<tr>
<td>CEMS6</td>
<td>Complex eigenvalues, matrix of size 6</td>
<td>368</td>
</tr>
<tr>
<td>DEMS5</td>
<td>Distinct eigenvalues, matrix of size 5</td>
<td>370</td>
</tr>
<tr>
<td>BDE</td>
<td>Building desired eigenvalues</td>
<td>381</td>
</tr>
<tr>
<td>SMS5</td>
<td>Similar matrices of size 5</td>
<td>390</td>
</tr>
<tr>
<td>SMS3</td>
<td>Similar matrices of size 3</td>
<td>391</td>
</tr>
<tr>
<td>EENS</td>
<td>Equal eigenvalues, not similar</td>
<td>392</td>
</tr>
<tr>
<td>DAB</td>
<td>Diagonalization of Archetype B</td>
<td>393</td>
</tr>
<tr>
<td>DMS3</td>
<td>Diagonalizing a matrix of size 3</td>
<td>395</td>
</tr>
<tr>
<td>NDMS4</td>
<td>A non-diagonalizable matrix of size 4</td>
<td>397</td>
</tr>
<tr>
<td>DEHD</td>
<td>Distinct eigenvalues, hence diagonalizable</td>
<td>398</td>
</tr>
<tr>
<td>HPDM</td>
<td>High power of a diagonalizable matrix</td>
<td>399</td>
</tr>
<tr>
<td>ALT</td>
<td>A linear transformation</td>
<td>406</td>
</tr>
<tr>
<td>NLT</td>
<td>Not a linear transformation</td>
<td>407</td>
</tr>
<tr>
<td>LTPM</td>
<td>Linear transformation, polynomials to matrices</td>
<td>407</td>
</tr>
<tr>
<td>LTPP</td>
<td>Linear transformation, polynomials to polynomials</td>
<td>408</td>
</tr>
<tr>
<td>LTM</td>
<td>Linear transformation from a matrix</td>
<td>409</td>
</tr>
<tr>
<td>MFLT</td>
<td>Matrix from a linear transformation</td>
<td>410</td>
</tr>
<tr>
<td>MOLT</td>
<td>Matrix of a linear transformation</td>
<td>412</td>
</tr>
<tr>
<td>LTDB1</td>
<td>Linear transformation defined on a basis</td>
<td>413</td>
</tr>
<tr>
<td>LTDB2</td>
<td>Linear transformation defined on a basis</td>
<td>414</td>
</tr>
<tr>
<td>LTDB3</td>
<td>Linear transformation defined on a basis</td>
<td>415</td>
</tr>
<tr>
<td>SPIAS</td>
<td>Sample pre-images, Archetype S</td>
<td>415</td>
</tr>
<tr>
<td>STLTT</td>
<td>Sum of two linear transformations</td>
<td>418</td>
</tr>
<tr>
<td>SMLT</td>
<td>Scalar multiple of a linear transformation</td>
<td>419</td>
</tr>
<tr>
<td>CTLT</td>
<td>Composition of two linear transformations</td>
<td>420</td>
</tr>
<tr>
<td>NIAQ</td>
<td>Not injective, Archetype Q</td>
<td>426</td>
</tr>
<tr>
<td>TAR</td>
<td>Injective, Archetype R</td>
<td>427</td>
</tr>
</tbody>
</table>
IAV Injective, Archetype V .............................................................. 428
NKAO Nontrivial kernel, Archetype O .............................................. 429
TKAP Trivial kernel, Archetype P ..................................................... 430
NIAQR Not injective, Archetype Q, revisited ...................................... 432
NIAO Not injective, Archetype O ..................................................... 432
IAP Injective, Archetype P ............................................................. 433
NIDAU Not injective by dimension, Archetype U ................................. 434

Section SLT
NSAQ Not surjective, Archetype Q .................................................. 440
SAR Surjective, Archetype R ........................................................... 441
SAV Surjective, Archetype V ........................................................... 442
RAO Range, Archetype O ............................................................... 444
FRAN Full range, Archetype N ......................................................... 445
NSAQQR Not surjective, Archetype Q, revisited .................................. 446
NSAO Not surjective, Archetype O .................................................... 447
SAN Surjective, Archetype N ........................................................... 447
BRLT A basis for the range of a linear transformation .......................... 448
NSDAT Not surjective by dimension, Archetype T .............................. 450

Section IVLT
AIVLT An invertible linear transformation ........................................ 456
ANILT A non-invertible linear transformation .................................... 457
IVSAV Isomorphic vector spaces, Archetype V ................................... 462

Section VR
VRC4 Vector representation in \( \mathbb{C}^4 \) .................................................. 474
VRP2 Vector representations in \( \mathbb{P}^2 \) .................................................. 476
TIVS Two isomorphic vector spaces .................................................. 478
CVSR Crazy vector space revealed .................................................... 479
ASC A subspace characterized ......................................................... 479
MIVS Multiple isomorphic vector spaces ......................................... 479
CP2 Coordinatizing in \( \mathbb{P}^2 \) ............................................................. 480
CM32 Coordinatization in \( \mathbb{M}_{32} \) ..................................................... 481

Section MR
OLTTR One linear transformation, three representations ..................... 485
ALTMM A linear transformation as matrix multiplication .................... 488
MPMR Matrix product of matrix representations ................................ 492
KVMR Kernel via matrix representation .......................................... 495
RVMR Range via matrix representation ............................................ 498
ILTVR Inverse of a linear transformation via a representation ........... 500

Section CB
ELTBM Eigenvectors of linear transformation between matrices ............ 515
ELTBP Eigenvectors of linear transformation between polynomials .......... 516
CBP Change of basis with polynomials ............................................. 517
CBCV Change of basis with column vectors ..................................... 520
MRCM Matrix representations and change-of-basis matrices .................. 522
MRBE Matrix representation with basis of eigenvectors ....................... 524
ELTT Eigenvectors of a linear transformation, twice ............................ 528
CELT Complex eigenvectors of a linear transformation ....................... 532

Section OD
ANM  A normal matrix  ..................................................................................  545

Section NLT
NM64  Nilpotent matrix, size 6, index 4  .......................................................  548
NM62  Nilpotent matrix, size 6, index 2  .......................................................  549
JB4  Jordan block, size 4  ...........................................................................  550
NJBJ5  Nilpotent Jordan block, size 5  .........................................................  550
NM83  Nilpotent matrix, size 8, index 3  .......................................................  551
KPNLT  Kernels of powers of a nilpotent linear transformation  .................  555
CFNLNT  Canonical form for a nilpotent linear transformation  .................  561

Section IS
TIS  Two invariant subspaces  .................................................................  565
EIS  Eigenspaces as invariant subspaces  ..................................................  567
ISJB  Invariant subspaces and Jordan blocks  ..........................................  568
GE4  Generalized eigenspaces, dimension 4 domain  ...............................  570
GE6  Generalized eigenspaces, dimension 6 domain  ...............................  571
LTRGE  Linear transformation restriction on generalized eigenspace  ........  573
ISMR4  Invariant subspaces, matrix representation, dimension 4 domain  .  575
ISMR6  Invariant subspaces, matrix representation, dimension 6 domain  .  576
GENR6  Generalized eigenspaces and nilpotent restrictions, dimension 6 domain  .  578

Section JCF
JCF10  Jordan canonical form, size 10  ....................................................  591

Section CNO
ACN  Arithmetic of complex numbers  ....................................................  611
CSCN  Conjugate of some complex numbers  ...........................................  613
MSCN  Modulus of some complex numbers  .............................................  614

Section SET
SETM  Set membership  ...........................................................................  615
SETT  Subset  ............................................................................................  615
CS  Cardinality and Size  ..........................................................................  616
SU  Set union  ............................................................................................  617
SI  Set intersection  ....................................................................................  617
SC  Set complement  ..................................................................................  618

Section PT
Section F
IM11  Integers mod 11  .............................................................................  719
VSIM5  Vector space over integers mod 5  .................................................  720
SM2Z7  Symmetric matrices of size 2 over \(\mathbb{Z}_7\)  .................................  720
FF8  Finite field of size 8  ..........................................................................  721

Section ROD
ROD2  Rank one decomposition, size 2  .................................................  728
ROD4  Rank one decomposition, size 4  .................................................  729

Section TD
TD4  Triangular decomposition, size 4  ....................................................  732
TDSSE  Triangular decomposition solves a system of equations  ...............  734
TDEE6  Triangular decomposition, entry by entry, size 6  .........................  736
Preface

This textbook is designed to teach the university mathematics student the basics of the subject of linear algebra and the techniques of formal mathematics. There are no prerequisites other than ordinary algebra, but it is probably best used by a student who has the “mathematical maturity” of a sophomore or junior. The text has two goals: to teach the fundamental concepts and techniques of matrix algebra and abstract vector spaces, and to teach the techniques associated with understanding the definitions and theorems forming a coherent area of mathematics. So there is an emphasis on worked examples of nontrivial size and on proving theorems carefully.

This book is copyrighted. This means that governments have granted the author a monopoly — the exclusive right to control the making of copies and derivative works for many years (too many years in some cases). It also gives others limited rights, generally referred to as “fair use,” such as the right to quote sections in a review without seeking permission. However, the author licenses this book to anyone under the terms of the GNU Free Documentation License (GFDL), which gives you more rights than most copyrights (see Appendix GFDL [71]). Loosely speaking, you may make as many copies as you like at no cost, and you may distribute these unmodified copies if you please. You may modify the book for your own use. The catch is that if you make modifications and you distribute the modified version, or make use of portions in excess of fair use in another work, then you must also license the new work with the GFDL. So the book has lots of inherent freedom, and no one is allowed to distribute a derivative work that restricts these freedoms. (See the license itself in the appendix for the exact details of the additional rights you have been given.)

Notice that initially most people are struck by the notion that this book is free (the French would say gratis, at no cost). And it is. However, it is more important that the book has freedom (the French would say liberté, liberty). It will never go “out of print” nor will there ever be trivial updates designed only to frustrate the used book market. Those considering teaching a course with this book can examine it thoroughly in advance. Adding new exercises or new sections has been purposely made very easy, and the hope is that others will contribute these modifications back for incorporation into the book, for the benefit of all.

Depending on how you received your copy, you may want to check for the latest version (and other news) at http://linear.ups.edu/.

Topics  The first half of this text (through Chapter M [163]) is basically a course in matrix algebra, though the foundation of some more advanced ideas is also being formed in these early sections. Vectors are presented exclusively as column vectors (since we also have the typographic freedom to avoid writing a column vector inline as the transpose of a row vector), and linear combinations are presented very early. Spans, null spaces and column spaces are also presented early, simply as sets, saving most of their vector space properties for later, so they are familiar objects before being scrutinized carefully.

You cannot do everything early, so in particular matrix multiplication comes later than usual. However, with a definition built on linear combinations of column vectors, it should seem more natural than the more frequent definition using dot products of rows with columns. And this delay emphasizes that linear algebra is built upon vector addition and scalar multiplication. Of course, matrix inverses must wait for matrix multiplication, but this does not prevent nonsingular matrices from occurring sooner. Vector space properties are hinted at when vector and matrix operations are first defined, but the notion of a vector space is saved for a more axiomatic treatment later.
(Chapter VS \[251\]). Once bases and dimension have been explored in the context of vector spaces, linear transformations and their matrix representations follow. The goal of the book is to go as far as Jordan canonical form in the Core (Part C \[2\]), with less central topics collected in the Topics (Part T \[718\]). A third part will contain contributed applications, with notation and theorems integrated with the earlier two parts (Part A \[??\]).

Linear algebra is an ideal subject for the novice mathematics student to learn how to develop a topic precisely, with all the rigor mathematics requires. Unfortunately, much of this rigor seems to have escaped the standard calculus curriculum, so for many university students this is their first exposure to careful definitions and theorems, and the expectation that they fully understand them, to say nothing of the expectation that they become proficient in formulating their own proofs. We have tried to make this text as helpful as possible with this transition. Every definition is stated carefully, set apart from the text. Likewise, every theorem is carefully stated, and almost every one has a complete proof. Theorems usually have just one conclusion, so they can be referenced precisely later. Definitions and theorems are cataloged in order of their appearance in the front of the book (Definitions \[viii\], Theorems \[x\]), and alphabetical order in the index at the back. Along the way, there are discussions of some more important ideas relating to formulating proofs (Proof Techniques \[??\]), which is part advice and part logic.

Origin and History This book is the result of the confluence of several related events and trends.

- At the University of Puget Sound we teach a one-semester, post-calculus linear algebra course to students majoring in mathematics, computer science, physics, chemistry and economics. Between January 1986 and June 2002, I taught this course seventeen times. For the Spring 2003 semester, I elected to convert my course notes to an electronic form so that it would be easier to incorporate the inevitable and nearly-constant revisions. Central to my new notes was a collection of stock examples that would be used repeatedly to illustrate new concepts. (These would become the Archetypes, Appendix A \[630\].) It was only a short leap to then decide to distribute copies of these notes and examples to the students in the two sections of this course. As the semester wore on, the notes began to look less like notes and more like a textbook.

- I used the notes again in the Fall 2003 semester for a single section of the course. Simultaneously, the textbook I was using came out in a fifth edition. A new chapter was added toward the start of the book, and a few additional exercises were added in other chapters. This demanded the annoyance of reworking my notes and list of suggested exercises to conform with the changed numbering of the chapters and exercises. I had an almost identical experience with the third course I was teaching that semester. I also learned that in the next academic year I would be teaching a course where my textbook of choice had gone out of print. I felt there had to be a better alternative to having the organization of my courses buffeted by the economics of traditional textbook publishing.

- I had used T\TeX\ and the Internet for many years, so there was little to stand in the way of typesetting, distributing and “marketing” a free book. With recreational and professional interests in software development, I had long been fascinated by the open-source software movement, as exemplified by the success of GNU and Linux, though public-domain T\TeX\ might also deserve mention. Obviously, this book is an attempt to carry over that model of creative endeavor to textbook publishing.

- As a sabbatical project during the Spring 2004 semester, I embarked on the current project of creating a freely-distributable linear algebra textbook. (Notice the implied financial support of the University of Puget Sound to this project.) Most of the material was written from scratch since changes in notation and approach made much of my notes of little use. By August 2004 I had written half the material necessary for our Math 232 course. The remaining half was written during the Fall 2004 semester as I taught another two sections of Math 232.
While in early 2005 the book was complete enough to build a course around, work continued for the next two years to fill out the narrative, exercises and supplements. In this time, I taught four sections of the course, while three of my colleagues at the University of Puget Sound taught another four sections.

However, much of my motivation for writing this book is captured by the sentiments expressed by H.M. Cundy and A.P. Rollet in their Preface to the First Edition of *Mathematical Models* (1952), especially the final sentence,

> This book was born in the classroom, and arose from the spontaneous interest of a Mathematical Sixth in the construction of simple models. A desire to show that even in mathematics one could have fun led to an exhibition of the results and attracted considerable attention throughout the school. Since then the Sherborne collection has grown, ideas have come from many sources, and widespread interest has been shown. It seems therefore desirable to give permanent form to the lessons of experience so that others can benefit by them and be encouraged to undertake similar work.

**How To Use This Book**  Chapters, Theorems, etc. are not numbered in this book, but are instead referenced by acronyms. This means that Theorem XYZ will always be Theorem XYZ, no matter if new sections are added, or if an individual decides to remove certain other sections. Within sections, the subsections are acronyms that begin with the acronym of the section. So Subsection XYZ.AB is the subsection AB in Section XYZ. Acronyms are unique within their type, so for example there is just one Definition B [294], but there is also a Section B [294]. At first, all the letters flying around may be confusing, but with time, you will begin to recognize the more important ones on sight. Furthermore, there are lists of theorems, examples, etc. in the front of the book, and an index that contains every acronym. If you are reading this in an electronic version (PDF or XML), you will see that all of the cross-references are hyperlinks, allowing you to click to a definition or example, and then use the back button to return. In printed versions, you must rely on the page numbers. However, note that page numbers are not permanent! Different editions, different margins, or different sized paper will affect what content is on each page. And in time, the addition of new material will affect the page numbering.

Chapter divisions are not critical to the organization of the book, as Sections are the main organizational unit. Sections are designed to be the subject of a single lecture or classroom session, though there is frequently more material than can be discussed and illustrated in a fifty-minute session. Consequently, the instructor will need to be selective about which topics to illustrate with other examples and which topics to leave to the student’s reading. Many of the examples are meant to be large, such as using five or six variables in a system of equations, so the instructor may just want to “walk” a class through these examples. The book has been written with the idea that some may work through it independently, so the hope is that students can learn some of the more mechanical ideas on their own.

The highest level division of the book is the three Parts: Core, Topics, Applications. The Core is meant to carefully describe the basic ideas required of a first exposure to linear algebra. In the final sections of the Core, one should ask the question: which previous Sections could be removed without destroying the logical development of the subject? Hopefully, the answer is “none.” The goal of the book is to finish the Core with the most general representations of linear transformations (Jordan and perhaps rational canonical forms). Of course, there will not be universal agreement on what should, or should not, constitute the Core, but the main idea will be to limit it to about forty sections. Topics is meant to contain those subjects that are important in linear algebra, and which would make profitable detours from the Core for those interested in pursuing them. Applications should illustrate the power and widespread applicability of linear algebra to as many fields as possible. The Archetypes [Appendix A [634]] cover many of the computational aspects of systems of linear equations, matrices and linear transformations. The student should consult them often, and this is encouraged by exercises that simply suggest the right properties to examine at the right time. But what is more important, they are a repository that contains enough variety to provide abundant examples of key theorems, while also providing counterexamples to hypotheses
or converses of theorems. The summary table at the start of this appendix should be especially useful.

I require my students to read each Section prior to the day’s discussion on that section. For some students this is a novel idea, but at the end of the semester a few always report on the benefits, both for this course and other courses where they have adopted the habit. To make good on this requirement, each section contains three Reading Questions. These sometimes only require parroting back a key definition or theorem, or they require performing a small example of a key computation, or they ask for musings on key ideas or new relationships between old ideas. Answers are emailed to me the evening before the lecture. Given the flavor and purpose of these questions, including solutions seems foolish.

Formulating interesting and effective exercises is as difficult, or more so, than building a narrative. But it is the place where a student really learns the material. As such, for the student’s benefit, complete solutions should be given. As the list of exercises expands, over time solutions will also be provided. Exercises and their solutions are referenced with a section name, followed by a dot, then a letter (C, M, or T) and a number. The letter ‘C’ indicates a problem that is mostly computational in nature, while the letter ‘T’ indicates a problem that is more theoretical in nature. A problem with a letter ‘M’ is somewhere in between (middle, mid-level, median, middling), probably a mix of computation and applications of theorems. So Solution MO.T13 [172] is a solution to an exercise in Section MO [163] that is theoretical in nature. The number ‘13’ has no intrinsic meaning.

More on Freedom  This book is freely-distributable under the terms of the GFDL, along with the underlying T\TeX code from which the book is built. This arrangement provides many benefits unavailable with traditional texts.

• No cost, or low cost, to students. With no physical vessel (i.e. paper, binding), no transportation costs (Internet bandwidth being a negligible cost) and no marketing costs (evaluation and desk copies are free to all), anyone with an Internet connection can obtain it, and a teacher could make available paper copies in sufficient quantities for a class. The cost to print a copy is not insignificant, but is just a fraction of the cost of a traditional textbook when printing is handled by a print-on-demand service over the Internet. Students will not feel the need to sell back their book (nor should there be much of a market for used copies), and in future years can even pick up a newer edition freely.

• The book will not go out of print. No matter what, a teacher can maintain their own copy and use the book for as many years as they desire. Further, the naming schemes for chapters, sections, theorems, etc. is designed so that the addition of new material will not break any course syllabi or assignment list.

• With many eyes reading the book and with frequent postings of updates, the reliability should become very high. Please report any errors you find that persist into the latest version.

• For those with a working installation of the popular typesetting program T\TeX, the book has been designed so that it can be customized. Page layouts, presence of exercises, solutions, sections or chapters can all be easily controlled. Furthermore, many variants of mathematical notation are achieved via T\TeX macros. So by changing a single macro, one’s favorite notation can be reflected throughout the text. For example, every transpose of a matrix is coded in the source as $\texttt{\textbackslash transpose(A)}$, which when printed will yield $A^t$. However by changing the definition of $\texttt{\textbackslash transpose\{ A\}}$, any desired alternative notation will then appear throughout the text instead.

• The book has also been designed to make it easy for others to contribute material. Would you like to see a section on symmetric bilinear forms? Consider writing one and contributing it to one of the Topics chapters. Does there need to be more exercises about the null space of a matrix? Send me some. Historical Notes? Contact me, and we will see about adding those in also.
• You have no legal obligation to pay for this book. It has been licensed with no expectation that you pay for it. You do not even have a moral obligation to pay for the book. Thomas Jefferson (1743 – 1826), the author of the United States Declaration of Independence, wrote,

If nature has made any one thing less susceptible than all others of exclusive property, it is the action of the thinking power called an idea, which an individual may exclusively possess as long as he keeps it to himself; but the moment it is divulged, it forces itself into the possession of every one, and the receiver cannot dispossess himself of it. Its peculiar character, too, is that no one possesses the less, because every other possesses the whole of it. He who receives an idea from me, receives instruction himself without lessening mine; as he who lights his taper at mine, receives light without darkening me. That ideas should freely spread from one to another over the globe, for the moral and mutual instruction of man, and improvement of his condition, seems to have been peculiarly and benevolently designed by nature, when she made them, like fire, expansible over all space, without lessening their density in any point, and like the air in which we breathe, move, and have our physical being, incapable of confinement or exclusive appropriation.

Letter to Isaac McPherson
August 13, 1813

However, if you feel a royalty is due the author, or if you would like to encourage the author, or if you wish to show others that this approach to textbook publishing can also bring financial compensation, then donations are gratefully received. Moreover, non-financial forms of help can often be even more valuable. A simple note of encouragement, submitting a report of an error, or contributing some exercises or perhaps an entire section for the Topics or Applications are all important ways you can acknowledge the freedoms accorded to this work by the copyright holder and other contributors.

Conclusion  Foremost, I hope that students find their time spent with this book profitable. I hope that instructors find it flexible enough to fit the needs of their course. And I hope that everyone will send me their comments and suggestions, and also consider the myriad ways they can help (as listed on the book’s website at [http://linear.ups.edu]).

Robert A. Beezer
Tacoma, Washington
December 2006
Acknowledgements

Many people have helped to make this book, and its freedoms, possible.

First, the time to create, edit and distribute the book has been provided implicitly and explicitly by the University of Puget Sound. A sabbatical leave Spring 2004 and a course release in Spring 2007 are two obvious examples of explicit support. The latter was provided by support from the Lind-VanEnkevort Fund. The university has also provided clerical support, computer hardware, network servers and bandwidth. Thanks to Dean Kris Bartanen and the chair of the Mathematics and Computer Science Department, Professor Martin Jackson, for their support, encouragement and flexibility.

My colleagues in the Mathematics and Computer Science Department have graciously taught our introductory linear algebra course using preliminary versions and have provided valuable suggestions that have improved the book immeasurably. Thanks to Professor Martin Jackson (v0.30), Professor David Scott (v0.70) and Professor Bryan Smith (v0.70, 0.80, v1.00).

University of Puget Sound librarians Lori Ricigliano, Elizabeth Knight and Jeanne Kimura provided valuable advice on production, and interesting conversations about copyrights.

Many aspects of the book have been influenced by insightful questions and creative suggestions from the students who have labored through the book in our courses. For example, the flashcards with theorems and definitions are a direct result of a student suggestion. I will single out a handful of students have been especially adept at finding and reporting mathematically significant typographical errors: Jake Linenthal, Christie Su, Kim Le, Sarah McQuate, Andy Zimmer, Travis Osborne, Andrew Tapay, Mark Shoemaker, Tasha Underhill, and Tim Zitzer.

I have tried to be as original as possible in the organization and presentation of this beautiful subject. However, I have been influenced by many years of teaching from another excellent textbook, Introduction to Linear Algebra by L.W. Johnson, R.D. Reiss and J.T. Arnold. When I have needed inspiration for the correct approach to particularly important proofs, I have learned to eventually consult two other textbooks. Sheldon Axler’s Linear Algebra Done Right is a highly original exposition, while Ben Noble’s Applied Linear Algebra frequently strikes just the right note between rigor and intuition. Noble’s excellent book is highly recommended, even though its publication dates to 1969.

Finally, in every possible case, the production and distribution of this book has been accomplished with open-source software. The range of individuals and projects is far too great to pretend to list them all. The book’s web site will maintain pointers to as many of these projects as possible.
Part C
Core
Chapter SLE
Systems of Linear Equations

We will motivate our study of linear algebra by studying solutions to systems of linear equations. While the focus of this chapter is on the practical matter of how to find, and describe, these solutions, we will also be setting ourselves up for more theoretical ideas that will appear later.

Section WILA
What is Linear Algebra?

Subsection LA
“Linear” + “Algebra”

The subject of linear algebra can be partially explained by the meaning of the two terms comprising the title. “Linear” is a term you will appreciate better at the end of this course, and indeed, attaining this appreciation could be taken as one of the primary goals of this course. However for now, you can understand it to mean anything that is “straight” or “flat.” For example in the $xy$-plane you might be accustomed to describing straight lines (is there any other kind?) as the set of solutions to an equation of the form $y = mx + b$, where the slope $m$ and the $y$-intercept $b$ are constants that together describe the line. In multivariate calculus, you may have discussed planes. Living in three dimensions, with coordinates described by triples $(x, y, z)$, they can be described as the set of solutions to equations of the form $ax + by + cz = d$, where $a, b, c, d$ are constants that together determine the plane. While we might describe planes as “flat,” lines in three dimensions might be described as “straight.” From a multivariate calculus course you will recall that lines are sets of points described by equations such as $x = 3t - 4$, $y = -7t + 2$, $z = 9t$, where $t$ is a parameter that can take on any value.

Another view of this notion of “flatness” is to recognize that the sets of points just described are solutions to equations of a relatively simple form. These equations involve addition and multiplication only. We will have a need for subtraction, and occasionally we will divide, but mostly you can describe “linear” equations as involving only addition and multiplication. Here are some examples of typical equations we will see in the next few sections:

\[
2x + 3y - 4z = 13 \quad 4x_1 + 5x_2 - x_3 + x_4 + x_5 = 0 \quad 9a - 2b + 7c + 2d = -7
\]

What we will not see are equations like:

\[
xy + 5yz = 13 \quad x_1 + x_2^2/x_4 - x_3x_4x_5^2 = 0 \quad \tan(ab) + \log(c - d) = -7
\]

The exception will be that we will on occasion need to take a square root.

You have probably heard the word “algebra” frequently in your mathematical preparation for this course. Most likely, you have spent a good ten to fifteen years learning the algebra of the
real numbers, along with some introduction to the very similar algebra of complex numbers (see Section CNO \[611\]). However, there are many new algebras to learn and use, and likely linear algebra will be your second algebra. Like learning a second language, the necessary adjustments can be challenging at times, but the rewards are many. And it will make learning your third and fourth algebras even easier. Perhaps you have heard of “groups” and “rings” (or maybe you have studied them already), which are excellent examples of other algebras with very interesting properties and applications. In any event, prepare yourself to learn a new algebra and realize that some of the old rules you used for the real numbers may no longer apply to this new algebra you will be learning!

The brief discussion above about lines and planes suggests that linear algebra has an inherently geometric nature, and this is true. Examples in two and three dimensions can be used to provide valuable insight into important concepts of this course. However, much of the power of linear algebra will be the ability to work with “flat” or “straight” objects in higher dimensions, without concerning ourselves with visualizing the situation. While much of our intuition will come from examples in two and three dimensions, we will maintain an algebraic approach to the subject, with the geometry being secondary. Others may wish to switch this emphasis around, and that can lead to a very fruitful and beneficial course, but here and now we are laying our bias bare.

**Subsection AA**

**An Application**

We conclude this section with a rather involved example that will highlight some of the power and techniques of linear algebra. Work through all of the details with pencil and paper, until you believe all the assertions made. However, in this introductory example, do not concern yourself with how some of the results are obtained or how you might be expected to solve a similar problem. We will come back to this example later and expose some of the techniques used and properties exploited. For now, use your background in mathematics to convince yourself that everything said here really is correct.

**Example TMP**

**Trail Mix Packaging**

Suppose you are the production manager at a food-packaging plant and one of your product lines is trail mix, a healthy snack popular with hikers and backpackers, containing raisins, peanuts and hard-shelled chocolate pieces. By adjusting the mix of these three ingredients, you are able to sell three varieties of this item. The fancy version is sold in half-kilogram packages at outdoor supply stores and has more chocolate and fewer raisins, thus commanding a higher price. The standard version is sold in one kilogram packages in grocery stores and gas station mini-markets. Since the standard version has roughly equal amounts of each ingredient, it is not as expensive as the fancy version. Finally, a bulk version is sold in bins at grocery stores for consumers to load into plastic bags in amounts of their choosing. To appeal to the shoppers that like bulk items for their economy and healthfulness, this mix has many more raisins (at the expense of chocolate) and therefore sells for less.

Your production facilities have limited storage space and early each morning you are able to receive and store 380 kilograms of raisins, 500 kilograms of peanuts and 620 kilograms of chocolate pieces. As production manager, one of your most important duties is to decide how much of each version of trail mix to make every day. Clearly, you can have up to 1500 kilograms of raw ingredients available each day, so to be the most productive you will likely produce 1500 kilograms of trail mix each day. Also, you would prefer not to have any ingredients leftover each day, so that your final product is as fresh as possible and so that you can receive the maximum delivery the next morning. But how should these ingredients be allocated to the mixing of the bulk, standard and fancy versions?

First, we need a little more information about the mixes. Workers mix the ingredients in 15 kilogram batches, and each row of the table below gives a recipe for a 15 kilogram batch. There
is some additional information on the costs of the ingredients and the price the manufacturer can charge for the different versions of the trail mix.

<table>
<thead>
<tr>
<th></th>
<th>Raisins (kg/batch)</th>
<th>Peanuts (kg/batch)</th>
<th>Chocolate (kg/batch)</th>
<th>Cost ($/kg)</th>
<th>Sale Price ($/kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bulk</td>
<td>7</td>
<td>6</td>
<td>2</td>
<td>3.69</td>
<td>4.99</td>
</tr>
<tr>
<td>Standard</td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>3.86</td>
<td>5.50</td>
</tr>
<tr>
<td>Fancy</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>4.45</td>
<td>6.50</td>
</tr>
<tr>
<td>Storage (kg)</td>
<td>380</td>
<td>500</td>
<td>620</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cost ($/kg)</td>
<td>2.55</td>
<td>4.65</td>
<td>4.80</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As production manager, it is important to realize that you only have three decisions to make — the amount of bulk mix to make, the amount of standard mix to make and the amount of fancy mix to make. Everything else is beyond your control or is handled by another department within the company. Principally, you are also limited by the amount of raw ingredients you can store each day. Let us denote the amount of each mix to produce each day, measured in kilograms, by the variable quantities \( b, s \) and \( f \). Your production schedule can be described as values of \( b, s \) and \( f \) that do several things. First, we cannot make negative quantities of each mix, so

\[
\begin{align*}
    b & \geq 0 \\
    s & \geq 0 \\
    f & \geq 0
\end{align*}
\]

Second, if we want to consume all of our ingredients each day, the storage capacities lead to three (linear) equations, one for each ingredient,

\[
\begin{align*}
    \frac{7}{15}b + \frac{6}{15}s + \frac{2}{15}f &= 380 & \text{(raisins)} \\
    \frac{6}{15}b + \frac{4}{15}s + \frac{5}{15}f &= 500 & \text{(peanuts)} \\
    \frac{2}{15}b + \frac{5}{15}s + \frac{8}{15}f &= 620 & \text{(chocolate)}
\end{align*}
\]

It happens that this system of three equations has just one solution. In other words, as production manager, your job is easy, since there is but one way to use up all of your raw ingredients making trail mix. This single solution is

\[
\begin{align*}
    b &= 300 \text{ kg} \\
    s &= 300 \text{ kg} \\
    f &= 900 \text{ kg}
\end{align*}
\]

We do not yet have the tools to explain why this solution is the only one, but it should be simple for you to verify that this is indeed a solution. (Go ahead, we will wait.) Determining solutions such as this, and establishing that they are unique, will be the main motivation for our initial study of linear algebra.

So we have solved the problem of making sure that we make the best use of our limited storage space, and each day use up all of the raw ingredients that are shipped to us. Additionally, as production manager, you must report weekly to the CEO of the company, and you know he will be more interested in the profit derived from your decisions than in the actual production levels. So you compute,

\[
300(4.99 - 3.69) + 300(5.50 - 3.86) + 900(6.50 - 4.45) = 2727.00
\]

for a daily profit of $2,727 from this production schedule. The computation of the daily profit is also beyond our control, though it is definitely of interest, and it too looks like a “linear” computation.

As often happens, things do not stay the same for long, and now the marketing department has suggested that your company’s trail mix products standardize on every mix being one-third peanuts. Adjusting the peanut portion of each recipe by also adjusting the chocolate portion, leads to revised recipes, and slightly different costs for the bulk and standard mixes, as given in the following table.
### Subsection WILA.AA An Application

<table>
<thead>
<tr>
<th></th>
<th>Raisins (kg/batch)</th>
<th>Peanuts (kg/batch)</th>
<th>Chocolate (kg/batch)</th>
<th>Cost ($/kg)</th>
<th>Sale Price ($/kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bulk</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>3.70</td>
<td>4.99</td>
</tr>
<tr>
<td>Standard</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3.85</td>
<td>5.50</td>
</tr>
<tr>
<td>Fancy</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>4.45</td>
<td>6.50</td>
</tr>
<tr>
<td>Storage (kg)</td>
<td>380</td>
<td>500</td>
<td>620</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cost ($/kg)</td>
<td>2.55</td>
<td>4.65</td>
<td>4.80</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In a similar fashion as before, we desire values of $b$, $s$ and $f$ so that

$$b \geq 0 \quad s \geq 0 \quad f \geq 0$$

and

$$\frac{7}{15}b + \frac{6}{15}s + \frac{2}{15}f = 380 \quad \text{(raisins)}$$

$$\frac{5}{15}b + \frac{5}{15}s + \frac{5}{15}f = 500 \quad \text{(peanuts)}$$

$$\frac{3}{15}b + \frac{4}{15}s + \frac{8}{15}f = 620 \quad \text{(chocolate)}$$

It now happens that this system of equations has infinitely many solutions, as we will now demonstrate. Let $f$ remain a variable quantity. Then if we make $f$ kilograms of the fancy mix, we will make $4f - 3300$ kilograms of the bulk mix and $-5f + 4800$ kilograms of the standard mix. Let us now verify that, for any choice of $f$, the values of $b = 4f - 3300$ and $s = -5f + 4800$ will yield a production schedule that exhausts all of the day’s supply of raw ingredients (right now, do not be concerned about how you might derive expressions like these for $b$ and $s$). Grab your pencil and paper and play along.

$$\frac{7}{15}(4f - 3300) + \frac{6}{15}(-5f + 4800) + \frac{2}{15}f = 0f + \frac{5700}{15} = 380$$

$$\frac{5}{15}(4f - 3300) + \frac{5}{15}(-5f + 4800) + \frac{5}{15}f = 0f + \frac{7500}{15} = 500$$

$$\frac{3}{15}(4f - 3300) + \frac{4}{15}(-5f + 4800) + \frac{8}{15}f = 0f + \frac{9300}{15} = 620$$

Convince yourself that these expressions for $b$ and $s$ allow us to vary $f$ and obtain an infinite number of possibilities for solutions to the three equations that describe our storage capacities. As a practical matter, there really are not an infinite number of solutions, since we are unlikely to want to end the day with a fractional number of bags of fancy mix, so our allowable values of $f$ should probably be integers. More importantly, we need to remember that we cannot make negative amounts of each mix! Where does this lead us? Positive quantities of the bulk mix requires that

$$b \geq 0 \quad \Rightarrow \quad 4f - 3300 \geq 0 \quad \Rightarrow \quad f \geq 825$$

Similarly for the standard mix,

$$s \geq 0 \quad \Rightarrow \quad -5f + 4800 \geq 0 \quad \Rightarrow \quad f \leq 960$$

So, as production manager, you really have to choose a value of $f$ from the finite set

$$\{825, 826, \ldots, 960\}$$

leaving you with 136 choices, each of which will exhaust the day’s supply of raw ingredients. Pause now and think about which you would choose.

Recalling your weekly meeting with the CEO suggests that you might want to choose a production schedule that yields the biggest possible profit for the company. So you compute an expression for the profit based on your as yet undetermined decision for the value of $f$,

$$(4f - 3300)(4.99 - 3.70) + (-5f + 4800)(5.50 - 3.85) + (6.50 - 4.45) = -1.04f + 3663$$
Since \( f \) has a negative coefficient it would appear that mixing fancy mix is detrimental to your profit and should be avoided. So you will make the decision to set daily fancy mix production at \( f = 825 \). This has the effect of setting \( b = 4(825) - 3300 = 0 \) and we stop producing bulk mix entirely. So the remainder of your daily production is standard mix at the level of \( s = -5(825) + 4800 = 675 \) kilograms and the resulting daily profit is \((-1.04)(825) + 3663 = 2805\). It is a pleasant surprise that daily profit has risen to $2,805, but this is not the most important part of the story. What is important here is that there are a large number of ways to produce trail mix that use all of the day’s worth of raw ingredients and you were able to easily choose the one that netted the largest profit. Notice too how all of the above computations look “linear.”

In the food industry, things do not stay the same for long, and now the sales department says that increased competition has led to the decision to stay competitive and charge just $5.25 for a kilogram of the standard mix, rather than the previous $5.50 per kilogram. This decision has no effect on the possibilities for the production schedule, but will affect the decision based on profit considerations. So you revisit just the profit computation, suitably adjusted for the new selling price of standard mix,

\[
(4f - 3300)(4.99 - 3.70) + (-5f + 4800)(5.25 - 3.85) + (f)(6.50 - 4.45) = 0.21f + 2463
\]

Now it would appear that fancy mix is beneficial to the company’s profit since the value of \( f \) has a positive coefficient. So you take the decision to make as much fancy mix as possible, setting \( f = 960 \). This leads to \( s = -5(960) + 4800 = 0 \) and the increased competition has driven you out of the standard mix market all together. The remainder of production is therefore bulk mix at a daily level of \( b = 4(960) - 3300 = 540 \) kilograms and the resulting daily profit is \( 0.21(960) + 2463 = 2664.60 \). A daily profit of $2,664.60 is less than it used to be, but as production manager, you have made the best of a difficult situation and shown the sales department that the best course is to pull out of the highly competitive standard mix market completely.

This example is taken from a field of mathematics variously known by names such as operations research, systems science, or management science. More specifically, this is a perfect example of problems that are solved by the techniques of “linear programming.”

There is a lot going on under the hood in this example. The heart of the matter is the solution to systems of linear equations, which is the topic of the next few sections, and a recurrent theme throughout this course. We will return to this example on several occasions to reveal some of the reasons for its behavior.

Subsection READ
Reading Questions

1. Is the equation \( x^2 + xy + \tan(y^3) = 0 \) linear or not? Why or why not?

2. Find all solutions to the system of two linear equations \( 2x + 3y = -8, \ x - y = 6 \).

3. Explain the importance of the procedures described in the trail mix application (Subsection [WILA.AA] [3]) from the point-of-view of the production manager.
C10  In Example TMP 3 the first table lists the cost (per kilogram) to manufacture each of the three varieties of trail mix (bulk, standard, fancy). For example, it costs $3.70 to make one kilogram of the bulk variety. Re-compute each of these three costs and notice that the computations are linear in character.
Contributed by Robert Beezer

M70  In Example TMP 3 two different prices were considered for marketing standard mix with the revised recipes (one-third peanuts in each recipe). Selling standard mix at $5.50 resulted in selling the minimum amount of the fancy mix and no bulk mix. At $5.25 it was best for profits to sell the maximum amount of fancy mix and then sell no standard mix. Determine a selling price for standard mix that allows for maximum profits while still selling some of each type of mix.
Contributed by Robert Beezer  Solution 8
If the price of standard mix is set at $5.292, then the profit function has a zero coefficient on the variable quantity $f$. So, we can set $f$ to be any integer quantity in $\{825, 826, \ldots, 960\}$. All but the extreme values ($f = 825, f = 960$) will result in production levels where some of every mix is manufactured. No matter what value of $f$ is chosen, the resulting profit will be the same, at $2,664.60$. 
Section SSLE
Solving Systems of Linear Equations

We will motivate our study of linear algebra by considering the problem of solving several linear equations simultaneously. The word “solve” tends to get abused somewhat, as in “solve this problem.” When talking about equations we understand a more precise meaning: find all of the values of some variable quantities that make an equation, or several equations, true.

Subsection SLE
Systems of Linear Equations

Example STNE
Solving two (nonlinear) equations
Suppose we desire the simultaneous solutions of the two equations,

\[
\begin{align*}
  x^2 + y^2 &= 1 \\
  -x + \sqrt{3}y &= 0
\end{align*}
\]

You can easily check by substitution that \( x = \frac{\sqrt{3}}{2}, \ y = \frac{1}{2} \) and \( x = -\frac{\sqrt{3}}{2}, \ y = -\frac{1}{2} \) are both solutions. We need to also convince ourselves that these are the only solutions. To see this, plot each equation on the \( xy \)-plane, which means to plot \((x, y)\) pairs that make an individual equation true. In this case we get a circle centered at the origin with radius 1 and a straight line through the origin with slope \( \frac{1}{\sqrt{3}} \). The intersections of these two curves are our desired simultaneous solutions, and so we believe from our plot that the two solutions we know already are indeed the only ones. We like to write solutions as sets, so in this case we write the set of solutions as

\[
S = \{ (\frac{\sqrt{3}}{2}, \frac{1}{2}), (-\frac{\sqrt{3}}{2}, -\frac{1}{2}) \}
\]

In order to discuss systems of linear equations carefully, we need a precise definition. And before we do that, we will introduce our periodic discussions about “Proof Techniques.” Linear algebra is an excellent setting for learning how to read, understand and formulate proofs. But this is a difficult step in your development as a mathematician, so we have included a series of vignettes containing advice and explanations to help you along. These can be found back in Section PT of Appendix P, and we will reference them as they become appropriate. Be sure to head back to the appendix to read this as they are introduced. With a definition next, now is the time for the first of our proof techniques. Head back to Section PT of Appendix P and study Technique D. We’ll be right here when you get back. See you in a bit.

Definition SLE
System of Linear Equations
A system of linear equations is a collection of \( m \) equations in the variable quantities \( x_1, \ x_2, \ x_3, \ldots, x_n \) of the form,

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
  a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
  &\vdots
\end{align*}
\]
where the values of $a_{ij}$, $b_i$ and $x_j$ are from the set of complex numbers, $\mathbb{C}$. $\triangle$

Don’t let the mention of the complex numbers, $\mathbb{C}$, rattle you. We will stick with real numbers exclusively for many more sections, and it will sometimes seem like we only work with integers! However, we want to leave the possibility of complex numbers open, and there will be occasions in subsequent sections where they are necessary. You can review the basic properties of complex numbers in Section CNO [611], but these facts will not be critical until we reach Section O [151]. For now, here is an example to illustrate using the notation introduced in Definition SLE [9].

Example NSE
Notation for a system of equations
Given the system of linear equations,
\begin{align*}
x_1 + 2x_2 + x_4 &= 7 \\
x_1 + x_2 + x_3 - x_4 &= 3 \\
3x_1 + x_2 + 5x_3 - 7x_4 &= 1
\end{align*}
we have $n = 4$ variables and $m = 3$ equations. Also,
\begin{align*}
a_{11} &= 1 & a_{12} &= 2 & a_{13} &= 0 & a_{14} &= 1 & b_1 &= 7 \\
a_{21} &= 1 & a_{22} &= 1 & a_{23} &= 1 & a_{24} &= -1 & b_2 &= 3 \\
a_{31} &= 3 & a_{32} &= 1 & a_{33} &= 5 & a_{34} &= -7 & b_3 &= 1
\end{align*}
Additionally, convince yourself that $x_1 = -2$, $x_2 = 4$, $x_3 = 2$, $x_4 = 1$ is one solution (but it is not the only one!). $\triangledown$

We will often shorten the term “system of linear equations” to “system of equations” leaving the linear aspect implied.

Subsection PSS
Possibilities for Solution Sets

The next example illustrates the possibilities for the solution set of a system of linear equations. We will not be too formal here, and the necessary theorems to back up our claims will come in subsequent sections. So read for feeling and come back later to revisit this example.

Example TTS
Three typical systems
Consider the system of two equations with two variables,
\begin{align*}
2x_1 + 3x_2 &= 3 \\
x_1 - x_2 &= 4
\end{align*}
If we plot the solutions to each of these equations separately on the $x_1x_2$-plane, we get two lines, one with negative slope, the other with positive slope. They have exactly one point in common, $(x_1, x_2) = (3, -1)$, which is the solution $x_1 = 3$, $x_2 = -1$. From the geometry, we believe that this is the only solution to the system of equations, and so we say it is unique.

Now adjust the system with a different second equation,
\begin{align*}
2x_1 + 3x_2 &= 3 \\
4x_1 + 6x_2 &= 6
\end{align*}
A plot of the solutions to these equations individually results in two lines, one on top of the other! There are infinitely many pairs of points that make both equations true. We will learn shortly
One more minor adjustment provides a third system of linear equations,

\[
\begin{align*}
2x_1 + 3x_2 &= 3 \\
4x_1 + 6x_2 &= 10
\end{align*}
\]

A plot now reveals two lines with identical slopes, i.e. parallel lines. They have no points in common, and so the system has a solution set that is empty, \( S = \emptyset \).

With all this talk about finding solution sets for systems of linear equations, you might be ready to begin learning how to find these solution sets yourself. We begin with our first definition that takes a common word and gives it a very precise meaning in the context of systems of linear equations.

**Definition ESYS Equivalent Systems**

Two systems of linear equations are **equivalent** if their solution sets are equal.

Notice here that the two systems of equations could *look* very different (i.e. not be equal), but still have equal solution sets, and we would then call the systems equivalent. Two linear equations in two variables might be plotted as two lines that intersect in a single point. A different system, with three equations in two variables might have a plot that is three lines, all intersecting at a common point, with this common point identical to the intersection point for the first system. By our definition, we could then say these two very different looking systems of equations are equivalent, since they have identical solution sets. It is really like a weaker form of equality, where we allow the systems to be different in some respects, but we use the term equivalent to highlight the situation when their solution sets are equal.

With this definition, we can begin to describe our strategy for solving linear systems. Given a system of linear equations that looks difficult to solve, we would like to have an equivalent system that is easy to solve. Since the systems will have equal solution sets, we can solve the “easy” system and get the solution set to the “difficult” system. Here come the tools for making this strategy viable.

**Definition EO Equation Operations**

Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an **equation operation**.

1. Swap the locations of two equations in the list of equations.
2. Multiply each term of an equation by a nonzero quantity.
3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.
These descriptions might seem a bit vague, but the proof or the examples that follow should make it clear what is meant by each. We will shortly prove a key theorem about equation operations and solutions to linear systems of equations. We are about to give a rather involved proof, so a discussion about just what a theorem really is would be timely. Head back and read Technique T 620. In the theorem we are about to prove, the conclusion is that two systems are equivalent. By Definition ESYS 11 this translates to requiring that solution sets be equal for the two systems. So we are being asked to show that two sets are equal. How do we do this? Well, there is a very standard technique, and we will use it repeatedly through the course. If you have not done so already, head to Section SET 615 and familiarize yourself with sets, their operations, and especially the notion of set equality, Definition SE 616 and the nearby discussion about its use.

**Theorem EOPSS**

**Equation Operations Preserve Solution Sets**

If we apply one of the three equation operations of Definition EO 11 to a system of linear equations (Definition SLE 9), then the original system and the transformed system are equivalent. □

**Proof** We take each equation operation in turn and show that the solution sets of the two systems are equal, using the definition of set equality (Definition SE 616).

1. It will not be our habit in proofs to resort to saying statements are “obvious,” but in this case, it should be. There is nothing about the order in which we write linear equations that affects their solutions, so the solution set will be equal if the systems only differ by a rearrangement of the order of the equations.

2. Suppose \( \alpha \neq 0 \) is a number. Let’s choose to multiply the terms of equation \( i \) by \( \alpha \) to build the new system of equations,

\[
\begin{align*}
& a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\
& a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\
& a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\
& \quad \vdots \\
& \alpha a_{i1}x_1 + \alpha a_{i2}x_2 + \alpha a_{i3}x_3 + \cdots + \alpha a_{in}x_n = \alpha b_i \\
& \quad \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m
\end{align*}
\]

Let \( S \) denote the solutions to the system in the statement of the theorem, and let \( T \) denote the solutions to the transformed system.

(a) Show \( S \subseteq T \). Suppose \((x_1, x_2, x_3, \ldots, x_n) = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in S \) is a solution to the original system. Ignoring the \( i \)-th equation for a moment, we know it makes all the other equations of the transformed system true. We also know that

\[
a_{i1}\beta_1 + a_{i2}\beta_2 + a_{i3}\beta_3 + \cdots + a_{in}\beta_n = b_i
\]

which we can multiply by \( \alpha \) to get

\[
\alpha a_{i1}\beta_1 + \alpha a_{i2}\beta_2 + \alpha a_{i3}\beta_3 + \cdots + \alpha a_{in}\beta_n = \alpha b_i
\]

This says that the \( i \)-th equation of the transformed system is also true, so we have established that \((\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in T \), and therefore \( S \subseteq T \).
3. Suppose $(x_1, x_2, x_3, \ldots, x_n) = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in T$ is a solution to the transformed system. Ignoring the $i$-th equation for a moment, we know it makes all the other equations of the original system true. We also know that

$$a_{i1}\beta_1 + a_{i2}\beta_2 + a_{i3}\beta_3 + \cdots + a_{in}\beta_n = \alpha b_i$$

which we can multiply by $\frac{1}{\alpha}$, since $\alpha \neq 0$, to get

$$a_{i1}\beta_1 + a_{i2}\beta_2 + a_{i3}\beta_3 + \cdots + a_{in}\beta_n = b_i$$

This says that the $i$-th equation of the original system is also true, so we have established that $(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in S$, and therefore $T \subseteq S$. Locate the key point where we required that $\alpha \neq 0$, and consider what would happen if $\alpha = 0$.

3. Suppose $\alpha$ is a number. Let’s choose to multiply the terms of equation $i$ by $\alpha$ and add them to equation $j$ in order to build the new system of equations,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Let $S$ denote the solutions to the system in the statement of the theorem, and let $T$ denote the solutions to the transformed system.

(a) Show $S \subseteq T$. Suppose $(x_1, x_2, x_3, \ldots, x_n) = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in S$ is a solution to the original system. Ignoring the $j$-th equation for a moment, we know this solution makes all the other equations of the transformed system true. Using the fact that the solution makes the $i$-th and $j$-th equations of the original system true, we find

$$(\alpha a_{i1} + a_{j1})\beta_1 + (\alpha a_{i2} + a_{j2})\beta_2 + \cdots + (\alpha a_{in} + a_{jn})\beta_n$$

$$= (\alpha a_{i1}\beta_1 + \alpha a_{i2}\beta_2 + \cdots + \alpha a_{in}\beta_n) + (a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n)$$

$$= \alpha(\beta_1) + a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n + (a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n)$$

$$= \alpha b_i + b_j.$$ 

This says that the $j$-th equation of the transformed system is also true, so we have established that $(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in T$, and therefore $S \subseteq T$.

(b) Now show $T \subseteq S$. Suppose $(x_1, x_2, x_3, \ldots, x_n) = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in T$ is a solution to the transformed system. Ignoring the $j$-th equation for a moment, we know it makes all the other equations of the original system true. We then find

$$a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n$$

$$= a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n + \alpha b_i - \alpha b_i$$

$$= a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n + (\alpha a_{i1}\beta_1 + \alpha a_{i2}\beta_2 + \cdots + \alpha a_{in}\beta_n) - \alpha b_i$$

$$= a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n + \alpha a_{i1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n + \alpha a_{in}\beta_n - \alpha b_i$$

$$= (\alpha a_{i1} + a_{j1})\beta_1 + (\alpha a_{i2} + a_{j2})\beta_2 + \cdots + (\alpha a_{in} + a_{jn})\beta_n - \alpha b_i$$

$$= \alpha b_i + b_j - \alpha b_i$$

$$= b_j$$

This says that the $j$-th equation of the original system is also true, so we have established that $(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in S$, and therefore $T \subseteq S$. 

Version 1.04
Why didn’t we need to require that $\alpha \neq 0$ for this row operation? In other words, how does the third statement of the theorem read when $\alpha = 0$? Does our proof require some extra care when $\alpha = 0$? Compare your answers with the similar situation for the second row operation. (See Exercise SSLE.T20\[18\].)

Theorem EOPSS\[12\] is the necessary tool to complete our strategy for solving systems of equations. We will use equation operations to move from one system to another, all the while keeping the solution set the same. With the right sequence of operations, we will arrive at a simpler equation to solve. The next two examples illustrate this idea, while saving some of the details for later.

Example US
Three equations, one solution
We solve the following system by a sequence of equation operations.

\[\begin{align*}
x_1 + 2x_2 + 2x_3 &= 4 \\
x_1 + 3x_2 + 3x_3 &= 5 \\
2x_1 + 6x_2 + 5x_3 &= 6
\end{align*}\]

$\alpha = -1$ times equation 1, add to equation 2:

\[\begin{align*}
x_1 + 2x_2 + 2x_3 &= 4 \\
0x_1 + 1x_2 + 1x_3 &= 1 \\
2x_1 + 6x_2 + 5x_3 &= 6
\end{align*}\]

$\alpha = -2$ times equation 1, add to equation 3:

\[\begin{align*}
x_1 + 2x_2 + 2x_3 &= 4 \\
0x_1 + 1x_2 + 1x_3 &= 1 \\
0x_1 + 2x_2 + 1x_3 &= -2
\end{align*}\]

$\alpha = -2$ times equation 2, add to equation 3:

\[\begin{align*}
x_1 + 2x_2 + 2x_3 &= 4 \\
0x_1 + 1x_2 + 1x_3 &= 1 \\
0x_1 + 0x_2 - 1x_3 &= -4
\end{align*}\]

$\alpha = -1$ times equation 3:

\[\begin{align*}
x_1 + 2x_2 + 2x_3 &= 4 \\
0x_1 + 1x_2 + 1x_3 &= 1 \\
0x_1 + 0x_2 + 1x_3 &= 4
\end{align*}\]

which can be written more clearly as

\[\begin{align*}
x_1 + 2x_2 + 2x_3 &= 4 \\
x_2 + x_3 &= 1 \\
x_3 &= 4
\end{align*}\]

This is now a very easy system of equations to solve. The third equation requires that $x_3 = 4$ to be true. Making this substitution into equation 2 we arrive at $x_2 = -3$, and finally, substituting these values of $x_2$ and $x_3$ into the first equation, we find that $x_1 = 2$. Note too that this is the
only solution to this final system of equations, since we were forced to choose these values to make
the equations true. Since we performed equation operations on each system to obtain the next one
in the list, all of the systems listed here are all equivalent to each other by Theorem EOPSS [12].
Thus \((x_1, x_2, x_3) = (2, -3, 4)\) is the unique solution to the original system of equations (and all of
the other intermediate systems of equations listed as we transformed one into another).

\[\alpha = -1 \text{ times equation 1, add to equation 2:}\]
\[
\begin{align*}
  x_1 + 2x_2 + 0x_3 + x_4 &= 7 \\
  0x_1 - x_2 + x_3 - 2x_4 &= -4 \\
  3x_1 + x_2 + 5x_3 - 7x_4 &= 1
\end{align*}
\]
\[\alpha = -3 \text{ times equation 1, add to equation 3:}\]
\[
\begin{align*}
  x_1 + 2x_2 + 0x_3 + x_4 &= 7 \\
  0x_1 - x_2 + x_3 - 2x_4 &= -4 \\
  0x_1 - 5x_2 + 5x_3 - 10x_4 &= -20
\end{align*}
\]
\[\alpha = -5 \text{ times equation 2, add to equation 3:}\]
\[
\begin{align*}
  x_1 + 2x_2 + 0x_3 + x_4 &= 7 \\
  0x_1 + x_2 - x_3 + 2x_4 &= 4 \\
  0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0
\end{align*}
\]
\[\alpha = -1 \text{ times equation 2:}\]
\[
\begin{align*}
  x_1 + 2x_2 + 0x_3 + x_4 &= 7 \\
  0x_1 + x_2 - x_3 + 2x_4 &= 4 \\
  0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0
\end{align*}
\]
\[\alpha = -2 \text{ times equation 2, add to equation 1:}\]
\[
\begin{align*}
  x_1 + 0x_2 + 2x_3 - 3x_4 &= -1 \\
  0x_1 + x_2 - x_3 + 2x_4 &= 4 \\
  0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0
\end{align*}
\]
which can be written more clearly as
\[
\begin{align*}
  x_1 + 2x_3 - 3x_4 &= -1 \\
  x_2 - x_3 + 2x_4 &= 4 \\
  0 &= 0
\end{align*}
\]

Example IS

Three equations, infinitely many solutions

The following system of equations made an appearance earlier in this section (Example NSE [10]),
where we listed one of its solutions. Now, we will try to find all of the solutions to this system. Don’t
concern yourself too much about why we choose this particular sequence of equation operations,
just believe that the work we do is all correct.

\[
\begin{align*}
  x_1 + 2x_2 + 0x_3 + x_4 &= 7 \\
  x_1 + x_2 + x_3 - x_4 &= 3 \\
  3x_1 + x_2 + 5x_3 - 7x_4 &= 1
\end{align*}
\]
What does the equation $0 = 0$ mean? We can choose any values for $x_1$, $x_2$, $x_3$, $x_4$ and this equation will be true, so we only need to consider further the first two equations, since the third is true no matter what. We can analyze the second equation without consideration of the variable $x_1$. It would appear that there is considerable latitude in how we can choose $x_2$, $x_3$, $x_4$ and make this equation true. Let’s choose $x_3$ and $x_4$ to be anything we please, say $x_3 = a$ and $x_4 = b$.

Now we can take these arbitrary values for $x_3$ and $x_4$, substitute them in equation 1, to obtain

$$x_1 + 2a - 3b = -1$$
$$x_1 = -1 - 2a + 3b$$

Similarly, equation 2 becomes

$$x_2 - a + 2b = 4$$
$$x_2 = 4 + a - 2b$$

So our arbitrary choices of values for $x_3$ and $x_4$ ($a$ and $b$) translate into specific values of $x_1$ and $x_2$. The lone solution given in Example NSE [10] was obtained by choosing $a = 2$ and $b = 1$. Now we can easily and quickly find many more (infinitely more). Suppose we choose $a = 5$ and $b = -2$, then we compute

$$x_1 = -1 - 2(5) + 3(-2) = -17$$
$$x_2 = 4 + 5 - 2(-2) = 13$$

and you can verify that $(x_1, x_2, x_3, x_4) = (-17, 13, 5, -2)$ makes all three equations true. The entire solution set is written as

$$S = \{ (-1 - 2a + 3b, 4 + a - 2b, a, b) \mid a \in \mathbb{C}, b \in \mathbb{C} \}$$

It would be instructive to finish off your study of this example by taking the general form of the solutions given in this set and substituting them into each of the three equations and verify that they are true in each case (Exercise SSLE.M40 [18]).

In the next section we will describe how to use equation operations to systematically solve any system of linear equations. But first, read one of our more important pieces of advice about speaking and writing mathematics. See Technique L [620].

Before attacking the exercises in this section, it will be helpful to read some advice on getting started on the construction of a proof. See Technique GS [621].

Subsection SSLE.READ  
Reading Questions

1. How many solutions does the system of equations $3x + 2y = 4$, $6x + 4y = 8$ have? Explain your answer.

2. How many solutions does the system of equations $3x + 2y = 4$, $6x + 4y = -2$ have? Explain your answer.

3. What do we mean when we say mathematics is a language?
Subsection EXC
Exercises

C10  Find a solution to the system in Example IS [15] where $x_3 = 6$ and $x_4 = 2$. Find two other solutions to the system. Find a solution where $x_1 = -17$ and $x_2 = 14$. How many possible answers are there to each of these questions?
Contributed by Robert Beezer

C20  Each archetype (Appendix A [630]) that is a system of equations begins by listing some specific solutions. Verify the specific solutions listed in the following archetypes by evaluating the system of equations with the solutions listed.

Archetype A [634]
Archetype B [638]
Archetype C [643]
Archetype D [647]
Archetype E [651]
Archetype F [654]
Archetype G [659]
Archetype H [663]
Archetype I [667]
Archetype J [671]
Contributed by Robert Beezer

C50  A three-digit number has two properties. The tens-digit and the ones-digit add up to 5. If the number is written with the digits in the reverse order, and then subtracted from the original number, the result is 792. Use a system of equations to find all of the three-digit numbers with these properties.
Contributed by Robert Beezer  Solution [19]

M10  Each sentence below has at least two meanings. Identify the source of the double meaning, and rewrite the sentence (at least twice) to clearly convey each meaning.

1. They are baking potatoes.
2. He bought many ripe pears and apricots.
3. She likes his sculpture.
4. I decided on the bus.

Contributed by Robert Beezer  Solution [19]

M11  Discuss the difference in meaning of each of the following three almost identical sentences, which all have the same grammatical structure. (These are due to Keith Devlin.)

1. She saw him in the park with a dog.
2. She saw him in the park with a fountain.
3. She saw him in the park with a telescope.

Contributed by Robert Beezer  Solution [19]

M12  The following sentence, due to Noam Chomsky, has a correct grammatical structure, but is meaningless. Critique its faults. “Colorless green ideas sleep furiously.” (Chomsky, Noam. 1957. Syntactic Structures. The Hague/Paris: Mouton. p. 15)
Contributed by Robert Beezer  Solution [19]
M13 Read the following sentence and form a mental picture of the situation.

The baby cried and the mother picked it up.

What assumptions did you make about the situation?

Contributed by Robert Beezer Solution

M30 This problem appears in a middle-school mathematics textbook: Together Dan and Diane have $20. Together Diane and Donna have $15. How much do the three of them have in total?


Contributed by David Beezer Solution

M40 Solutions to the system in Example IS are given as

\((x_1, x_2, x_3, x_4) = (-1 - 2a + 3b, 4 + a - 2b, a, b)\)

Evaluate the three equations of the original system with these expressions in \(a\) and \(b\) and verify that each equation is true, no matter what values are chosen for \(a\) and \(b\).

Contributed by Robert Beezer

M70 We have seen in this section that systems of linear equations have limited possibilities for solution sets, and we will shortly prove Theorem PSSLS that describes these possibilities exactly. This exercise will show that if we relax the requirement that our equations be linear, then the possibilities expand greatly. Consider a system of two equations in the two variables \(x\) and \(y\), where the departure from linearity involves simply squaring the variables.

\[
\begin{align*}
x^2 - y^2 &= 1 \\
x^2 + y^2 &= 4
\end{align*}
\]

After solving this system of non-linear equations, replace the second equation in turn by \(x^2 + 2x + y^2 = 3\), \(x^2 + y^2 = 1\), \(x^2 - x + y^2 = 0\), \(4x^2 + 4y^2 = 1\) and solve each resulting system of two equations in two variables.

Contributed by Robert Beezer Solution

T10 Technique D asks you to formulate a definition of what it means for a whole number to be odd. What is your definition? (Don’t say “the opposite of even.”) Is 6 odd? Is 11 odd? Justify your answers by using your definition.

Contributed by Robert Beezer Solution

T20 Explain why the second equation operation in Definition EO requires that the scalar be nonzero, while in the third equation operation this restriction on the scalar is not present.

Contributed by Robert Beezer Solution
Subsection SOL
Solutions

C50 Contributed by Robert Beezer Statement [17]
Let $a$ be the hundreds digit, $b$ the tens digit, and $c$ the ones digit. Then the first condition says that $b + c = 5$. The original number is $100a + 10b + c$, while the reversed number is $100c + 10b + a$. So the second condition is

$$792 = (100a + 10b + c) - (100c + 10b + a) = 99a - 99c$$

So we arrive at the system of equations

$$b + c = 5$$
$$99a - 99c = 792$$

Using equation operations, we arrive at the equivalent system

$$a - c = 8$$
$$b + c = 5$$

We can vary $c$ and obtain infinitely many solutions. However, $c$ must be a digit, restricting us to ten values (0 – 9). Furthermore, if $c > 1$, then the first equation forces $a > 9$, an impossibility. Setting $c = 0$, yields 850 as a solution, and setting $c = 1$ yields 941 as another solution.

M10 Contributed by Robert Beezer Statement [17]
1. Is “baking” a verb or an adjective?
   Potatoes are being baked.
   Those are baking potatoes.

2. Are the apricots ripe, or just the pears? Parentheses could indicate just what the adjective “ripe” is meant to modify. Were there many apricots as well, or just many pears?
   He bought many pears and many ripe apricots.
   He bought apricots and many ripe pears.

3. Is “sculpture” a single physical object, or the sculptor’s style expressed over many pieces and many years?
   She likes his sculpture of the girl.
   She likes his sculptural style.

4. Was a decision made while in the bus, or was the outcome of a decision to choose the bus. Would the sentence “I decided on the car,” have a similar double meaning?
   I made my decision while on the bus.
   I decided to ride the bus.

M11 Contributed by Robert Beezer Statement [17]
We know the dog belongs to the man, and the fountain belongs to the park. It is not clear if the telescope belongs to the man, the woman, or the park.

M12 Contributed by Robert Beezer Statement [17]
In adjacent pairs the words are contradictory or inappropriate. Something cannot be both green and colorless, ideas do not have color, ideas do not sleep, and it is hard to sleep furiously.

M13 Contributed by Robert Beezer Statement [18]
Did you assume that the baby and mother are human?
Did you assume that the baby is the child of the mother?
Did you assume that the mother picked up the baby as an attempt to stop the crying?

If \( x, y \) and \( z \) represent the money held by Dan, Diane and Donna, then \( y = 15 - z \) and \( x = 20 - y = 20 - (15 - z) = 5 + z \). We can let \( z \) take on any value from 0 to 15 without any of the three amounts being negative, since presumably middle-schoolers are too young to assume debt.

Then the total capital held by the three is \( x + y + z = (5 + z) + (15 - z) + z = 20 + z \). So their combined holdings can range anywhere from \$20\) (Donna is broke) to \$35\) (Donna is flush).

We will have more to say about this situation in Section TSS, and specifically Theorem CMVEI.

The equation \( x^2 - y^2 = 1 \) has a solution set by itself that has the shape of a hyperbola when plotted. The five different second equations have solution sets that are circles when plotted individually. Where the hyperbola and circle intersect are the solutions to the system of two equations. As the size and location of the circle varies, the number of intersections varies from four to none (in the order given). Sketching the relevant equations would be instructive, as was discussed in Example STNE.

The exact solution sets are (according to the choice of the second equation),

- \( x^2 + y^2 = 4 \) : \( \left\{ \left( \frac{\sqrt{5}}{2}, \frac{\sqrt{3}}{2} \right), \left( -\frac{\sqrt{5}}{2}, \frac{\sqrt{3}}{2} \right), \left( \frac{\sqrt{5}}{2}, -\frac{\sqrt{3}}{2} \right), \left( -\frac{\sqrt{5}}{2}, -\frac{\sqrt{3}}{2} \right) \right\} \)
- \( x^2 + 2x + y^2 = 3 \) : \( \{(1,0), (-2, \sqrt{3}), (-2, -\sqrt{3})\} \)
- \( x^2 + y^2 = 1 \) : \( \{(1,0), (-1,0)\} \)
- \( x^2 - x + y^2 = 0 \) : \( \{(1,0)\} \)
- \( 4x^2 + 4y^2 = 1 \) : \( \emptyset \)

We can say that an integer is odd if when it is divided by 2 there is a remainder of 1. So 6 is not odd since \( 6 = 3 \times 2 + 0 \), while 11 is odd since \( 11 = 5 \times 2 + 1 \).

Definition EO is engineered to make Theorem EOPSS true. If we were to allow a zero scalar to multiply an equation then that equation would be transformed to the equation \( 0 = 0 \), which is true for any possible values of the variables. Any restrictions on the solution set imposed by the original equation would be lost.

However, in the third operation, it is allowed to choose a zero scalar, multiply an equation by this scalar and add the transformed equation to a second equation (leaving the first unchanged). The result? Nothing. The second equation is the same as it was before. So the theorem is true in this case, the two systems are equivalent. But in practice, this would be a silly thing to actually ever do! We still allow it though, in order to keep our theorem as general as possible.

Notice the location in the proof of Theorem EOPSS where the expression \( \frac{1}{\alpha} \) appears — this explains the prohibition on \( \alpha = 0 \) in the second equation operation.
Section RREF
Reduced Row-Echelon Form

After solving a few systems of equations, you will recognize that it doesn’t matter so much what we call our variables, as opposed to what numbers act as their coefficients. A system in the variables $x_1, x_2, x_3$ would behave the same if we changed the names of the variables to $a, b, c$ and kept all the constants the same and in the same places. In this section, we will isolate the key bits of information about a system of equations into something called a matrix, and then use this matrix to systematically solve the equations. Along the way we will obtain one of our most important and useful computational tools.

Subsection MVNSE
Matrix and Vector Notation for Systems of Equations

Definition M
Matrix
An $m \times n$ matrix is a rectangular layout of numbers from $\mathbb{C}$ having $m$ rows and $n$ columns. We will use upper-case Latin letters from the start of the alphabet (A, B, C, . . . ) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix $A$, the notation $[A]_{ij}$ will refer to the complex number in row $i$ and column $j$ of $A$.

Be careful with this notation for individual entries, since it is easy to think that $[A]_{ij}$ refers to the whole matrix. It does not. It is just a number, but is a convenient way to talk about all the entries at once. This notation will get a heavy workout once we get to Chapter M [163].

Example AM
A matrix

$$B = \begin{bmatrix} -1 & 2 & 5 & 3 \\ 1 & 0 & -6 & 1 \\ -4 & 2 & 2 & -2 \end{bmatrix}$$

is a matrix with $m = 3$ rows and $n = 4$ columns. We can say that $[B]_{2,3} = -6$ while $[B]_{3,4} = -2$.

A calculator or computer language can be a convenient way to perform calculations with matrices. But first you have to enter the matrix. See: Computation ME.MMA [604] Computation ME.TI86 [608] Computation ME.TI83 [609] . When we do equation operations on system of equations, the names of the variables really aren’t very important. $x_1, x_2, x_3$, or $a, b, c$, or $x, y, z$, it really doesn’t matter. In this subsection we will describe some notation that will make it easier to describe linear systems, solve the systems and describe the solution sets. Here is a list of definitions, laden with notation.

Definition CV
Column Vector
A column vector of size $m$ is an ordered list of $m$ numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a vector. Column vectors will be written in bold, usually with lower case Latin letter from
the end of the alphabet such as $u, v, w, x, y, z$. Some books like to write vectors with arrows, such as $\vec{u}$. Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in $\vec{u}$. To refer to the entry or component that is number $i$ in the list that is the vector $v$ we write $[v]_i$.

Be careful with this notation. While the symbols $[v]_i$ might look somewhat substantial, as an object this represents just one component of a vector, which is just a single complex number.

**Definition ZCV**

**Zero Column Vector**
The zero vector of size $m$ is the column vector of size $m$ where each entry is the number zero,

$$0 = \begin{bmatrix}
0 \\
0 \\

\vdots \\
0
\end{bmatrix}$$

or more compactly, $[0]_i = 0$ for $1 \leq i \leq m$.

**Definition CM**

**Coefficient Matrix**

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

the coefficient matrix is the $m \times n$ matrix

$$A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}$$

**Definition VOC**

**Vector of Constants**

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$
the vector of constants is the column vector of size m

\[ b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix} \]

**Definition SV**

**Solution Vector**

For a system of linear equations,

\[
\begin{aligned}
    a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
    a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
    \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
\]

the solution vector is the column vector of size n

\[ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \]

The solution vector may do double-duty on occasion. It might refer to a list of variable quantities at one point, and subsequently refer to values of those variables that actually form a particular solution to that system.

**Definition LSMR**

**Matrix Representation of a Linear System**

If \( A \) is the coefficient matrix of a system of linear equations and \( b \) is the vector of constants, then we will write \( \mathcal{L}(A, b) \) as a shorthand expression for the system of linear equations, which we will refer to as the **matrix representation** of the linear system.

(This definition contains Notation LSMR.)

**Example NSLE**

**Notation for systems of linear equations**

The system of linear equations

\[
\begin{aligned}
    2x_1 + 4x_2 - 3x_3 + 5x_4 + x_5 &= 9 \\
    3x_1 + x_2 + x_4 - 3x_5 &= 0 \\
    -2x_1 + 7x_2 - 5x_3 + 2x_4 + 2x_5 &= -3
\end{aligned}
\]

has coefficient matrix

\[ A = \begin{bmatrix} 2 & 4 & -3 & 5 & 1 \\ 3 & 1 & 0 & 1 & -3 \\ -2 & 7 & -5 & 2 & 2 \end{bmatrix} \]

and vector of constants

\[ b = \begin{bmatrix} 9 \\ 0 \\ -3 \end{bmatrix} \]
Subsection RREF.RO Row Operations

and so will be referenced as \( \mathcal{L}(A, b) \).

**Definition AM**

**Augmented Matrix**

Suppose we have a system of \( m \) equations in \( n \) variables, with coefficient matrix \( A \) and vector of constants \( b \). Then the **augmented matrix** of the system of equations is the \( m \times (n + 1) \) matrix whose first \( n \) columns are the columns of \( A \) and whose last column (number \( n + 1 \)) is the column vector \( b \). This matrix will be written as \([A|b]\).

(This definition contains Notation AM.)

The augmented matrix represents all the important information in the system of equations, since the names of the variables have been ignored, and the only connection with the variables is the location of their coefficients in the matrix. It is important to realize that the augmented matrix is just that, a matrix, and **not** a system of equations. In particular, the augmented matrix does not have any “solutions,” though it will be useful for finding solutions to the system of equations that it is associated with. (Think about your objects, and review Technique L [620].) However, notice that an augmented matrix always belongs to some system of equations, and vice versa, so it is tempting to try and blur the distinction between the two. Here’s a quick example.

**Example AMAA**

**Augmented matrix for Archetype A**

Archetype A [634] is the following system of 3 equations in 3 variables.

\[
\begin{align*}
    x_1 - x_2 + 2x_3 &= 1 \\
    2x_1 + x_2 + x_3 &= 8 \\
    x_1 + x_2 &= 5
\end{align*}
\]

Here is its augmented matrix.

\[
\begin{bmatrix}
    1 & -1 & 2 & 1 \\
    2 & 1 & 1 & 8 \\
    1 & 1 & 0 & 5
\end{bmatrix}
\]

Subsection RO Row Operations

An augmented matrix for a system of equations will save us the tedium of continually writing down the names of the variables as we solve the system. It will also release us from any dependence on the actual names of the variables. We have seen how certain operations we can perform on equations (Definition EO [11]) will preserve their solutions (Theorem EOPSS [12]). The next two definitions and the following theorem carry over these ideas to augmented matrices.

**Definition RO**

**Row Operations**

The following three operations will transform an \( m \times n \) matrix into a different matrix of the same size, and each is known as a **row operation**.

1. Swap the locations of two rows.
2. Multiply each entry of a single row by a nonzero quantity.
3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.
We will use a symbolic shorthand to describe these row operations:

1. \( R_i \leftrightarrow R_j \): Swap the location of rows \( i \) and \( j \).
2. \( \alpha R_i \): Multiply row \( i \) by the nonzero scalar \( \alpha \).
3. \( \alpha R_i + R_j \): Multiply row \( i \) by the scalar \( \alpha \) and add to row \( j \).

(This definition contains Notation RO.)

**Definition REM**

**Row-Equivalent Matrices**

Two matrices, \( A \) and \( B \), are **row-equivalent** if one can be obtained from the other by a sequence of row operations.

**Example TREM**

**Two row-equivalent matrices**

The matrices

\[
A = \begin{bmatrix}
2 & -1 & 3 & 4 \\
5 & 2 & -2 & 3 \\
1 & 1 & 0 & 6
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 & 1 & 0 & 6 \\
3 & 0 & -2 & -9 \\
2 & -1 & 3 & 4
\end{bmatrix}
\]

are row-equivalent as can be seen from

\[
\begin{bmatrix}
2 & -1 & 3 & 4 \\
5 & 2 & -2 & 3 \\
1 & 1 & 0 & 6
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 & 6 \\
5 & 2 & -2 & 3 \\
2 & -1 & 3 & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 & 6 \\
3 & 0 & -2 & -9 \\
2 & -1 & 3 & 4
\end{bmatrix}
\]

We can also say that any pair of these three matrices are row-equivalent.

Notice that each of the three row operations is reversible (Exercise RREF.T10 [36]), so we do not have to be careful about the distinction between “\( A \) is row-equivalent to \( B \)” and “\( B \) is row-equivalent to \( A \).” The preceding definitions are designed to make the following theorem possible. It says that row-equivalent matrices represent systems of linear equations that have identical solution sets.

**Theorem REMES**

**Row-Equivalent Matrices represent Equivalent Systems**

Suppose that \( A \) and \( B \) are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

**Proof** If we perform a single row operation on an augmented matrix, it will have the same effect as if we did the analogous equation operation on the corresponding system of equations. By exactly the same methods as we used in the proof of Theorem EOPSS [12] we can see that each of these row operations will preserve the set of solutions for the corresponding system of equations.

So at this point, our strategy is to begin with a system of equations, represent it by an augmented matrix, perform row operations (which will preserve solutions for the corresponding systems) to get a “simpler” augmented matrix, convert back to a “simpler” system of equations and then solve that system, knowing that its solutions are those of the original system. Here’s a rehash of Example US [14] as an exercise in using our new tools.

**Example USR**

**Three equations, one solution, reprised**

We solve the following system using augmented matrices and row operations. This is the same system of equations solved in Example US [14] using equation operations.

\[
x_1 + 2x_2 + 2x_3 = 4
\]
$x_1 + 3x_2 + 3x_3 = 5$

$2x_1 + 6x_2 + 5x_3 = 6$

Form the augmented matrix,

$$A = \begin{bmatrix}
1 & 2 & 2 & 4 \\
1 & 3 & 3 & 5 \\
2 & 6 & 5 & 6
\end{bmatrix}$$

and apply row operations,

$$-1R_1 + R_2 \Rightarrow \begin{bmatrix}
1 & 2 & 2 & 4 \\
0 & 1 & 1 & 1 \\
2 & 6 & 5 & 6
\end{bmatrix}$$

$$-2R_2 + R_3 \Rightarrow \begin{bmatrix}
1 & 2 & 2 & 4 \\
0 & 1 & 1 & 1 \\
0 & 0 & -1 & -4
\end{bmatrix}$$

$$-1R_3 \Rightarrow \begin{bmatrix}
1 & 2 & 2 & 4 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 4
\end{bmatrix}$$

So the matrix

$$B = \begin{bmatrix}
1 & 2 & 2 & 4 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 4
\end{bmatrix}$$

is row equivalent to $A$ and by Theorem REMES the system of equations below has the same solution set as the original system of equations.

$$x_1 + 2x_2 + 2x_3 = 4$$

$$x_2 + x_3 = 1$$

$$x_3 = 4$$

Solving this “simpler” system is straightforward and is identical to the process in Example US.

Subsection RREF

Reduced Row-Echelon Form

The preceding example amply illustrates the definitions and theorems we have seen so far. But it still leaves two questions unanswered. Exactly what is this “simpler” form for a matrix, and just how do we get it? Here’s the answer to the first question, a definition of reduced row-echelon form.

Definition RREF

Reduced Row-Echelon Form

A matrix is in reduced row-echelon form if it meets all of the following conditions:

1. A row where every entry is zero lies below any row that contains a nonzero entry.

2. The leftmost nonzero entry of a row is equal to 1.

3. The leftmost nonzero entry of a row is the only nonzero entry in its column.

4. Consider any two different leftmost nonzero entries, one located in row $i$, column $j$ and the other located in row $s$, column $t$. If $s > i$, then $t > j$.

A row of only zero entries will be called a zero row and the leftmost nonzero entry of a nonzero row will be called a leading 1. The number of nonzero rows will be denoted by $r$.

A column containing a leading 1 will be called a pivot column. The set of column indices for all of the pivot columns will be denoted by $D = \{d_1, d_2, d_3, \ldots, d_r\}$ where $d_1 < d_2 < d_3 < \cdots < d_r$, \[\text{Version 1.04}\]
while the columns that are not pivot columns will be denoted as $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

(This definition contains Notation RREFA.)

△

The principal feature of reduced row-echelon form is the pattern of leading 1’s guaranteed by conditions (2) and (4), reminiscent of a flight of geese, or steps in a staircase, or water cascading down a mountain stream.

There are a number of new terms and notation introduced in this definition, which should make you suspect that this is an important definition. Given all there is to digest here, we will save the use of $D$ and $F$ until Section TSS [42].

Example RREF

A matrix in reduced row-echelon form

The matrix $C$ is in reduced row-echelon form.

$$
\begin{bmatrix}
1 & -3 & 0 & 6 & 0 & 0 & -5 & 9 \\
0 & 0 & 0 & 1 & 0 & 3 & -7 \\
0 & 0 & 0 & 0 & 1 & 7 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

This matrix has two zero rows and three leading 1’s. $r = 3$. Columns 1, 5, and 6 are pivot columns.

Example NRREF

A matrix not in reduced row-echelon form

The matrix $D$ is not in reduced row-echelon form, as it fails each of the four requirements once.

$$
\begin{bmatrix}
1 & 0 & -3 & 0 & 6 & 0 & 7 & -5 & 9 \\
0 & 0 & 0 & 5 & 0 & 1 & 0 & 3 & -7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -4 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 7 & 3 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Our next theorem has a “constructive” proof. Learn about the meaning of this term in Technique C [621].

Theorem REMEF

Row-Equivalent Matrix in Echelon Form

Suppose $A$ is a matrix. Then there is a matrix $B$ so that

1. $A$ and $B$ are row-equivalent.
2. $B$ is in reduced row-echelon form.

Proof Suppose that $A$ has $m$ rows and $n$ columns. We will describe a process for converting $A$ into $B$ via row operations. This procedure is known as Gauss–Jordan elimination. Tracing through this procedure will be easier if you recognize that $i$ refers to a row that is being converted, $j$ refers to a column that is being converted, and $r$ keeps track of the number of nonzero rows. Here we go.

1. Set $j = 0$ and $r = 0$.
2. Increase $j$ by 1. If $j$ now equals $n + 1$, then stop.
3. Examine the entries of $A$ in column $j$ located in rows $r + 1$ through $m$.
   If all of these entries are zero, then go to Step 2.

4. Choose a row from rows $r + 1$ through $m$ with a nonzero entry in column $j$.
   Let $i$ denote the index for this row.

5. Increase $r$ by 1.

6. Use the first row operation to swap rows $i$ and $r$.

7. Use the second row operation to convert the entry in row $r$ and column $j$ to a 1.

8. Use the third row operation with row $r$ to convert every other entry of column $j$ to zero.

9. Go to Step 2.

The result of this procedure is that the matrix $A$ is converted to a matrix in reduced row-echelon form, which we will refer to as $B$. We need to now prove this claim by showing that the converted matrix has the requisite properties of Definition RREF [26]. First, the matrix is only converted through row operations (Step 6, Step 7, Step 8), so $A$ and $B$ are row-equivalent (Definition REM [25]).

It is a bit more work to be certain that $B$ is in reduced row-echelon form. We claim that as we begin Step 2, the first $j$ columns of the matrix are in reduced row-echelon form with $r$ nonzero rows. Certainly this is true at the start when $j = 0$, since the matrix has no columns and so vacuously meets the conditions of Definition RREF [26] with $r = 0$ nonzero rows.

In Step 2 we increase $j$ by 1 and begin to work with the next column. There are two possible outcomes for Step 3. Suppose that every entry of column $j$ in rows $r + 1$ through $m$ is zero. Then with no changes we recognize that the first $j$ columns of the matrix has its first $r$ rows still in reduced-row echelon form, with the final $m - r$ rows still all zero.

Suppose instead that the entry in row $i$ of column $j$ is nonzero. Notice that since $r + 1 \leq i \leq m$, we know the first $j - 1$ entries of this row are all zero. Now, in Step 5 we increase $r$ by 1, and then embark on building a new nonzero row. In Step 6 we swap row $r$ and row $i$. In the first $j$ columns, the first $r - 1$ rows remain in reduced row-echelon form after the swap. In Step 7 we multiply row $r$ by a nonzero scalar, creating a 1 in the entry in column $j$ of row $i$, and not changing any other rows. This new leading 1 is the first nonzero entry in its row, and is located to the right of all the leading 1’s in the preceding $r - 1$ rows. With Step 8 we insure that every entry in the column with this new leading 1 is now zero, as required for reduced row-echelon form. Also, rows $r + 1$ through $m$ are now all zeros in the first $j$ columns, so we now only have one new nonzero row, consistent with our increase of $r$ by one. Furthermore, since the first $j - 1$ entries of row $r$ are zero, the employment of the third row operation does not destroy any of the necessary features of rows 1 through $r - 1$ and rows $r + 1$ through $m$, in columns 1 through $j - 1$.

So at this stage, the first $j$ columns of the matrix are in reduced row-echelon form. When Step 2 finally increases $j$ to $n + 1$, then the procedure is completed and the full $n$ columns of the matrix are in reduced row-echelon form, with the value of $r$ correctly recording the number of nonzero rows.

The procedure given in the proof of Theorem REMEF [27] can be more precisely described using a pseudo-code version of a computer program, as follows:

```plaintext
input m, n and A
r ← 0
for j ← 1 to n
   i ← r + 1
   while i ≤ m and [A]_{ij} = 0
      i ← i + 1
   if i ≠ m + 1
      r ← r + 1
```

Version 1.04
swap rows $i$ and $r$ of $A$ (row op 1)
scale entry in row $r$, column $j$ of $A$ to a leading 1 (row op 2)
for $k ← 1$ to $m, k ≠ r$
zero out entry in row $k$, column $j$ of $A$ (row op 3 using row $r$)
output $r$ and $A$

Notice that as a practical matter the “and” used in the conditional statement of the while statement should be of the “short-circuit” variety so that the array access that follows is not out-of-bounds.

So now we can put it all together. Begin with a system of linear equations (Definition SLE [9]), and represent the system by its augmented matrix (Definition AM [24]). Use row operations (Definition RO [24]) to convert this matrix into reduced row-echelon form (Definition RREF [26]), using the procedure outlined in the proof of Theorem REMEF [27]. Theorem REMEF [27] also tells us we can always accomplish this, and that the result is row-equivalent (Definition REM [25]) to the original augmented matrix. Since the matrix in reduced-row echelon form has the same solution set, we can analyze the row-reduced version instead of the original matrix, viewing it as the augmented matrix of a different system of equations. The beauty of augmented matrices in reduced row-echelon form is that the solution sets to their corresponding systems can be easily determined, as we will see in the next few examples and in the next section.

We will see through the course that almost every interesting property of a matrix can be discerned by looking at a row-equivalent matrix in reduced row-echelon form. For this reason it is important to know that the matrix $B$ guaranteed to exist by Theorem REMEF [27] is also unique. We could prove this result right now, but the proof will be much easier to state and understand a few sections from now when we have a few more definitions. However, the proof we will provide does not explicitly require any more theorems than we have right now, so we can, and will, make use of the uniqueness of $B$ between now and then by citing Theorem RREFU [96]. You might want to jump forward now to read the statement of this important theorem and save studying its proof for later, once the rest of us get there.

We will now run through some examples of using these definitions and theorems to solve some systems of equations. From now on, when we have a matrix in reduced row-echelon form, we will mark the leading 1’s with a small box. In your work, you can box ’em, circle ’em or write ’em in a different color — just identify ’em somehow. This device will prove very useful later and is a very good habit to start developing right now.

**Example SAB**

**Solutions for Archetype B**

Let’s find the solutions to the following system of equations,

$$
-7x_1 - 6x_2 - 12x_3 = -33 \\
5x_1 + 5x_2 + 7x_3 = 24 \\
x_1 + 4x_3 = 5
$$

First, form the augmented matrix,

$$
\begin{bmatrix}
-7 & -6 & -12 & -33 \\
5 & 5 & 7 & 24 \\
1 & 0 & 4 & 5
\end{bmatrix}
$$

and work to reduced row-echelon form, first with $i = 1$,

$$
\begin{bmatrix}
1 & 0 & 4 & 5 \\
5 & 5 & 7 & 24 \\
-7 & -6 & -12 & -33
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 4 & 5 \\
0 & 5 & -13 & -1 \\
0 & -6 & 16 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 4 & 5 \\
0 & 5 & -13 & -1 \\
0 & -6 & 16 & 2
\end{bmatrix}
$$
Now, with $i = 2$,
\[
\begin{bmatrix}
    1 & 0 & 4 & 5 \\
    0 & -6 & 16 & 2
\end{bmatrix}
\]
\[
\begin{bmatrix}
    1 & 0 & 4 & 5 \\
    0 & 1 & -\frac{13}{5} & -\frac{1}{2}
\end{bmatrix}
\]
\[
\begin{bmatrix}
    1 & 0 & 4 & 5 \\
    0 & 1 & -\frac{13}{5} & -\frac{1}{2}
\end{bmatrix}
\]

And finally, with $i = 3$,
\[
\begin{bmatrix}
    1 & 0 & 0 & -3 \\
    0 & 1 & 0 & 5 \\
    0 & 0 & 1 & 2
\end{bmatrix}
\]
\[
\begin{bmatrix}
    1 & 0 & 0 & 5 \\
    0 & 1 & 0 & 2
\end{bmatrix}
\]

This is now the augmented matrix of a very simple system of equations, namely $x_1 = -3$, $x_2 = 5$, $x_3 = 2$, which has an obvious solution. Furthermore, we can see that this is the only solution to this system, so we have determined the entire solution set,

\[
S = \left\{ \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \right\}
\]

You might compare this example with the procedure we used in Example US [14].

Archetypes A and B are meant to contrast each other in many respects. So let’s solve Archetype A now.

**Example SAA**

**Solutions for Archetype A**

Let’s find the solutions to the following system of equations,

\[
\begin{align*}
    x_1 - x_2 + 2x_3 &= 1 \\
    2x_1 + x_2 + x_3 &= 8 \\
    x_1 + x_2 &= 5
\end{align*}
\]

First, form the augmented matrix,

\[
\begin{bmatrix}
    1 & -1 & 2 & 1 \\
    2 & 1 & 1 & 8 \\
    1 & 1 & 0 & 5
\end{bmatrix}
\]

and work to reduced row-echelon form, first with $i = 1$,

\[
\begin{bmatrix}
    1 & -1 & 2 & 1 \\
    0 & 3 & -3 & 6 \\
    1 & 1 & 0 & 5
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1 & -1 & 2 & 1 \\
    0 & 3 & -3 & 6 \\
    0 & 2 & -2 & 4
\end{bmatrix}
\]

Now, with $i = 2$,

\[
\begin{bmatrix}
    1 & -1 & 2 & 1 \\
    0 & 1 & -1 & 2 \\
    0 & 2 & -2 & 4
\end{bmatrix}
\]

\[
\begin{bmatrix}
    1 & 0 & 1 & 3 \\
    0 & 1 & -1 & 2 \\
    0 & 2 & -2 & 4
\end{bmatrix}
\]
The system of equations represented by this augmented matrix needs to be considered a bit differently than that for Archetype B. First, the last row of the matrix is the equation $0 = 0$, which is always true, so it imposes no restrictions on our possible solutions and therefore we can safely ignore it as we analyze the other two equations. These equations are,

\[
x_1 + x_3 = 3 \\
x_2 - x_3 = 2.
\]

While this system is fairly easy to solve, it also appears to have a multitude of solutions. For example, choose $x_3 = 1$ and see that then $x_1 = 2$ and $x_2 = 3$ will together form a solution. Or choose $x_3 = 0$, and then discover that $x_1 = 3$ and $x_2 = 2$ lead to a solution. Try it yourself: pick any value of $x_3$ you please, and figure out what $x_1$ and $x_2$ should be to make the first and second equations (respectively) true. We’ll wait while you do that. Because of this behavior, we say that $x_3$ is a “free” or “independent” variable. But why do we vary $x_3$ and not some other variable?

For now, notice that the third column of the augmented matrix does not have any leading 1’s in its column. With this idea, we can rearrange the two equations, solving each for the variable that corresponds to the leading 1 in that row.

\[
x_1 = 3 - x_3 \\
x_2 = 2 + x_3
\]

To write the set of solution vectors in set notation, we have

\[
S = \left\{ \begin{bmatrix} 3 - x_3 \\ 2 + x_3 \\ x_3 \end{bmatrix} \mid x_3 \in \mathbb{C} \right\}
\]

We’ll learn more in the next section about systems with infinitely many solutions and how to express their solution sets. Right now, you might look back at Example IS [15].

Example SAE
Solutions for Archetype E

Let’s find the solutions to the following system of equations,

\[
2x_1 + x_2 + 7x_3 - 7x_4 = 2 \\
-3x_1 + 4x_2 - 5x_3 - 6x_4 = 3 \\
x_1 + x_2 + 4x_3 - 5x_4 = 2
\]

First, form the augmented matrix,

\[
\begin{bmatrix}
2 & 1 & 7 & -7 & 2 \\
-3 & 4 & -5 & -6 & 3 \\
1 & 1 & 4 & -5 & 2
\end{bmatrix}
\]

and work to reduced row-echelon form, first with $i = 1$,

\[
R_1 \rightarrow R_3, \begin{bmatrix}
1 & 1 & 4 & -5 & 2 \\
-3 & 4 & -5 & -6 & 3 \\
2 & 1 & 7 & -7 & 2
\end{bmatrix}
\]

\[
3R_1 + R_2, \begin{bmatrix}
1 & 1 & 4 & -5 & 2 \\
0 & 7 & 7 & -21 & 9 \\
2 & 1 & 7 & -7 & 2
\end{bmatrix}
\]

\[
-2R_1 + R_3, \begin{bmatrix}
1 & 1 & 4 & -5 & 2 \\
0 & 7 & 7 & -21 & 9 \\
0 & -1 & -1 & 3 & -2
\end{bmatrix}
\]

Now, with $i = 2$,

\[
R_2 \rightarrow R_3, \begin{bmatrix}
1 & 1 & 4 & -5 & 2 \\
0 & -1 & -1 & 3 & -2 \\
0 & 7 & 7 & -21 & 9
\end{bmatrix}
\]

\[
-1R_2, \begin{bmatrix}
1 & 1 & 4 & -5 & 2 \\
0 & 1 & 1 & -3 & 2 \\
0 & 7 & 7 & -21 & 9
\end{bmatrix}
\]
And finally, with \( i = 3 \),

\[
- \begin{bmatrix} 1 & 0 & 3 & -2 & 0 \\ 0 & 1 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

Let’s analyze the equations in the system represented by this augmented matrix. The third equation will read 0 = 1. This is patently false, all the time. No choice of values for our variables will ever make it true. We’re done. Since we cannot even make the last equation true, we have no hope of making all of the equations simultaneously true. So this system has no solutions, and its solution set is the empty set, \( \emptyset = \{ \} \) (Definition ES \[615\]).

Notice that we could have reached this conclusion sooner. After performing the row operation \(-7R_2 + R_3\), we can see that the third equation reads 0 = -5, a false statement. Since the system represented by this matrix has no solutions, none of the systems represented has any solutions. However, for this example, we have chosen to bring the matrix fully to reduced row-echelon form for the practice.

These three examples (Example SAB \[29\], Example SAA \[30\], Example SAE \[31\]) illustrate the full range of possibilities for a system of linear equations — no solutions, one solution, or infinitely many solutions. In the next section we’ll examine these three scenarios more closely.

**Definition RR**

**Row-Reducing**

To **row-reduce** the matrix \( A \) means to apply row operations to \( A \) and arrive at a row-equivalent matrix \( B \) in reduced row-echelon form.

So the term **row-reduce** is used as a verb. **Theorem REMEF \[27\]** tells us that this process will always be successful and **Theorem RREFU \[96\]** tells us that the result will be unambiguous. Typically, the analysis of \( A \) will proceed by analyzing \( B \) and applying theorems whose hypotheses include the row-equivalence of \( A \) and \( B \).

After some practice by hand, you will want to use your favorite computing device to do the computations required to bring a matrix to reduced row-echelon form (Exercise RREF.C30 \[35\]). See: Computation RR.MMA \[604\] Computation RR.TI86 \[609\] Computation RR.TI83 \[610\].

**Subsection READ**

**Reading Questions**

1. Is the matrix below in reduced row-echelon form? Why or why not?

\[
\begin{bmatrix} 1 & 5 & 0 & 6 & 8 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

2. Use row operations to convert the matrix below to reduced row-echelon form and report the final matrix.

\[
\begin{bmatrix} 2 & 1 & 8 \\ -1 & 1 & -1 \\ -2 & 5 & 4 \end{bmatrix}
\]
3. Find all the solutions to the system below by using an augmented matrix and row operations. Report your final matrix in reduced row-echelon form and the set of solutions.

\[
\begin{align*}
2x_1 + 3x_2 - x_3 &= 0 \\
x_1 + 2x_2 + x_3 &= 3 \\
x_1 + 3x_2 + 3x_3 &= 7
\end{align*}
\]
Subsection EXC
Exercises

C05 Each archetype below is a system of equations. Form the augmented matrix of the system of equations, convert the matrix to reduced row-echelon form by using equation operations and then describe the solution set of the original system of equations.

Archetype A
Archetype B
Archetype C
Archetype D
Archetype E
Archetype F
Archetype G
Archetype H
Archetype I
Archetype J
Contributed by Robert Beezer

For problems C10–C18, find all solutions to the system of linear equations. Write the solutions as a set, using correct set notation.

C10

\[
\begin{align*}
2x_1 - 3x_2 + x_3 + 7x_4 &= 14 \\
2x_1 + 8x_2 - 4x_3 + 5x_4 &= -1 \\
x_1 + 3x_2 - 3x_3 &= 4 \\
-5x_1 + 2x_2 + 3x_3 + 4x_4 &= -19
\end{align*}
\]

Contributed by Robert Beezer Solution 37

C11

\[
\begin{align*}
3x_1 + 4x_2 - x_3 + 2x_4 &= 6 \\
x_1 - 2x_2 + 3x_3 + x_4 &= 2 \\
10x_2 - 10x_3 - x_4 &= 1
\end{align*}
\]

Contributed by Robert Beezer Solution 37

C12

\[
\begin{align*}
2x_1 + 4x_2 + 5x_3 + 7x_4 &= -26 \\
x_1 + 2x_2 + x_3 - x_4 &= -4 \\
-2x_1 - 4x_2 + x_3 + 11x_4 &= -10
\end{align*}
\]

Contributed by Robert Beezer Solution 37

C13

\[
\begin{align*}
x_1 + 2x_2 + 8x_3 - 7x_4 &= -2 \\
3x_1 + 2x_2 + 12x_3 - 5x_4 &= 6 \\
-x_1 + x_2 + x_3 - 5x_4 &= -10
\end{align*}
\]

Contributed by Robert Beezer Solution 37
C14

\[\begin{align*}
2x_1 + x_2 + 7x_3 - 2x_4 &= 4 \\
3x_1 - 2x_2 + 11x_4 &= 13 \\
x_1 + x_2 + 5x_3 - 3x_4 &= 1
\end{align*}\]

Contributed by Robert Beezer Solution 38

C15

\[\begin{align*}
2x_1 + 3x_2 - x_3 - 9x_4 &= -16 \\
x_1 + 2x_2 + x_3 &= 0 \\
-x_1 + 2x_2 + 3x_3 + 4x_4 &= 8
\end{align*}\]

Contributed by Robert Beezer Solution 38

C16

\[\begin{align*}
2x_1 + 3x_2 + 19x_3 - 4x_4 &= 2 \\
x_1 + 2x_2 + 12x_3 - 3x_4 &= 1 \\
-x_1 + 2x_2 + 8x_3 - 5x_4 &= 1
\end{align*}\]

Contributed by Robert Beezer Solution 38

C17

\[\begin{align*}
-x_1 + 5x_2 &= -8 \\
-2x_1 + 5x_2 + 5x_3 + 2x_4 &= 9 \\
-3x_1 - x_2 + 3x_3 + x_4 &= 3 \\
7x_1 + 6x_2 + 5x_3 + x_4 &= 30
\end{align*}\]

Contributed by Robert Beezer Solution 39

C18

\[\begin{align*}
x_1 + 2x_2 - 4x_3 - x_4 &= 32 \\
x_1 + 3x_2 - 7x_3 - x_5 &= 33 \\
x_1 + 2x_3 - 2x_4 + 3x_5 &= 22
\end{align*}\]

Contributed by Robert Beezer Solution 39

For problems C30–C32, row-reduce the matrix without the aid of a calculator, indicating the row operations you are using at each step using the notation of Definition RO 24.

C30

\[
\begin{bmatrix}
2 & 1 & 5 & 10 \\
1 & -3 & -1 & -2 \\
4 & -2 & 6 & 12
\end{bmatrix}
\]

Contributed by Robert Beezer Solution 39
C31

\[
\begin{bmatrix}
1 & 2 & -4 \\
-3 & -1 & -3 \\
-2 & 1 & -7
\end{bmatrix}
\]

Contributed by Robert Beezer Solution 40

C32

\[
\begin{bmatrix}
1 & 1 & 1 \\
-4 & -3 & -2 \\
3 & 2 & 1
\end{bmatrix}
\]

Contributed by Robert Beezer Solution 40

M50  A parking lot has 66 vehicles (cars, trucks, motorcycles and bicycles) in it. There are four times as many cars as trucks. The total number of tires (4 per car or truck, 2 per motorcycle or bicycle) is 252. How many cars are there? How many bicycles?

Contributed by Robert Beezer Solution 40

T10  Prove that each of the three row operations (Definition RO 24) is reversible. More precisely, if the matrix $B$ is obtained from $A$ by application of a single row operation, show that there is a single row operation that will transform $B$ back into $A$.

Contributed by Robert Beezer Solution 40

T11  Suppose that $A$, $B$ and $C$ are $m \times n$ matrices. Use the definition of row-equivalence (Definition REM 25) to prove the following three facts.

1. $A$ is row-equivalent to $A$.

2. If $A$ is row-equivalent to $B$, then $B$ is row-equivalent to $A$.

3. If $A$ is row-equivalent to $B$, and $B$ is row-equivalent to $C$, then $A$ is row-equivalent to $C$.

A relationship that satisfies these three properties is known as an equivalence relation, an important idea in the study of various algebras. This is a formal way of saying that a relationship behaves like equality, without requiring the relationship to be as strict as equality itself. We’ll see it again in Theorem SER 391.

Contributed by Robert Beezer

T12  Suppose that $B$ is an $m \times n$ matrix in reduced row-echelon form. Build a new, likely smaller, $k \times \ell$ matrix $C$ as follows. Keep any collection of $k$ adjacent rows, $k \leq m$. From these rows, keep columns 1 through $\ell$, $\ell \leq n$. Prove that $C$ is in reduced row-echelon form.

Contributed by Robert Beezer

Version 1.04
Subsection SOL
Solutions

C10 Contributed by Robert Beezer Statement 34
The augmented matrix row-reduces to
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & -3 \\
0 & 0 & 1 & 0 & -4 \\
0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]
and we see from the locations of the leading 1’s that the system is consistent (Theorem RCLS 45) and that \( n - r = 4 - 4 = 0 \) and so the system has no free variables (Theorem CSRN 46) and hence has a unique solution. This solution is
\[
S = \left\{ \begin{bmatrix}
1 \\
-3 \\
-4 \\
1 \\
\end{bmatrix} \right\}
\]

C11 Contributed by Robert Beezer Statement 34
The augmented matrix row-reduces to
\[
\begin{bmatrix}
1 & 0 & 1 & 4/5 & 0 \\
0 & 1 & -1 & -1/10 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
and a leading 1 in the last column tells us that the system is inconsistent (Theorem RCLS 45). So the solution set is \( \emptyset = \{\} \).

C12 Contributed by Robert Beezer Statement 34
The augmented matrix row-reduces to
\[
\begin{bmatrix}
1 & 2 & 0 & -4 & 2 \\
0 & 0 & 1 & 3 & -6 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
(Theorem RCLS 45) and (Theorem CSRN 46) tells us the system is consistent and the solution set can be described with \( n - r = 4 - 2 = 2 \) free variables, namely \( x_2 \) and \( x_4 \). Solving for the dependent variables \( (D = \{x_1, x_3\}) \) the first and second equations represented in the row-reduced matrix yields,
\[
x_1 = 2 - 2x_2 + 4x_4 \\
x_3 = -6 - 3x_4
\]
As a set, we write this as
\[
\left\{ \begin{bmatrix}
x_2 + 4x_4 \\
-6 - 3x_4 \\
x_4 \\
\end{bmatrix} \right| x_2, x_4 \in \mathbb{C}
\]

C13 Contributed by Robert Beezer Statement 34
The augmented matrix of the system of equations is
\[
\begin{bmatrix}
1 & 2 & 8 & -7 & -2 \\
3 & 2 & 12 & -5 & 6 \\
-1 & 1 & 1 & -5 & -10 \\
\end{bmatrix}
\]
which row-reduces to
\[
\begin{bmatrix}
1 & 0 & 2 & 1 & 0 \\
0 & 1 & 3 & -4 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

With a leading one in the last column Theorem RCLS tells us the system of equations is inconsistent, so the solution set is the empty set, ∅.

C14 Contributed by Robert Beezer Statement
The augmented matrix of the system of equations is
\[
\begin{bmatrix}
2 & 1 & 7 & -2 & 4 \\
3 & -2 & 0 & 11 & 13 \\
1 & 1 & 5 & -3 & 1
\end{bmatrix}
\]
which row-reduces to
\[
\begin{bmatrix}
1 & 0 & 2 & 1 & 3 \\
0 & 1 & 3 & -4 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Then \(D = \{1, 2\}\) and \(F = \{3, 4, 5\}\), so the system is consistent (5 \(\notin D\)) and can be described by the two free variables \(x_3\) and \(x_4\). Rearranging the equations represented by the two nonzero rows to gain expressions for the dependent variables \(x_1\) and \(x_2\), yields the solution set,

\[
S = \left\{ \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \in \mathbb{C} \mid \begin{bmatrix} 3 - 2x_3 - x_4 \\ -2 - 3x_3 + 4x_4 \end{bmatrix} \right\}
\]

C15 Contributed by Robert Beezer Statement
The augmented matrix of the system of equations is
\[
\begin{bmatrix}
2 & 3 & -1 & -9 & -16 \\
1 & 2 & 1 & 0 & 0 \\
-1 & 2 & 3 & 4 & 8
\end{bmatrix}
\]
which row-reduces to
\[
\begin{bmatrix}
1 & 0 & 0 & 2 & 3 \\
0 & 1 & 0 & -3 & -5 \\
0 & 0 & 1 & 4 & 7
\end{bmatrix}
\]

Then \(D = \{1, 2, 3\}\) and \(F = \{4, 5\}\), so the system is consistent (5 \(\notin D\)) and can be described by the one free variable \(x_4\). Rearranging the equations represented by the three nonzero rows to gain expressions for the dependent variables \(x_1\), \(x_2\) and \(x_3\), yields the solution set,

\[
S = \left\{ \begin{bmatrix} x_4 \end{bmatrix} \in \mathbb{C} \mid \begin{bmatrix} 3 - 2x_4 \\ -5 + 3x_4 \\ 7 - 4x_4 \end{bmatrix} \right\}
\]

C16 Contributed by Robert Beezer Statement
The augmented matrix of the system of equations is
\[
\begin{bmatrix}
2 & 3 & 19 & -4 & 2 \\
1 & 2 & 12 & -3 & 1 \\
-1 & 2 & 8 & -5 & 1
\end{bmatrix}
\]
which row-reduces to
\[
\begin{bmatrix}
1 & 0 & 2 & 1 & 0 \\
0 & 1 & 5 & -2 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
With a leading one in the last column, Theorem RCLS tells us the system of equations is inconsistent, so the solution set is the empty set, \( \emptyset = \{ \} \).

**C17 Contributed by Robert Beezer**

Statement

We row-reduce the augmented matrix of the system of equations,

\[
\begin{bmatrix}
-1 & 5 & 0 & 0 & -8 \\
-2 & 5 & 5 & 2 & 9 \\
-3 & -1 & 3 & 1 & 3 \\
7 & 6 & 5 & 1 & 30
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 & 3 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 5
\end{bmatrix}
\]

The reduced row-echelon form of the matrix is the augmented matrix of the system

\[
x_1 = 3,
\]

\[
x_2 = -1,
\]

\[
x_3 = 2,
\]

\[
x_4 = 5,
\]

which has a unique solution. As a set of column vectors, the solution set is

\[
S = \left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix} \right\}
\]

**C18 Contributed by Robert Beezer**

Statement

We row-reduce the augmented matrix of the system of equations,

\[
\begin{bmatrix}
1 & 2 & -4 & -1 & 0 & 32 \\
1 & 3 & -7 & 0 & -1 & 33 \\
1 & 0 & 2 & -2 & 3 & 22
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 2 & 0 & 5 & 6 \\
0 & 1 & -3 & 0 & -2 & 9 \\
0 & 0 & 0 & 1 & 1 & -8
\end{bmatrix}
\]

With no leading 1 in the final column, we recognize the system as consistent (Theorem RCLS). Since the system is consistent, we compute the number of free variables as \( n - r = 5 - 3 = 2 \), and we see that columns 3 and 5 are not pivot columns, so \( x_3 \) and \( x_5 \) are free variables. We convert each row of the reduced row-echelon form of the matrix into an equation, and solve it for the lone dependent variable, as in expression in the two free variables.

\[
x_1 + 2x_3 + 5x_5 = 6 \quad \rightarrow \quad x_1 = 6 - 2x_3 - 5x_5
\]

\[
x_2 - 3x_3 - 2x_5 = 9 \quad \rightarrow \quad x_2 = 9 + 3x_3 + 2x_5
\]

\[
x_4 + x_5 = -8 \quad \rightarrow \quad x_4 = -8 - x_5
\]

These expressions give us a convenient way to describe the solution set, \( S \).

\[
S = \left\{ \begin{bmatrix} 6 - 2x_3 - 5x_5 \\ 9 + 3x_3 + 2x_5 \\ x_3 \\ -8 - x_5 \\ x_5 \end{bmatrix} \mid x_3, x_5 \in \mathbb{C} \right\}
\]

**C30 Contributed by Robert Beezer**

Statement

We row-reduce the augmented matrix of the system of equations,

\[
\begin{bmatrix}
2 & 1 & 5 & 10 \\
1 & -3 & -1 & -2 \\
4 & -2 & 6 & 12
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -3 & -1 & -2 \\
2 & 1 & 5 & 10 \\
4 & -2 & 6 & 12
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -3 & -1 & -2 \\
0 & 7 & 7 & 14 \\
4 & -2 & 6 & 12
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -3 & -1 & -2 \\
0 & 7 & 7 & 14 \\
0 & 10 & 10 & 20
\end{bmatrix}
\]

\[
\frac{1}{2}R_2
\]
\[-10R_2 + R_3, \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

**C31** Contributed by Robert Beezer Statement 36

\[
\begin{bmatrix} 1 & 2 & -4 \\ -3 & -1 & -3 \\ -2 & 1 & -7 \end{bmatrix} \xrightarrow{3R_1 + R_2} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 5 & -15 \\ -2 & 1 & -7 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 2 & -4 \\ 0 & 5 & -15 \\ 0 & 5 & -15 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -3 \\ 0 & 5 & -15 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 5 & -15 \end{bmatrix} \xrightarrow{-5R_2 + R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}
\]

**C32** Contributed by Robert Beezer Statement 36
Following the algorithm of Theorem RREF [27], and working to create pivot columns from left to right, we have

\[
\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -3 & -2 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{4R_1 + R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{-3R_1 + R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{-1R_2 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

**M50** Contributed by Robert Beezer Statement 36

Let \(c, t, m, b\) denote the number of cars, trucks, motorcycles, and bicycles. Then the statements from the problem yield the equations:

\[
c + t + m + b = 66
\]

\[
c - 4t = 0
\]

\[
4c + 4t + 2m + 2b = 252
\]

The augmented matrix for this system is

\[
\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -4 & 0 & 0 & 0 \\ 4 & 4 & 2 & 2 & 252 \end{bmatrix}
\]

which row-reduces to

\[
\begin{bmatrix} 1 & 0 & 0 & 0 & 48 \\ 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 1 & 1 & 6 \end{bmatrix}
\]

\(c = 48\) is the first equation represented in the row-reduced matrix so there are 48 cars. \(m + b = 6\) is the third equation represented in the row-reduced matrix so there are anywhere from 0 to 6 bicycles. We can also say that \(b\) is a free variable, but the context of the problem limits it to 7 integer values since cannot have a negative number of motorcycles.

**T10** Contributed by Robert Beezer Statement 36

If we can reverse each row operation individually, then we can reverse a sequence of row operations. The operations that reverse each operation are listed below, using our shorthand notation,

\[R_i \leftrightarrow R_j \quad R_i \leftrightarrow R_j\]
\[ \alpha R_i, \alpha \neq 0 \quad \frac{1}{\alpha} R_i \]
\[ \alpha R_i + R_j \quad -\alpha R_i + R_j \]
Section TSS
Types of Solution Sets

We will now be more careful about analyzing the reduced row-echelon form derived from the augmented matrix of a system of linear equations. In particular, we will see how to systematically handle the situation when we have infinitely many solutions to a system, and we will prove that every system of linear equations has either zero, one or infinitely many solutions. With these tools, we will be able to solve any system by a well-described method.

Subsection CS
Consistent Systems

The computer scientist Donald Knuth said, “Science is what we understand well enough to explain to a computer. Art is everything else.” In this section we’ll remove solving systems of equations from the realm of art, and into the realm of science. We begin with a definition.

Definition CS
Consistent System
A system of linear equations is consistent if it has at least one solution. Otherwise, the system is called inconsistent.

We will want to first recognize when a system is inconsistent or consistent, and in the case of consistent systems we will be able to further refine the types of solutions possible. We will do this by analyzing the reduced row-echelon form of a matrix, using the value of $r$, and the sets of column indices, $D$ and $F$, first defined back in Definition RREF [26].

Use of the notation for the elements of $D$ and $F$ can be a bit confusing, since we have subscripted variables that are in turn equal to integers used to index the matrix. However, many questions about matrices and systems of equations can be answered once we know $r$, $D$ and $F$. The choice of the letters $D$ and $F$ refer to our upcoming definition of dependent and free variables (Definition IDV [44]). An example will help us begin to get comfortable with this aspect of reduced row-echelon form.

Example RREFN
Reduced row-echelon form notation
For the $5 \times 9$ matrix

\[
B = \begin{bmatrix}
1 & 5 & 0 & 0 & 2 & 8 & 0 & 5 & -1 \\
0 & 0 & 1 & 0 & 4 & 7 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & 3 & 9 & 0 & 3 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

in reduced row-echelon form we have

\[
\begin{align*}
r &= 4 \\
d_1 &= 1 & d_2 &= 3 & d_3 &= 4 & d_4 &= 7 \\
f_1 &= 2 & f_2 &= 5 & f_3 &= 6 & f_4 &= 8 & f_5 &= 9.
\end{align*}
\]

Notice that the sets $D = \{d_1, d_2, d_3, d_4\} = \{1, 3, 4, 7\}$ and $F = \{f_1, f_2, f_3, f_4, f_5\} = \{2, 5, 6, 8, 9\}$ have nothing in common and together account for all of the columns of $B$ (we say it is a partition of the set of column indices).

The number $r$ is the single most important piece of information we can get from the reduced row-echelon form of a matrix. It is defined as the number of non-zero rows, but since each non-zero
row has a leading 1, it is also the number of leading 1’s present. For each leading 1, we have a pivot column, so \( r \) is also the number of pivot columns. Repeating ourselves, \( r \) is the number of leading 1’s, the number of non-zero rows and the number of pivot columns. Across different situations, each of these interpretations of the meaning of \( r \) will be useful.

Before proving some theorems about the possibilities for solution sets to systems of equations, let’s analyze one particular system with an infinite solution set very carefully as an example. We’ll use this technique frequently, and shortly we’ll refine it slightly.

Archetypes I and J are both fairly large for doing computations by hand (though not impossibly large). Their properties are very similar, so we will frequently analyze the situation in Archetype I, and leave you the joy of analyzing Archetype J yourself. So work through Archetype I with us, and we’ll use this technique frequently, and shortly we’ll refine it slightly.

Archetypes I and J are both fairly large for doing computations by hand (though not impossibly large). Their properties are very similar, so we will frequently analyze the situation in Archetype I, and leave you the joy of analyzing Archetype J yourself. So work through Archetype I with us, and we’ll use this technique frequently, and shortly we’ll refine it slightly.

Example ISSI
Describing infinite solution sets, Archetype I

Archetype I \([667]\) is the system of \( m = 4 \) equations in \( n = 7 \) variables.

\[
\begin{align*}
&x_1 + 4x_2 - x_4 + 7x_6 - 9x_7 = 3 \\
&2x_1 + 8x_2 - x_3 + 3x_4 + 9x_5 - 13x_6 + 7x_7 = 9 \\
&2x_3 - 3x_4 - 4x_5 + 12x_6 - 8x_7 = 1 \\
&-x_1 - 4x_2 + 2x_3 + 4x_4 + 8x_5 - 31x_6 + 37x_7 = 4
\end{align*}
\]

This system has a \( 4 \times 8 \) augmented matrix that is row-equivalent to the following matrix (check this!), and which is in reduced row-echelon form (the existence of this matrix is guaranteed by Theorem REMEF \([27]\)),

\[
\begin{bmatrix}
1 & 4 & 0 & 0 & 2 & 1 & -3 & 4 \\
0 & 0 & 1 & 0 & 1 & -3 & 5 & 2 \\
0 & 0 & 0 & 1 & 2 & -6 & 6 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

So we find that \( r = 3 \) and

\[
D = \{d_1, d_2, d_3\} = \{1, 3, 4\} \quad F = \{f_1, f_2, f_3, f_4, f_5\} = \{2, 5, 6, 7, 8\}.
\]

Let \( i \) denote one of the \( r = 3 \) non-zero rows, and then we see that we can solve the corresponding equation represented by this row for the variable \( x_{d_i} \) and write it as a linear function of the variables \( x_{f_1}, x_{f_2}, x_{f_3}, x_{f_4} \) (notice that \( f_5 = 8 \) does not reference a variable). We’ll do this now, but you can already see how the subscripts upon subscripts takes some getting used to.

\[
\begin{align*}
(i = 1) & \quad x_{d_1} = x_1 = 4 - 4x_2 - 2x_5 - x_6 + 3x_7 \\
(i = 2) & \quad x_{d_2} = x_3 = 2 - x_5 + 3x_6 - 5x_7 \\
(i = 3) & \quad x_{d_3} = x_4 = 1 - 2x_5 + 6x_6 - 6x_7
\end{align*}
\]

Each element of the set \( F = \{f_1, f_2, f_3, f_4, f_5\} = \{2, 5, 6, 7, 8\} \) is the index of a variable, except for \( f_5 = 8 \). We refer to \( x_{f_1} = x_2, x_{f_2} = x_5, x_{f_3} = x_6 \) and \( x_{f_4} = x_7 \) as “free” (or “independent”) variables since they are allowed to assume any possible combination of values that we can imagine and we can continue on to build a solution to the system by solving individual equations for the values of the other (“dependent”) variables.

Each element of the set \( D = \{d_1, d_2, d_3\} = \{1, 3, 4\} \) is the index of a variable. We refer to the variables \( x_{d_1} = x_1, x_{d_2} = x_3 \) and \( x_{d_3} = x_4 \) as “dependent” variables since they depend on the independent variables. More precisely, for each possible choice of values for the independent variables we get exactly one set of values for the dependent variables that combine to form a solution of the system.
To express the solutions as a set, we write
\[
\begin{bmatrix}
4 - 4x_2 - 2x_5 - x_6 + 3x_7 \\
x_2 \\
2 - x_5 + 3x_6 - 5x_7 \\
x_5 \\
1 - 2x_3 + 6x_6 - 6x_7 \\
x_6 \\
x_7 \\
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_5 \\
x_6 \\
x_7 \\
\end{bmatrix}
\bigg|\begin{bmatrix}
x_2, x_5, x_6, x_7 \in \mathbb{C}
\end{bmatrix}
\]

The condition that \(x_2, x_5, x_6, x_7 \in \mathbb{C}\) is how we specify that the variables \(x_2, x_5, x_6, x_7\) are “free” to assume any possible values.

This systematic approach to solving a system of equations will allow us to create a precise description of the solution set for any consistent system once we have found the reduced row-echelon form of the augmented matrix. It will work just as well when the set of free variables is empty and we get just a single solution. And we could program a computer to do it! Now have a whack at Archetype J (Exercise TSS.T10 [50]), mimicking the discussion in this example. We’ll still be here when you get back.

Using the reduced row-echelon form of the augmented matrix of a system of equations to determine the nature of the solution set of the system is a very key idea. So let’s look at one more example like the last one. But first a definition, and then the example. We mix our metaphors a bit when we call variables free versus dependent. Maybe we should call dependent variables “enslaved”?

**Definition IDV**

**Independent and Dependent Variables**

Suppose \(A\) is the augmented matrix of a consistent system of linear equations and \(B\) is a row-equivalent matrix in reduced row-echelon form. Suppose \(j\) is the index of a column of \(B\) that contains the leading 1 for some row (i.e. column \(j\) is a pivot column), and this column is not the last column. Then the variable \(x_j\) is dependent. A variable that is not dependent is called independent or free.

**Example FDV**

**Free and dependent variables**

Consider the system of five equations in five variables,

\[
\begin{align*}
 x_1 - x_2 - 2x_3 + x_4 + 11x_5 &= 13 \\
 x_1 - x_2 + x_3 + x_4 + 5x_5 &= 16 \\
 2x_1 - 2x_2 + x_4 + 10x_5 &= 21 \\
 2x_1 - 2x_2 - x_3 + 3x_4 + 20x_5 &= 38 \\
 2x_1 - 2x_2 + x_3 + x_4 + 8x_5 &= 22
\end{align*}
\]

whose augmented matrix row-reduces to

\[
\begin{bmatrix}
1 & -1 & 0 & 0 & 3 & 6 \\
0 & 0 & 1 & 0 & -2 & 1 \\
0 & 0 & 0 & 1 & 4 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

There are leading 1’s in columns 1, 3 and 4, so \(D = \{1, 3, 4\}\). From this we know that the variables \(x_1, x_3\) and \(x_4\) will be dependent variables, and each of the \(r = 3\) nonzero rows of the row-reduced matrix will yield an expression for one of these three variables. The set \(F\) is all the remaining column indices, \(F = \{2, 5, 6\}\). Since \(6 \in F\) we know there is no leading 1 in the final column, so the system is consistent by [Theorem RCLS] [45]. The remaining indices in \(F\) will correspond to...
free variables, so $x_2$ and $x_5$ are our free variables. The resulting three equations that describe our solution set are then,

\[
\begin{align*}
(x_{d_1} &= x_1) & x_1 &= 6 + x_2 - 3x_5 \\
(x_{d_2} &= x_3) & x_3 &= 1 + 2x_5 \\
(x_{d_3} &= x_4) & x_4 &= 9 - 4x_5
\end{align*}
\]

Make sure you understand where these three equations came from, and notice how the location of the leading 1’s determined the variables on the left-hand side of each equation. We can compactly describe the solution set as,

\[ S = \left\{ \begin{bmatrix} 6 + x_2 - 3x_5 \\ x_2 \\ 1 + 2x_5 \\ 9 - 4x_5 \\ x_5 \end{bmatrix} \mid x_2, x_5 \in \mathbb{C} \right\} \]

Notice how we express the freedom for $x_2$ and $x_5$: $x_2, x_5 \in \mathbb{C}$.

Sets are an important part of algebra, and we’ve seen a few already. Being comfortable with sets is important for understanding and writing proofs. If you haven’t already, pay a visit now to Section SET [615].

We can now use the values of $m$, $n$, $r$, and the independent and dependent variables to categorize the solution sets for linear systems through a sequence of theorems. Through the following sequence of proofs, you will want to consult three proof techniques. See Technique E [622]. See Technique N [622]. See Technique CP [623].

First we have a theorem that explores the distinction between consistent and inconsistent linear systems.

**Theorem RCLS**

**Recognizing Consistency of a Linear System**

Suppose $A$ is the augmented matrix of a system of linear equations with $m$ equations in $n$ variables. Suppose also that $B$ is a row-equivalent matrix in reduced row-echelon form with $r$ rows that are not zero rows. Then the system of equations is inconsistent if and only if the leading 1 of row $r$ is located in column $n + 1$ of $B$. □

**Proof** (⇐) The first half of the proof begins with the assumption that the leading 1 of row $r$ is located in column $n + 1$ of $B$. Then row $r$ of $B$ begins with $n$ consecutive zeros, finishing with the leading 1. This is a representation of the equation $0 = 1$, which is false. Since this equation is false for any collection of values we might choose for the variables, there are no solutions for the system of equations, and it is inconsistent.

(⇒) For the second half of the proof, we wish to show that if we assume the system is inconsistent, then the final leading 1 is located in the last column. But instead of proving this directly, we’ll form the logically equivalent statement that is the contrapositive, and prove that instead (see Technique CP [623]). Turning the implication around, and negating each portion, we arrive at the logically equivalent statement: If the leading 1 of row $r$ is not in column $n + 1$, then the system of equations is consistent.

If the leading 1 for row $r$ is located somewhere in columns 1 through $n$, then every preceding row’s leading 1 is also located in columns 1 through $n$. In other words, since the last leading 1 is not in the last column, no leading 1 for any row is in the last column, due to the echelon layout of the leading 1’s. Let $b_{i,n+1}$, $1 \leq i \leq r$, denote the entries of the last column of $B$ for the first $r$ rows. Employ our notation for columns of the reduced row-echelon form of a matrix (see Notation RREFA [27]) to $B$ and set $x_{f_i} = 0$, $1 \leq i \leq n - r$ and then set $x_{d_i} = b_{i,n+1}$, $1 \leq i \leq r$. In other words, set the dependent variables equal to the corresponding values in the final column and set all the free variables to zero. These values for the variables make the equations represented by the first $r$ rows all true (convince yourself of this). Rows $r + 1$ through $m$ (if any) are all zero rows,
hence represent the equation $0 = 0$ and are also all true. We have now identified one solution to the system, so we can say the system is consistent.

The beauty of this theorem being an equivalence is that we can unequivocally test to see if a system is consistent or inconsistent by looking at just a single entry of the reduced row-echelon form matrix. We could program a computer to do it!

Notice that for a consistent system the row-reduced augmented matrix has $n + 1 \in F$, so the largest element of $F$ does not refer to a variable. Also, for an inconsistent system, $n + 1 \in D$, and it then does not make much sense to discuss whether or not variables are free or dependent since there is no solution. With the characterization of Theorem RCLS [45], we can explore the relationships between $r$ and $n$ in light of the consistency of a system of equations. First, a situation where we can quickly conclude the inconsistency of a system.

**Theorem ISRN**

**Inconsistent Systems, $r$ and $n$**

Suppose $A$ is the augmented matrix of a system of linear equations with $m$ equations in $n$ variables. Suppose also that $B$ is a row-equivalent matrix in reduced row-echelon form with $r$ rows that are not completely zeros. If $r = n + 1$, then the system of equations is inconsistent.

**Proof** If $r = n + 1$, then $D = \{1, 2, 3, \ldots, n, n+1\}$ and every column of $B$ contains a leading 1 and is a pivot column. In particular, the entry of column $n + 1$ for row $r = n + 1$ is a leading 1. Theorem RCLS [45] then says that the system is inconsistent.

Do not confuse Theorem ISRN [46] with its converse! Go check out Technique CV [623] right now.

Next, if a system is consistent, we can distinguish between a unique solution and infinitely many solutions, and furthermore, we recognize that these are the only two possibilities.

**Theorem CSRN**

**Consistent Systems, $r$ and $n$**

Suppose $A$ is the augmented matrix of a consistent system of linear equations with $m$ equations in $n$ variables. Suppose also that $B$ is a row-equivalent matrix in reduced row-echelon form with $r$ rows that are not zero rows. Then $r \leq n$. If $r = n$, then the system has a unique solution, and if $r < n$, then the system has infinitely many solutions.

**Proof** This theorem contains three implications that we must establish. Notice first that $B$ has $n + 1$ columns, so there can be at most $n + 1$ pivot columns, i.e. $r \leq n + 1$. If $r = n + 1$, then Theorem ISRN [46] tells us that the system is inconsistent, contrary to our hypothesis. We are left with $r \leq n$.

When $r = n$, we find $n - r = 0$ free variables (i.e. $F = \{n + 1\}$) and any solution must equal the unique solution given by the first $n$ entries of column $n + 1$ of $B$.

When $r < n$, we have $n - r > 0$ free variables, corresponding to columns of $B$ without a leading 1, excepting the final column, which also does not contain a leading 1 by Theorem RCLS [45]. By varying the values of the free variables suitably, we can demonstrate infinitely many solutions. ■

**Subsection FV**

**Free Variables**

The next theorem simply states a conclusion from the final paragraph of the previous proof, allowing us to state explicitly the number of free variables for a consistent system.

**Theorem FVCS**

**Free Variables for Consistent Systems**

Suppose $A$ is the augmented matrix of a consistent system of linear equations with $m$ equations in $n$ variables. Suppose also that $B$ is a row-equivalent matrix in reduced row-echelon form with $r$
rows that are not completely zeros. Then the solution set can be described with $n - r$ free variables.

\[ \square \]

**Proof** See the proof of Theorem CSRN \[46\].

**Example CFV**

Counting free variables

For each archetype that is a system of equations, the values of $n$ and $r$ are listed. Many also contain a few sample solutions. We can use this information profitably, as illustrated by four examples.

1. **Archetype A** \[634\] has $n = 3$ and $r = 2$. It can be seen to be consistent by the sample solutions given. Its solution set then has $n - r = 1$ free variables, and therefore will be infinite.

2. **Archetype B** \[638\] has $n = 3$ and $r = 3$. It can be seen to be consistent by the single sample solution given. Its solution set can then be described with $n - r = 0$ free variables, and therefore will have just the single solution.

3. **Archetype H** \[663\] has $n = 2$ and $r = 3$. In this case, $r = n + 1$, so Theorem ISRN \[46\] says the system is inconsistent. We should not try to apply Theorem FVCS \[46\] to count free variables, since the theorem only applies to consistent systems. (What would happen if you did?)

4. **Archetype E** \[651\] has $n = 4$ and $r = 3$. However, by looking at the reduced row-echelon form of the augmented matrix, we find a leading 1 in row 3, column 4. By Theorem RCLS \[45\] we recognize the system is then inconsistent. (Why doesn’t this example contradict Theorem ISRN \[46\]?)

We have accomplished a lot so far, but our main goal has been the following theorem, which is now very simple to prove. The proof is so simple that we ought to call it a corollary, but the result is important enough that it deserves to be called a theorem. (See Technique LC \[627\].) Notice that this theorem was presaged first by Example TTS \[10\] and further foreshadowed by other examples.

**Theorem PSSLS**

Possible Solution Sets for Linear Systems

A system of linear equations has no solutions, a unique solution or infinitely many solutions.

\[ \square \]

**Proof** By definition, a system is either inconsistent or consistent. The first case describes systems with no solutions. For consistent systems, we have the remaining two possibilities as guaranteed by, and described in, Theorem CSRN \[46\].

We have one more theorem to round out our set of tools for determining solution sets to systems of linear equations.

**Theorem CMVEI**

Consistent, More Variables than Equations, Infinite solutions

Suppose a consistent system of linear equations has $m$ equations in $n$ variables. If $n > m$, then the system has infinitely many solutions.

\[ \square \]

**Proof** Suppose that the augmented matrix of the system of equations is row-equivalent to $B$, a matrix in reduced row-echelon form with $r$ nonzero rows. Because $B$ has $m$ rows in total, the number that are nonzero rows is less. In other words, $r \leq m$. Follow this with the hypothesis that $n > m$ and we find that the system has a solution set described by at least one free variable because

\[ n - r \geq n - m > 0. \]

A consistent system with free variables will have an infinite number of solutions, as given by Theorem CSRN \[46\].

Notice that to use this theorem we need only know that the system is consistent, together with the values of $m$ and $n$. We do not necessarily have to compute a row-equivalent reduced row-echelon
form matrix, even though we discussed such a matrix in the proof. This is the substance of the following example.

Example OSGMD
One solution gives many, Archetype D
Archetype D is the system of \( m = 3 \) equations in \( n = 4 \) variables,

\[
\begin{align*}
2x_1 + x_2 + 7x_3 - 7x_4 &= 8 \\
-3x_1 + 4x_2 - 5x_3 - 6x_4 &= -12 \\
x_1 + x_2 + 4x_3 - 5x_4 &= 4
\end{align*}
\]

and the solution \( x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 1 \) can be checked easily by substitution. Having been handed this solution, we know the system is consistent. This, together with \( n > m \), allows us to apply Theorem CMVEI \[47\] and conclude that the system has infinitely many solutions. 

These theorems give us the procedures and implications that allow us to completely solve any system of linear equations. The main computational tool is using row operations to convert an augmented matrix into reduced row-echelon form. Here’s a broad outline of how we would instruct a computer to solve a system of linear equations.

1. Represent a system of linear equations by an augmented matrix (an array is the appropriate data structure in most computer languages).
2. Convert the matrix to a row-equivalent matrix in reduced row-echelon form using the procedure from the proof of Theorem REMEF \[27\].
3. Determine \( r \) and locate the leading 1 of row \( r \). If it is in column \( n + 1 \), output the statement that the system is inconsistent and halt.
4. With the leading 1 of row \( r \) not in column \( n + 1 \), there are two possibilities:
   
   (a) \( r = n \) and the solution is unique. It can be read off directly from the entries in rows 1 through \( n \) of column \( n + 1 \).
   
   (b) \( r < n \) and there are infinitely many solutions. If only a single solution is needed, set all the free variables to zero and read off the dependent variable values from column \( n + 1 \), as in the second half of the proof of Theorem RCLS \[45\]. If the entire solution set is required, figure out some nice compact way to describe it, since your finite computer is not big enough to hold all the solutions (we’ll have such a way soon).

The above makes it all sound a bit simpler than it really is. In practice, row operations employ division (usually to get a leading entry of a row to convert to a leading 1) and that will introduce round-off errors. Entries that should be zero sometimes end up being very, very small nonzero entries, or small entries lead to overflow errors when used as divisors. A variety of strategies can be employed to minimize these sorts of errors, and this is one of the main topics in the important subject known as numerical linear algebra.

Solving a linear system is such a fundamental problem in so many areas of mathematics, and its applications, that any computational device worth using for linear algebra will have a built-in routine to do just that. See: Computation LS.MMA \[604\]. In this section we’ve gained a foolproof procedure for solving any system of linear equations, no matter how many equations or variables. We also have a handful of theorems that allow us to determine partial information about a solution set without actually constructing the whole set itself. Donald Knuth would be proud.

Subsection READ
Reading Questions

1. How do we recognize when a system of linear equations is inconsistent?
2. Suppose we have converted the augmented matrix of a system of equations into reduced row-echelon form. How do we then identify the dependent and independent (free) variables?

3. What are the possible solution sets for a system of linear equations?
Subsection EXC

Exercises

C10  In the spirit of Example ISSI [43], describe the infinite solution set for Archetype J [671].
Contributed by Robert Beezer

M45  Prove that Archetype J [671] has infinitely many solutions without row-reducing the augmented matrix.
Contributed by Robert Beezer  Solution

For Exercises M51–M57 say as much as possible about each system’s solution set. Be sure to make it clear which theorems you are using to reach your conclusions.

M51  A consistent system of 8 equations in 6 variables.
Contributed by Robert Beezer  Solution

M52  A consistent system of 6 equations in 8 variables.
Contributed by Robert Beezer  Solution

M53  A system of 5 equations in 9 variables.
Contributed by Robert Beezer  Solution

M54  A system with 12 equations in 35 variables.
Contributed by Robert Beezer  Solution

M56  A system with 6 equations in 12 variables.
Contributed by Robert Beezer  Solution

M57  A system with 8 equations and 6 variables. The reduced row-echelon form of the augmented matrix of the system has 7 pivot columns.
Contributed by Robert Beezer  Solution

M60  Without doing any computations, and without examining any solutions, say as much as possible about the form of the solution set for each archetype that is a system of equations.
Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671
Contributed by Robert Beezer

T10  An inconsistent system may have $r > n$. If we try (incorrectly!) to apply Theorem FVCS [46] to such a system, how many free variables would we discover?
Contributed by Robert Beezer  Solution

T40  Suppose that the coefficient matrix of a system of linear equations has two columns that are identical. Prove that the system has infinitely many solutions.
Contributed by Robert Beezer  Solution
Subsection SOL
Solutions

M45 Contributed by Robert Beezer Statement
Demonstrate that the system is consistent by verifying any one of the four sample solutions provided. Then because \( n = 9 > 6 = m \), Theorem CMVEI gives us the conclusion that the system has infinitely many solutions.

Notice that we only know the system will have at least \( 9 - 6 = 3 \) free variables, but very well could have more. We do not know that \( r = 6 \), only that \( r \leq 6 \).

M51 Contributed by Robert Beezer Statement
Consistent means there is at least one solution (Definition CS). It will have either a unique solution or infinitely many solutions (Theorem PSSLS).

M52 Contributed by Robert Beezer Statement
With 6 rows in the augmented matrix, the row-reduced version will have \( r \leq 6 \). Since the system is consistent, apply Theorem CSRN to see that \( n - r \geq 2 \) implies infinitely many solutions.

M53 Contributed by Robert Beezer Statement
The system could be inconsistent. If it is consistent, then because it has more variables than equations Theorem CMVEI implies that there would be infinitely many solutions. So, of all the possibilities in Theorem PSSLS, only the case of a unique solution can be ruled out.

M54 Contributed by Robert Beezer Statement
The system could be inconsistent. If it is consistent, then Theorem CMVEI tells us the solution set will be infinite. So we can be certain that there is not a unique solution.

M56 Contributed by Robert Beezer Statement
The system could be inconsistent. If it is consistent, and since \( 12 > 6 \), then Theorem CMVEI says we will have infinitely many solutions. So there are two possibilities. Theorem PSSLS allows to state equivalently that a unique solution is an impossibility.

M57 Contributed by Robert Beezer Statement
7 pivot columns implies that there are \( r = 7 \) nonzero rows (so row 8 is all zeros in the reduced row-echelon form). Then \( n + 1 = 6 + 1 = 7 = r \) and Theorem ISRN allows to conclude that the system is inconsistent.

T10 Contributed by Robert Beezer Statement
Theorem FVCS will indicate a negative number of free variables, but we can say even more. If \( r > n \), then the only possibility is that \( r = n + 1 \), and then we compute \( n - r = n - (n + 1) = -1 \) free variables.

T40 Contributed by Robert Beezer Statement
Since the system is consistent, we know there is either a unique solution, or infinitely many solutions (Theorem PSSLS). If we perform row operations on the augmented matrix of the system, the two equal columns of the coefficient matrix will suffer the same fate, and remain equal in the final reduced row-echelon form. Suppose both of these columns are pivot columns (Definition RREF). Then there is single row containing the two leading 1’s of the two pivot columns, a violation of reduced row-echelon form (Definition RREF). So at least one of these columns is not a pivot column, and the column index indicates a free variable in the description of the solution set (Definition IDV). With a free variable, we arrive at an infinite solution set (Theorem FVCS).
Section HSE
Homogeneous Systems of Equations

In this section we specialize to systems of linear equations where every equation has a zero as its constant term. Along the way, we will begin to express more and more ideas in the language of matrices and begin a move away from writing out whole systems of equations. The ideas initiated in this section will carry through the remainder of the course.

Subsection SHS
Solutions of Homogeneous Systems

As usual, we begin with a definition.

Definition HS
Homogeneous System
A system of linear equations, \( \mathcal{L}(A, b) \) is homogeneous if the vector of constants is the zero vector, in other words, \( b = 0 \).

Example AHSAC
Archetype C as a homogeneous system
For each archetype that is a system of equations, we have formulated a similar, yet different, homogeneous system of equations by replacing each equation’s constant term with a zero. To wit, for Archetype C, we can convert the original system of equations into the homogeneous system,

\[
\begin{align*}
2x_1 - 3x_2 + x_3 - 6x_4 &= 0 \\
4x_1 + x_2 + 2x_3 + 9x_4 &= 0 \\
3x_1 + x_2 + x_3 + 8x_4 &= 0
\end{align*}
\]

Can you quickly find a solution to this system without row-reducing the augmented matrix? ☞

As you might have discovered by studying Example AHSAC, setting each variable to zero will always be a solution of a homogeneous system. This is the substance of the following theorem.

Theorem HSC
Homogeneous Systems are Consistent
Suppose that a system of linear equations is homogeneous. Then the system is consistent. ☐

Proof Set each variable of the system to zero. When substituting these values into each equation, the left-hand side evaluates to zero, no matter what the coefficients are. Since a homogeneous system has zero on the right-hand side of each equation as the constant term, each equation is true. With one demonstrated solution, we can call the system consistent.

Since this solution is so obvious, we now define it as the trivial solution.

Definition TSHSE
Trivial Solution to Homogeneous Systems of Equations
Suppose a homogeneous system of linear equations has \( n \) variables. The solution \( x_1 = 0, x_2 = 0, \ldots, x_n = 0 \) (i.e. \( \mathbf{x} = 0 \)) is called the trivial solution. ☐

Here are three typical examples, which we will reference throughout this section. Work through the row operations as we bring each to reduced row-echelon form. Also notice what is similar in each example, and what differs.
Example HUSAB
Homogeneous, unique solution, Archetype B
Archetype B can be converted to the homogeneous system,

\[-11x_1 + 2x_2 - 14x_3 = 0 \]
\[23x_1 - 6x_2 + 33x_3 = 0 \]
\[14x_1 - 2x_2 + 17x_3 = 0 \]

whose augmented matrix row-reduces to

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

By Theorem HSC [52], the system is consistent, and so the computation \( n - r = 3 - 3 = 0 \) means the solution set contains just a single solution. Then, this lone solution must be the trivial solution.

Example HISAA
Homogeneous, infinite solutions, Archetype A
Archetype A [634] can be converted to the homogeneous system,

\[x_1 - x_2 + 2x_3 = 0 \]
\[2x_1 + x_2 + x_3 = 0 \]
\[x_1 + x_2 = 0 \]

whose augmented matrix row-reduces to

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

By Theorem HSC [52], the system is consistent, and so the computation \( n - r = 3 - 2 = 1 \) means the solution set contains one free variable by Theorem FVCS [46], and hence has infinitely many solutions. We can describe this solution set using the free variable \( x_3 \),

\[ S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid x_1 = -x_3, \ x_2 = x_3 \right\} = \left\{ \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} \mid x_3 \in \mathbb{C} \right\} \]

Geometrically, these are points in three dimensions that lie on a line through the origin.

Example HISAD
Homogeneous, infinite solutions, Archetype D
Archetype D [647] (and identically, Archetype E [651]) can be converted to the homogeneous system,

\[2x_1 + x_2 + 7x_3 - 7x_4 = 0 \]
\[-3x_1 + 4x_2 - 5x_3 - 6x_4 = 0 \]
\[x_1 + x_2 + 4x_3 - 5x_4 = 0 \]

whose augmented matrix row-reduces to

\[
\begin{bmatrix}
1 & 0 & 3 & -2 & 0 \\
0 & 1 & 1 & -3 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
By [Theorem HSC][52], the system is consistent, and so the computation \( n - r = 4 - 2 = 2 \) means the solution set contains two free variables by [Theorem FVCS][46], and hence has infinitely many solutions. We can describe this solution set using the free variables \( x_3 \) and \( x_4 \),

\[
S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \ \mid \ x_1 = -3x_3 + 2x_4, \ x_2 = -x_3 + 3x_4 \right\} \\
= \left\{ \begin{bmatrix} -3x_3 + 2x_4 \\ -x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix} \ \mid \ x_3, \ x_4 \in \mathbb{C} \right\}
\]

After working through these examples, you might perform the same computations for the slightly larger example, [Archetype J][671].

Notice that when we do row operations on the augmented matrix of a homogeneous system of linear equations the last column of the matrix is all zeros. Any one of the three allowable row operations will convert zeros to zeros and thus, the final column of the matrix in reduced row-echelon form will also be all zeros. So in this case, we may be as likely to reference only the coefficient matrix and presume that we remember that the final column begins with zeros, and after any number of row operations is still zero.

[Example HISAD][53] suggests the following theorem.

**Theorem HMVEI**

**Homogeneous, More Variables than Equations, Infinite solutions**

Suppose that a homogeneous system of linear equations has \( m \) equations and \( n \) variables with \( n > m \). Then the system has infinitely many solutions. \( \square \)

**Proof** We are assuming the system is homogeneous, so [Theorem HSC][52] says it is consistent. Then the hypothesis that \( n > m \), together with [Theorem CMVEI][47], gives infinitely many solutions. \( \blacksquare \)

[Example HUSAB][52] and [Example HISAA][53] are concerned with homogeneous systems where \( n = m \) and expose a fundamental distinction between the two examples. One has a unique solution, while the other has infinitely many. These are exactly the only two possibilities for a homogeneous system and illustrate that each is possible (unlike the case when \( n > m \) where [Theorem HMVEI][54] tells us that there is only one possibility for a homogeneous system).

### Subsection NSM

#### Null Space of a Matrix

The set of solutions to a homogeneous system (which by [Theorem HSC][52] is never empty) is of enough interest to warrant its own name. However, we define it as a property of the coefficient matrix, not as a property of some system of equations.

**Definition NSM**

**Null Space of a Matrix**

The **null space** of a matrix \( A \), denoted \( \mathcal{N}(A) \), is the set of all the vectors that are solutions to the homogeneous system \( \mathcal{L}S(A, 0) \).

(This definition contains Notation NSM.) \( \triangle \)

In the Archetypes (Appendix A[630]) each example that is a system of equations also has a corresponding homogeneous system of equations listed, and several sample solutions are given.
These solutions will be elements of the null space of the coefficient matrix. We’ll look at one example.

**Example NSEAI**

**Null space elements of Archetype I**

The write-up for Archetype I \[ \text{Archetype I} \] lists several solutions of the corresponding homogeneous system. Here are two, written as solution vectors. We can say that they are in the null space of the coefficient matrix for the system of equations in Archetype I \[ \text{Archetype I} \].

\[
\begin{bmatrix}
3 \\
0 \\
-5 \\
-6 \\
0 \\
0 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
-4 \\
1 \\
-3 \\
-2 \\
1 \\
1 \\
1
\end{bmatrix}
\]

However, the vector

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

is not in the null space, since it is not a solution to the homogeneous system. For example, it fails to even make the first equation true.

Here are two (prototypical) examples of the computation of the null space of a matrix. Notice that we will now begin writing solutions as vectors.

**Example CNS1**

**Computing a null space, #1**

Let’s compute the null space of

\[ A = \begin{bmatrix} 2 & -1 & 7 & -3 & -8 \\ 1 & 0 & 2 & 4 & 9 \\ 2 & 2 & -2 & -1 & 8 \end{bmatrix} \]

which we write as \( \mathcal{N}(A) \). Translating Definition NSM \[ \text{Definition NSM} \], we simply desire to solve the homogeneous system \( \mathcal{L}S(A, 0) \). So we row-reduce the augmented matrix to obtain

\[
\begin{bmatrix}
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 1 & -3 & 0 & 4 & 0 \\
0 & 0 & 0 & 1 & 2 & 0
\end{bmatrix}
\]

The variables (of the homogeneous system) \( x_3 \) and \( x_5 \) are free (since columns 1, 2 and 4 are pivot columns), so we arrange the equations represented by the matrix in reduced row-echelon form to

\[
\begin{align*}
&x_1 = -2x_3 - x_5 \\
&x_2 = 3x_3 - 4x_5 \\
&x_4 = -2x_5
\end{align*}
\]

So we can write the infinite solution set as sets using column vectors,

\[
\mathcal{N}(A) = \left\{ \begin{bmatrix} -2x_3 - x_5 \\ 3x_3 - 4x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{bmatrix} \middle| \begin{bmatrix} x_3 \\ x_5 \end{bmatrix} \in \mathbb{C} \right\}
\]
Example CNS2  
**Computing a null space, #2**  
Let’s compute the null space of 

\[
C = \begin{bmatrix}
-4 & 6 & 1 \\
-1 & 4 & 1 \\
5 & 6 & 7 \\
4 & 7 & 1 \\
\end{bmatrix}
\]

which we write as \( N(C) \). Translating [Definition NSM 54](#), we simply desire to solve the homogeneous system \( LS(C, 0) \). So we row-reduce the augmented matrix to obtain 

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

There are no free variables in the homogenous system represented by the row-reduced matrix, so there is only the trivial solution, the zero vector, \( \mathbf{0} \). So we can write the (trivial) solution set as 

\[
N(C) = \{ \mathbf{0} \} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}
\]

**Subsection READ**  
**Reading Questions**

1. What is *always* true of the solution set for a homogeneous system of equations?  
2. Suppose a homogeneous system of equations has 13 variables and 8 equations. How many solutions will it have? Why?  
3. Describe in words (not symbols) the null space of a matrix.
Subsection HSE.EXC Exercises

C10 Each archetype (Appendix A 630) that is a system of equations has a corresponding homogeneous system with the same coefficient matrix. Compute the set of solutions for each. Notice that these solution sets are the null spaces of the coefficient matrices.

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671
Contributed by Robert Beezer

C20 Archetype K 676 and Archetype L 680 are simply 5 $\times$ 5 matrices (i.e. they are not systems of equations). Compute the null space of each matrix.

Contributed by Robert Beezer

C30 Compute the null space of the matrix $A$, $\mathcal{N}(A)$.

\[
A = \begin{bmatrix}
2 & 4 & 1 & 3 & 8 \\
-1 & -2 & -1 & -1 & 1 \\
2 & 4 & 0 & -3 & 4 \\
2 & 4 & -1 & -7 & 4
\end{bmatrix}
\]

Contributed by Robert Beezer Solution 59

C31 Find the null space of the matrix $B$, $\mathcal{N}(B)$.

\[
B = \begin{bmatrix}
-6 & 4 & -36 & 6 \\
2 & -1 & 10 & -1 \\
-3 & 2 & -18 & 3
\end{bmatrix}
\]

Contributed by Robert Beezer Solution 59

M45 Without doing any computations, and without examining any solutions, say as much as possible about the form of the solution set for corresponding homogeneous system of equations of each archetype that is a system of equations.

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671
Contributed by Robert Beezer

For Exercises M50–M52 say as much as possible about each system’s solution set. Be sure to make it clear which theorems you are using to reach your conclusions.

M50 A homogeneous system of 8 equations in 8 variables.
Contributed by Robert Beezer Solution 59

M51 A homogeneous system of 8 equations in 9 variables.
Contributed by Robert Beezer Solution 60

Version 1.04
M52  A homogeneous system of 8 equations in 7 variables.
Contributed by Robert Beezer  Solution 60

T10  Prove or disprove: A system of linear equations is homogeneous if and only if the system has the zero vector as a solution.
Contributed by Martin Jackson  Solution 60

T20  Consider the homogeneous system of linear equations $\mathbf{LS}(A, \mathbf{0})$, and suppose that $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}$ is one solution to the system of equations. Prove that $\mathbf{v} = \begin{bmatrix} 4u_1 \\ 4u_2 \\ 4u_3 \\ \vdots \\ 4u_n \end{bmatrix}$ is also a solution to $\mathbf{LS}(A, \mathbf{0})$.
Contributed by Robert Beezer  Solution 60
Subsection HSE.SOL
Solutions

C30 Contributed by Robert Beezer Statement 57
Definition NSM 54 tells us that the null space of $A$ is the solution set to the homogeneous system $LS(A, 0)$. The augmented matrix of this system is

$$\begin{bmatrix}
2 & 4 & 1 & 3 & 8 & 0 \\
-1 & -2 & -1 & -1 & 1 & 0 \\
2 & 4 & 0 & -3 & 4 & 0 \\
2 & 4 & -1 & -7 & 4 & 0
\end{bmatrix}$$

To solve the system, we row-reduce the augmented matrix and obtain,

$$\begin{bmatrix}
1 & 2 & 0 & 0 & 5 & 0 \\
0 & 0 & 1 & 0 & -8 & 0 \\
0 & 0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

This matrix represents a system with equations having three dependent variables ($x_1, x_3$, and $x_4$) and two independent variables ($x_2$ and $x_5$). These equations rearrange to

$$x_1 = -2x_2 - 5x_5 \quad x_3 = 8x_5 \quad x_4 = -2x_5$$

So we can write the solution set (which is the requested null space) as

$$N(A) = \left\{ \begin{bmatrix}
-2x_2 - 5x_5 \\
x_2 \\
8x_5 \\
-2x_5 \\
x_5
\end{bmatrix} \mid x_2, x_5 \in \mathbb{C} \right\}$$

C31 Contributed by Robert Beezer Statement 57
We form the augmented matrix of the homogeneous system $LS(B, 0)$ and row-reduce the matrix,

$$\begin{bmatrix}
-6 & 4 & -36 & 6 & 0 \\
2 & -1 & 10 & -1 & 0 \\
-3 & 2 & -18 & 3 & 0
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 2 & 1 & 0 \\
0 & 1 & -6 & 3 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

We knew ahead of time that this system would be consistent (Theorem HSC 52), but we can now see there are $n - r = 4 - 2 = 2$ free variables, namely $x_3$ and $x_4$ (Theorem FVCS 46). Based on this analysis, we can rearrange the equations associated with each nonzero row of the reduced row-echelon form into an expression for the lone dependent variable as a function of the free variables.

We arrive at the solution set to the homogeneous system, which is the null space of the matrix by Definition NSM 54,

$$N(B) = \left\{ \begin{bmatrix}
-2x_3 - x_4 \\
6x_3 - 3x_4 \\
x_3 \\
x_4
\end{bmatrix} \mid x_3, x_4 \in \mathbb{C} \right\}$$

M50 Contributed by Robert Beezer Statement 57
Since the system is homogeneous, we know it has the trivial solution (Theorem HSC 52). We cannot say anymore based on the information provided, except to say that there is either a unique
solution or infinitely many solutions (Theorem PSSLS [47]). See Archetype A [634] and Archetype B [638] to understand the possibilities.

**M51** Contributed by Robert Beezer Statement [57]
Since there are more variables than equations, Theorem HMVEI [54] applies and tells us that the solution set is infinite. From the proof of Theorem HSC [52] we know that the zero vector is one solution.

**M52** Contributed by Robert Beezer Statement [58]
By Theorem HSC [52], we know the system is consistent because the zero vector is always a solution of a homogeneous system. There is no more that we can say, since both a unique solution and infinitely many solutions are possibilities.

**T10** Contributed by Robert Beezer Statement [58]
This is a true statement. A proof is:

$(\Rightarrow)$ Suppose we have a homogeneous system $\mathcal{L}S(A, 0)$. Then by substituting the scalar zero for each variable, we arrive at true statements for each equation. So the zero vector is a solution. This is the content of Theorem HSC [52].

$(\Leftarrow)$ Suppose now that we have a generic (i.e. not necessarily homogeneous) system of equations, $\mathcal{L}S(A, \mathbf{b})$ that has the zero vector as a solution. Upon substituting this solution into the system, we discover that each component of $\mathbf{b}$ must also be zero. So $\mathbf{b} = \mathbf{0}$.

**T20** Contributed by Robert Beezer Statement [58]
Suppose that a single equation from this system (the $i$-th one) has the form,

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n = 0$$

Evaluate the left-hand side of this equation with the components of the proposed solution vector $\mathbf{v}$,

$$a_{i1}(4u_1) + a_{i2}(4u_2) + a_{i3}(4u_3) + \cdots + a_{in}(4u_n) = 4a_{i1}u_1 + 4a_{i2}u_2 + 4a_{i3}u_3 + \cdots + 4a_{in}u_n$$

Commutativity

$$= 4(a_{i1}u_1 + a_{i2}u_2 + a_{i3}u_3 + \cdots + a_{in}u_n)$$

Distributivity

$$= 4(0)$$

$\mathbf{u}$ solution to $\mathcal{L}S(A, 0)$

$$= 0$$

So $\mathbf{v}$ makes each equation true, and so is a solution to the system.

Notice that this result is not true if we change $\mathcal{L}S(A, 0)$ from a homogeneous system to a non-homogeneous system. Can you create an example of a (non-homogeneous) system with a solution $\mathbf{u}$ such that $\mathbf{v}$ is not a solution?
In this section we specialize and consider matrices with equal numbers of rows and columns, which when considered as coefficient matrices lead to systems with equal numbers of equations and variables. We will see in the second half of the course (Chapter D [333], Chapter E [356], Chapter LT [405], Chapter R [473]) that these matrices are especially important.

Subsection NM
Nonsingular Matrices

Our theorems will now establish connections between systems of equations (homogeneous or otherwise), augmented matrices representing those systems, coefficient matrices, constant vectors, the reduced row-echelon form of matrices (augmented and coefficient) and solution sets. Be very careful in your reading, writing and speaking about systems of equations, matrices and sets of vectors. A system of equations is not a matrix, a matrix is not a solution set, and a solution set is not a system of equations. Now would be a great time to review the discussion about speaking and writing mathematics in Technique L [620].

Definition SQM
Square Matrix
A matrix with \( m \) rows and \( n \) columns is square if \( m = n \). In this case, we say the matrix has size \( n \). To emphasize the situation when a matrix is not square, we will call it rectangular. △

We can now present one of the central definitions of linear algebra.

Definition NM
Nonsingular Matrix
Suppose \( A \) is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations \( LS(A, 0) \) is \( \{0\} \), i.e. the system has only the trivial solution. Then we say that \( A \) is a nonsingular matrix. Otherwise we say \( A \) is a singular matrix. △

We can investigate whether any square matrix is nonsingular or not, no matter if the matrix is derived somehow from a system of equations or if it is simply a matrix. The definition says that to perform this investigation we must construct a very specific system of equations (homogeneous, with the matrix as the coefficient matrix) and look at its solution set. We will have theorems in this section that connect nonsingular matrices with systems of equations, creating more opportunities for confusion. Convince yourself now of two observations, (1) we can decide nonsingularity for any square matrix, and (2) the determination of nonsingularity involves the solution set for a certain homogenous system of equations.

Notice that it makes no sense to call a system of equations nonsingular (the term does not apply to a system of equations), nor does it make any sense to call a \( 5 \times 7 \) matrix singular (the matrix is not square).

Example S
A singular matrix, Archetype A
Example HISAA [53] shows that the coefficient matrix derived from Archetype A [634], specifically the \( 3 \times 3 \) matrix,

\[
A = \begin{bmatrix}
1 & -1 & 2 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]

is a singular matrix since there are nontrivial solutions to the homogeneous system \( LS(A, 0) \). ♦
A nonsingular matrix, Archetype B
Example HUSAB 52 shows that the coefficient matrix derived from Archetype B 638, specifically the \(3 \times 3\) matrix,
\[
B = \begin{bmatrix}
-7 & -6 & -12 \\
5 & 5 & 7 \\
1 & 0 & 4
\end{bmatrix}
\]
is a nonsingular matrix since the homogeneous system, \(LS(B, 0)\), has only the trivial solution. ☐

Notice that we will not discuss Example HISAD 53 as being a singular or nonsingular coefficient matrix since the matrix is not square.

The next theorem combines with our main computational technique (row-reducing a matrix) to make it easy to recognize a nonsingular matrix. But first a definition.

**Definition IM**

**Identity Matrix**
The \(m \times m\) identity matrix, \(I_m\), is defined by
\[
[I_m]_{ij} = \begin{cases} 
1 & i = j \\
0 & i \neq j
\end{cases}
\]
(This definition contains Notation IM.) △

**Example IM**

**An identity matrix**
The \(4 \times 4\) identity matrix is
\[
I_4 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Notice that an identity matrix is square, and in reduced row-echelon form. So in particular, if we were to arrive at the identity matrix while bringing a matrix to reduced row-echelon form, then it would have all of the diagonal entries circled as leading 1’s.

**Theorem NMRRI**

**Nonsingular Matrices Row Reduce to the Identity matrix**
Suppose that \(A\) is a square matrix and \(B\) is a row-equivalent matrix in reduced row-echelon form. Then \(A\) is nonsingular if and only if \(B\) is the identity matrix. □

**Proof**  
(\(\Leftarrow\)) Suppose \(B\) is the identity matrix. When the augmented matrix \([A | 0]\) is row-reduced, the result is \([B | 0] = [I_n | 0]\). The number of nonzero rows is equal to the number of variables in the linear system of equations \(LS(A, 0)\), so \(n = r\) and [Theorem FVCS 46] gives \(n - r = 0\) free variables. Thus, the homogeneous system \(LS(A, 0)\) has just one solution, which must be the trivial solution. This is exactly the definition of a nonsingular matrix.

(\(\Rightarrow\)) If \(A\) is nonsingular, then the homogeneous system \(LS(A, 0)\) has a unique solution, and has no free variables in the description of the solution set. The homogeneous system is consistent (Theorem HSC 52) so [Theorem FVCS 46] applies and tells us there are \(n - r\) free variables. Thus, \(n - r = 0\), and so \(n = r\). So \(B\) has \(n\) pivot columns among its total of \(n\) columns. This is enough to force \(B\) to be the \(n \times n\) identity matrix \(I_n\).

Notice that since this theorem is an equivalence it will always allow us to determine if a matrix is either nonsingular or singular. Here are two examples of this, continuing our study of Archetype A and Archetype B.

**Example SRR**

**Singular matrix, row-reduced**
The coefficient matrix for Archetype A \[634\] is

\[
A = \begin{bmatrix}
1 & -1 & 2 \\
2 & 1 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
\]

which when row-reduced becomes the row-equivalent matrix

\[
B = \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

Since this matrix is not the \(3 \times 3\) identity matrix, Theorem NMRRRI \[02\] tells us that \(A\) is a singular matrix.

Example NSR
Nonsingular matrix, row-reduced
The coefficient matrix for Archetype B \[638\] is

\[
A = \begin{bmatrix}
-7 & -6 & -12 \\
5 & 5 & 7 \\
1 & 0 & 4 \\
\end{bmatrix}
\]

which when row-reduced becomes the row-equivalent matrix

\[
B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

Since this matrix is the \(3 \times 3\) identity matrix, Theorem NMRRRI \[62\] tells us that \(A\) is a nonsingular matrix.

Subsection NSNM
Null Space of a Nonsingular Matrix

Nonsingular matrices and their null spaces are intimately related, as the next two examples illustrate.

Example NSS
Null space of a singular matrix
Given the coefficient matrix from Archetype A \[634\],

\[
A = \begin{bmatrix}
1 & -1 & 2 \\
2 & 1 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
\]

the null space is the set of solutions to the homogeneous system of equations \(\mathbf{LS}(A, \mathbf{0})\) has a solution set and null space constructed in Example HISAA \[53\] as

\[
N(A) = \left\{ \begin{bmatrix}
-x_3 \\
x_3 \\
x_3 \\
\end{bmatrix} \mid x_3 \in \mathbb{C} \right\}
\]

Example NSNM
Null space of a nonsingular matrix
Given the coefficient matrix from Archetype B, 
\[ A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \]
the homogeneous system \( \mathcal{L}S(A, 0) \) has a solution set constructed in Example HUSAB that contains only the trivial solution, so the null space has only a single element,
\[ \mathcal{N}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \]
These two examples illustrate the next theorem, which is another equivalence.

**Theorem NMTNS**
**Nonsingular Matrices have Trivial Null Spaces**
Suppose that \( A \) is a square matrix. Then \( A \) is nonsingular if and only if the null space of \( A, \mathcal{N}(A) \), contains only the zero vector, i.e. \( \mathcal{N}(A) = \{0\} \).

**Proof**  
The null space of a square matrix, \( A \), is equal to the set of solutions to the homogeneous system, \( \mathcal{L}S(A, 0) \). A matrix is nonsingular if and only if the set of solutions to the homogeneous system, \( \mathcal{L}S(A, 0) \), has only a trivial solution. These two observations may be chained together to construct the two proofs necessary for each half of this theorem.

The next theorem pulls a lot of ideas together. Two proof techniques are applicable to the proof. So first, head out and read two more proof techniques: Technique CD and Technique U. **Theorem NMUS** tells us that we can learn a lot about solutions to a system of linear equations with a square coefficient matrix by examining a similar homogeneous system.

**Theorem NMUS**
**Nonsingular Matrices and Unique Solutions**
Suppose that \( A \) is a square matrix. \( A \) is a nonsingular matrix if and only if the system \( \mathcal{L}S(A, b) \) has a unique solution for every choice of the constant vector \( b \).

**Proof**  
(\( \Rightarrow \)) The hypothesis for this half of the proof is that the system \( \mathcal{L}S(A, b) \) has a unique solution for every choice of the constant vector \( b \). We will make a very specific choice for \( b \): \( b = 0 \). Then we know that the system \( \mathcal{L}S(A, 0) \) has a unique solution. But this is precisely the definition of what it means for \( A \) to be nonsingular. That almost seems too easy! Notice that we have not used the full power of our hypothesis, but there is nothing that says we must use a hypothesis to its fullest.

If the first half of the proof seemed easy, perhaps we’ll have to work a bit harder to get the implication in the opposite direction. We provide two different proofs for the second half. The first is suggested by Asa Scherer and relies on the uniqueness of the reduced row-echelon form of a matrix, a result that we could have proven earlier, but we have decided to delay until later. The second proof is lengthier and more involved, but does not rely on the uniqueness of the reduced row-echelon form of a matrix, a result we have not proven yet. It is also a good example of the types of proofs we will encounter throughout the course.

(\( \Rightarrow \), Round 1) We assume that \( A \) is nonsingular, so we know there is a sequence of row operations that will convert \( A \) into the identity matrix \( I_n \). Form the augmented matrix \( A' = [A | b] \) and apply this same sequence of row operations to \( A' \). The result will be the matrix \( B' = [I_n | c] \), which is in reduced row-echelon form. It should be clear that \( c \) is a solution to \( \mathcal{L}S(A, b) \). Furthermore, since \( B' \) is unique, the vector \( c \) must be unique, and therefore is a unique solution of \( \mathcal{L}S(A, b) \).

(\( \Rightarrow \), Round 2) We will assume \( A \) is nonsingular, and try to solve the system \( \mathcal{L}S(A, b) \) without making any assumptions about \( b \). To do this we will begin by constructing a new homogeneous
linear system of equations that looks very much like the original. Suppose $A$ has size $n$ (why must it be square?) and write the original system as,

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
  a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
  \vdots \\
  a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{align*}
\]  

(*)

form the new, homogeneous system in $n$ equations with $n + 1$ variables, by adding a new variable $y$, whose coefficients are the negatives of the constant terms,

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n - b_1y &= 0 \\
  a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n - b_2y &= 0 \\
  a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n - b_3y &= 0 \\
  \vdots \\
  a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n - b_ny &= 0
\end{align*}
\]  

(**)

Since this is a homogeneous system with more variables than equations ($m = n + 1 > n$), Theorem [HMVEI] says that the system has infinitely many solutions. We will choose one of these solutions, any one of these solutions, so long as it is not the trivial solution. Write this solution as

\[
\begin{align*}
  x_1 &= c_1 \\
  x_2 &= c_2 \\
  x_3 &= c_3 \\
  \cdots \\
  x_n &= c_n \\
  y &= c_{n+1}
\end{align*}
\]

We know that at least one value of the $c_i$ is nonzero, but we will now show that in particular $c_{n+1} \neq 0$. We do this using a proof by contradiction (Technique CD [623]). So suppose the $c_i$ form a solution as described, and in addition that $c_{n+1} = 0$. Then we can write the $i$-th equation of system (**) as,

\[
a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3 + \cdots + a_{in}c_n - b_i(0) = 0
\]

which becomes

\[
a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3 + \cdots + a_{in}c_n = 0
\]

Since this is true for each $i$, we have that $x_1 = c_1$, $x_2 = c_2$, $x_3 = c_3$, \ldots, $x_n = c_n$ is a solution to the homogeneous system $\text{LS}(A, \mathbf{0})$ formed with a nonsingular coefficient matrix. This means that the only possible solution is the trivial solution, so $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, \ldots, $c_n = 0$. So, assuming simply that $c_{n+1} = 0$, we conclude that all of the $c_i$ are zero. But this contradicts our choice of the $c_i$ as not being the trivial solution to the system (**) . So $c_{n+1} \neq 0$.

We now propose and verify a solution to the original system (*). Set

\[
\begin{align*}
  x_1 &= \frac{c_1}{c_{n+1}} \\
  x_2 &= \frac{c_2}{c_{n+1}} \\
  x_3 &= \frac{c_3}{c_{n+1}} \\
  \cdots \\
  x_n &= \frac{c_n}{c_{n+1}}
\end{align*}
\]

Notice how it was necessary that we know that $c_{n+1} \neq 0$ for this step to succeed. Now, evaluate the $i$-th equation of system (*) with this proposed solution, and recognize in the third line that $c_1$ through $c_{n+1}$ appear as if they were substituted into the left-hand side of the $i$-th equation of system (**),

\[
\begin{align*}
  a_{i1} \frac{c_1}{c_{n+1}} + a_{i2} \frac{c_2}{c_{n+1}} + a_{i3} \frac{c_3}{c_{n+1}} + \cdots + a_{in} \frac{c_n}{c_{n+1}} &= \frac{1}{c_{n+1}} \left( a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3 + \cdots + a_{in}c_n \right)
\end{align*}
\]
Then, system \((\ast)\) has two solutions, 
\[
\begin{align*}
    x_1 & = d_1 & x_2 & = d_2 & x_3 & = d_3 & \ldots & x_n & = d_n \\
    x_1 & = e_1 & x_2 & = e_2 & x_3 & = e_3 & \ldots & x_n & = e_n \\
\end{align*}
\]
Then, 
\[
\begin{align*}
    (a_{i1}(d_1 - e_1) + a_{i2}(d_2 - e_2) + a_{i3}(d_3 - e_3) + \cdots + a_{in}(d_n - e_n)) \\
    = (a_{i1}d_1 + a_{i2}d_2 + a_{i3}d_3 + \cdots + a_{in}d_n) - (a_{i1}e_1 + a_{i2}e_2 + a_{i3}e_3 + \cdots + a_{in}e_n) \\
    = b_i - b_i \\
    = 0
\end{align*}
\]
This is the \(i\)-th equation of the homogeneous system \(\mathcal{L}\mathcal{S}(A, 0)\) evaluated with \(x_j = d_j - e_j, 1 \leq j \leq n\). Since \(A\) is nonsingular, we must conclude that this solution is the trivial solution, and so \(0 = d_j - e_j, 1 \leq j \leq n\). That is, \(d_j = e_j\) for all \(j\) and the two solutions are identical, meaning any solution to \((\ast)\) is unique.

This important theorem deserves several comments. First, notice that the proposed solution \((x_i = \frac{1}{c_{n+1}})\) appeared in the Round 2 proof with no motivation whatsoever. This is just fine in a proof. A proof should convince you that a theorem is true. It is your job to read the proof and be convinced of every assertion. Questions like “Where did that come from?” or “How would I think of that?” have no bearing on the validity of the proof.

Second, this theorem helps to explain part of our interest in nonsingular matrices. If a matrix is nonsingular, then no matter what vector of constants we pair it with, using the matrix as the coefficient matrix will always yield a linear system of equations with a solution, and the solution is unique. To determine if a matrix has this property (non-singularity) it is enough to just solve one linear system, the homogeneous system with the matrix as coefficient matrix and the zero vector as the vector of constants (or any other vector of constants, see Exercise MM.T10 [184]).

Finally, formulating the negation of the second part of this theorem is a good exercise. A singular matrix has the property that for some value of the vector \(b\), the system \(\mathcal{L}\mathcal{S}(A, b)\) does not have a unique solution (which means that it has no solution or infinitely many solutions). We will be able to say more about this case later (see the discussion following Theorem PSPHS [93]). Square matrices that are nonsingular have a long list of interesting properties, which we will start to catalog in the following, recurring, theorem. Of course, singular matrices will then have all of the opposite properties. The following theorem is a list of equivalences. We want to understand just what is involved with understanding and proving a theorem that says several conditions are equivalent. So have a look at Technique ME 624 before studying the first in this series of theorems.

**Theorem NME1**

**Nonsingular Matrix Equivalences, Round 1**

Suppose that \(A\) is a square matrix. The following are equivalent.

1. \(A\) is nonsingular.
2. \(A\) row-reduces to the identity matrix.
3. The null space of \(A\) contains only the zero vector, \(\mathcal{N}(A) = \{ 0 \} \).
4. The linear system $\mathcal{L}(A, b)$ has a unique solution for every possible choice of $b$.

**Proof** That $A$ is nonsingular is equivalent to each of the subsequent statements by, in turn, Theorem NMRRI [62], Theorem NMTNS [64] and Theorem NMUS [64]. So the statement of this theorem is just a convenient way to organize all these results.

---

**Subsection READ**

**Reading Questions**

1. What is the definition of a nonsingular matrix?
2. What is the easiest way to recognize a nonsingular matrix?
3. Suppose we have a system of equations and its coefficient matrix is nonsingular. What can you say about the solution set for this system?
Subsection EXC
Exercises

In Exercises C30–C33 determine if the matrix is nonsingular or singular. Give reasons for your answer.

C30

\[
\begin{bmatrix}
-3 & 1 & 2 & 8 \\
2 & 0 & 3 & 4 \\
1 & 2 & 7 & -4 \\
5 & -1 & 2 & 0
\end{bmatrix}
\]

Contributed by Robert Beezer  Solution 70

C31

\[
\begin{bmatrix}
2 & 3 & 1 & 4 \\
1 & 1 & 1 & 0 \\
-1 & 2 & 3 & 5 \\
1 & 2 & 1 & 3
\end{bmatrix}
\]

Contributed by Robert Beezer  Solution 70

C32

\[
\begin{bmatrix}
9 & 3 & 2 & 4 \\
5 & -6 & 1 & 3 \\
4 & 1 & 3 & -5
\end{bmatrix}
\]

Contributed by Robert Beezer  Solution 70

C33

\[
\begin{bmatrix}
-1 & 2 & 0 & 3 \\
1 & -3 & -2 & 4 \\
-2 & 0 & 4 & 3 \\
-3 & 1 & -2 & 3
\end{bmatrix}
\]

Contributed by Robert Beezer  Solution 70

C40  Each of the archetypes below is a system of equations with a square coefficient matrix, or is itself a square matrix. Determine if these matrices are nonsingular, or singular. Comment on the null space of each matrix.

Archetype A 634
Archetype B 638
Archetype F 654
Archetype K 676
Archetype L 680

Contributed by Robert Beezer

M30  Let \( A \) be the coefficient matrix of the system of equations below. Is \( A \) nonsingular or singular? Explain what you could infer about the solution set for the system based only on what you have learned about \( A \) being singular or nonsingular.

\[
-x_1 + 5x_2 = -8 \\
-2x_1 + 5x_2 + 5x_3 + 2x_4 = 9 \\
-3x_1 - x_2 + 3x_3 + x_4 = 3 \\
7x_1 + 6x_2 + 5x_3 + x_4 = 30
\]

Contributed by Robert Beezer  Solution 70
For Exercises M51–M52 say **as much as possible** about each system’s solution set. Be sure to make it clear which theorems you are using to reach your conclusions.

**M51** 6 equations in 6 variables, singular coefficient matrix.
Contributed by [Robert Beezer](#)  Solution [70]

**M52** A system with a nonsingular coefficient matrix, not homogeneous.
Contributed by [Robert Beezer](#)  Solution [70]

**T10** Suppose that $A$ is a singular matrix, and $B$ is a matrix in reduced row-echelon form that is row-equivalent to $A$. Prove that the last row of $B$ is a zero row.
Contributed by [Robert Beezer](#)  Solution [71]
The matrix row-reduces to
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
which is the $4\times4$ identity matrix. By Theorem NMRRI, the original matrix must be nonsingular.

Row-reducing the matrix yields,
\[
\begin{bmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
Since this is not the $4\times4$ identity matrix, Theorem NMRRI tells us the matrix is singular.

The matrix is not square, so neither term is applicable. See Definition NM, which is stated for just square matrices.

Theorem NMRRI tells us we can answer this question by simply row-reducing the matrix. Doing this we obtain,
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Since the reduced row-echelon form of the matrix is the $4\times4$ identity matrix $I_4$, we know that $B$ is nonsingular.

We row-reduce the coefficient matrix of the system of equations,
\[
\begin{bmatrix}
-1 & 5 & 0 & 0 \\
-2 & 5 & 5 & 2 \\
-3 & -1 & 3 & 1 \\
7 & 6 & 5 & 1
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Since the row-reduced version of the coefficient matrix is the $4\times4$ identity matrix, $I_4$, by Theorem NMRRI, we know the coefficient matrix is nonsingular. According to Theorem NMUS, we know that the system is guaranteed to have a unique solution, based only on the extra information that the coefficient matrix is nonsingular.

Theorem NMRRI tells us that the coefficient matrix will not row-reduce to the identity matrix. So if were to row-reduce the augmented matrix of this system of equations, we would not get a unique solution. So by Theorem PSSLS, there remaining possibilities are no solutions, or infinitely many.

Any system with a nonsingular coefficient matrix will have a unique solution by Theorem NMUS.
If the system is not homogeneous, the solution cannot be the zero vector (Exercise HSE.T10).

Contributed by Robert Beezer Statement

Let $n$ denote the size of the square matrix $A$. By Theorem NMRRI the hypothesis that $A$ is singular implies that $B$ is not the identity matrix $I_n$. If $B$ has $n$ pivot columns, then it would have to be $I_n$, so $B$ must have fewer than $n$ pivot columns. But the number of nonzero rows in $B$ is equal to the number of pivot columns as well. So the $n$ rows of $B$ have fewer than $n$ nonzero rows, and $B$ must contain at least one zero row. By Definition RREF, this row must be at the bottom of $B$. 
Chapter V
Vectors

We have worked extensively in the last chapter with matrices, and some with vectors. In this chapter we will develop the properties of vectors, while preparing to study vector spaces. Initially we will depart from our study of systems of linear equations, but in Section LC we will forge a connection between linear combinations and systems of linear equations in Theorem SLSLC. This connection will allow us to understand systems of linear equations at a higher level, while consequently discussing them less frequently.

Section VO
Vector Operations

In this section we define some new operations involving vectors, and collect some basic properties of these operations. Begin by recalling our definition of a column vector as an ordered list of complex numbers, written vertically (Definition CV). The collection of all possible vectors of a fixed size is a commonly used set, so we start with its definition.

Definition VSCV
Vector Space of Column Vectors
The vector space $\mathbb{C}^m$ is the set of all column vectors of size $m$ with entries from the set of complex numbers, $\mathbb{C}$.
(This definition contains Notation VSCV.)

When a set similar to this is defined using only column vectors where all the entries are from the real numbers, it is written as $\mathbb{R}^m$ and is known as Euclidean $m$-space.

The term “vector” is used in a variety of different ways. We have defined it as an ordered list written vertically. It could simply be an ordered list of numbers, and written as $(2, 3, -1, 6)$. Or it could be interpreted as a point in $m$ dimensions, such as $(3, 4, -2)$ representing a point in three dimensions relative to $x$, $y$ and $z$ axes. With an interpretation as a point, we can construct an arrow from the origin to the point which is consistent with the notion that a vector has direction and magnitude.

All of these ideas can be shown to be related and equivalent, so keep that in mind as you connect the ideas of this course with ideas from other disciplines. For now, we’ll stick with the idea that a vector is a just a list of numbers, in some particular order.

Subsection VEASM
Vector Equality, Addition, Scalar Multiplication

We start our study of this set by first defining what it means for two vectors to be the same.
Definition CVE

Column Vector Equality
Suppose that \( u, v \in \mathbb{C}^m \). Then \( u \) and \( v \) are \textbf{equal}, written \( u = v \) if

\[
[u]_i = [v]_i \quad \text{for} \quad 1 \leq i \leq m
\]

(This definition contains Notation CVE.)

Now this may seem like a silly (or even stupid) thing to say so carefully. Of course two vectors are equal if they are equal for each corresponding entry! Well, this is not as silly as it appears. We will see a few occasions later where the obvious definition is \textit{not} the right one. And besides, in doing mathematics we need to be very careful about making all the necessary definitions and making them unambiguous. And we’ve done that here.

Notice now that the symbol ‘\( = \)’ is now doing triple-duty. We know from our earlier education what it means for two numbers (real or complex) to be equal, and we take this for granted. In Definition SE we defined what it meant for two sets to be equal. Now we have defined what it means for two vectors to be equal, and that definition builds on our definition for when two numbers are equal when we use the condition \( u_i = v_i \) for all \( 1 \leq i \leq m \). So think carefully about your objects when you see an equal sign and think about just which notion of equality you have encountered. This will be especially important when you are asked to construct proofs whose conclusion states that two objects are equal.

OK, let’s do an example of vector equality that begins to hint at the utility of this definition.

Example VESE

Vector equality for a system of equations
Consider the system of linear equations in Archetype B,

\[
\begin{align*}
-7x_1 - 6x_2 - 12x_3 &= -33 \\
5x_1 + 5x_2 + 7x_3 &= 24 \\
x_1 + 4x_3 &= 5
\end{align*}
\]

Note the use of three equals signs — each indicates an equality of numbers (the linear expressions are numbers when we evaluate them with fixed values of the variable quantities). Now write the vector equality,

\[
\begin{bmatrix}
-7x_1 - 6x_2 - 12x_3 \\
5x_1 + 5x_2 + 7x_3 \\
x_1 + 4x_3
\end{bmatrix} =
\begin{bmatrix}
-33 \\
24 \\
5
\end{bmatrix}.
\]

By Definition CVE, this \textit{single} equality (of two column vectors) translates into \textit{three} simultaneous equalities of numbers that form the system of equations. So with this new notion of vector equality we can become less reliant on referring to systems of simultaneous equations. There’s more to vector equality than just this, but this is a good example for starters and we will develop it further.

We will now define two operations on the set \( \mathbb{C}^m \). By this we mean well-defined procedures that somehow convert vectors into other vectors. Here are two of the most basic definitions of the entire course.

Definition CVA

Column Vector Addition
Suppose that \( u, v \in \mathbb{C}^m \). The \textbf{sum} of \( u \) and \( v \) is the vector \( u + v \) defined by

\[
[u + v]_i = [u]_i + [v]_i \quad \text{for} \quad 1 \leq i \leq m
\]

(This definition contains Notation CVA.)

So vector addition takes two vectors of the same size and combines them (in a natural way!) to create a new vector of the same size. Notice that this definition is required, even if we agree
that this is the obvious, right, natural or correct way to do it. Notice too that the symbol ‘+’ is being recycled. We all know how to add numbers, but now we have the same symbol extended to double-duty and we use it to indicate how to add two new objects, vectors. And this definition of our new meaning is built on our previous meaning of addition via the expressions $u_i + v_i$. Think about your objects, especially when doing proofs. Vector addition is easy, here’s an example from $\mathbb{C}^4$.

**Example VA**

**Addition of two vectors in** $\mathbb{C}^4$

If

$$u = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} -1 \\ 5 \\ 2 \\ -7 \end{bmatrix}$$

then

$$u + v = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \\ 2 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 + (-1) \\ 2 + 5 \\ 4 + 2 \\ 2 + (-7) \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 6 \\ -5 \end{bmatrix}.$$  

Our second operation takes two objects of different types, specifically a number and a vector, and combines them to create another vector. In this context we call a number a **scalar** in order to emphasize that it is not a vector.

**Definition CVSM**

**Column Vector Scalar Multiplication**

Suppose $u \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$, then the **scalar multiple** of $u$ by $\alpha$ is the vector $\alpha u$ defined by

$$[\alpha u]_i = \alpha [u]_i \quad 1 \leq i \leq m$$

(This definition contains Notation CVSM.)

Notice that we are doing a kind of multiplication here, but we are **defining** a new type, perhaps in what appears to be a natural way. We use juxtaposition (smashing two symbols together side-by-side) to denote this operation rather than using a symbol like we did with vector addition. So this can be another source of confusion. When two symbols are next to each other, are we doing regular old multiplication, the kind we’ve done for years, or are we doing scalar vector multiplication, the operation we just defined? Think about your objects — if the first object is a scalar, and the second is a vector, then it **must** be that we are doing our new operation, and the **result** of this operation will be another vector.

Notice how consistency in notation can be an aid here. If we write scalars as lower case Greek letters from the start of the alphabet (such as $\alpha$, $\beta$, . . .) and write vectors in bold Latin letters from the end of the alphabet ($u$, $v$, . . .), then we have some hints about what type of objects we are working with. This can be a blessing and a curse, since when we go read another book about linear algebra, or read an application in another discipline (physics, economics, . . .) the types of notation employed may be very different and hence unfamiliar.

Again, computationally, vector scalar multiplication is very easy.

**Example CVSM**

**Scalar multiplication in** $\mathbb{C}^5$

If

$$u = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 4 \\ -1 \end{bmatrix}$$

Version 1.04
and $\alpha = 6$, then

$$\alpha u = 6 \begin{bmatrix} 3 \\ 1 \\ -2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 6(3) \\ 6(1) \\ 6(-2) \\ 6(4) \\ 6(-1) \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \\ -12 \\ 24 \\ -6 \end{bmatrix}.$$

Vector addition and scalar multiplication are the most natural and basic operations to perform on vectors, so it should be easy to have your computational device form a linear combination. See: Computation VLC.MMA 605, Computation VLC.TI86 609, Computation VLC.TI83 610.

Subsection VSP
Vector Space Properties

With definitions of vector addition and scalar multiplication we can state, and prove, several properties of each operation, and some properties that involve their interplay. We now collect ten of them here for later reference.

Theorem VSPCV
Vector Space Properties of Column Vectors
Suppose that $\mathbb{C}^m$ is the set of column vectors of size $m$ (Definition VSCV 72) with addition and scalar multiplication as defined in Definition CVA 73 and Definition CVSM 74. Then

- **ACC** Additive Closure, Column Vectors
  If $u, v \in \mathbb{C}^m$, then $u + v \in \mathbb{C}^m$.

- **SCC** Scalar Closure, Column Vectors
  If $\alpha \in \mathbb{C}$ and $u \in \mathbb{C}^m$, then $\alpha u \in \mathbb{C}^m$.

- **CC** Commutativity, Column Vectors
  If $u, v \in \mathbb{C}^m$, then $u + v = v + u$.

- **AAC** Additive Associativity, Column Vectors
  If $u, v, w \in \mathbb{C}^m$, then $u + (v + w) = (u + v) + w$.

- **ZC** Zero Vector, Column Vectors
  There is a vector, $0$, called the zero vector, such that $u + 0 = u$ for all $u \in \mathbb{C}^m$.

- **AIC** Additive Inverses, Column Vectors
  If $u \in \mathbb{C}^m$, then there exists a vector $-u \in \mathbb{C}^m$ so that $u + (-u) = 0$.

- **SMAC** Scalar Multiplication Associativity, Column Vectors
  If $\alpha, \beta \in \mathbb{C}$ and $u \in \mathbb{C}^m$, then $\alpha(\beta u) = (\alpha\beta)u$.

- **DVAC** Distributivity across Vector Addition, Column Vectors
  If $\alpha \in \mathbb{C}$ and $u, v \in \mathbb{C}^m$, then $\alpha(u + v) = \alpha u + \alpha v$.

- **DSAC** Distributivity across Scalar Addition, Column Vectors
  If $\alpha, \beta \in \mathbb{C}$ and $u \in \mathbb{C}^m$, then $(\alpha + \beta)u = \alpha u + \beta u$.

- **OC** One, Column Vectors
  If $u \in \mathbb{C}^m$, then $1u = u$. 
Proof While some of these properties seem very obvious, they all require proof. However, the proofs are not very interesting, and border on tedious. We’ll prove one version of distributivity very carefully, and you can test your proof-building skills on some of the others. We need to establish an equality, so we will do so by beginning with one side of the equality, apply various definitions and theorems (listed to the right of each step) to massage the expression from the left into the expression on the right. Now would be a good time to read Technique PI, just below. Here we go with a proof of Property DSAC. For $1 \leq i \leq m$,

$\begin{align*}
[(\alpha + \beta)\mathbf{u}]_i &= (\alpha + \beta) [\mathbf{u}]_i \\
&= \alpha [\mathbf{u}]_i + \beta [\mathbf{u}]_i \\
&= [\alpha \mathbf{u}]_i + [\beta \mathbf{u}]_i \\
&= [\alpha \mathbf{u} + \beta \mathbf{u}]_i
\end{align*}$

Definition CVSM
Distributivity in $\mathbb{C}$
Definition CVSM
Definition CVA

Since the individual components of the vectors $(\alpha + \beta)\mathbf{u}$ and $\alpha \mathbf{u} + \beta \mathbf{u}$ are equal for all $i$, $1 \leq i \leq m$, Definition CVE tells us the vectors are equal.

Many of the conclusions of our theorems can be characterized as “identities,” especially when we are establishing basic properties of operations such as those in this section. So some advice about the style we use for proving identities is appropriate right now. Have a look at Technique PI.

Be careful with the notion of the vector $-\mathbf{u}$. This is a vector that we add to $\mathbf{u}$ so that the result is the particular vector $\mathbf{0}$. This is basically a property of vector addition. It happens that we can compute $-\mathbf{u}$ using the other operation, scalar multiplication. We can prove this directly by writing that

$[-\mathbf{u}]_i = -[\mathbf{u}]_i = (-1) [\mathbf{u}]_i = [(−1)\mathbf{u}]_i$

We will see later how to derive this property as a consequence of several of the ten properties listed in Theorem VSPCV.

Subsection READ
Reading Questions

1. Where have you seen vectors used before in other courses? How were they different?
2. In words, when are two vectors equal?
3. Perform the following computation with vector operations

$$
2 \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}
$$
Subsection EXC
Exercises

C10  Compute

\[
\begin{bmatrix}
2 & 1 & -1 \\
-3 & 2 & 3 \\
4 & -5 & 0 \\
1 & 2 & 1 \\
0 & 4 & 2
\end{bmatrix}
\]

Contributed by Robert Beezer  Solution 78

T13  Prove Property CC 75 of Theorem VSPCV 75. Write your proof in the style of the proof of Property DSAC 75 given in this section.
Contributed by Robert Beezer  Solution 78

T17  Prove Property SMAC 75 of Theorem VSPCV 75. Write your proof in the style of the proof of Property DSAC 75 given in this section.
Contributed by Robert Beezer

T18  Prove Property DVAC 75 of Theorem VSPCV 75. Write your proof in the style of the proof of Property DSAC 75 given in this section.
Contributed by Robert Beezer
Subsection SOL
Solutions

\[ \begin{bmatrix} 5 \\ -13 \\ 26 \\ 1 \end{bmatrix} \]

Contributed by Robert Beezer

For all \( 1 \leq i \leq m \),

\[
[u + v]_i = [u]_i + [v]_i \\
= [v]_i + [u]_i \\
= [v + u]_i
\]

With equality of each component of the vectors \( u + v \) and \( v + u \) being equal, Definition CVE tells us the two vectors are equal.

Definition CVA

Commutativity in \( \mathbb{C} \)

Definition CVA
In Section VO we defined vector addition and scalar multiplication. These two operations combine nicely to give us a construction known as a linear combination, a construct that we will work with throughout this course.

Definition LCCV
Linear Combination of Column Vectors
Given \( n \) vectors \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n \) from \( \mathbb{C}^m \) and \( n \) scalars \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \), their linear combination is the vector

\[
\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n.
\]

So this definition takes an equal number of scalars and vectors, combines them using our two new operations (scalar multiplication and vector addition) and creates a single brand-new vector, of the same size as the original vectors. When a definition or theorem employs a linear combination, think about the nature of the objects that go into its creation (lists of scalars and vectors), and the type of object that results (a single vector). Computationally, a linear combination is pretty easy.

Example TLC
Two linear combinations in \( \mathbb{C}^6 \)
Suppose that

\[
\alpha_1 = 1 \quad \alpha_2 = -4 \quad \alpha_3 = 2 \quad \alpha_4 = -1
\]

and

\[
\mathbf{u}_1 = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 1 \\ 2 \\ 9 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 6 \\ 3 \\ -2 \\ 1 \\ 1 \\ 4 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} -5 \\ 2 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{u}_4 = \begin{bmatrix} 3 \\ 2 \\ 7 \\ 1 \end{bmatrix}
\]

then their linear combination is

\[
\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \alpha_4 \mathbf{u}_4 = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 1 \\ 2 \\ 9 \end{bmatrix} + (-4) \begin{bmatrix} 6 \\ 3 \\ -2 \\ 1 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} -5 \\ 2 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 2 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} -35 \\ -6 \end{bmatrix}.
\]
A different linear combination, of the same set of vectors, can be formed with different scalars. Take

\[ \beta_1 = 3 \quad \beta_2 = 0 \quad \beta_3 = 5 \quad \beta_4 = -1 \]

and form the linear combination

\[ \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \beta_3 \mathbf{u}_3 + \beta_4 \mathbf{u}_4 = (3) \begin{bmatrix} 2 \\ 4 \\ -3 \\ 1 \\ 9 \\ 12 \end{bmatrix} + (0) \begin{bmatrix} 6 \\ 3 \\ 0 \\ 1 \\ 4 \\ 0 \end{bmatrix} + (5) \begin{bmatrix} -5 \\ -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 2 \\ 1 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ -9 \\ -25 \\ 6 \\ 27 \end{bmatrix}. \]

Notice how we could keep our set of vectors fixed, and use different sets of scalars to construct different vectors. You might build a few new linear combinations of \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \) right now. We’ll be right here when you get back. What vectors were you able to create? Do you think you could create the vector \( \mathbf{w} = \begin{bmatrix} 13 \\ 15 \\ 5 \\ -17 \\ 2 \\ 25 \end{bmatrix} \) with a “suitable” choice of four scalars? Do you think you could create any possible vector from \( \mathbb{C}^6 \) by choosing the proper scalars? These last two questions are very fundamental, and time spent considering them now will prove beneficial later.

Our next two examples are key ones, and a discussion about decompositions is timely. Have a look at [Technique DC 625] before studying the next two examples.

**Example ABLC**

**Archetype B as a linear combination**

In this example we will rewrite [Archetype B 638] in the language of vectors, vector equality and linear combinations. In [Example VESE 73] we wrote the system of \( m = 3 \) equations as the vector equality

\[ \begin{bmatrix} -7x_1 - 6x_2 - 12x_3 \\ 5x_1 + 5x_2 + 7x_3 \\ x_1 + 4x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}. \]

Now we will bust up the linear expressions on the left, first using vector addition,

\[ \begin{bmatrix} -7x_1 \\ 5x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} -6x_2 \\ 5x_2 \\ 0x_2 \end{bmatrix} + \begin{bmatrix} -12x_3 \\ 7x_3 \\ 4x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}. \]

Now we can rewrite each of these \( n = 3 \) vectors as a scalar multiple of a fixed vector, where the scalar is one of the unknown variables, converting the left-hand side into a linear combination

\[ x_1 \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}. \]
We can now interpret the problem of solving the system of equations as determining values for the scalar multiples that make the vector equation true. In the analysis of Archetype B, we were able to determine that it had only one solution. A quick way to see this is to row-reduce the coefficient matrix to the $3 \times 3$ identity matrix and apply Theorem NMRRRI to determine that the coefficient matrix is nonsingular. Then Theorem NMUS tells us that the system of equations has a unique solution. This solution is

$$x_1 = -3 \quad x_2 = 5 \quad x_3 = 2.$$  

So, in the context of this example, we can express the fact that these values of the variables are a solution by writing the linear combination,

$$(-3) \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix} + (5) \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}. $$

Furthermore, these are the only three scalars that will accomplish this equality, since they come from a unique solution.

Notice how the three vectors in this example are the columns of the coefficient matrix of the system of equations. This is our first hint of the important interplay between the vectors that form the columns of a matrix, and the matrix itself.

With any discussion of Archetype A or Archetype B we should be sure to contrast with the other.

**Example AALC**

**Archetype A as a linear combination**

As a vector equality, Archetype A can be written as

$$\begin{bmatrix} x_1 - x_2 + 2x_3 \\ 2x_1 + x_2 + x_3 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix}. $$

Now bust up the linear expressions on the left, first using vector addition,

$$\begin{bmatrix} x_1 \\ 2x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2x_3 \\ x_3 \\ 0x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix}. $$

Rewrite each of these $n = 3$ vectors as a scalar multiple of a fixed vector, where the scalar is one of the unknown variables, converting the left-hand side into a linear combination

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix}. $$

Row-reducing the augmented matrix for Archetype A leads to the conclusion that the system is consistent and has free variables, hence infinitely many solutions. So for example, the two solutions

$$x_1 = 2 \quad \quad x_2 = 3 \quad \quad x_3 = 1$$
$$x_1 = 3 \quad \quad x_2 = 2 \quad \quad x_3 = 0$$

can be used together to say that,

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + (3) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix} = (3) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (0) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}. $$
Ignore the middle of this equation, and move all the terms to the left-hand side,

\[
(2) \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \\ 0 \\ -3 \\ 2 \\ 1 \\ 1 \\ 0 \\ -2 \\ 1 \\ 1 \\ 0 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (3) \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ -2 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + (0) \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

Regrouping gives

\[
(\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}) + (1) \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

Notice that these three vectors are the columns of the coefficient matrix for the system of equations in [Archetype A] 634. This equality says there is a linear combination of those columns that equals the vector of all zeros. Give it some thought, but this says that

\[
x_1 = -1 \quad x_2 = 1 \quad x_3 = 1
\]

is a nontrivial solution to the homogeneous system of equations with the coefficient matrix for the original system in [Archetype A] 634. In particular, this demonstrates that this coefficient matrix is singular.

There’s a lot going on in the last two examples. Come back to them in a while and make some connections with the intervening material. For now, we will summarize and explain some of this behavior with a theorem.

**Theorem SLSLC**

**Solutions to Linear Systems are Linear Combinations**

Denote the columns of the \(m \times n\) matrix \(A\) as the vectors \(A_1, A_2, A_3, \ldots, A_n\). Then \(x\) is a solution to the linear system of equations \(LS(A, b)\) if and only if

\[
[x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n = b
\]

**Proof** The proof of this theorem is as much about a change in notation as it is about making logical deductions. Write the system of equations \(LS(A, b)\) as

\[
\begin{align*}
 a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
 a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
 &\vdots \\
 a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m.
\end{align*}
\]

Notice then that the entry of the coefficient matrix \(A\) in row \(i\) and column \(j\) has two names: \(a_{ij}\) as the coefficient of \(x_j\) in equation \(i\) of the system and \([A_j]_i\) as the \(i\)-th entry of the column vector in column \(j\) of the coefficient matrix \(A\). Likewise, entry \(i\) of \(b\) has two names: \(b_i\) from the linear system and \([b]_i\) as an entry of a vector. Our theorem is an equivalence [Technique E 622] so we need to prove both “directions.”

\((\Leftarrow)\) Suppose we have the vector equality between \(b\) and the linear combination of the columns of \(A\). Then for \(1 \leq i \leq n\),

\[
b_i = [b]_i = \begin{cases} 
[x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n \\
[x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n \\
[x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n \\
[x]_1 a_{11} + [x]_2 a_{12} + [x]_3 a_{13} + \cdots + [x]_n a_{1n}
\end{cases}
\]

**Notation**

Hypothesis

[Definition CVA 73]

[Definition CVSM 74]
This says that the entries of \( \mathbf{x} \) form a solution to equation \( i \) of \( \mathcal{L}\mathcal{S}(\mathbf{A}, \mathbf{b}) \) for all \( 1 \leq i \leq n \), i.e. \( \mathbf{x} \) is a solution to \( \mathcal{L}\mathcal{S}(\mathbf{A}, \mathbf{b}) \).

\[
\begin{align*}
[b]_i &= b_i \\
&= a_{i1} [x]_1 + a_{i2} [x]_2 + a_{i3} [x]_3 + \cdots + a_{in} [x]_n \\
&= [x]_1 a_{i1} + [x]_2 a_{i2} + [x]_3 a_{i3} + \cdots + [x]_n a_{in} \\
&= [x]_1 [\mathbf{A}_1]_i + [x]_2 [\mathbf{A}_2]_i + [x]_3 [\mathbf{A}_3]_i + \cdots + [x]_n [\mathbf{A}_n]_i \\
&= [[x]_1 \mathbf{A}_1]_i + [[x]_2 \mathbf{A}_2]_i + [[x]_3 \mathbf{A}_3]_i + \cdots + [[x]_n \mathbf{A}_n]_i
\end{align*}
\]

Sinc the components of \( \mathbf{b} \) and the linear combination of the columns of \( \mathbf{A} \) agree for all \( 1 \leq i \leq n \), \text{Definition CVE} [73] tells us that the vectors are equal.

In other words, this theorem tells us that solutions to systems of equations are linear combinations of the column vectors of the coefficient matrix \( (\mathbf{A}_i) \) which yield the constant vector \( \mathbf{b} \). Or said another way, a solution to a system of equations \( \mathcal{L}\mathcal{S}(\mathbf{A}, \mathbf{b}) \) is an answer to the question “How can I form the vector \( \mathbf{b} \) as a linear combination of the columns of \( \mathbf{A} \)?” Look through the archetypes that are systems of equations and examine a few of the advertised solutions. In each case use the solution to form a linear combination of the columns of the coefficient matrix and verify that the result equals the constant vector (see \text{Exercise LC.C21} [98]).
in each row yields the vector equality,

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} = \begin{bmatrix}
    4 - 3x_3 + 2x_4 \\
    -x_3 + 3x_4 \\
    x_3 \\
    x_4
\end{bmatrix}
\]

Now we will use the definitions of column vector addition and scalar multiplication to express this vector as a linear combination,

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} = \begin{bmatrix}
    4 \\
    0 \\
    0 \\
    0
\end{bmatrix} + x_3 \begin{bmatrix}
    -3 \\
    -1 \\
    1 \\
    0
\end{bmatrix} + x_4 \begin{bmatrix}
    2 \\
    3 \\
    0 \\
    1
\end{bmatrix}
\]

We will develop the same linear combination a bit quicker, using three steps. While the method above is instructive, the method below will be our preferred approach.

Step 1. Write the vector of variables as a fixed vector, plus a linear combination of \(n - r\) vectors, using the free variables as the scalars.

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} = \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} = \begin{bmatrix}
    4 \\
    0 \\
    0 \\
    0
\end{bmatrix} + x_3 \begin{bmatrix}
    -3 \\
    -1 \\
    1 \\
    0
\end{bmatrix} + x_4 \begin{bmatrix}
    2 \\
    3 \\
    0 \\
    1
\end{bmatrix}
\]

Step 2. Use 0’s and 1’s to ensure equality for the entries of the vectors with indices in \(F\) (corresponding to the free variables).

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} = \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0
\end{bmatrix} + x_3 \begin{bmatrix}
    1 \\
    0 \\
    0 \\
    1
\end{bmatrix} + x_4 \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0
\end{bmatrix}
\]

Step 3. For each dependent variable, use the augmented matrix to formulate an equation expressing the dependent variable as a constant plus multiples of the free variables. Convert this equation into entries of the vectors that ensure equality for each dependent variable, one at a time.

\[
x_1 = 4 - 3x_3 + 2x_4 \quad \Rightarrow \quad \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} = \begin{bmatrix}
    4 \\
    0 \\
    -3 \\
    -1
\end{bmatrix} + x_3 \begin{bmatrix}
    1 \\
    0 \\
    1 \\
    0
\end{bmatrix} + x_4 \begin{bmatrix}
    2 \\
    0 \\
    3 \\
    1
\end{bmatrix}
\]

\[
x_2 = 0 - 1x_3 + 3x_4 \quad \Rightarrow \quad \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4
\end{bmatrix} = \begin{bmatrix}
    4 \\
    0 \\
    -3 \\
    -1
\end{bmatrix} + x_3 \begin{bmatrix}
    1 \\
    0 \\
    1 \\
    0
\end{bmatrix} + x_4 \begin{bmatrix}
    2 \\
    0 \\
    3 \\
    1
\end{bmatrix}
\]

This final form of a typical solution is especially pleasing and useful. For example, we can build solutions quickly by choosing values for our free variables, and then compute a linear combination.
Such as
\[ x_3 = 2, \quad x_4 = -5 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + (-5) \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ -17 \\ 2 \\ -5 \end{bmatrix} \]
or,
\[ x_3 = 1, \quad x_4 = 3 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + (3) \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \\ 3 \end{bmatrix} . \]

You’ll find the second solution listed in the write-up for Archetype D [647], and you might check the first solution by substituting it back into the original equations.

While this form is useful for quickly creating solutions, its even better because it tells us exactly what every solution looks like. We know the solution set is infinite, which is pretty big, but now we can say that a solution is some multiple of \( \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} \) plus a multiple of \( \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \) plus the fixed vector \( \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \). Period. So it only takes us three vectors to describe the entire infinite solution set, provided we also agree on how to combine the three vectors into a linear combination.

This is such an important and fundamental technique, we’ll do another example.

**Example VFS**

**Vector form of solutions**

Consider a linear system of \( m = 5 \) equations in \( n = 7 \) variables, having the augmented matrix \( A \).

\[
A = \begin{bmatrix}
  2 & 1 & -1 & -2 & 2 & 1 & 5 \\
  1 & 1 & -3 & 1 & 1 & 1 & 2 \\
  1 & 2 & -8 & 5 & 1 & 1 & -6 \\
  3 & 3 & -9 & 3 & 6 & 5 & 2 \\
 -2 & -1 & 1 & 2 & 1 & 1 & -9 \\
\end{bmatrix}
\]

Row-reducing we obtain the matrix

\[
B = \begin{bmatrix}
  1 & 0 & 2 & -3 & 0 & 0 & 9 \\
  0 & 0 & 0 & 0 & 1 & 0 & -10 \\
  0 & 0 & 0 & 0 & 0 & 0 & 11 \\
  0 & 0 & 0 & 0 & 0 & 0 & -21 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

and we see \( r = 4 \) nonzero rows. Also, \( D = \{1, 2, 5, 6\} \) so the dependent variables are then \( x_1, x_2, x_5, \) and \( x_6 \). \( F = \{3, 4, 7, 8\} \) so the \( n - r = 3 \) free variables are \( x_3, x_4 \) and \( x_7 \). We will express a generic solution for the system by two different methods: both a decomposition and a construction.

First, we will decompose (Technique DC [625]) a solution vector. Rearranging each equation represented in the row-reduced form of the augmented matrix by solving for the dependent variable...
Subsection LC.VFSS Vector Form of Solution Sets

in each row yields the vector equality,

\[
\begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5 \\
 x_6 \\
 x_7
\end{bmatrix} = \begin{bmatrix}
 15 - 2x_3 + 3x_4 - 9x_7 \\
 -10 + 5x_3 - 4x_4 + 8x_7 \\
 x_3 \\
 x_4 \\
 11 + 6x_7 \\
 -21 - 7x_7 \\
 x_7
\end{bmatrix}
\]

Now we will use the definitions of column vector addition and scalar multiplication to decompose this generic solution vector as a linear combination,

\[
\begin{bmatrix}
 15 \\
 -10 \\
 0 \\
 0 \\
 11 \\
 -21 \\
 0
\end{bmatrix} + x_3 \begin{bmatrix}
 -2x_3 \\
 5x_3 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
\end{bmatrix} + x_4 \begin{bmatrix}
 3x_4 \\
 -4x_4 \\
 0 \\
 0 \\
 6x_7 \\
 0 \\
 0
\end{bmatrix} + x_7 \begin{bmatrix}
 -9x_7 \\
 8x_7 \\
 0 \\
 0 \\
 -7x_7 \\
 0 \\
 x_7
\end{bmatrix}
\]

\[
\begin{bmatrix}
 15 \\
 -10 \\
 0 \\
 0 \\
 11 \\
 -21 \\
 0
\end{bmatrix} \begin{bmatrix}
 -2 \\
 5 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
\end{bmatrix} + x_3 \begin{bmatrix}
 3 \\
 -4 \\
 0 \\
 0 \\
 6 \\
 0 \\
 0
\end{bmatrix} + x_4 \begin{bmatrix}
 0 \\
 1 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0
\end{bmatrix} + x_7 \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 1
\end{bmatrix}
\]

We will now develop the same linear combination a bit quicker, using three steps. While the method above is instructive, the method below will be our preferred approach.

Step 1. Write the vector of variables as a fixed vector, plus a linear combination of \( n-r \) vectors, using the free variables as the scalars.

\[
x = \begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4 \\
 x_5 \\
 x_6 \\
 x_7
\end{bmatrix} = \begin{bmatrix}
 x_3 \\
 x_4 \\
 x_7
\end{bmatrix} + x_3 \begin{bmatrix}
 1 \\
 0 \\
 0
\end{bmatrix} + x_4 \begin{bmatrix}
 0 \\
 1 \\
 0
\end{bmatrix} + x_7 \begin{bmatrix}
 0 \\
 0 \\
 1
\end{bmatrix}
\]

Step 2. Use 0’s and 1’s to ensure equality for the entries of the the vectors with indices in \( F \) (corresponding to the free variables).

Step 3. For each dependent variable, use the augmented matrix to formulate an equation expressing the dependent variable as a constant plus multiples of the free variables. Convert this equation...
into entries of the vectors that ensure equality for each dependent variable, one at a time.

\[
x_1 = 15 - 2x_3 + 3x_4 - 9x_7 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -2 \\ 3 \\ -9 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
x_2 = -10 + 5x_3 - 4x_4 + 8x_7 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -2 \\ 3 \\ -9 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
x_5 = 11 + 6x_7 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -2 \\ 3 \\ -9 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
x_6 = -21 - 7x_7 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -2 \\ 3 \\ -9 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

This final form of a typical solution is especially pleasing and useful. For example, we can build solutions quickly by choosing values for our free variables, and then compute a linear combination. For example

\[
x_3 = 2, \ x_4 = -4, \ x_7 = 3 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -2 \\ 3 \\ -9 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

or perhaps,

\[
x_3 = 5, \ x_4 = 2, \ x_7 = 1 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -2 \\ 3 \\ -9 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]
or even,

\[ x_3 = 0, \ x_4 = 0, \ x_7 = 0 \quad \Rightarrow \]

\[
\begin{bmatrix}
0 & 15 & -2 & 3 & -9 & 15 \\
-10 & 5 & -4 & 8 & -10 \\
0 & 0 & 1 & 0 & 0 & 0 \\
11 & 0 & 0 & 6 & 11 \\
-21 & 0 & 0 & -7 & -21 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

So we can compactly express all of the solutions to this linear system with just 4 fixed vectors, provided we agree how to combine them in a linear combinations to create solution vectors.

Suppose you were told that the vector \( \mathbf{w} \) below was a solution to this system of equations. Could you turn the problem around and write \( \mathbf{w} \) as a linear combination of the four vectors \( \mathbf{c}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \)? (See Exercise LC.M11 [99].)

\[
\begin{bmatrix}
100 \\
-75 \\
7 \\
9 \\
-37 \\
35 \\
-8
\end{bmatrix}
= 
\begin{bmatrix}
15 \\
-10 \\
0 \\
0 \\
11 \\
-21 \\
0
\end{bmatrix}
\begin{bmatrix}
-2 \\
5 \\
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
3 \\
-4 \\
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
-9 \\
8 \\
0 \\
0 \\
6 \\
-7 \\
1
\end{bmatrix}
\]

Did you think a few weeks ago that you could so quickly and easily list all the solutions to a linear system of 5 equations in 7 variables?

We’ll now formalize the last two (important) examples as a theorem.

**Theorem VFSLS**

**Vector Form of Solutions to Linear Systems**

Suppose that \( |A| \ |b| \) is the augmented matrix for a consistent linear system \( LS(A, b) \) of \( m \) equations in \( n \) variables. Let \( B \) be a row-equivalent \( m \times (n+1) \) matrix in reduced row-echelon form. Suppose that \( B \) has \( r \) nonzero rows, columns without leading 1’s with indices \( F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\} \), and columns with leading 1’s (pivot columns) having indices \( D = \{d_1, d_2, d_3, \ldots, d_r\} \). Define vectors \( \mathbf{c}, \mathbf{u}_j, 1 \leq j \leq n-r \) of size \( n \) by

\[
[c]_i = \begin{cases} 
0 & \text{if } i \in F \\
[B]_{k,n+1} & \text{if } i \in D, \ i = d_k 
\end{cases}
\]

\[
[u]_{j,i} = \begin{cases} 
1 & \text{if } i \in F, \ i \neq f_j \\
0 & \text{if } i \in F, \ i = f_j \\
-[B]_{k,f_j} & \text{if } i \in D, \ i = d_k 
\end{cases}
\]

Then the set of solutions to the system of equations \( LS(A, b) \) is

\( S = \{ \mathbf{c} + x_{f_1} \mathbf{u}_1 + x_{f_2} \mathbf{u}_2 + x_{f_3} \mathbf{u}_3 + \cdots + x_{f_{n-r}} \mathbf{u}_{n-r} \mid x_{f_1}, x_{f_2}, x_{f_3}, \ldots, x_{f_{n-r}} \in \mathbb{C} \} \)

**Proof** We are being asked to prove that the solution set has a particular form. First, \( LS(A, b) \) is equivalent to the linear system of equations that has the matrix \( B \) as its augmented matrix (Theorem REMES [25]), so we need only show that \( S \) is the solution set for the system with \( B \) as its augmented matrix.
We begin by showing that every element of $S$ is a solution to the system. Let $x_{f_1} = \alpha_1, x_{f_2} = \alpha_2, \ldots, x_{f_{n-r}} = \alpha_{n-r}$ be one choice of the values of $x_{f_1}, x_{f_2}, x_{f_3}, \ldots, x_{f_{n-r}}$. So a proposed solution is

$$x = c + \alpha_{f_1}u_1 + \alpha_{f_2}u_2 + \alpha_{f_3}u_3 + \cdots + \alpha_{f_{n-r}}u_{n-r}$$

So we evaluate equation $\ell$ of the system represented by $B$ with the solution vector $x$,

$$\beta = [B]_{\ell 1} x_1 + [B]_{\ell 2} x_2 + [B]_{\ell 3} x_3 + \cdots + [B]_{\ell n} x_n$$

When $r + 1 \leq \ell \leq m$, row $\ell$ of the matrix $B$ is a zero row, so the equation represented by that row is always true, no matter which solution vector we propose. So assume $1 \leq \ell \leq r$. Then $[B]_{\ell i} = 0$ for all $1 \leq i \leq r$, except that $[B]_{\ell d_r} = 1$, so $\beta$ simplifies to

$$\beta = [x]_{d_r} + [B]_{\ell f_1} x_{f_1} + [B]_{\ell f_2} x_{f_2} + [B]_{\ell f_3} x_{f_3} + \cdots + [B]_{\ell f_{n-r}} x_{f_{n-r}}$$

Notice that for $1 \leq \ell \leq n - r$

$$[x]_{f_i} = [c]_{f_i} + \alpha_{f_1} [u_1]_{f_i} + \alpha_{f_2} [u_2]_{f_i} + \alpha_{f_3} [u_3]_{f_i} + \cdots + \alpha_{f_{n-r}} [u_{n-r}]_{f_i}$$

$$= 0 + \alpha_{f_1}(0) + \alpha_{f_2}(0) + \alpha_{f_3}(0) + \cdots + \alpha_{f_{n-r}}(0)$$

$$= \alpha_{f_i}$$

So $\beta$ simplifies further to

$$\beta = [x]_{d_r} + [B]_{\ell f_1} \alpha_{f_1} + [B]_{\ell f_2} \alpha_{f_2} + [B]_{\ell f_3} \alpha_{f_3} + \cdots + [B]_{\ell f_{n-r}} \alpha_{f_{n-r}}$$

Now examine the $[x]_{d_r}$ term of $\beta$,

$$[x]_{d_r} = [c]_{d_r} + \alpha_{f_1} [u_1]_{d_r} + \alpha_{f_2} [u_2]_{d_r} + \alpha_{f_3} [u_3]_{d_r} + \cdots + \alpha_{f_{n-r}} [u_{n-r}]_{d_r}$$

$$= [B]_{\ell,n+1} + \alpha_{f_1}(-[B]_{\ell,f_1}) + \alpha_{f_2}(-[B]_{\ell,f_2}) + \alpha_{f_3}(-[B]_{\ell,f_3}) + \cdots + \alpha_{f_{n-r}}(-[B]_{\ell,f_{n-r}})$$

Replacing this term into the expression for $\beta$, we obtain

$$\beta = [x]_{d_r} + [B]_{\ell f_1} \alpha_{f_1} + [B]_{\ell f_2} \alpha_{f_2} + [B]_{\ell f_3} \alpha_{f_3} + \cdots + [B]_{\ell f_{n-r}} \alpha_{f_{n-r}}$$

$$= [B]_{\ell,n+1} + \alpha_{f_1}(-[B]_{\ell,f_1}) + \alpha_{f_2}(-[B]_{\ell,f_2}) + \alpha_{f_3}(-[B]_{\ell,f_3}) + \cdots + \alpha_{f_{n-r}}(-[B]_{\ell,f_{n-r}})+$$

$$[B]_{\ell f_1} \alpha_{f_1} + [B]_{\ell f_2} \alpha_{f_2} + [B]_{\ell f_3} \alpha_{f_3} + \cdots + [B]_{\ell f_{n-r}} \alpha_{f_{n-r}}$$

$$= [B]_{\ell,n+1}$$

So $\beta$ began as the left-hand side of equation $\ell$ from the system represented by $B$ and we now know it equals $[B]_{\ell,n+1}$, the constant term for equation $\ell$. So this arbitrarily chosen vector from $S$ makes every equation true, and therefore is a solution to the system.

For the second half of the proof, assume that $x_1 = \alpha_1, x_2 = \alpha_2, x_3 = \alpha_3, \ldots, x_n = \alpha_n$ are the components of a solution vector for the system having $B$ as its augmented matrix, and show that this solution vector is an element of the set $S$. Begin with the observation that this solution makes equation $\ell$ of the system true for $1 \leq \ell \leq m$,

$$[B]_{\ell 1} \alpha_1 + [B]_{\ell 2} \alpha_2 + [B]_{\ell 3} \alpha_3 + \cdots + [B]_{\ell n} \alpha_n = [B]_{\ell,n+1}$$

Since $B$ is in reduced row-echelon form, when $\ell > r$ we know that all the entries of $B$ in row $\ell$ are all zero and this equation is true. For $\ell \leq r$, we can further exploit the knowledge of the structure of $B$, specifically recalling that $B$ has no leading 1’s in the final column since the system is consistent (Theorem RCLS [45]). Equation $\ell$ then reduces to

$$(1)\alpha_{d_r} + [B]_{\ell f_1} \alpha_{f_1} + [B]_{\ell f_2} \alpha_{f_2} + [B]_{\ell f_3} \alpha_{f_3} + \cdots + [B]_{\ell f_{n-r}} \alpha_{f_{n-r}} = [B]_{\ell,n+1}$$

Rearranging, this becomes,

$$\alpha_{d_r} = [B]_{\ell,n+1} - [B]_{\ell f_1} \alpha_{f_1} - [B]_{\ell f_2} \alpha_{f_2} - [B]_{\ell f_3} \alpha_{f_3} - \cdots - [B]_{\ell f_{n-r}} \alpha_{f_{n-r}}$$
= \alpha f_i = 1 \alpha f_j \\
= 0 + 0 \alpha f_i + 0 \alpha f_j + 0 \alpha f_k + \cdots + 0 \alpha f_{j-1} + 1 \alpha f_j + 0 \alpha f_{j+1} + \cdots + 0 \alpha f_{n-r} \\
= [c], i + \alpha f_i [u_1]_i + \alpha f_2 [u_2]_i + \alpha f_3 [u_3]_i + \cdots + \alpha f_{n-r} [u_{n-r}]_i \\
= [c + \alpha f_1 u_1 + \alpha f_2 u_2 + \cdots + \alpha f_{n-r} u_{n-r}]_i

So our solution vector is also of the right form in the remaining slots, and hence qualifies for membership in the set $S$. 

Theorem VFSLS \ref{thm:vfsls} formalizes what happened in the three steps of Example VFSAD \ref{example:vfsad}. The theorem will be useful in proving other theorems, and it is useful since it tells us an exact procedure for simply describing an infinite solution set. We could program a computer to implement it, once we have the augmented matrix row-reduced and have checked that the system is consistent. By Knuth’s definition, this completes our conversion of linear equation solving from art into science. Notice that it even applies (but is overkill) in the case of a unique solution. However, as a practical matter, I prefer the three-step process of Example VFSAD \ref{example:vfsad} when I need to describe an infinite solution set. So let’s practice some more, but with a bigger example.

Example VFSAI

Vector form of solutions for Archetype I

Archetype I \ref{archetype:i} is a linear system of $m = 4$ equations in $n = 7$ variables. Row-reducing the augmented matrix yields

$$
\begin{bmatrix}
1 & 4 & 0 & 0 & 2 & 1 & -3 & 4 \\
0 & 0 & 1 & 0 & 1 & -3 & 5 & 2 \\
0 & 0 & 0 & 1 & 2 & -6 & 6 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

and we see $r = 3$ nonzero rows. The columns with leading 1’s are $D = \{1, 3, 4\}$ so the $r$ dependent variables are $x_1, x_3, x_4$. The columns without leading 1’s are $F = \{2, 5, 6, 7, 8\}$, so the $n - r = 4$ free variables are $x_2, x_5, x_6, x_7$.

Step 1. Write the vector of variables ($x$) as a fixed vector ($c$), plus a linear combination of $n - r = 4$ vectors ($u_1, u_2, u_3, u_4$), using the free variables as the scalars.

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = + x_2 \begin{bmatrix} 2 \end{bmatrix} + x_5 \begin{bmatrix} 1 \end{bmatrix} + x_6 \begin{bmatrix} 0 \end{bmatrix} + x_7 \begin{bmatrix} 0 \end{bmatrix}
$$

Step 2. For each free variable, use 0’s and 1’s to ensure equality for the corresponding entry of the the vectors. Take note of the pattern of 0’s and 1’s at this stage, because this is the best look you’ll have at it. We’ll state an important theorem in the next section and the proof will essentially rely
on this observation.

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}
\]

Step 3. For each dependent variable, use the augmented matrix to formulate an equation expressing the dependent variable as a constant plus multiples of the free variables. Convert this equation into entries of the vectors that ensure equality for each dependent variable, one at a time.

\[
x_1 = 4 - 4x_2 - 2x_5 - 1x_6 + 3x_7 \quad \Rightarrow \quad
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix} =
\begin{bmatrix}
4 & -4 & -2 & -1 & 3 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}
\]

\[
x_3 = 2 + 0x_2 - x_5 + 3x_6 - 5x_7 \quad \Rightarrow \quad
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix} =
\begin{bmatrix}
4 & -4 & -2 & -1 & 3 \\
0 & 1 & 0 & 0 & 0 \\
2 & 0 & -1 & 3 & -5 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}
\]

\[
x_4 = 1 + 0x_2 - 2x_5 + 6x_6 - 6x_7 \quad \Rightarrow \quad
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix} =
\begin{bmatrix}
4 & -4 & -2 & -1 & 3 \\
0 & 1 & 0 & 0 & 0 \\
2 & 0 & -1 & 3 & -5 \\
1 & 0 & -2 & 6 & 7 & -6 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}
\]

We can now use this final expression to quickly build solutions to the system. You might try to recreate each of the solutions listed in the write-up for Archetype I [667]. (Hint: look at the values of the free variables in each solution, and notice that the vector \( \mathbf{c} \) has 0’s in these locations.)

Even better, we have a description of the infinite solution set, based on just 5 vectors, which we combine in linear combinations to produce solutions.

Whenever we discuss Archetype I [667] you know that’s your cue to go work through Archetype II [671] by yourself. Remember to take note of the 0/1 pattern at the conclusion of Step 2. Have fun — we won’t go anywhere while you’re away.

This technique is so important, that we’ll do one more example. However, an important distinction will be that this system is homogeneous.

Example VFSAL

Vector form of solutions for Archetype L
Archetype L $L_680$ is presented simply as the $5 \times 5$ matrix

$$L = \begin{bmatrix}
-2 & -1 & -2 & -4 & 4 \\
-6 & -5 & -4 & -4 & 6 \\
10 & 7 & 7 & 10 & -13 \\
-7 & -5 & -6 & -9 & 10 \\
-4 & -3 & -4 & -6 & 6
\end{bmatrix}$$

We’ll interpret it here as the coefficient matrix of a homogeneous system and reference this matrix as $L$. So we are solving the homogeneous system $LS(L, 0)$ having $m = 5$ equations in $n = 5$ variables. If we built the augmented matrix, we would add a sixth column to $L$ containing all zeros. As we did row operations, this sixth column would remain all zeros. So instead we will row-reduce the coefficient matrix, and mentally remember the missing sixth column of zeros. This row-reduced matrix is

$$\begin{bmatrix}
1 & 0 & 0 & 1 & -2 \\
0 & 1 & 0 & -2 & 2 \\
0 & 0 & 1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

and we see $r = 3$ nonzero rows. The columns with leading 1’s are $D = \{1, 2, 3\}$ so the $r$ dependent variables are $x_1, x_2, x_3$. The columns without leading 1’s are $F = \{4, 5\}$, so the $n - r = 2$ free variables are $x_4, x_5$. Notice that if we had included the all-zero vector of constants to form the augmented matrix for the system, then the index 6 would have appeared in the set $F$, and subsequently would have been ignored when listing the free variables.

Step 1. Write the vector of variables ($x$) as a fixed vector ($c$), plus a linear combination of $n - r = 2$ vectors ($u_1, u_2$), using the free variables as the scalars.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \end{bmatrix} + x_4 \begin{bmatrix} \end{bmatrix} + x_5 \begin{bmatrix} \end{bmatrix}$$

Step 2. For each free variable, use 0’s and 1’s to ensure equality for the corresponding entry of the the vectors. Take note of the pattern of 0’s and 1’s at this stage, even if it is not as illuminating as in other examples.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \end{bmatrix} + x_4 \begin{bmatrix} \end{bmatrix} + x_5 \begin{bmatrix} \end{bmatrix}$$

Step 3. For each dependent variable, use the augmented matrix to formulate an equation expressing the dependent variable as a constant plus multiples of the free variables. Don’t forget about the “missing” sixth column being full of zeros. Convert this equation into entries of the vectors that ensure equality for each dependent variable, one at a time.

$$x_1 = 0 - 1x_4 + 2x_5 \quad \Rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$x_2 = 0 + 2x_4 - 2x_5 \quad \Rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$
The vector \( \mathbf{c} \) will always have 0’s in the entries corresponding to free variables. However, since we are solving a homogeneous system, the row-reduced augmented matrix has zeros in column \( n + 1 = 6 \), and hence all the entries of \( \mathbf{c} \) are zero. So we can write

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ x_4 \\ 0 + x_4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} x_4 + \begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}
\]

It will always happen that the solutions to a homogeneous system has \( \mathbf{c} = \mathbf{0} \) (even in the case of a unique solution?). So our expression for the solutions is a bit more pleasing. In this example it says that the solutions are all possible linear combinations of the two vectors \( \mathbf{u}_1 = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \) and

\[
\mathbf{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \text{ with no mention of any fixed vector entering into the linear combination.}
\]

This observation will motivate our next section and the main definition of that section, and after that we will conclude the section by formalizing this situation.  

**Subsection PSNSH**

**Particular Solutions, Homogeneous Solutions**

The next theorem tells us that in order to find all of the solutions to a linear system of equations, it is sufficient to find just one solution, and then find all of the solutions to the corresponding homogeneous system. This explains part of our interest in the null space, the set of all solutions to a homogeneous system.

**Theorem PSNSH**

**Particular Solution Plus Homogeneous Solutions**

Suppose that \( \mathbf{w} \) is one solution to the linear system of equations \( \mathcal{L} \mathbf{A}(\mathbf{b}) \). Then \( \mathbf{y} \) is a solution to \( \mathcal{L} \mathbf{A}(\mathbf{b}) \) if and only if \( \mathbf{y} = \mathbf{w} + \mathbf{z} \) for some vector \( \mathbf{z} \in \mathcal{N}(\mathbf{A}) \).

**Proof** Let \( \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n \) be the columns of the coefficient matrix \( \mathbf{A} \).

\((\Rightarrow)\) Suppose \( \mathbf{y} = \mathbf{w} + \mathbf{z} \) and \( \mathbf{z} \in \mathcal{N}(\mathbf{A}) \). Then

\[
\mathbf{b} = [\mathbf{w}]_1 \mathbf{A}_1 + [\mathbf{w}]_2 \mathbf{A}_2 + [\mathbf{w}]_3 \mathbf{A}_3 + \cdots + [\mathbf{w}]_n \mathbf{A}_n
\]

...
Applying Theorem SLSLC \[82\] we see that \( y \) is a solution to \( \mathcal{L}S(A, b) \).

\[ (\Rightarrow) \text{ Suppose } y \text{ is a solution to } \mathcal{L}S(A, b). \text{ Then} \]

\[
0 = b - b = [y]_1 A_1 + [y]_2 A_2 + [y]_3 A_3 + \cdots + [y]_n A_n - ([w]_1 A_1 + [w]_2 A_2 + [w]_3 A_3 + \cdots + [w]_n A_n)
\]

\[
= (([y]_1 - [w]_1) A_1 + ([y]_2 - [w]_2) A_2 + \cdots + ([y]_n - [w]_n) A_n)
\]

\[
= [y - w]_1 A_1 + [y - w]_2 A_2 + [y - w]_3 A_3 + \cdots + [y - w]_n A_n
\]

By Theorem SLSLC \[82\] we see that \( y - w \) is a solution to the homogenous system \( \mathcal{L}S(A, 0) \) and by Definition NSM \[54\], \( y - w \in \mathcal{N}(A) \). In other words, \( y - w = z \) for some vector \( z \in \mathcal{N}(A) \). Rewritten, this is \( y = w + z \), as desired.

After proving Theorem NMUS \[64\] we commented (insufficiently) on the negation of one half of the theorem. Nonsingular coefficient matrices lead to unique solutions for every choice of the vector of constants. What does this say about singular matrices? A singular matrix \( A \) has a nontrivial null space Theorem NMTNS \[69\]. For a given vector of constants, \( b \), the system \( \mathcal{L}S(A, b) \) could be inconsistent, meaning there are no solutions. But if there is at least one solution (\( w \)), then Theorem PSPHS \[93\] tells us there will be infinitely many solutions because of the role of the infinite null space for a singular matrix. So a system of equations with a singular coefficient matrix never has a unique solution. Either there are no solutions, or infinitely many solutions, depending on the choice of the vector of constants (\( b \)).

Example PSHS

Particular solutions, homogeneous solutions, Archetype D

Archetype D \[647\] is a consistent system of equations with a nontrivial null space. Let \( A \) denote the coefficient matrix of this system. The write-up for this system begins with three solutions,

\[
y_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \quad y_2 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad y_3 = \begin{bmatrix} 7 \\ 8 \\ 1 \\ 3 \end{bmatrix}
\]

We will choose to have \( y_1 \) play the role of \( w \) in the statement of Theorem PSPHS \[93\], any one of the three vectors listed here (or others) could have been chosen. To illustrate the theorem, we should be able to write each of these three solutions as the vector \( w \) plus a solution to the corresponding homogenous system of equations. Since \( 0 \) is always a solution to a homogeneous system we can easily write

\[
y_1 = w = w + 0.
\]

The vectors \( y_2 \) and \( y_3 \) will require a bit more effort. Solutions to the homogeneous system \( \mathcal{L}S(A, 0) \) are exactly the elements of the null space of the coefficient matrix, which by an application of Theorem VFSLS \[88\] is

\[
\mathcal{N}(A) = \left\{ x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix} \mid x_3, x_4 \in \mathbb{C} \right\}
\]

Then

\[
y_2 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix} = w + z_2
\]
where
\[ z_2 = \begin{bmatrix} 4 \\ -1 \\ -2 \\ -1 \end{bmatrix} = (-2) \begin{bmatrix} -3 \\ -1 \\ -1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \]
is obviously a solution of the homogeneous system since it is written as a linear combination of the vectors describing the null space of the coefficient matrix (or as a check, you could just evaluate the equations in the homogeneous system with \( z_2 \)).

Again
\[ y_3 = \begin{bmatrix} 7 \\ 8 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = w + z_3 \]

where
\[ z_3 = \begin{bmatrix} 7 \\ 7 \\ -1 \\ 2 \end{bmatrix} = (-1) \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \]
is obviously a solution of the homogeneous system since it is written as a linear combination of the vectors describing the null space of the coefficient matrix (or as a check, you could just evaluate the equations in the homogeneous system with \( z_3 \)).

Here’s another view of this theorem, in the context of this example. Grab two new solutions of the original system of equations, say
\[ y_4 = \begin{bmatrix} 11 \\ 0 \\ -3 \\ -1 \end{bmatrix} \quad y_5 = \begin{bmatrix} -4 \\ 2 \\ 4 \\ 2 \end{bmatrix} \]
and form their difference,
\[ u = \begin{bmatrix} 11 \\ 0 \\ -3 \\ -1 \end{bmatrix} - \begin{bmatrix} -4 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 15 \\ -2 \\ -7 \\ -3 \end{bmatrix} \]

It is no accident that \( u \) is a solution to the homogeneous system (check this!). In other words, the difference between any two solutions to a linear system of equations is an element of the null space of the coefficient matrix. This is an equivalent way to state \text{Theorem PSPHS}\ [93]. (See Exercise MM.T50 [184].)

The ideas of this subsection will appear again in Chapter LT [405] when we discuss pre-images of linear transformations (Definition PI [415]).

Subsection URREF
Uniqueness of Reduced Row-Echelon Form

We are now in a position to establish that the reduced row-echelon form of a matrix is unique. Going forward, we will emphasize the point-of-view that a matrix is a collection of columns. But there are two occasions when we need to work carefully with the rows of a matrix. This is the first such occasion. We could define something called a \textbf{row vector} that would equal a given row of a matrix, and might be written as a horizontal list. Then we could define vector equality, the basic operations of vector addition and scalar multiplication, followed by a definition of a linear combination of row vectors. We will not incur the overhead of stating all these definitions, but will instead convert the rows of a matrix to column vectors and use our definitions that are already in
place. This was our reason for delaying this proof until now. Remind yourself as you work through this proof that it only relies only on the definition of equivalent matrices, reduced row-echelon form and linear combinations. So in particular, we are not guilty of circular reasoning. Should we have defined vector operations and linear combinations just prior to discussing reduced row-echelon form, then the following proof of uniqueness could have been presented at that time. OK, here we go.

**Theorem RREFU**

**Reduced Row-Echelon Form is Unique**

Suppose that $A$ is an $m \times n$ matrix and that $B$ and $C$ are $m \times n$ matrices that are row-equivalent to $A$ and in reduced row-echelon form. Then $B = C$. □

**Proof**

Denote the pivot columns of $B$ as $D = \{d_1, d_2, \ldots, d_r\}$ and the pivot columns of $C$ as $D' = \{d'_1, d'_2, d'_3, \ldots, d'_{r'}\}$ (Notation RREFA [27]). We begin by showing that $D = D'$. For both $B$ and $C$, we can take the elements of a row of the matrix and use them to construct a column vector. We will denote these by $b_i$ and $c_i$, respectively, $1 \leq i \leq m$. Since $B$ and $C$ are both row-equivalent to $A$, there is a sequence of row operations that will convert $B$ to $C$, and vice-versa, since row operations are reversible. If we can convert $B$ into $C$ via a sequence of row operations, then any row of $C$ expressed as a column vector, say $c_k$, is a linear combination of the column vectors derived from the rows of $B$, $\{b_1, b_2, b_3, \ldots, b_m\}$. Similarly, any row of $B$ is a linear combination of the set of rows of $C$. Our principal device in this proof is to carefully analyze individual entries of vector equalities between a single row of either $B$ or $C$ and a linear combination of the rows of the other matrix.

Let’s first show that $d_1 = d'_1$. Suppose that $d_1 < d'_1$. We can write the first row of $B$ as a linear combination of the rows of $C$, that is, there are scalars $a_1, a_2, a_3, \ldots, a_m$ such that

$$b_1 = a_1c_1 + a_2c_2 + a_3c_3 + \cdots + a_mc_m$$

Consider the entry in location $d_1$ on both sides of this equality. Since $B$ is in reduced row-echelon form (Definition RREF [26]) we find a one in $b_1$ on the left. Since $d_1 < d'_1$, and $C$ is in reduced row-echelon form (Definition RREF [26]) each vector $c_i$ has a zero in location $d_1$, and therefore the linear combination on the right also has a zero in location $d_1$. This is a contradiction, so we know that $d_1 \geq d'_1$. By an entirely similar argument, we could conclude that $d_1 \leq d'_1$. This means that $d_1 = d'_1$.

Suppose that we have determined that $d_1 = d'_1$, $d_2 = d'_2$, $d_3 = d'_3$, \ldots, $d_k = d'_k$. Let’s now show that $d_{k+1} = d'_{k+1}$. To achieve a contradiction, suppose that $d_{k+1} < d'_{k+1}$. Row $k+1$ of $B$ is a linear combination of the rows of $C$, so there are scalars $a_1, a_2, a_3, \ldots, a_m$ such that

$$b_{k+1} = a_1c_1 + a_2c_2 + a_3c_3 + \cdots + a_mc_m$$

Since $B$ is in reduced row-echelon form (Definition RREF [26]), the entries of $b_{k+1}$ in locations $d_{k+1}, d_2, d_3, \ldots, d_k$ are all zero. Since $C$ is in reduced row-echelon form (Definition RREF [26]), location $d_i$ of $c_i$ is one for each $1 \leq i \leq k$. The equality of these vectors in locations $d_{k+1}, d_2, d_3, \ldots, d_k$ then implies that $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, \ldots, $a_k = 0$.

Now consider location $d_{k+1}$ in this vector equality. The vector $b_{k+1}$ on the left is one in this location since $B$ is in reduced row-echelon form (Definition RREF [26]). Vectors $c_1, c_2, c_3, \ldots, c_k$, are multiplied by zero scalars in the linear combination on the right. The remaining vectors, $c_{k+1}, c_{k+2}, c_{k+3}, \ldots, c_m$ each has a zero in location $d_{k+1}$ since $d_{k+1} < d'_{k+1}$ and $C$ is in reduced row-echelon form (Definition RREF [26]). So the right hand side of the vector equality is zero in location $d_{k+1}$, a contradiction. Thus $d_{k+1} \geq d'_{k+1}$. By an entirely similar argument, we could conclude that $d_{k+1} \leq d'_{k+1}$, and therefore $d_{k+1} = d'_{k+1}$.

Now we establish that $r = r'$. Suppose that $r < r'$. By the arguments above we can show that $d_1 = d'_1$, $d_2 = d'_2$, $d_3 = d'_3$, \ldots, $d_r = d'_r$. Row $r'$ of $C$ is a linear combination of the $r$ non-zero rows of $B$, so there are scalars $a_1, a_2, a_3, \ldots, a_r$ so that

$$c_r = a_1b_1 + a_2b_2 + a_3b_3 + \cdots + a_rb_r$$
Locations $d_1, d_2, d_3, \ldots, d_r$ of $c_{r'}$ are all zero since $r < r'$ and $C$ is in reduced row-echelon form (Definition RREF [26]). For a given index $i$, $1 \leq i \leq r$, the vectors $b_1, b_2, b_3, \ldots, b_r$ have zeros in location $d_i$, except that the vector $b_i$ is one in location $d_i$ since $B$ is in reduced row-echelon form (Definition RREF [26]). This consideration of location $d_i$ implies that $a_i = 0$, $1 \leq i \leq r$. With all the scalars in the linear combination equal to zero, we conclude that $c_{r'} = 0$, contradicting the existence of a leading 1 in $c_{r'}$. So $r \geq r'$. By a similar argument, we conclude that $r \leq r'$ and therefore $r = r'$. Thus $D = D'$.

To finally show that $B = C$, we will show that the rows of the two matrices are equal. Row $k$ of $C$, $c_k$, is a linear combination of the $r$ non-zero rows of $B$, so there are scalars $a_1, a_2, a_3, \ldots, a_r$ such that

$$c_k = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_r b_r.$$

Because $C$ is in reduced row-echelon form (Definition RREF [26]), location $d_i$ of $c_k$ is zero for $1 \leq i \leq r$, except in location $d_k$ where the entry is one. In the linear combination on the right of the vector equality, the vectors $b_1, b_2, b_3, \ldots, b_r$ have zeros in location $d_i$, except that $b_k$ has a one in location $d_i$, since $B$ is in reduced row-echelon form (Definition RREF [26]). This implies that $a_1 = 0$, $a_2 = 0$, $\ldots$, $a_{k-1} = 0$, $a_{k+1} = 0$, $a_{k+2} = 0$, $\ldots$, $a_r = 0$ and $a_k = 1$. Then the vector equality reduces to simply $c_k = b_k$. Since $k$ was arbitrary, $B$ and $C$ have equal rows and so are equal matrices.

Subsection READ
Reading Questions

1. Earlier, a reading question asked you to solve the system of equations

$$2x_1 + 3x_2 - x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 3$$
$$x_1 + 3x_2 + 3x_3 = 7$$

Use a linear combination to rewrite this system of equations as a vector equality.

2. Find a linear combination of the vectors

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -5 \end{bmatrix} \right\}$$

that equals the vector $\begin{bmatrix} 1 \\ -9 \\ 11 \end{bmatrix}$.

3. The matrix below is the augmented matrix of a system of equations, row-reduced to reduced row-echelon form. Write the vector form of the solutions to the system.

$$\begin{bmatrix}
1 & 3 & 0 & 6 & 0 & 9 \\
0 & 0 & 1 & -2 & 0 & -8 \\
0 & 0 & 0 & 0 & 1 & 3
\end{bmatrix}$$
Subsection EXC
Exercises

C21  Consider each archetype that is a system of equations. For individual solutions listed (both for the original system and the corresponding homogeneous system) express the vector of constants as a linear combination of the columns of the coefficient matrix, as guaranteed by Theorem SLSLC. Verify this equality by computing the linear combination. For systems with no solutions, recognize that it is then impossible to write the vector of constants as a linear combination of the columns of the coefficient matrix. Note too, for homogeneous systems, that the solutions give rise to linear combinations that equal the zero vector.

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671

Contributed by Robert Beezer  Solution 100

C22  Consider each archetype that is a system of equations. Write elements of the solution set in vector form, as guaranteed by Theorem VFSLS.

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671

Contributed by Robert Beezer  Solution 100

C40  Find the vector form of the solutions to the system of equations below.

\[
\begin{align*}
2x_1 - 4x_2 + 3x_3 + x_5 &= 6 \\
x_1 - 2x_2 - 2x_3 + 14x_4 - 4x_5 &= 15 \\
x_1 - 2x_2 + x_3 + 2x_4 + x_5 &= -1 \\
-2x_1 + 4x_2 - 12x_4 + x_5 &= -7
\end{align*}
\]

Contributed by Robert Beezer  Solution 100

C41  Find the vector form of the solutions to the system of equations below.

\[
\begin{align*}
-2x_1 - x_2 - 8x_3 + 8x_4 + 4x_5 - 9x_6 - x_7 - x_8 - 18x_9 &= 3 \\
3x_1 - 2x_2 + 5x_3 + 2x_4 - 2x_5 - 5x_6 + x_7 + 2x_8 + 15x_9 &= 10 \\
4x_1 - 2x_2 + 8x_3 + 2x_5 - 14x_6 - 2x_8 + 2x_9 &= 36 \\
-x_1 + 2x_2 + x_3 - 6x_4 + 7x_5 - x_7 - 3x_9 &= -8
\end{align*}
\]
\[
\begin{align*}
3x_1 + 2x_2 + 13x_3 - 14x_4 - 1x_5 + 5x_6 - 1x_8 + 12x_9 &= 15 \\
-2x_1 + 2x_2 - 2x_3 - 4x_4 + 1x_5 + 6x_6 - 2x_7 - 2x_8 - 15x_9 &= -7
\end{align*}
\]

Contributed by Robert Beezer  Solution 100

**M10** Example TLC 79 asks if the vector
\[
w = \begin{bmatrix}
13 \\
15 \\
5 \\
-17 \\
2 \\
25
\end{bmatrix}
\]
can be written as a linear combination of the four vectors
\[
\begin{align*}
u_1 &= \begin{bmatrix} 2 \\ 4 \\ -3 \\ 1 \\ 2 \\ 9 \end{bmatrix} & u_2 &= \begin{bmatrix} 6 \\ 3 \\ 0 \\ -2 \\ 1 \\ 4 \end{bmatrix} & u_3 &= \begin{bmatrix} -5 \\ 2 \\ 1 \\ -3 \\ 0 \\ 3 \end{bmatrix} & u_4 &= \begin{bmatrix} 3 \\ 2 \\ 2 \\ -5 \\ 1 \\ 3 \end{bmatrix}
\end{align*}
\]
Can it? Can any vector in \(\mathbb{C}^6\) be written as a linear combination of the four vectors \(u_1, u_2, u_3, u_4\)?

Contributed by Robert Beezer  Solution 101

**M11** At the end of Example VFS 85, the vector \(w\) is claimed to be a solution to the linear system under discussion. Verify that \(w\) really is a solution. Then determine the four scalars that express \(w\) as a linear combination of \(c, u_1, u_2, u_3\).

Contributed by Robert Beezer  Solution 101

**T30** Suppose that \(x\) is a solution to \(\mathcal{L}S(A, b)\) and that \(z\) is a solution to the homogeneous system \(\mathcal{L}S(A, 0)\). Prove that \(x + z\) is a solution to \(\mathcal{L}S(A, b)\).

Contributed by Robert Beezer  Solution 101
Subsection SOL
Solutions

C21 Contributed by Robert Beezer Statement 98
Solutions for Archetype A 634 and Archetype B 638 are described carefully in Example AALC 81 and Example ABLC 80.

C22 Contributed by Robert Beezer Statement 98
Solutions for Archetype D 647 and Archetype I 667 are described carefully in Example VFSAD 83 and Example VFSAI 90. The technique described in these examples is probably more useful than carefully deciphering the notation of Theorem VFSLS 88. The solution for each archetype is contained in its description. So now you can check-off the box for that item.

C40 Contributed by Robert Beezer Statement 98
Row-reduce the augmented matrix representing this system, to find
\[
\begin{bmatrix}
1 & -2 & 0 & 6 & 0 & 1 \\
0 & 0 & 1 & -4 & 0 & 3 \\
0 & 0 & 0 & 0 & 1 & -5 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
The system is consistent (no leading one in column 6, Theorem RCLS 45). \(x_2\) and \(x_4\) are the free variables. Now apply Theorem VFSLS 88 directly, or follow the three-step process of Example VFS 85, Example VFSAD 83, Example VFSAI 90, or Example VFSAL 91 to obtain
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} =
\begin{bmatrix}
1 \\
0 \\
3 \\
0 \\
1
\end{bmatrix} +
\begin{bmatrix}
x_2 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} +
\begin{bmatrix}
x_4 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} +
\begin{bmatrix}
x_5 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

C41 Contributed by Robert Beezer Statement 98
Row-reduce the augmented matrix representing this system, to find
\[
\begin{bmatrix}
1 & 0 & 3 & -2 & 0 & -1 & 0 & 0 & 3 & 6 \\
0 & 1 & 2 & -4 & 0 & 3 & 0 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & -1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
The system is consistent (no leading one in column 10, Theorem RCLS 45). \(F = \{3, 4, 6, 9, 10\}\), so the free variables are \(x_3, x_4, x_6\) and \(x_9\). Now apply Theorem VFSLS 88 directly, or follow the three-step process of Example VFS 85, Example VFSAD 83, Example VFSAI 90, or Example VFSAL 91 to obtain the solution set
\[
S = \left\{ \begin{bmatrix}
6 \\
-1 \\
0 \\
0 \\
-2 \\
0
\end{bmatrix} + \begin{bmatrix}
-3 \\
-2 \\
1 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
2 \\
4 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
1 \\
-3 \\
0 \\
0 \\
1 \\
1
\end{bmatrix} \bigg| x_3, x_4, x_6, x_9 \in \mathbb{C} \right\}
\]
M10 Contributed by Robert Beezer Statement 99
No, it is not possible to create \( \mathbf{w} \) as a linear combination of the four vectors \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \). By creating the desired linear combination with unknowns as scalars, Theorem SLSLC 82 provides a system of equations that has no solution. This one computation is enough to show us that it is not possible to create all the vectors of \( \mathbb{C}^6 \) through linear combinations of the four vectors \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \).

M11 Contributed by Robert Beezer Statement 99
The coefficient of \( \mathbf{c} \) is 1. The coefficients of \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \) lie in the third, fourth and seventh entries of \( \mathbf{w} \). Can you see why? (Hint: \( F = \{3, 4, 7, 8\} \), so the free variables are \( x_3, x_4 \) and \( x_7 \).)

T30 Contributed by Robert Beezer Statement 99
Write the columns of \( A \) as \( A_1, A_2, A_3, \ldots, A_n \). Then

\[
\mathbf{b} = [x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n \\
= [x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n + 0 \\
= [x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n \\
+ [z]_1 A_1 + [z]_2 A_2 + [z]_3 A_3 + \cdots + [z]_n A_n \\
= ([x]_1 + [z]_1) A_1 + ([x]_2 + [z]_2) A_2 + \cdots + ([x]_n + [z]_n) A_n \\
= [x + z]_1 A_1 + [x + z]_2 A_2 + \cdots + [x + z]_n A_n
\]

This equation then allows us to employ Theorem SLSLC 82 and conclude that \( \mathbf{x} + \mathbf{z} \) is a solution to \( \mathcal{L}(A, \mathbf{b}) \).
Section SS  
Spanning Sets

In this section we will describe a compact way to indicate the elements of an infinite set of vectors, making use of linear combinations. This will give us a convenient way to describe the elements of a set of solutions to a linear system, or the elements of the null space of a matrix, or many other sets of vectors.

Subsection SSV  
Span of a Set of Vectors

In Example VFSAL we saw the solution set of a homogeneous system described as all possible linear combinations of two particular vectors. This happens to be a useful way to construct or describe infinite sets of vectors, so we encapsulate this idea in a definition.

Definition SSCV  
Span of a Set of Column Vectors
Given a set of vectors \( S = \{u_1, u_2, u_3, \ldots, u_p\} \), their span, \( \langle S \rangle \), is the set of all possible linear combinations of \( u_1, u_2, u_3, \ldots, u_p \). Symbolically,

\[
\langle S \rangle = \{ \sum_{i=1}^{p} \alpha_i u_i \mid \alpha_i \in \mathbb{C}, 1 \leq i \leq p \}
\]

(This definition contains Notation SSV.)

The span is just a set of vectors, though in all but one situation it is an infinite set. (Just when is it not infinite?) So we start with a finite collection of vectors \( S \) (of them to be precise), and use this finite set to describe an infinite set of vectors, \( \langle S \rangle \). Confusing the finite set \( S \) with the infinite set \( \langle S \rangle \) is one of the most pervasive problems in understanding introductory linear algebra. We will see this construction repeatedly, so let’s work through some examples to get comfortable with it. The most obvious question about a set is if a particular item of the correct type is in the set, or not.

Example ABS  
A basic span
Consider the set of 5 vectors, \( S \), from \( \mathbb{C}^4 \)

\[
S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 9 \\ 0 \end{bmatrix} \right\}
\]

and consider the infinite set of vectors \( \langle S \rangle \) formed from all possible linear combinations of the elements of \( S \). Here are four vectors we definitely know are elements of \( \langle S \rangle \), since we will construct them in accordance with Definition SSCV.

\[
\begin{align*}
w &= (2) \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix} + (2) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix} + (3) \begin{bmatrix} -1 \\ 0 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 28 \\ 10 \end{bmatrix}
\end{align*}
\]
The purpose of a set is to collect objects with some common property, and to exclude objects without that property. So the most fundamental question about a set is if a given object is an element of the set, and which are not. Let’s learn more about \( \langle S \rangle \) by investigating which vectors are an element of the set, and which are not.

First, is \( \mathbf{u} = \begin{bmatrix} -15 \\ -6 \\ 19 \\ 5 \end{bmatrix} \) an element of \( \langle S \rangle \)? We are asking if there are scalars \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \) such that

\[
\begin{bmatrix}
    1 \\
    3 \\
    1 \\
    2 \\
    -1
\end{bmatrix} + \alpha_2 \begin{bmatrix}
    2 \\
    1 \\
    2 \\
    1 \\
    -1
\end{bmatrix} + \alpha_3 \begin{bmatrix}
    7 \\
    3 \\
    5 \\
    -5
\end{bmatrix} + \alpha_4 \begin{bmatrix}
    1 \\
    1 \\
    -1 \\
    2
\end{bmatrix} + \alpha_5 \begin{bmatrix}
    -1 \\
    1 \\
    1 \\
    0
\end{bmatrix} = \mathbf{u} = \begin{bmatrix}
    -15 \\
    -6 \\
    19 \\
    5
\end{bmatrix}
\]

Applying [Theorem SLSLC][82], we recognize the search for these scalars as a solution to a linear system of equations with augmented matrix

\[
\begin{bmatrix}
    1 & 2 & 7 & 1 & -1 & -15 \\
    1 & 1 & 3 & 1 & 0 & -6 \\
    3 & 2 & 5 & -1 & 9 & 19 \\
    1 & -1 & -5 & 2 & 0 & 5
\end{bmatrix}
\]

which row-reduces to

\[
\begin{bmatrix}
    1 & 0 & -1 & 0 & 3 & 10 \\
    0 & 1 & 4 & 0 & -1 & -9 \\
    0 & 0 & 0 & 1 & -2 & -7 \\
    0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

At this point, we see that the system is consistent (no a leading 1 in the last column, [Theorem RCLS][45]), so we know there is a solution for the five scalars \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \). This is enough evidence for us to say that \( \mathbf{u} \in \langle S \rangle \). If we wished further evidence, we could compute an actual solution, say

\[
\alpha_1 = 2 \quad \alpha_2 = 1 \quad \alpha_3 = -2 \quad \alpha_4 = -3 \quad \alpha_5 = 2
\]

This particular solution allows us to write

\[
(2) \begin{bmatrix}
    1 \\
    3 \\
    1
\end{bmatrix} + (1) \begin{bmatrix}
    2 \\
    1 \\
    2 \\
\end{bmatrix} + (-2) \begin{bmatrix}
    7 \\
    3 \\
    5 \\
\end{bmatrix} + (-3) \begin{bmatrix}
    1 \\
    1 \\
    -1 \\
\end{bmatrix} + (2) \begin{bmatrix}
    -1 \\
    1 \\
    2 \\
\end{bmatrix} = \mathbf{u} = \begin{bmatrix}
    -15 \\
    -6 \\
    19 \\
    5
\end{bmatrix}
\]

making it even more obvious that \( \mathbf{u} \in \langle S \rangle \).
Let’s do it again. Is \( \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \) an element of \( \langle S \rangle \)? We are asking if there are scalars \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \) such that

\[
\begin{align*}
\alpha_1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 7 \\ 3 \\ -5 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \alpha_5 \begin{bmatrix} -1 \\ 0 \\ 9 \end{bmatrix} &= \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}
\end{align*}
\]

Applying Theorem SLC [82] we recognize the search for these scalars as a solution to a linear system of equations with augmented matrix

\[
\begin{bmatrix}
1 & 2 & 7 & 1 & -1 & 3 \\
1 & 1 & 3 & 1 & 0 & 1 \\
3 & 2 & 5 & -1 & 9 & 2 \\
1 & -1 & -5 & 2 & 0 & -1
\end{bmatrix}
\]

which row-reduces to

\[
\begin{bmatrix}
1 & 0 & -1 & 0 & 3 & 0 \\
0 & 1 & 4 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

At this point, we see that the system is inconsistent (a leading 1 in the last column, Theorem RCLS [45]), so we know there is not a solution for the five scalars \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \). This is enough evidence for us to say that \( \mathbf{v} \notin \langle S \rangle \). End of story.

Example SCAA
Span of the columns of Archetype A
Begin with the finite set of three vectors of size 3

\[
S = \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}
\]

and consider the infinite set \( \langle S \rangle \). The vectors of \( S \) could have been chosen to be anything, but for reasons that will become clear later, we have chosen the three columns of the coefficient matrix in Archetype A 634. First, as an example, note that

\[
\mathbf{v} = (5) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (-3) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + (7) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 22 \\ 14 \\ 2 \end{bmatrix}
\]

is in \( \langle S \rangle \), since it is a linear combination of \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \). We write this succinctly as \( \mathbf{v} \in \langle S \rangle \). There is nothing magical about the scalars \( \alpha_1 = 5, \alpha_2 = -3, \alpha_3 = 7 \), they could have been chosen to be anything. So repeat this part of the example yourself, using different values of \( \alpha_1, \alpha_2, \alpha_3 \). What happens if you choose all three scalars to be zero?

So we know how to quickly construct sample elements of the set \( \langle S \rangle \). A slightly different question arises when you are handed a vector of the correct size and asked if it is an element of \( \langle S \rangle \). For example, is \( \mathbf{w} = \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix} \) in \( \langle S \rangle \)? More succinctly, \( \mathbf{w} \in \langle S \rangle \)?

To answer this question, we will look for scalars \( \alpha_1, \alpha_2, \alpha_3 \) so that

\[
\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 = \mathbf{w}.
\]
By Theorem SLSLC [82], solutions to this vector equality are solutions to the system of equations

\[
\begin{align*}
\alpha_1 - \alpha_2 + 2\alpha_3 &= 1 \\
2\alpha_1 + \alpha_2 + \alpha_3 &= 8 \\
\alpha_1 + \alpha_2 &= 5.
\end{align*}
\]

Building the augmented matrix for this linear system, and row-reducing, gives

\[
\left[
\begin{array}{ccc|c}
1 & 0 & 1 & 3 \\
0 & 1 & -1 & 2 \\
0 & 0 & 0 & 0
\end{array}
\right].
\]

This system has infinitely many solutions (there’s a free variable in \(x_3\)), but all we need is one solution vector. The solution,

\[
\alpha_1 = 2 \quad \alpha_2 = 3 \quad \alpha_3 = 1
\]

tells us that

\[(2)u_1 + (3)u_2 + (1)u_3 = w\]

so we are convinced that \(w\) really is in \(S\). Notice that there are an infinite number of ways to answer this question affirmatively. We could choose a different solution, this time choosing the free variable to be zero,

\[
\alpha_1 = 3 \quad \alpha_2 = 2 \quad \alpha_3 = 0
\]

shows us that

\[(3)u_1 + (2)u_2 + (0)u_3 = w\]

Verifying the arithmetic in this second solution maybe makes it seem obvious that \(w\) is in this span? And of course, we now realize that there are an infinite number of ways to realize \(w\) as element of \(S\). Let’s ask the same type of question again, but this time with \(y = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}\), i.e. is \(y \in \langle S \rangle\)?

So we’ll look for scalars \(\alpha_1, \alpha_2, \alpha_3\) so that

\[\alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3 = y.\]

By Theorem SLSLC [82] this linear combination becomes the system of equations

\[
\begin{align*}
\alpha_1 - \alpha_2 + 2\alpha_3 &= 2 \\
2\alpha_1 + \alpha_2 + \alpha_3 &= 4 \\
\alpha_1 + \alpha_2 &= 3.
\end{align*}
\]

Building the augmented matrix for this linear system, and row-reducing, gives

\[
\left[
\begin{array}{ccc|c}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1
\end{array}
\right].
\]

This system is inconsistent (there’s a leading 1 in the last column, Theorem RCLS [45]), so there are no scalars \(\alpha_1, \alpha_2, \alpha_3\) that will create a linear combination of \(u_1, u_2, u_3\) that equals \(y\). More precisely, \(y \not\in \langle S \rangle\).

There are three things to observe in this example. (1) It is easy to construct vectors in \(\langle S \rangle\). (2) It is possible that some vectors are in \(\langle S \rangle\) (e.g. \(w\)), while others are not (e.g. \(y\)). (3) Deciding
if a given vector is in $\langle S \rangle$ leads to solving a linear system of equations and asking if the system is consistent.

With a computer program in hand to solve systems of linear equations, could you create a program to decide if a vector was, or wasn’t, in the span of a given set of vectors? Is this art or science?

This example was built on vectors from the columns of the coefficient matrix of Archetype A [634]. Study the determination that $v \in \langle S \rangle$ and see if you can connect it with some of the other properties of Archetype A [634].

Having analyzed Archetype A [634] in Example SCAA [104], we will of course subject Archetype B [638] to a similar investigation.

Example SCAB
Span of the columns of Archetype B
Begin with the finite set of three vectors of size 3 that are the columns of the coefficient matrix in Archetype B [638],

$$R = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix} \right\}$$

and consider the infinite set $V = \langle R \rangle$. First, as an example, note that

$$x = (2) \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix} + (4) \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ -10 \end{bmatrix}$$

is in $\langle R \rangle$, since it is a linear combination of $v_1, v_2, v_3$. In other words, $x \in \langle R \rangle$. Try some different values of $\alpha_1, \alpha_2, \alpha_3$ yourself, and see what vectors you can create as elements of $\langle R \rangle$.

Now ask if a given vector is an element of $\langle R \rangle$. For example, is $z = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}$ in $\langle R \rangle$? Is $z \in \langle R \rangle$?

To answer this question, we will look for scalars $\alpha_1, \alpha_2, \alpha_3$ so that

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = z.$$

By Theorem SLSLC [82] this linear combination becomes the system of equations

$$
\begin{align*}
-7\alpha_1 - 6\alpha_2 - 12\alpha_3 &= -33 \\
5\alpha_1 + 5\alpha_2 + 7\alpha_3 &= 24 \\
\alpha_1 + 4\alpha_3 &= 5
\end{align*}
$$

Building the augmented matrix for this linear system, and row-reducing, gives

$$
\begin{bmatrix}
1 & 0 & 0 & -3 \\
0 & 1 & 0 & 5 \\
0 & 0 & 1 & 2
\end{bmatrix}.
$$

This system has a unique solution,

$$\alpha_1 = -3 \quad \quad \alpha_2 = 5 \quad \quad \alpha_3 = 2$$

telling us that

$$(-3)v_1 + (5)v_2 + (2)v_3 = z$$

so we are convinced that $z$ really is in $\langle R \rangle$. Notice that in this case we have only one way to answer the question affirmatively since the solution is unique.

Let’s ask about another vector, say is $x = \begin{bmatrix} -7 \\ 8 \\ -3 \end{bmatrix}$ in $\langle R \rangle$? Is $x \in \langle R \rangle$?
We desire scalars $\alpha_1, \alpha_2, \alpha_3$ so that
\[ \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = x. \]

By Theorem SLSLC \[82\] this linear combination becomes the system of equations
\[ \begin{align*}
-7\alpha_1 - 6\alpha_2 - 12\alpha_3 &= -7 \\
5\alpha_1 + 5\alpha_2 + 7\alpha_3 &= 8 \\
\alpha_1 + 4\alpha_3 &= -3.
\end{align*} \]

Building the augmented matrix for this linear system, and row-reducing, gives
\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

This system has a unique solution,
\[ \begin{align*}
\alpha_1 &= 1 \\
\alpha_2 &= 2 \\
\alpha_3 &= -1
\end{align*} \]

telling us that
\[ (1)v_1 + (2)v_2 + (-1)v_3 = x \]
so we are convinced that $x$ really is in $(R)$. Notice that in this case we again have only one way to answer the question affirmatively since the solution is again unique.

We could continue to test other vectors for membership in $(R)$, but there is no point. A question about membership in $(R)$ inevitably leads to a system of three equations in the three variables $\alpha_1, \alpha_2, \alpha_3$ with a coefficient matrix whose columns are the vectors $v_1, v_2, v_3$. This particular coefficient matrix is nonsingular, so by Theorem NMUS \[64\], it is guaranteed to have a solution. (This solution is unique, but that’s not critical here.) So no matter which vector we might have chosen for $z$, we would have been certain to discover that it was an element of $(R)$. Stated differently, every vector of size 3 is in $(R)$, or $(R) = \mathbb{C}^3$.

Compare this example with Example SCAA \[104\], and see if you can connect $z$ with some aspects of the write-up for Archetype B \[638\].

Subsection SSNS
Spanning Sets of Null Spaces

We saw in Example VFSAL \[91\] that when a system of equations is homogeneous the solution set can be expressed in the form described by Theorem VFSL5 \[88\] where the vector $c$ is the zero vector. We can essentially ignore this vector, so that the remainder of the typical expression for a solution looks like an arbitrary linear combination, where the scalars are the free variables and the vectors are $u_1, u_2, u_3, \ldots, u_{n-r}$. This sounds a lot like a span. This is the substance of the next theorem.

Theorem SSNS
Spanning Sets for Null Spaces

Suppose that $A$ is an $m \times n$ matrix, and $B$ is a row-equivalent matrix in reduced row-echelon form with $r$ nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the column indices where $B$ has leading 1’s (pivot columns) and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the set of column indices where $B$ does not have leading 1’s. Construct the $n - r$ vectors $z_j, 1 \leq j \leq n - r$ of size $n$ as
\[
[z_j]_{ik} = \begin{cases} 
1 & \text{if } i \in F, i = f_j \\
0 & \text{if } i \in F, i \neq f_j \\
-B_{ik,f_j} & \text{if } i \in D, i = d_k
\end{cases} \]
Then the null space of $A$ is given by
\[ \mathcal{N}(A) = \langle \{z_1, z_2, z_3, \ldots, z_{n-r}\} \rangle. \]

**Proof** Consider the homogeneous system with $A$ as a coefficient matrix, $\mathcal{L}(A, \mathbf{0})$. Its set of solutions, $S$, is by **Definition NSM** [54], the null space of $A$, $\mathcal{N}(A)$. Let $B'$ denote the result of row-reducing the augmented matrix of this homogeneous system. Since the system is homogeneous, the final column of the augmented matrix will be all zeros, and after any number of row operations (**Definition RO** [24]), the column will still be all zeros. So $B'$ has a final column that is totally zeros.

Now apply **Theorem VFSLS** [88] to $B'$, after noting that our homogeneous system must be consistent (**Theorem HSC** [52]). The vector $c$ has zeros for each entry that corresponds to an index in $F$. For entries that correspond to an index in $D$, the value is $−[B']_{k,n+1}$, but for $B'$ any entry in the final column (index $n + 1$) is zero. So $c = \mathbf{0}$. The vectors $z_j$, $1 \leq j \leq n - r$ are identical to the vectors $u_j$, $1 \leq j \leq n - r$ described in **Theorem VFSLS** [88]. Putting it all together and applying **Definition SSCV** [102] in the final step,

\[ \mathcal{N}(A) = S = \{ c + x_1 u_1 + x_2 u_2 + x_3 u_3 + \cdots + x_{n-r} u_{n-r} \mid x_1, x_2, x_3, \ldots, x_{n-r} \in \mathbb{C} \} = \{ x_1 z_1 + x_2 z_2 + x_3 z_3 + \cdots + x_{n-r} z_{n-r} \mid x_1, x_2, x_3, \ldots, x_{n-r} \in \mathbb{C} \} = \langle \{z_1, z_2, z_3, \ldots, z_{n-r}\} \rangle \]

**Example SSNS**

**Spanning set of a null space**

Find a set of vectors, $S$, so that the null space of the matrix $A$ below is the span of $S$, that is, $\langle S \rangle = \mathcal{N}(A)$.

\[ A = \begin{bmatrix} 1 & 3 & 3 & -1 & -5 \\ 2 & 5 & 7 & 1 & 1 \\ 1 & 1 & 5 & 1 & 5 \\ -1 & -4 & -2 & 0 & 4 \end{bmatrix} \]

The null space of $A$ is the set of all solutions to the homogeneous system $\mathcal{L}(A, \mathbf{0})$. If we find the vector form of the solutions to this homogenous system (**Theorem VFSLS** [88]) then the vectors $u_j$, $1 \leq j \leq n - r$ in the linear combination are exactly the vectors $z_j$, $1 \leq j \leq n - r$ described in **Theorem SSNS** [107]. So we can mimic **Example VFSAL** [91] to arrive at these vectors (rather than being a slave to the formulas in the statement of the theorem).

Begin by row-reducing $A$. The result is

\[ \begin{bmatrix} 1 & 0 & 6 & 0 & 4 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

With $D = \{1, 2, 4\}$ and $F = \{3, 5\}$ we recognize that $x_3$ and $x_5$ are free variables and we can express each nonzero row as an expression for the dependent variables $x_1, x_2, x_4$ (respectively) in the free variables $x_3$ and $x_5$. With this we can write the vector form of a solution vector as

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -6x_3 - 4x_5 \\ x_3 + 2x_5 \\ x_3 \\ -3x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} \]
Then in the notation of \textit{Theorem SSNS} [107],

\[ z_1 = \begin{pmatrix} -6 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad z_2 = \begin{pmatrix} -4 \\ 2 \\ 0 \\ -3 \end{pmatrix} \]

and

\[ N(A) = \langle \{z_1, z_2\} \rangle = \langle \begin{pmatrix} -6 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ 1 \\ 0 \\ -3 \\ 1 \end{pmatrix} \rangle \]

\textbf{Example NSDS}

\textbf{Null space directly as a span}

Let’s express the null space of \( A \) as the span of a set of vectors, applying \textit{Theorem SSNS} [107] as economically as possible, without reference to the underlying homogeneous system of equations (in contrast to \textit{Example SSNS} [108]).

\[ A = \begin{bmatrix} 2 & 1 & 5 & 1 & 5 & 1 \\ 1 & 1 & 3 & 1 & 6 & -1 \\ -1 & 1 & -1 & 0 & 4 & -3 \\ -3 & 2 & -4 & -4 & -7 & 0 \\ 3 & -1 & 5 & 2 & 2 & 3 \end{bmatrix} \]

\textit{Theorem SSNS} [107] creates vectors for the span by first row-reducing the matrix in question. The row-reduced version of \( A \) is

\[ B = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 2 \\ 0 & 1 & 1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

I usually find it easier to envision the construction of the homogenous system of equations represented by this matrix, solve for the dependent variables and then unravel the equations into a linear combination. But we can just as well mechanically follow the prescription of \textit{Theorem SSNS} [107]. Here we go, in two big steps.

First, the indices of the non-pivot columns have indices \( F = \{3, 5, 6\} \), so we will construct the \( n - r = 6 - 3 = 3 \) vectors with a pattern of zeros and ones corresponding to the indices in \( F \). This is the realization of the first two lines of the three-case definition of the vectors \( z_j, 1 \leq j \leq n - r \).

\[ z_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad z_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad z_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

Each of these vectors arises due to the presence of a column that is not a pivot column. The remaining entries of each vector are the entries of the corresponding non-pivot column, negated, and distributed into the empty slots in order (these slots have indices in the set \( D \) and correspond...
to pivot columns). This is the realization of the third line of the three-case definition of the vectors $z_j$, $1 \leq j \leq n - r$.

$$z_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \quad z_3 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So, by [Theorem SSNS 107], we have

$$N(A) = \langle \{z_1, z_2, z_3\} \rangle = \langle \begin{bmatrix} -2 \\ 1 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rangle$$

We know that the null space of $A$ is the solution set of the homogeneous system $LS(A, 0)$, but nowhere in this application of [Theorem SSNS 107] have we found occasion to reference the variables or equations of this system.

More advanced computational devices will compute the null space of a matrix. See: [Computation NS.MMA 605]. Here’s an example that will simultaneously exercise the span construction and [Theorem SSNS 107], while also pointing the way to the next section.

**Example SCAD**

**Span of the columns of Archetype D**

Begin with the set of four vectors of size 3

$$T = \{w_1, w_2, w_3, w_4\} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 7 \\ -5 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ -5 \\ -5 \end{bmatrix} \right\}$$

and consider the infinite set $W = \langle T \rangle$. The vectors of $T$ have been chosen as the four columns of the coefficient matrix in [Archetype D 647]. Check that the vector

$$z_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

is a solution to the homogeneous system $LS(D, 0)$ (it is the vector $z_2$ provided by the description of the null space of the coefficient matrix $D$ from [Theorem SSNS 107]). Applying [Theorem SLSLC 82], we can write the linear combination,

$$2w_1 + 3w_2 + 0w_3 + 1w_4 = 0$$

which we can solve for $w_4$,

$$w_4 = (-2)w_1 + (-3)w_2.$$
\[ = 11w_1 + 5w_2 + 6w_3. \]

So what began as a linear combination of the vectors \( w_1, w_2, w_3, w_4 \) has been reduced to a linear combination of the vectors \( w_1, w_2, w_3 \). A careful proof using our definition of set equality (Definition SE [616]) would now allow us to conclude that this reduction is possible for any vector in \( W \), so

\[ W = \langle \{w_1, w_2, w_3\} \rangle. \]

So the span of our set of vectors, \( W \), has not changed, but we have described it by the span of a set of three vectors, rather than four. Furthermore, we can achieve yet another, similar, reduction.

Check that the vector

\[
z_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}
\]

is a solution to the homogeneous system \( LS(D, 0) \) (it is the vector \( z_1 \) provided by the description of the null space of the coefficient matrix \( D \) from Theorem SSNS [107]). Applying Theorem SLSLC [82], we can write the linear combination,

\[ (-3)w_1 + (-1)w_2 + 1w_3 = 0 \]

which we can solve for \( w_3 \),

\[ w_3 = 3w_1 + 1w_2. \]

This equation says that whenever we encounter the vector \( w_3 \), we can replace it with a specific linear combination of the vectors \( w_1 \) and \( w_2 \). So, as before, the vector \( w_3 \) is not needed in the description of \( W \), provided we have \( w_1 \) and \( w_2 \) available. In particular, a careful proof would show that

\[ W = \langle \{w_1, w_2\} \rangle. \]

So \( W \) began life as the span of a set of four vectors, and we have now shown (utilizing solutions to a homogeneous system) that \( W \) can also be described as the span of a set of just two vectors. Convince yourself that we cannot go any further. In other words, it is not possible to dismiss either \( w_1 \) or \( w_2 \) in a similar fashion and winnow the set down to just one vector.

What was it about the original set of four vectors that allowed us to declare certain vectors as surplus? And just which vectors were we able to dismiss? And why did we have to stop once we had two vectors remaining? The answers to these questions motivate “linear independence,” our next section and next definition, and so are worth considering carefully now.

It is possible to have your computational device crank out the vector form of the solution set to a linear system of equations. See: Computation VFSS.MMA [606].

Subsection READ
Reading Questions

1. Let \( S \) be the set of three vectors below.

\[
S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \right\}
\]

Let \( W = \langle S \rangle \) be the span of \( S \). Is the vector \[
\begin{bmatrix} -1 \\ -8 \\ -4 \end{bmatrix}
\]
in \( W \)? Give an explanation of the reason for your answer.
2. Use $S$ and $W$ from the previous question. Is the vector \[
\begin{bmatrix}
6 \\
5 \\
-1
\end{bmatrix}
\] in $W$? Give an explanation of the reason for your answer.

3. For the matrix $A$ below, find a set $S$ so that $\langle S \rangle = \mathcal{N}(A)$, where $\mathcal{N}(A)$ is the null space of $A$. (See Theorem SSNS [107].)

\[
A = \begin{bmatrix}
1 & 3 & 1 & 9 \\
2 & 1 & -3 & 8 \\
1 & 1 & -1 & 5
\end{bmatrix}
\]
C22  For each archetype that is a system of equations, consider the corresponding homogeneous system of equations. Write elements of the solution set to these homogeneous systems in vector form, as guaranteed by Theorem VFSLS. Then write the null space of the coefficient matrix of each system as the span of a set of vectors, as described in Theorem SSNS.

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671

Contributed by Robert Beezer  Solution 115

C23  Archetype K 676 and Archetype L 680 are defined as matrices. Use Theorem SSNS directly to find a set $S$ so that $\langle S \rangle$ is the null space of the matrix. Do not make any reference to the associated homogeneous system of equations in your solution.

Contributed by Robert Beezer  Solution 115

C40  Suppose that $S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\}$. Let $W = \langle S \rangle$ and let $x = \begin{bmatrix} 5 \\ 8 \\ -12 \\ -5 \end{bmatrix}$. Is $x \in W$? If so, provide an explicit linear combination that demonstrates this.

Contributed by Robert Beezer  Solution 115

C41  Suppose that $S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\}$. Let $W = \langle S \rangle$ and let $y = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 5 \end{bmatrix}$. Is $y \in W$? If so, provide an explicit linear combination that demonstrates this.

Contributed by Robert Beezer  Solution 115

C42  Suppose $R = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ -1 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 4 \\ -1 \\ -2 \end{bmatrix} \right\}$. Is $y = \begin{bmatrix} -8 \\ -1 \\ 0 \end{bmatrix}$ in $\langle R \rangle$?

Contributed by Robert Beezer  Solution 116

C43  Suppose $R = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ -1 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 4 \\ -1 \\ -2 \end{bmatrix} \right\}$. Is $z = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$ in $\langle R \rangle$?

Contributed by Robert Beezer  Solution 116

C44  Suppose that $S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \\ -6 \\ 5 \\ 5 \\ 4 \\ 1 \end{bmatrix} \right\}$. Let $W = \langle S \rangle$ and let $y = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$. Is $x \in W$? If so, provide an explicit linear combination that demonstrates this.

Contributed by Robert Beezer  Solution 117
C45  Suppose that $S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 1 \end{bmatrix} \right\}$. Let $W = \langle S \rangle$ and let $w = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. Is $x \in W$? If so, provide an explicit linear combination that demonstrates this.

Contributed by Robert Beezer  Solution [117]

C50  Let $A$ be the matrix below.

(a) Find a set $S$ so that $\mathcal{N}(A) = \langle S \rangle$.

(b) If $z = \begin{bmatrix} 3 \\ -5 \\ 1 \\ 2 \end{bmatrix}$, then show directly that $z \in \mathcal{N}(A)$.

(c) Write $z$ as a linear combination of the vectors in $S$.

$$A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 2 & 1 & 3 \\ -1 & 0 & 1 & 1 \end{bmatrix}$$

Contributed by Robert Beezer  Solution [118]

C60  For the matrix $A$ below, find a set of vectors $S$ so that the span of $S$ equals the null space of $A$, $\langle S \rangle = \mathcal{N}(A)$.

$$A = \begin{bmatrix} 1 & 1 & 6 & -8 \\ 1 & -2 & 0 & 1 \\ -2 & 1 & -6 & 7 \end{bmatrix}$$

Contributed by Robert Beezer  Solution [118]

M20  In Example SCAD [110] we began with the four columns of the coefficient matrix of Archetype D [647], and used these columns in a span construction. Then we methodically argued that we could remove the last column, then the third column, and create the same set by just doing a span construction with the first two columns. We claimed we could not go any further, and had removed as many vectors as possible. Provide a convincing argument for why a third vector cannot be removed.

Contributed by Robert Beezer

M21  In the spirit of Example SCAD [110], begin with the four columns of the coefficient matrix of Archetype C [643], and use these columns in a span construction to build the set $S$. Argue that $S$ can be expressed as the span of just three of the columns of the coefficient matrix (saying exactly which three) and in the spirit of Exercise SS.M20 [114] argue that no one of these three vectors can be removed and still have a span construction create $S$.

Contributed by Robert Beezer  Solution [119]

T10  Suppose that $v_1, v_2 \in \mathbb{C}^m$. Prove that

$$\langle \{v_1, v_2\} \rangle = \langle \{v_1, v_2, 5v_1 + 3v_2\} \rangle$$

Contributed by Robert Beezer  Solution [119]

T20  Suppose that $S$ is a set of vectors from $\mathbb{C}^m$. Prove that the zero vector, $0$, is an element of $\langle S \rangle$.

Contributed by Robert Beezer  Solution [119]

T21  Suppose that $S$ is a set of vectors from $\mathbb{C}^m$ and $x, y \in \langle S \rangle$. Prove that $x + y \in \langle S \rangle$.

Contributed by Robert Beezer

T22  Suppose that $S$ is a set of vectors from $\mathbb{C}^m$, $\alpha \in \mathbb{C}$, and $x \in \langle S \rangle$. Prove that $\alpha x \in \langle S \rangle$.

Contributed by Robert Beezer
Subsection SS.SOL Solutions

C22 Contributed by Robert Beezer Statement 113

The vector form of the solutions obtained in this manner will involve precisely the vectors described in Theorem SSNS 107 as providing the null space of the coefficient matrix of the system as a span. These vectors occur in each archetype in a description of the null space. Studying Example VFSAL 91 may be of some help.

C23 Contributed by Robert Beezer Statement 113

Study Example NSDS 109 to understand the correct approach to this question. The solution for each is listed in the Archetypes (Appendix A 630) themselves.

C40 Contributed by Robert Beezer Statement 113

Rephrasing the question, we want to know if there are scalars $\alpha_1$ and $\alpha_2$ such that

$$\begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ -12 \\ -5 \end{bmatrix}$$

Theorem SLSLC 82 allows us to rephrase the question again as a quest for solutions to the system of four equations in two unknowns with an augmented matrix given by

$$\begin{bmatrix} 2 & 3 & 5 \\ -1 & 2 & 8 \\ 3 & -2 & -12 \\ 4 & 1 & -5 \end{bmatrix}$$

This matrix row-reduces to

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From the form of this matrix, we can see that $\alpha_1 = -2$ and $\alpha_2 = 3$ is an affirmative answer to our question. More convincingly,

$$\begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix} + (3) \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ -12 \\ -5 \end{bmatrix}$$

C41 Contributed by Robert Beezer Statement 113

Rephrasing the question, we want to know if there are scalars $\alpha_1$ and $\alpha_2$ such that

$$\begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 5 \end{bmatrix}$$

Theorem SLSLC 82 allows us to rephrase the question again as a quest for solutions to the system of four equations in two unknowns with an augmented matrix given by

$$\begin{bmatrix} 2 & 3 & 5 \\ -1 & 2 & 8 \\ 3 & -2 & 3 \\ 4 & 1 & 5 \end{bmatrix}$$

Version 1.04
This matrix row-reduces to
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
With a leading 1 in the last column of this matrix (Theorem RCLS [45]) we can see that the system of equations has no solution, so there are no values for \(\alpha_1\) and \(\alpha_2\) that will allow us to conclude that \(y\) is in \(W\). So \(y \not\in W\).

C42 Contributed by Robert Beezer Statement [113]
Form a linear combination, with unknown scalars, of \(R\) that equals \(y\),
\[
\begin{align*}
a_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 3 \\ -2 \end{bmatrix} &= \begin{bmatrix} 1 \\ -1 \\ -8 \\ -4 \\ -3 \end{bmatrix}
\end{align*}
\]
We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in \((R)\). By Theorem SLSLC [82] any such values will also be solutions to the linear system represented by the augmented matrix,
\[
\begin{bmatrix}
2 & 1 & 3 & 1 \\
-1 & 1 & -1 & -1 \\
3 & 2 & 0 & -8 \\
4 & 2 & 3 & -4 \\
0 & -1 & -2 & -3
\end{bmatrix}
\]
Row-reducing the matrix yields,
\[
\begin{bmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
From this we see that the system of equations is consistent (Theorem RCLS [45]), and has a unique solution. This solution will provide a linear combination of the vectors in \(R\) that equals \(y\). So \(y \in R\).

C43 Contributed by Robert Beezer Statement [113]
Form a linear combination, with unknown scalars, of \(R\) that equals \(z\),
\[
\begin{align*}
a_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 3 \\ -2 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \\ 5 \\ 3 \\ 1 \end{bmatrix}
\end{align*}
\]
We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in \((R)\). By Theorem SLSLC [82] any such values will also be solutions to the linear system represented by the augmented matrix,
\[
\begin{bmatrix}
2 & 1 & 3 & 1 \\
-1 & 1 & -1 & -1 \\
3 & 2 & 0 & 5 \\
4 & 2 & 3 & 3 \\
0 & -1 & -2 & 1
\end{bmatrix}
\]
Row-reducing the matrix yields,
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

With a leading 1 in the last column, the system is inconsistent (Theorem RCLS 45), so there are no scalars \(a_1, a_2, a_3\) that will create a linear combination of the vectors in \(R\) that equal \(z\). So \(z \notin R\).

C44 Contributed by Robert Beezer Statement 113
Form a linear combination, with unknown scalars, of \(S\) that equals \(y\),
\[
\begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix} + \begin{bmatrix}
2 \\
3 \\
1
\end{bmatrix} + \begin{bmatrix}
3 \\
1 \\
5
\end{bmatrix} + \begin{bmatrix}
1 \\
5 \\
4
\end{bmatrix} + \begin{bmatrix}
-6 \\
5 \\
1
\end{bmatrix} = \begin{bmatrix}
-5 \\
3 \\
0
\end{bmatrix}
\]

We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in \(\langle S \rangle\). By Theorem SLSLC 82 any such values will also be solutions to the linear system represented by the augmented matrix,
\[
\begin{bmatrix}
-1 & 3 & 1 & -6 & -5 \\
2 & 1 & 5 & 5 & 3 \\
1 & 2 & 4 & 1 & 0
\end{bmatrix}
\]

Row-reducing the matrix yields,
\[
\begin{bmatrix}
1 & 0 & 2 & 3 & 2 \\
0 & 1 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

From this we see that the system of equations is consistent (Theorem RCLS 45), and has an infinitely many solutions. Any solution will provide a linear combination of the vectors in \(R\) that equals \(y\). So \(y \in S\), for example,
\[
(-10)\begin{bmatrix}
-1 \\
2 \\
1
\end{bmatrix} + (-2)\begin{bmatrix}
3 \\
1 \\
2
\end{bmatrix} + (3)\begin{bmatrix}
1 \\
5 \\
4
\end{bmatrix} + (2)\begin{bmatrix}
-6 \\
5 \\
1
\end{bmatrix} = \begin{bmatrix}
-5 \\
3 \\
0
\end{bmatrix}
\]

C45 Contributed by Robert Beezer Statement 114
Form a linear combination, with unknown scalars, of \(S\) that equals \(w\),
\[
\begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix} + \begin{bmatrix}
3 \\
1 \\
2
\end{bmatrix} + \begin{bmatrix}
3 \\
5 \\
4
\end{bmatrix} + \begin{bmatrix}
1 \\
5 \\
1
\end{bmatrix} + \begin{bmatrix}
-6 \\
5 \\
1
\end{bmatrix} = \begin{bmatrix}
2 \\
1 \\
3
\end{bmatrix}
\]

We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in \(\langle S \rangle\). By Theorem SLSLC 82 any such values will also be solutions to the linear system represented by the augmented matrix,
\[
\begin{bmatrix}
-1 & 3 & 1 & -6 & 2 \\
2 & 1 & 5 & 5 & 1 \\
1 & 2 & 4 & 1 & 3
\end{bmatrix}
\]

Row-reducing the matrix yields,
\[
\begin{bmatrix}
1 & 0 & 2 & 3 & 0 \\
0 & 1 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
With a leading 1 in the last column, the system is inconsistent (Theorem RCLS), so there are no scalars $a_1, a_2, a_3, a_4$ that will create a linear combination of the vectors in $S$ that equal $w$. So $w \notin \langle S \rangle$.

(a) Theorem SSNS provides formulas for a set $S$ with this property, but first we must row-reduce $A$

$$A \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_3$ and $x_4$ would be the free variables in the homogeneous system $\mathcal{L}S(A, 0)$ and Theorem SSNS provides the set $S = \{z_1, z_2\}$ where

$$z_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

(b) Simply employ the components of the vector $z$ as the variables in the homogeneous system $\mathcal{L}S(A, 0)$. The three equations of this system evaluate as follows,

$$\begin{align*}
2(3) + 3(-5) + 1(1) + 4(2) &= 0 \\
1(3) + 2(-5) + 1(1) + 3(2) &= 0 \\
-1(3) + 0(-5) + 1(1) + 1(2) &= 0
\end{align*}$$

Since each result is zero, $z$ qualifies for membership in $N(A)$.

(c) By Theorem SSNS we know this must be possible (that is the moral of this exercise). Find scalars $\alpha_1$ and $\alpha_2$ so that

$$\alpha_1 z_1 + \alpha_2 z_2 = \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 1 \\ 2 \end{bmatrix} = \mathbf{z}$$

Theorem SLSLC allows us to convert this question into a question about a system of four equations in two variables. The augmented matrix of this system row-reduces to

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A solution is $\alpha_1 = 1$ and $\alpha_2 = 2$. (Notice too that this solution is unique!)
Then with $S$ given by

$$
S = \begin{bmatrix}
-4 & 5 \\
-2 & 3 \\
1 & 0 \\
0 & 1
\end{bmatrix}
$$

Theorem SSNS [107] guarantees that

$$
\mathcal{N}(A) = \langle S \rangle = \begin{bmatrix}
-4 & 5 \\
-2 & 3 \\
1 & 0 \\
0 & 1
\end{bmatrix}
$$

M21 Contributed by Robert Beezer Statement [114]

If the columns of the coefficient matrix from Archetype C [643] are named $u_1, u_2, u_3, u_4$ then we can discover the equation

$$
(-2)u_1 + (-3)u_2 + u_3 + u_4 = 0
$$

by building a homogeneous system of equations and viewing a solution to the system as scalars in a linear combination via Theorem SLSLC [82]. This particular vector equation can be rearranged to read

$$
u_4 = (2)u_1 + (3)u_2 + (-1)u_3
$$

This can be interpreted to mean that $u_4$ is unnecessary in $\langle \{u_1, u_2, u_3, u_4\} \rangle$, so that

$$
\langle \{u_1, u_2, u_3, u_4\} \rangle = \langle \{u_1, u_2, u_3\} \rangle
$$

If we try to repeat this process and find a linear combination of $u_1, u_2, u_3$ that equals the zero vector, we will fail. The required homogeneous system of equations (via Theorem SLSLC [82]) has only a trivial solution, which will not provide the kind of equation we need to remove one of the three remaining vectors.

T10 Contributed by Robert Beezer Statement [114]

This is an equality of sets, so Definition SE [616] applies.

First show that $X = \langle \{v_1, v_2\} \rangle \subseteq \langle \{v_1, v_2, 5v_1 + 3v_2\} \rangle = Y$.

Choose $x \in X$. Then $x = a_1v_1 + a_2v_2$ for some scalars $a_1$ and $a_2$. Then,

$$
x = a_1v_1 + a_2v_2 = a_1v_1 + a_2v_2 + 0(5v_1 + 3v_2)
$$

which qualifies $x$ for membership in $Y$, as it is a linear combination of $v_1, v_2, 5v_1 + 3v_2$.

Now show the opposite inclusion, $Y = \langle \{v_1, v_2, 5v_1 + 3v_2\} \rangle \subseteq \langle \{v_1, v_2\} \rangle = X$.

Choose $y \in Y$. Then there are scalars $a_1, a_2, a_3$ such that

$$
y = a_1v_1 + a_2v_2 + a_3(5v_1 + 3v_2)
$$

Rearranging, we obtain,

$$
y = a_1v_1 + a_2v_2 + a_3(5v_1 + 3v_2)
$$

$$
= a_1v_1 + a_2v_2 + 5a_3v_1 + 3a_3v_2
$$

$$
= a_1v_1 + 5a_3v_1 + a_2v_2 + 3a_3v_2
$$

$$
= (a_1 + 5a_3)v_1 + (a_2 + 3a_3)v_2
$$

This is an expression for $y$ as a linear combination of $v_1$ and $v_2$, earning $y$ membership in $X$. Since $X$ is a subset of $Y$, and vice versa, we see that $X = Y$, as desired.

T20 Contributed by Robert Beezer Statement [114]

No matter what the elements of the set $S$ are, we can choose the scalars in a linear combination to all be zero. Suppose that $S = \{v_1, v_2, v_3, \ldots, v_p\}$. Then compute

$$
0v_1 + 0v_2 + 0v_3 + \cdots + 0v_p = 0 + 0 + 0 + \cdots + 0
$$
= 0

But what if we choose $S$ to be the empty set? The *convention* is that the empty sum in Definition SSCVI [102] evaluates to “zero,” in this case this is the zero vector.
Section LI
Linear Independence

Subsection LISV
Linearly Independent Sets of Vectors

Theorem SLSLC \[82\] tells us that a solution to a homogeneous system of equations is a linear combination of the columns of the coefficient matrix that equals the zero vector. We used just this situation to our advantage (twice!) in Example SCAD \[110\] where we reduced the set of vectors used in a span construction from four down to two, by declaring certain vectors as surplus. The next two definitions will allow us to formalize this situation.

Definition RLDCV
Relation of Linear Dependence for Column Vectors
Given a set of vectors \( S = \{u_1, u_2, u_3, \ldots, u_n\} \), a true statement of the form
\[
\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_n u_n = 0
\]
is a relation of linear dependence on \( S \). If this statement is formed in a trivial fashion, i.e. \( \alpha_i = 0, 1 \leq i \leq n \), then we say it is the trivial relation of linear dependence on \( S \).

Definition LICV
Linear Independence of Column Vectors
The set of vectors \( S = \{u_1, u_2, u_3, \ldots, u_n\} \) is linearly dependent if there is a relation of linear dependence on \( S \) that is not trivial. In the case where the only relation of linear dependence on \( S \) is the trivial one, then \( S \) is a linearly independent set of vectors.

Notice that a relation of linear dependence is an equation. Though most of it is a linear combination, it is not a linear combination (that would be a vector). Linear independence is a property of a set of vectors. It is easy to take a set of vectors, and an equal number of scalars, all zero, and form a linear combination that equals the zero vector. When the easy way is the only way, then we say the set is linearly independent. Here’s a couple of examples.

Example LDS
Linearly dependent set in \( \mathbb{C}^5 \)
Consider the set of \( n = 4 \) vectors from \( \mathbb{C}^5 \),
\[
S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -3 \\ -1 \\ 0 \end{bmatrix} \right\}.
\]
To determine linear independence we first form a relation of linear dependence,
\[
\alpha_1 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} -6 \\ 7 \\ -3 \\ -1 \\ 0 \end{bmatrix} = 0.
\]
We know that \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \) is a solution to this equation, but that is of no interest whatsoever. That is always the case, no matter what four vectors we might have chosen. We are curious to know if there are other, nontrivial, solutions. Theorem SLSLC \[82\] tells us that we can
find such solutions as solutions to the homogeneous system \( LS(A, \mathbf{0}) \) where the coefficient matrix has these four vectors as columns,

\[
A = \begin{bmatrix}
2 & 1 & 2 & -6 \\
-1 & 2 & 1 & 7 \\
3 & -1 & -3 & -1 \\
1 & 5 & 6 & 0 \\
2 & 2 & 1 & 1
\end{bmatrix}.
\]

Row-reducing this coefficient matrix yields,

\[
\begin{bmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

We could solve this homogeneous system completely, but for this example all we need is one nontrivial solution. Setting the lone free variable to any nonzero value, such as \( x_4 = 1 \), yields the nontrivial solution

\[
x = \begin{bmatrix}
2 \\
-4 \\
3 \\
1
\end{bmatrix}.
\]

completing our application of Theorem SLSLC \[82\], we have

\[
\begin{bmatrix}
2 \\
-1 \\
3 \\
1
\end{bmatrix} + (-4) \begin{bmatrix}
1 \\
-1 \\
5 \\
2
\end{bmatrix} + 3 \begin{bmatrix}
2 \\
1 \\
6 \\
1
\end{bmatrix} + 1 \begin{bmatrix}
-6 \\
2 \\
-3 \\
1
\end{bmatrix} = \mathbf{0}.
\]

This is a relation of linear dependence on \( S \) that is not trivial, so we conclude that \( S \) is linearly dependent.

\[\top\]

Example LIS

Linearly independent set in \( \mathbb{C}^5 \)

Consider the set of \( n = 4 \) vectors from \( \mathbb{C}^5 \),

\[
T = \left\{ \begin{bmatrix}
2 \\
-1 \\
3 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
2 \\
-1 \\
5
\end{bmatrix}, \begin{bmatrix}
2 \\
1 \\
6 \\
2
\end{bmatrix}, \begin{bmatrix}
-6 \\
7 \\
-1 \\
1
\end{bmatrix} \right\}.
\]

To determine linear independence we first form a relation of linear dependence,

\[
\alpha_1 \begin{bmatrix}
2 \\
-1 \\
3 \\
1
\end{bmatrix} + \alpha_2 \begin{bmatrix}
1 \\
2 \\
-1 \\
5
\end{bmatrix} + \alpha_3 \begin{bmatrix}
2 \\
1 \\
-3 \\
6
\end{bmatrix} + \alpha_4 \begin{bmatrix}
-6 \\
7 \\
-1 \\
1
\end{bmatrix} = \mathbf{0}.
\]

We know that \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \) is a solution to this equation, but that is of no interest whatsoever. That is always the case, no matter what four vectors we might have chosen. We are curious to know if there are other, nontrivial, solutions. Theorem SLSLC \[82\] tells us that we can
find such solutions as solution to the homogeneous system $LS(B, 0)$ where the coefficient matrix
has these four vectors as columns,

$$B = \begin{bmatrix}
2 & 1 & 2 & -6 \\
-1 & 2 & 1 & 7 \\
3 & -1 & -3 & -1 \\
1 & 5 & 6 & 1 \\
2 & 2 & 1 & 1
\end{bmatrix}.$$ 

Row-reducing this coefficient matrix yields,

$$\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$ 

From the form of this matrix, we see that there are no free variables, so the solution is unique, and
because the system is homogeneous, this unique solution is the trivial solution. So we now know
that there is but one way to combine the four vectors of $T$ into a relation of linear dependence,
and that one way is the easy and obvious way. In this situation we say that the set, $T$, is linearly
independent. 

**Example LDS** [121] and **Example LIS** [122] relied on solving a homogeneous system of equations
to determine linear independence. We can codify this process in a time-saving theorem.

**Theorem LIVHS**

**Linearly Independent Vectors and Homogeneous Systems**

Suppose that $A$ is an $m \times n$ matrix and $S = \{A_1, A_2, A_3, \ldots, A_n\}$ is the set of vectors in $\mathbb{C}^m$ that
are the columns of $A$. Then $S$ is a linearly independent set if and only if the homogeneous system
$LS(A, 0)$ has a unique solution. 

**Proof** ($\Leftarrow$) Suppose that $LS(A, 0)$ has a unique solution. Since it is a homogeneous system, this
solution must be the trivial solution $x = 0$. By **Theorem SLSLC** [82], this means that the only
relation of linear dependence on $S$ is the trivial one. So $S$ is linearly independent.

($\Rightarrow$) We will prove the contrapositive. Suppose that $LS(A, 0)$ does not have a unique solution.
Since it is a homogeneous system, it is consistent (**Theorem HSC** [52]), and so must have infinitely
many solutions (**Theorem PSSLS** [47]). One of these infinitely many solutions must be nontrivial
(in fact, almost all of them are), so choose one. By **Theorem SLSLC** [82] this nontrivial solution
will give a nontrivial relation of linear dependence on $S$, so we can conclude that $S$ is a linearly
dependent set.

Since **Theorem LIVHS** [123] is an equivalence, we can use it to determine the linear independence
or dependence of any set of column vectors, just by creating a corresponding matrix and analyzing
the row-reduced form. Let’s illustrate this with two more examples.

**Example LIHS**

**Linearly independent, homogeneous system**

Is the set of vectors

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

linearly independent or linearly dependent?
Theorem LIVHS \[123\] suggests we study the matrix whose columns are the vectors in \( S \),
\[
A = \begin{bmatrix}
2 & 6 & 4 \\
-1 & 2 & 3 \\
3 & -1 & -4 \\
4 & 3 & 5 \\
2 & 4 & 1
\end{bmatrix}
\]

Specifically, we are interested in the size of the solution set for the homogeneous system \( \mathcal{L}S(A, 0) \).
Row-reducing \( A \), we obtain
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Now, \( r = 3 \), so there are \( n - r = 3 - 3 = 0 \) free variables and we see that \( \mathcal{L}S(A, 0) \) has a unique solution (Theorem HSC \[52\], Theorem FVCS \[46\]). By Theorem LIVHS \[123\], the set \( S \) is linearly independent. \( \square \)

Example LDHS
Linearly dependent, homogeneous system
Is the set of vectors
\[
S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -4 \\ -1 \\ 2 \end{bmatrix} \right\}
\]
linearly independent or linearly dependent?

Theorem LIVHS \[123\] suggests we study the matrix whose columns are the vectors in \( S \),
\[
A = \begin{bmatrix}
2 & 6 & 4 \\
-1 & 2 & 3 \\
3 & -1 & -4 \\
4 & 3 & -1 \\
2 & 4 & 2
\end{bmatrix}
\]

Specifically, we are interested in the size of the solution set for the homogeneous system \( \mathcal{L}S(A, 0) \).
Row-reducing \( A \), we obtain
\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Now, \( r = 2 \), so there are \( n - r = 3 - 2 = 1 \) free variables and we see that \( \mathcal{L}S(A, 0) \) has infinitely many solutions (Theorem HSC \[52\], Theorem FVCS \[46\]). By Theorem LIVHS \[123\], the set \( S \) is linearly dependent. \( \square \)

As an equivalence, Theorem LIVHS \[123\] gives us a straightforward way to determine if a set of vectors is linearly independent or dependent.

Review Example LIHS \[123\] and Example LDHS \[124\]. They are very similar, differing only in the last two slots of the third vector. This resulted in slightly different matrices when row-reduced, and slightly different values of \( r \), the number of nonzero rows. Notice, too, that we are less interested in the actual solution set, and more interested in its form or size. These observations allow us to make a slight improvement in Theorem LIVHS \[123\].
Theorem LIVRN
Linearly Independent Vectors, \( r \) and \( n \)

Suppose that \( A \) is an \( m \times n \) matrix and \( S = \{A_1, A_2, A_3, \ldots, A_n\} \) is the set of vectors in \( \mathbb{C}^n \) that are the columns of \( A \). Let \( B \) be a matrix in reduced row-echelon form that is row-equivalent to \( A \) and let \( r \) denote the number of non-zero rows in \( B \). Then \( S \) is linearly independent if and only if \( n = r \). □

Proof Theorem LIVHS \[123\] says the linear independence of \( S \) is equivalent to the homogeneous linear system \( \mathcal{L}(A, 0) \) having a unique solution. Since \( \mathcal{L}(A, 0) \) is consistent (Theorem HSC \[52\]) we can apply Theorem CSRN \[46\] to see that the solution is unique exactly when \( n = r \). ■

So now here’s an example of the most straightforward way to determine if a set of column vectors is linearly independent or linearly dependent. While this method can be quick and easy, don’t forget the logical progression from the definition of linear independence through homogeneous system of equations which makes it possible.

Example LDRN
Linearly dependent, \( r < n \)

Is the set of vectors

\[
S = \left\{ \begin{bmatrix} 2 & 9 & 1 & -3 & 6 \\ -1 & -6 & 1 & 1 & 2 \\ 3 & -2 & 1 & 4 & 1 \\ 1 & 3 & 0 & 2 & 4 \\ 0 & 2 & 0 & 1 & 3 \\ 3 & 1 & 1 & 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 7 & 0 & 5 & -2 \\ 1 & 2 & 4 & -2 & 1 \\ 1 & -3 & -1 & 2 & 4 \\ 3 & 1 & -2 & 9 & 3 \end{bmatrix} \right\}
\]

linearly independent or linearly dependent? Theorem LIVHS \[123\] suggests we place these vectors into a matrix as columns and analyze the row-reduced version of the matrix:

\[
\begin{bmatrix} 2 & 9 & 1 & -3 & 6 \\ -1 & -6 & 1 & 1 & 2 \\ 3 & -2 & 1 & 4 & 1 \\ 1 & 3 & 0 & 2 & 4 \\ 0 & 2 & 0 & 1 & 3 \\ 3 & 1 & 1 & 2 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

Now we need only compute that \( r = 4 < 5 = n \) to recognize, via Theorem LIVHS \[123\] that \( S \) is a linearly dependent set. Boom! ✤

Example LLDS
Large linearly dependent set in \( \mathbb{C}^4 \)

Consider the set of \( n = 9 \) vectors from \( \mathbb{C}^4 \),

\[
R = \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ -3 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \end{bmatrix} \right\}
\]

To employ Theorem LIVHS \[123\], we form a \( 4 \times 9 \) coefficient matrix, \( C \),

\[
C = \begin{bmatrix} -1 & 7 & 1 & 0 & 5 & 2 & 3 & 1 & -6 \\ 3 & 1 & 2 & 4 & -2 & 1 & 0 & 1 & -1 \\ 1 & -3 & -1 & 2 & 4 & -6 & -3 & 5 & 1 \\ 2 & 6 & -2 & 9 & 3 & 4 & 1 & 3 & 1 \end{bmatrix}
\]

To determine if the homogeneous system \( \mathcal{L}(C, 0) \) has a unique solution or not, we would normally row-reduce this matrix. But in this particular example, we can do better. Theorem HMVEI \[54\] tells us that since the system is homogeneous with \( n = 9 \) variables in \( m = 4 \) equations, and \( n > m \),
there must be infinitely many solutions. Since there is not a unique solution, Theorem LIVHS says the set is linearly dependent.

The situation in Example LLDS is slick enough to warrant formulating as a theorem.

**Theorem MVSLD**

**More Vectors than Size implies Linear Dependence**

Suppose that \( S = \{ u_1, u_2, u_3, \ldots, u_n \} \) is the set of vectors in \( \mathbb{C}^m \), and that \( n > m \). Then \( S \) is a linearly dependent set. □

**Proof** Form the \( m \times n \) coefficient matrix \( A \) that has the column vectors \( u_i, 1 \leq i \leq n \) as its columns. Consider the homogeneous system \( LS(A, 0) \). By Theorem HMVEI, this system has infinitely many solutions. Since the system does not have a unique solution, Theorem LIVHS says the columns of \( A \) form a linearly dependent set, which is the desired conclusion. ■

---

**Subsection LINM**

**Linear Independence and Nonsingular Matrices**

We will now specialize to sets of \( n \) vectors from \( \mathbb{C}^n \). This will put Theorem MVSLD off-limits, while Theorem LIVHS will involve square matrices. Let’s begin by contrasting Archetype A and Archetype B.

**Example LDCAA**

**Linearly dependent columns in Archetype A**

Archetype A is a system of linear equations with coefficient matrix,

\[
A = \begin{bmatrix}
1 & -1 & 2 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}.
\]

Do the columns of this matrix form a linearly independent or dependent set? By Example S we know that \( A \) is singular. According to the definition of nonsingular matrices, the homogeneous system \( LS(A, 0) \) has infinitely many solutions. So by Theorem LIVHS, the columns of \( A \) form a linearly dependent set. ■

**Example LICAB**

**Linearly independent columns in Archetype B**

Archetype B is a system of linear equations with coefficient matrix,

\[
B = \begin{bmatrix}
-7 & -6 & -12 \\
5 & 5 & 7 \\
1 & 0 & 4
\end{bmatrix}.
\]

Do the columns of this matrix form a linearly independent or dependent set? By Example NM we know that \( B \) is nonsingular. According to the definition of nonsingular matrices, the homogeneous system \( LS(A, 0) \) has a unique solution. So by Theorem LIVHS, the columns of \( B \) form a linearly independent set. ■

That Archetype A and Archetype B have opposite properties for the columns of their coefficient matrices is no accident. Here’s the theorem, and then we will update our equivalences for nonsingular matrices.

**Theorem NMLIC**

**Nonsingular Matrices have Linearly Independent Columns**

Suppose that \( A \) is a square matrix. Then \( A \) is nonsingular if and only if the columns of \( A \) form a linearly independent set. □

**Proof** This is a proof where we can chain together equivalences, rather than proving the two halves separately.

\( A \) nonsingular \( \iff \) \( LS(A, 0) \) has a unique solution Definition NM
Here’s an update to Theorem NME1 [66].

**Theorem NME2**

**Nonsingular Matrix Equivalences, Round 2**

Suppose that $A$ is a square matrix. The following are equivalent.

1. $A$ is nonsingular.
2. $A$ row-reduces to the identity matrix.
3. The null space of $A$ contains only the zero vector, $N(A) = \{0\}$.
4. The linear system $\mathcal{LS}(A, b)$ has a unique solution for every possible choice of $b$.
5. The columns of $A$ form a linearly independent set.

**Proof** Theorem NMLIC [126] is yet another equivalence for a nonsingular matrix, so we can add it to the list in Theorem NME1 [66].

---

**Subsection NSSLI**

**Null Spaces, Spans, Linear Independence**

In Subsection SS.SSNS [107] we proved Theorem SSNS [107] which provided $n-r$ vectors that could be used with the span construction to build the entire null space of a matrix. As we have hinted in Example SCAD [110], and as we will see again going forward, linearly dependent sets carry redundant vectors with them when used in building a set as a span. Our aim now is to show that the vectors provided by Theorem SSNS [107] form a linearly independent set, so in one sense they are as efficient as possible a way to describe the null space. Notice that the vectors $z_j$, $1 \leq j \leq n-r$ first appear in the vector form of solutions to arbitrary linear systems (Theorem VFSLS [88]). The exact same vectors appear again in the span construction in the conclusion of Theorem SSNS [107]. Since this second theorem specializes to homogeneous systems the only real difference is that the vector $c$ in Theorem VFSLS [88] is the zero vector for a homogeneous system. Finally, Theorem BNS [128] will now show that these same vectors are a linearly independent set. we’ll set the stage for the proof of this theorem with a moderately large example. Study the example carefully, as it will make it easier to understand the proof.

**Example LINSB**

**Linearly independence of null space basis**

Suppose that we are interested in the null space of the a $3 \times 7$ matrix, $A$, which row-reduces to

$$B = \begin{bmatrix} 1 & 0 & -2 & 4 & 0 & 3 & 9 \\ 0 & 1 & 5 & 6 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 1 & 8 & -5 \end{bmatrix}$$

The set $F = \{3, 4, 6, 7\}$ is the set of indices for our four free variables that would be used in a description of the solution set for the homogeneous system $\text{homosystem}A$. Applying Theorem SSNS [107] we can begin to construct a set of four vectors whose span is the null space of $A$, a set
of vectors we will reference as $T$.

$$\mathcal{N}(A) = \langle T \rangle = \langle \{z_1, z_2, z_3, z_4\} \rangle = \langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rangle$$

So far, we have constructed as much of these individual vectors as we can, based just on the knowledge of the contents of the set $F$. This has allowed us to determine the entries in slots 3, 4, 6 and 7, while we have left slots 1, 2 and 5 blank. Without doing any more, let’s ask if $T$ is linearly independent? Begin with a relation of linear dependence on $T$, and see what we can learn about the scalars,

$$0 = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 + \alpha_4 z_4$$

Applying Definition CVE [73] to the two ends of this chain of equalities, we see that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. So the only relation of linear dependence on the set $T$ is a trivial one. By Definition LICV [121] the set $T$ is linearly independent. The important feature of this example is how the “pattern of zeros and ones” in the four vectors led to the conclusion of linear independence.

The proof of Theorem BNS [128] is really quite straightforward, and relies on the “pattern of zeros and ones” that arise in the vectors $z_i$, $1 \leq i \leq n-r$ in the entries that correspond to the free variables. Play along with Example LINSB [127] as you study the proof. Also, take a look at Example VFSAD [83], Example VFSAI [90] and Example VFSAL [91], especially at the conclusion of Step 2 (temporarily ignore the construction of the constant vector, $c$). This proof is also a good first example of how to prove a conclusion that states a set is linearly independent.

Theorem BNS
Basis for Null Spaces

Suppose that $A$ is an $m \times n$ matrix, and $B$ is a row-equivalent matrix in reduced row-echelon form with $r$ nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where $B$ does and does not (respectively) have leading 1’s. Construct the $n - r$ vectors $z_j$, $1 \leq j \leq n - r$ of size $n$ as

$$[z_j]_i = \begin{cases} 1 & \text{if } i \in F, i = f_j \\ 0 & \text{if } i \in F, i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D, i = d_k \end{cases}$$

Define the set $S = \{z_1, z_2, z_3, \ldots, z_{n-r}\}$. Then
1. \( \mathcal{N}(A) = \langle S \rangle \).

2. \( S \) is a linearly independent set.

**Proof** Notice first that the vectors \( z_j, 1 \leq j \leq n - r \) are exactly the same as the \( n - r \) vectors defined in [Theorem SSNS][107]. Also, the hypotheses of [Theorem SSNS][107] are the same as the hypotheses of the theorem we are currently proving. So it is then simply the conclusion of [Theorem SSNS][107] that tells us that \( \mathcal{N}(A) = \langle S \rangle \). That was the easy half, but the second part is not much harder. What is new here is the claim that \( S \) is a linearly independent set.

To prove the linear independence of a set, we need to start with a relation of linear dependence and somehow conclude that the scalars involved must all be zero, i.e. that the relation of linear dependence only happens in the trivial fashion. So to establish the linear independence of \( S \), we start with

\[
\alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 + \cdots + \alpha_{n-r} z_{n-r} = 0.
\]

For each \( j, 1 \leq j \leq n - r \), consider the equality of the individual entries of the vectors on both sides of this equality in position \( f_j \),

\[
0 = [0]_{f_j} = [\alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 + \cdots + \alpha_{n-r} z_{n-r}]_{f_j} = [\alpha_1 z_1]_{f_j} + [\alpha_2 z_2]_{f_j} + [\alpha_3 z_3]_{f_j} + \cdots + [\alpha_{n-r} z_{n-r}]_{f_j}
\]

\[
= \alpha_1 [z_1]_{f_j} + \alpha_2 [z_2]_{f_j} + \alpha_3 [z_3]_{f_j} + \cdots + \alpha_{j-1} [z_{j-1}]_{f_j} + \alpha_j [z_j]_{f_j} + \alpha_{j+1} [z_{j+1}]_{f_j} + \cdots + \alpha_{n-r} [z_{n-r}]_{f_j}
\]

\[
= \alpha_1(0) + \alpha_2(0) + \alpha_3(0) + \cdots + \alpha_{j-1}(0) + \alpha_j(1) + \alpha_{j+1}(0) + \cdots + \alpha_{n-r}(0)
\]

\[
= \alpha_j
\]

So for all \( j, 1 \leq j \leq n - r \), we have \( \alpha_j = 0 \), which is the conclusion that tells us that the only relation of linear dependence on \( S = \{z_1, z_2, z_3, \ldots, z_{n-r}\} \) is the trivial one, hence the set is linearly independent, as desired.

**Example NSLIL**

**Null space spanned by linearly independent set, Archetype L**

In [Example VFSAL][91] we previewed [Theorem SSNS][107] by finding a set of two vectors such that their span was the null space for the matrix in [Archetype L][680]. Writing the matrix as \( L \), we have

\[
\mathcal{N}(L) = \left\langle \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\rangle.
\]

Solving the homogeneous system \( LS(L, 0) \) resulted in recognizing \( x_4 \) and \( x_5 \) as the free variables. So look in entries 4 and 5 of the two vectors above and notice the pattern of zeros and ones that provides the linear independence of the set.
1. Let $S$ be the set of three vectors below.

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Is $S$ linearly independent or linearly dependent? Explain why.

2. Let $S$ be the set of three vectors below.

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -4 \end{bmatrix} \right\}$$

Is $S$ linearly independent or linearly dependent? Explain why.

3. Based on your answer to the previous question, is the matrix below singular or nonsingular? Explain.

$$\begin{bmatrix} 1 & 3 & 4 \\ -1 & 2 & 3 \\ 0 & 2 & -4 \end{bmatrix}$$
Subsection EXC
Exercises

Determine if the sets of vectors in Exercises C20–C25 are linearly independent or linearly dependent.

C20 \[ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -1 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} \]

Contributed by Robert Beezer Solution 134

C21 \[ \begin{bmatrix} -1 \\ 2 \\ 4 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -1 \\ 3 \\ 8 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} \]

Contributed by Robert Beezer Solution 134

C22 \[ \begin{bmatrix} 1 \\ 5 \\ 1 \\ 6 \\ -1 \\ 2 \\ 9 \\ -3 \\ 8 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} \]

Contributed by Robert Beezer Solution 134

C23 \[ \begin{bmatrix} 1 \\ -2 \\ 2 \\ -4 \\ 3 \\ 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 5 \\ 1 \\ -4 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \]

Contributed by Robert Beezer Solution 134

C24 \[ \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 2 \\ 2 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 1 \\ 2 \\ -2 \\ -3 \\ -1 \end{bmatrix} \]

Contributed by Robert Beezer Solution 134

C25 \[ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ -2 \\ 3 \\ 2 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 10 \\ -7 \\ 10 \end{bmatrix} \]

Contributed by Robert Beezer Solution 135

C30 For the matrix B below, find a set S that is linearly independent and spans the null space of B, that is, \( \mathcal{N}(B) = \langle S \rangle \).

\[ B = \begin{bmatrix} -3 & 1 & -2 & 7 \\ -1 & 2 & 1 & 4 \\ 1 & 1 & 2 & -1 \end{bmatrix} \]

Contributed by Robert Beezer Solution 135

C31 For the matrix A below, find a linearly independent set S so that the null space of A is spanned by S, that is, \( \mathcal{N}(A) = \langle S \rangle \).

\[ A = \begin{bmatrix} -1 & -2 & 2 & 1 & 5 \\ 1 & 2 & 1 & 1 & 5 \\ 3 & 6 & 1 & 2 & 7 \\ 2 & 4 & 0 & 1 & 2 \end{bmatrix} \]
C50  Consider each archetype that is a system of equations and consider the solutions listed for
the homogeneous version of the archetype.  (If only the trivial solution is listed, then assume this is
the only solution to the system.)  From the solution set, determine if the columns of the coefficient
matrix form a linearly independent or linearly dependent set.  In the case of a linearly dependent
set, use one of the sample solutions to provide a nontrivial relation of linear dependence on the set
of columns of the coefficient matrix (Definition RLD 280).  Indicate when Theorem MVSLD 126
applies and connect this with the number of variables and equations in the system of equations.

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671

C51  For each archetype that is a system of equations consider the homogeneous version.  Write
elements of the solution set in vector form (Theorem VFSLS 88) and from this extract the vectors
\( \mathbf{z}_j \) described in Theorem BNS 128.  These vectors are used in a span construction to describe
the null space of the coefficient matrix for each archetype.  What does it mean when we write a null
space as \( \langle \{ \} \rangle \)?

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671

C52  For each archetype that is a system of equations consider the homogeneous version.  Sample
solutions are given and a linearly independent spanning set is given for the null space of the
coefficient matrix.  Write each of the sample solutions individually as a linear combination of the
vectors in the spanning set for the null space of the coefficient matrix.

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671

C60  For the matrix \( A \) below, find a set of vectors \( S \) so that (1) \( S \) is linearly independent, and
(2) the span of $S$ equals the null space of $A$, $\langle S \rangle = N(A)$. (See Exercise SS.C60 [114].)

$$A = \begin{bmatrix} 1 & 1 & 6 & -8 \\ 1 & -2 & 0 & 1 \\ -2 & 1 & -6 & 7 \end{bmatrix}$$

Contributed by Robert Beezer Solution [136]

**M50** Consider the set of vectors from $\mathbb{C}^3$, $W$, given below. Find a set $T$ that contains three vectors from $W$ and such that $W = \langle T \rangle$.

$$W = \langle \{v_1, v_2, v_3, v_4, v_5\} \rangle = \left\langle \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\} \right\rangle$$

Contributed by Robert Beezer Solution [136]

**T10** Prove that if a set of vectors contains the zero vector, then the set is linearly dependent. (Ed. “The zero vector is death to linearly independent sets.”)

Contributed by Martin Jackson

**T15** Suppose that $\{v_1, v_2, v_3, \ldots, v_n\}$ is a set of vectors. Prove that $\{v_1 - v_2, v_2 - v_3, v_3 - v_4, \ldots, v_n - v_1\}$ is a linearly dependent set.

Contributed by Robert Beezer Solution [137]

**T20** Suppose that $\{v_1, v_2, v_3, v_4\}$ is a linearly independent set in $\mathbb{C}^3$. Prove that $\{v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4\}$ is a linearly independent set.

Contributed by Robert Beezer Solution [137]

**T50** Suppose that $A$ is matrix with linearly independent columns and the linear system $LS(A, b)$ is consistent. Show that this system has a unique solution. (Notice that we are not requiring $A$ to be square.)

Contributed by Robert Beezer Solution [138]
Subsection SOL
Solutions

C20 Contributed by Robert Beezer Statement 131
With three vectors from $\mathbb{C}^3$, we can form a square matrix by making these three vectors the columns of a matrix. We do so, and row-reduce to obtain,

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

the $3 \times 3$ identity matrix. So by Theorem NME2 127 the original matrix is nonsingular and its columns are therefore a linearly independent set.

C21 Contributed by Robert Beezer Statement 131
Theorem LIVRN 125 says we can answer this question by putting these vectors into a matrix as columns and row-reducing. Doing this we obtain,

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

With $n = 3$ (3 vectors, 3 columns) and $r = 3$ (3 leading 1’s) we have $n = r$ and the theorem says the vectors are linearly independent.

C22 Contributed by Robert Beezer Statement 131
Five vectors from $\mathbb{C}^3$, Theorem MVSLD 126 says the set is linearly dependent. Boom.

C23 Contributed by Robert Beezer Statement 131
Theorem LIVRN 125 suggests we analyze a matrix whose columns are the vectors of $S$,

$$
A = \begin{bmatrix}
1 & 3 & 2 & 1 \\
-2 & 3 & 1 & 0 \\
2 & 1 & 2 & 1 \\
5 & 2 & -1 & 2 \\
3 & -4 & 1 & 2
\end{bmatrix}
$$

Row-reducing the matrix $A$ yields,

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

We see that $r = 4 = n$, where $r$ is the number of nonzero rows and $n$ is the number of columns. By Theorem LIVRN 125, the set $S$ is linearly independent.

C24 Contributed by Robert Beezer Statement 131
Theorem LIVRN 125 suggests we analyze a matrix whose columns are the vectors from the set,

$$
A = \begin{bmatrix}
1 & 3 & 4 & -1 \\
2 & 2 & 4 & 2 \\
-1 & -1 & -2 & -1 \\
0 & 2 & 2 & -2 \\
1 & 2 & 3 & 0
\end{bmatrix}
$$
Row-reducing the matrix $A$ yields,

\[
\begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

We see that $r = 2 \neq 4 = n$, where $r$ is the number of nonzero rows and $n$ is the number of columns. By [Theorem LIVRN][125], the set $S$ is linearly dependent.

**C25** Contributed by Robert Beezer  Statement [131]

Theorem LIVRN [125] suggests we analyze a matrix whose columns are the vectors from the set,

\[
A = \begin{bmatrix}
2 & 4 & 10 \\
1 & -2 & -7 \\
3 & 1 & 0 \\
-1 & 3 & 10 \\
2 & 2 & 4
\end{bmatrix}
\]

Row-reducing the matrix $A$ yields,

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 3 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

We see that $r = 2 \neq 3 = n$, where $r$ is the number of nonzero rows and $n$ is the number of columns. By [Theorem LIVRN][125], the set $S$ is linearly dependent.

**C30** Contributed by Robert Beezer  Statement [131]

The requested set is described by Theorem BNS [128]. It is easiest to find by using the procedure of Example VFSAL [91]. Begin by row-reducing the matrix, viewing it as the coefficient matrix of a homogeneous system of equations. We obtain,

\[
\begin{bmatrix}
1 & 0 & 1 & -2 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Now build the vector form of the solutions to this homogeneous system (Theorem VFSLS [88]). The free variables are $x_3$ and $x_4$, corresponding to the columns without leading 1’s,

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = x_3 \begin{bmatrix}-1 \\ -1 \\ 1 \\ 0\end{bmatrix} + x_4 \begin{bmatrix}2 \\ -1 \\ 0 \\ 1\end{bmatrix}
\]

The desired set $S$ is simply the constant vectors in this expression, and these are the vectors $z_1$ and $z_2$ described by Theorem BNS [128].

\[
S = \left\{ \begin{bmatrix}-1 \\ -1 \\ 1 \\ 0\end{bmatrix}, \begin{bmatrix}2 \\ -1 \\ 0 \\ 1\end{bmatrix} \right\}
\]

**C31** Contributed by Robert Beezer  Statement [131]

Theorem BNS [128] provides formulas for $n - r$ vectors that will meet the requirements of this question. These vectors are the same ones listed in Theorem VFSLS [88] when we solve the homogeneous system $LS(A, 0)$, whose solution set is the null space (Definition NSM [54]).
To apply Theorem BNS \[128\] or Theorem VFSLS \[88\] we first row-reduce the matrix, resulting in

\[
B = \begin{bmatrix}
1 & 2 & 0 & 0 & 3 \\
0 & 0 & 1 & 0 & 6 \\
0 & 0 & 0 & 1 & -4 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

So we see that \(n - r = 5 - 3 = 2\) and \(F = \{2, 5\}\), so the vector form of a generic solution vector is

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix} = x_2 \begin{bmatrix}
-2 \\
1 \\
0 \\
0 \\
0 \\
\end{bmatrix} + x_5 \begin{bmatrix}
-3 \\
0 \\
-6 \\
4 \\
1 \\
\end{bmatrix}
\]

So we have

\[
N(A) = \left\langle \begin{bmatrix}
-2 \\
1 \\
0 \\
0 \\
\end{bmatrix}, \begin{bmatrix}
-3 \\
0 \\
-6 \\
4 \\
1 \\
\end{bmatrix} \right\rangle
\]

Theorem BNS \[128\] says that if we find the vector form of the solutions to the homogeneous system \(LS(A, 0)\), then the fixed vectors (one per free variable) will have the desired properties. Row-reduce \(A\), viewing it as the augmented matrix of a homogeneous system with an invisible columns of zeros as the last column,

\[
\begin{bmatrix}
1 & 0 & 4 & -5 \\
0 & 1 & 2 & -3 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Moving to the vector form of the solutions (Theorem VFSLS \[88\]), with free variables \(x_3\) and \(x_4\), solutions to the consistent system (it is homogeneous, Theorem HSC \[52\]) can be expressed as

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = x_3 \begin{bmatrix}
-4 \\
-2 \\
1 \\
0 \\
\end{bmatrix} + x_4 \begin{bmatrix}
5 \\
3 \\
0 \\
1 \\
\end{bmatrix}
\]

Then with \(S\) given by

\[
S = \left\langle \begin{bmatrix}
-4 \\
-2 \\
1 \\
0 \\
\end{bmatrix}, \begin{bmatrix}
5 \\
3 \\
0 \\
1 \\
\end{bmatrix} \right\rangle
\]

Theorem BNS \[128\] guarantees the set has the desired properties.
From this we that solutions can be obtained employing the free variables $x_4$ and $x_5$. With appropriate choices we will be able to conclude that vectors $v_4$ and $v_5$ are unnecessary for creating $W$ via a span. By **Theorem SLSLC** \[82\] the choice of free variables below lead to solutions and linear combinations, which are then rearranged.

$$
\begin{align*}
x_4 = 1, x_5 = 0 &\Rightarrow (-2)v_1 + (-1)v_2 + (0)v_3 + (1)v_4 + (0)v_5 = 0 &\Rightarrow v_4 = 2v_1 + v_2 \\
x_4 = 0, x_5 = 1 &\Rightarrow (1)v_1 + (2)v_2 + (0)v_3 + (0)v_4 + (1)v_5 = 0 &\Rightarrow v_5 = -v_1 - 2v_2
\end{align*}
$$

Since $v_4$ and $v_5$ can be expressed as linear combinations of $v_1$ and $v_2$ we can say that $v_4$ and $v_5$ are not needed for the linear combinations used to build $W$ (a claim that we could establish carefully with a pair of set equality arguments). Thus

$$W = \langle \{v_1, v_2, v_3\} \rangle = \left\langle \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\} \right\rangle$$

That the $\{v_1, v_2, v_3\}$ is linearly independent set can be established quickly with **Theorem LIVRN** \[125\].

There are other answers to this question, but notice that any nontrivial linear combination of $v_1, v_2, v_3, v_4, v_5$ will have a zero coefficient on $v_3$, so this vector can never be eliminated from the set used to build the span.

**T15** Contributed by Robert Beezer  Statement \[133\]

Consider the following linear combination

$$1(v_1 - v_2) + 1(v_2 - v_3) + 1(v_3 - v_4) + \cdots + 1(v_n - v_1)$$

$$= v_1 - v_2 + v_2 - v_3 + v_3 - v_4 + \cdots + v_n - v_1$$

$$= v_1 + 0 + 0 + \cdots + 0 - v_1$$

$$= 0$$

This is a nontrivial relation of linear dependence \[Definition RLDCV \[121\]\], so by **Definition LICV** \[121\] the set is linearly dependent.

**T20** Contributed by Robert Beezer  Statement \[133\]

Our hypothesis and our conclusion use the term linear independence, so it will get a workout. To establish linear independence, we begin with the definition \[Definition LICV \[121\]\] and write a relation of linear dependence \[Definition RLDCV \[121\]\],

$$\alpha_1(v_1) + \alpha_2(v_1 + v_2) + \alpha_3(v_1 + v_2 + v_3) + \alpha_4(v_1 + v_2 + v_3 + v_4) = 0$$

Using the distributive and commutative properties of vector addition and scalar multiplication (**Theorem VSPCV** \[75\]) this equation can be rearranged as

$$(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)v_1 + (\alpha_2 + \alpha_3 + \alpha_4)v_2 + (\alpha_3 + \alpha_4)v_3 + (\alpha_4)v_4 = 0$$

However, this is a relation of linear dependence \[Definition RLDCV \[121\]\] on a linearly independent set, $\{v_1, v_2, v_3, v_4\}$ (this was our lone hypothesis). By the definition of linear independence \[Definition LICV \[121\]\] the scalars must all be zero. This is the homogeneous system of equations,

$$\begin{align*}
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0 \\
\alpha_2 + \alpha_3 + \alpha_4 &= 0 \\
\alpha_3 + \alpha_4 &= 0 \\
\alpha_4 &= 0
\end{align*}$$

Row-reducing the coefficient matrix of this system (or backsolving) gives the conclusion

$$\begin{align*}
\alpha_1 &= 0 \\
\alpha_2 &= 0 \\
\alpha_3 &= 0 \\
\alpha_4 &= 0
\end{align*}$$
This means, by Definition LICV \[121\], that the original set
\[
\{v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4\}
\]
is linearly independent.

\[70\] Contributed by Robert Beezer Statement \[133\]
Let \(A = [A_1 | A_2 | A_3 | \ldots | A_n]\). \(LS(A, b)\) is consistent, so we know the system has at least one solution (Definition CS \[42\]). We would like to show that there are no more than one solution to the system. Employing Technique U \[624\], suppose that \(x\) and \(y\) are two solution vectors for \(LS(A, b)\). By Theorem SLSLC \[82\] we know we can write,
\[
b = [x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n
\]
\[
b = [y]_1 A_1 + [y]_2 A_2 + [y]_3 A_3 + \cdots + [y]_n A_n
\]
Then
\[
0 = b - b
\]
\[
= ([x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n) -
\]
\[
([y]_1 A_1 + [y]_2 A_2 + [y]_3 A_3 + \cdots + [y]_n A_n)
\]
\[
= ([x]_1 - [y]_1) A_1 + ([x]_2 - [y]_2) A_2 + \cdots + ([x]_n - [y]_n) A_n
\]
This is a relation of linear dependence (Definition RLDCV \[121\]) on a linearly independent set (the columns of \(A\)). So the scalars must all be zero,
\[
[x]_1 - [y]_1 = 0 \quad [x]_2 - [y]_2 = 0 \quad \ldots \quad [x]_n - [y]_n = 0
\]
Rearranging these equations yields the statement that \([x]_i = [y]_i\), for \(1 \leq i \leq n\). However, this is exactly how we define vector equality (Definition CVE \[73\]), so \(x = y\).
Section LDS
Linear Dependence and Spans

In any linearly dependent set there is always one vector that can be written as a linear combination of the others. This is the substance of the upcoming Theorem DLDS [139]. Perhaps this will explain the use of the word “dependent.” In a linearly dependent set, at least one vector “depends” on the others (via a linear combination).

Indeed, because Theorem DLDS [139] is an equivalence (Technique E [622]) some authors use this condition as a definition (Technique D [619]) of linear dependence. Then linear independence is defined as the logical opposite of linear dependence. Of course, we have chosen to take Definition LICV [121] as our definition, and then present Theorem DLDS [139] as a theorem.

Subsection LDSS
Linearly Dependent Sets and Spans

If we use a linearly dependent set to construct a span, then we can always create the same infinite set with a starting set that is one vector smaller in size. We will illustrate this behavior in Example RSC5 [140]. However, this will not be possible if we build a span from a linearly independent set. So in a certain sense, using a linearly independent set to formulate a span is the best possible way to go about it — there aren’t any extra vectors being used to build up all the necessary linear combinations. OK, here’s the theorem, and then the example.

Theorem DLDS
Dependency in Linearly Dependent Sets

Suppose that \( S = \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n \} \) is a set of vectors. Then \( S \) is a linearly dependent set if and only if there is an index \( t, 1 \leq t \leq n \) such that \( \mathbf{u}_t \) is a linear combination of the vectors \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{t-1}, \mathbf{u}_{t+1}, \ldots, \mathbf{u}_n \).

Proof (⇒) Suppose that \( S \) is linearly dependent, so there is a nontrivial relation of linear dependence,

\[
\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{0}.
\]

Since the \( \alpha_i \) cannot all be zero, choose one, say \( \alpha_t \), that is nonzero. Then,

\[
-\alpha_t \mathbf{u}_t = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_{t-1} \mathbf{u}_{t-1} + \alpha_{t+1} \mathbf{u}_{t+1} + \cdots + \alpha_n \mathbf{u}_n
\]

and we can multiply by \( \frac{-1}{\alpha_t} \) since \( \alpha_t \neq 0 \),

\[
\mathbf{u}_t = \frac{-\alpha_1}{\alpha_t} \mathbf{u}_1 + \frac{-\alpha_2}{\alpha_t} \mathbf{u}_2 + \frac{-\alpha_3}{\alpha_t} \mathbf{u}_3 + \cdots + \frac{-\alpha_{t-1}}{\alpha_t} \mathbf{u}_{t-1} + \frac{-\alpha_{t+1}}{\alpha_t} \mathbf{u}_{t+1} + \cdots + \frac{-\alpha_n}{\alpha_t} \mathbf{u}_n.
\]

Since the values of \( \frac{\alpha_i}{\alpha_t} \) are again scalars, we have expressed \( \mathbf{u}_t \) as the desired linear combination.

(⇐) Suppose that the vector \( \mathbf{u}_t \) is a linear combination of the other vectors in \( S \). Write this linear combination as

\[
\beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \beta_3 \mathbf{u}_3 + \cdots + \beta_{t-1} \mathbf{u}_{t-1} + \beta_{t+1} \mathbf{u}_{t+1} + \cdots + \beta_n \mathbf{u}_n = \mathbf{u}_t
\]

and move \( \mathbf{u}_t \) to the other side of the equality

\[
\beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \beta_3 \mathbf{u}_3 + \cdots + \beta_{t-1} \mathbf{u}_{t-1} + (-1) \mathbf{u}_t + \beta_{t+1} \mathbf{u}_{t+1} + \cdots + \beta_n \mathbf{u}_n = \mathbf{0}.
\]
Then the scalars $\beta_1, \beta_2, \beta_3, \ldots, \beta_{t-1}, \beta_t = -1, \beta_{t+1}, \ldots, \beta_n$ provide a nontrivial linear combination of the vectors in $S$, thus establishing that $S$ is a linearly dependent set.

This theorem can be used, sometimes repeatedly, to whittle down the size of a set of vectors used in a span construction. We have seen some of this already in Example SCAD [110], but in the next example we will detail some of the subtleties.

Example RSC5
Reducing a span in $\mathbb{C}^5$
Consider the set of $n = 4$ vectors from $\mathbb{C}^5$,

$$ R = \{v_1, v_2, v_3, v_4\} = \left\{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -7 \\ -11 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -11 \\ 1 \\ -2 \end{bmatrix}\right\} $$

and define $V = \langle R \rangle$.

To employ Theorem LIVHS [123], we form a $5 \times 4$ coefficient matrix, $D$,

$$ D = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 1 & -7 & 1 \\ -1 & 3 & 6 & 2 \\ 3 & 1 & -11 & 1 \\ 2 & 2 & -2 & 6 \end{bmatrix} $$

and row-reduce to understand solutions to the homogeneous system $LS(D, 0)$,

$$ \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

We can find infinitely many solutions to this system, most of them nontrivial, and we choose any one we like to build a relation of linear dependence on $R$. Let’s begin with $x_4 = 1$, to find the solution

$$ \begin{bmatrix} -4 \\ 0 \\ -1 \\ 1 \end{bmatrix} $$

So we can write the relation of linear dependence,

$$( -4)v_1 + 0v_2 + ( -1)v_3 + 1v_4 = 0. $$

Theorem DLDS [139] guarantees that we can solve this relation of linear dependence for some vector in $R$, but the choice of which one is up to us. Notice however that $v_2$ has a zero coefficient.

In this case, we cannot choose to solve for $v_2$. Maybe some other relation of linear dependence would produce a nonzero coefficient for $v_2$ if we just had to solve for this vector. Unfortunately, this example has been engineered to always produce a zero coefficient here, as you can see from solving the homogeneous system. Every solution has $x_2 = 0$!

OK, if we are convinced that we cannot solve for $v_2$, let’s instead solve for $v_3$,

$$ v_3 = ( -4)v_1 + 0v_2 + 1v_4 = ( -4)v_1 + 1v_4. $$

We now claim that this particular equation will allow us to write

$$ V = \langle R \rangle = \langle \{v_1, v_2, v_4\}\rangle = \langle \{v_1, v_2\}\rangle $$
in essence declaring $v_4$ as surplus for the task of building $V$ as a span. This claim is an equality of two sets, so we will use Definition SE\[616\] to establish it carefully. Let $R' = \{v_1, v_2, v_4\}$ and $V' = \langle R' \rangle$. We want to show that $V = V'$.

First show that $V' \subseteq V$. Since every vector of $R'$ is in $R$, any vector we can construct in $V'$ as a linear combination of vectors from $R'$ can also be constructed as a vector in $V$ by the same linear combination of the same vectors in $R$. That was easy, now turn it around.

Next show that $V \subseteq V'$. Choose any $v$ from $V$. Then there are scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ so that

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4$$

$$= \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 (-4v_1 + v_4) + \alpha_4 v_4$$

$$= \alpha_1 v_1 + \alpha_2 v_2 + ((-4\alpha_3) v_1 + \alpha_3 v_4) + \alpha_4 v_4$$

$$= (\alpha_1 - 4\alpha_3) v_1 + \alpha_2 v_2 + (\alpha_3 + \alpha_4) v_4.$$  

This equation says that $v$ can then be written as a linear combination of the vectors in $R'$ and hence qualifies for membership in $V'$. So $V \subseteq V'$ and we have established that $V = V'$.

If $R'$ was also linearly dependent (it’s not), we could reduce the set even further. Notice that we could have chosen to eliminate any one of $v_1$, $v_3$ or $v_4$, but somehow $v_2$ is essential to the creation of $V$ since it cannot be replaced by any linear combination of $v_1$, $v_3$ or $v_4$.

**Subsection COV**

**Casting Out Vectors**

In Example RSC5\[140\] we used four vectors to create a span. With a relation of linear dependence in hand, we were able to “toss-out” one of these four vectors and create the same span from a subset of just three vectors from the original set of four. We did have to take some care as to just which vector we tossed-out. In the next example, we will be more methodical about just how we choose to eliminate vectors from a linearly dependent set while preserving a span.

**Example COV**

**Casting out vectors**

We begin with a set $S$ containing seven vectors from $\mathbb{C}^4$,

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} 7 \\ 9 \\ -4 \\ 8 \end{bmatrix}, \begin{bmatrix} -9 \\ 7 \\ -13 \\ -31 \end{bmatrix} \right\}$$

and define $W = \langle S \rangle$. The set $S$ is obviously linearly dependent by Theorem MVSLD\[126\], since we have $n = 7$ vectors from $\mathbb{C}^4$. So we can slim down $S$ some, and still create $W$ as the span of a smaller set of vectors. As a device for identifying relations of linear dependence among the vectors of $S$, we place the seven column vectors of $S$ into a matrix as columns,

$$A = [A_1|A_2|A_3|\ldots|A_7] = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}$$

By Theorem SLSLC\[82\], a nontrivial solution to $LS(A, \mathbf{0})$ will give us a nontrivial relation of linear dependence (Definition RLDCV\[121\]) on the columns of $A$ (which are the elements of the set $S$). The row-reduced form for $A$ is the matrix

$$B = \begin{bmatrix} 1 & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 1 & 2 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Version 1.04
so we can easily create solutions to the homogeneous system $\mathcal{L}S(A, 0)$ using the free variables $x_2, x_5, x_6, x_7$. Any such solution will correspond to a relation of linear dependence on the columns of $I$. These solutions will allow us to solve for one column vector as a linear combination of some others, in the spirit of Theorem DLDS \[139\], and remove that vector from the set. We’ll set about forming these linear combinations methodically. Set the free variable $x_2$ to one, and set the other free variables to zero. Then a solution to $\mathcal{L}S(A, 0)$ is

$$\mathbf{x} = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which can be used to create the linear combination

$$(−4)A_1 + 1A_2 + 0A_3 + 0A_4 + 0A_5 + 0A_6 + 0A_7 = 0$$

This can then be arranged and solved for $A_2$, resulting in $A_2$ expressed as a linear combination of \{A_1, A_3, A_4\},

$$A_2 = 4A_1 + 0A_3 + 0A_4$$

This means that $A_2$ is surplus, and we can create $W$ just as well with a smaller set with this vector removed,

$$W = \langle\{A_1, A_3, A_4, A_5, A_6, A_7\}\rangle$$

Technically, this set equality for $W$ requires a proof, in the spirit of Example RSC5 \[140\], but we will bypass this requirement here, and in the next few paragraphs.

Now, set the free variable $x_5$ to one, and set the other free variables to zero. Then a solution to $\mathcal{L}S(I, 0)$ is

$$\mathbf{x} = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

which can be used to create the linear combination

$$(−2)A_1 + 0A_2 + (−1)A_3 + (−2)A_4 + 1A_5 + 0A_6 + 0A_7 = 0$$

This can then be arranged and solved for $A_5$, resulting in $A_5$ expressed as a linear combination of \{A_1, A_3, A_4\},

$$A_5 = 2A_1 + 1A_3 + 2A_4$$

This means that $A_5$ is surplus, and we can create $W$ just as well with a smaller set with this vector removed,

$$W = \langle\{A_1, A_3, A_4, A_6, A_7\}\rangle$$

Do it again, set the free variable $x_6$ to one, and set the other free variables to zero. Then a solution to $\mathcal{L}S(I, 0)$ is

$$\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$
which can be used to create the linear combination

\((-1)\mathbf{A}_1 + 0\mathbf{A}_2 + 3\mathbf{A}_3 + 6\mathbf{A}_4 + 0\mathbf{A}_5 + 1\mathbf{A}_6 + 0\mathbf{A}_7 = 0\)

This can then be arranged and solved for \(\mathbf{A}_6\), resulting in \(\mathbf{A}_6\) expressed as a linear combination of \(\{\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4\}\),

\[\mathbf{A}_6 = 1\mathbf{A}_1 + (-3)\mathbf{A}_3 + (-6)\mathbf{A}_4\]

This means that \(\mathbf{A}_6\) is surplus, and we can create \(W\) just as well with a smaller set with this vector removed,

\[W = \langle\{\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4\}\rangle\]

Set the free variable \(x_7\) to one, and set the other free variables to zero. Then a solution to \(\text{LS}(I, \mathbf{0})\) is

\[\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}\]

which can be used to create the linear combination

\[3\mathbf{A}_1 + 0\mathbf{A}_2 + (-5)\mathbf{A}_3 + (-6)\mathbf{A}_4 + 0\mathbf{A}_5 + 0\mathbf{A}_6 + 1\mathbf{A}_7 = 0\]

This can then be arranged and solved for \(\mathbf{A}_7\), resulting in \(\mathbf{A}_7\) expressed as a linear combination of \(\{\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4\}\),

\[\mathbf{A}_7 = (-3)\mathbf{A}_1 + 5\mathbf{A}_3 + 6\mathbf{A}_4\]

This means that \(\mathbf{A}_7\) is surplus, and we can create \(W\) just as well with a smaller set with this vector removed,

\[W = \langle\{\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4\}\rangle\]

You might think we could keep this up, but we have run out of free variables. And not coincidentally, the set \(\{\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4\}\) is linearly independent (check this!). It should be clear how each free variable was used to eliminate the corresponding column from the set used to span the column space, as this will be the essence of the proof of the next theorem. The column vectors in \(S\) were not chosen entirely at random, they are the columns of Archetype I \[667\]. See if you can mimic this example using the columns of Archetype J \[671\]. Go ahead, we’ll go grab a cup of coffee and be back before you finish up.

For extra credit, notice that the vector

\[\mathbf{b} = \begin{bmatrix} 3 \\ 9 \\ 1 \\ 4 \end{bmatrix}\]

is the vector of constants in the definition of Archetype I \[667\]. Since the system \(\text{LS}(I, \mathbf{b})\) is consistent, we know by Theorem SLSLC \[82\] that \(\mathbf{b}\) is a linear combination of the columns of \(A\), or stated equivalently, \(\mathbf{b} \in W\). This means that \(\mathbf{b}\) must also be a linear combination of just the three columns \(\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4\). Can you find such a linear combination? Did you notice that there is just a single (unique) answer? Hmmmm.

Example COV \[141\] deserves your careful attention, since this important example motivates the following very fundamental theorem.

Theorem BS

**Basis of a Span**

Suppose that \(S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}\) is a set of column vectors. Define \(W = \langle S \rangle\) and let \(A\) be
the matrix whose columns are the vectors from \( S \). Let \( B \) be the reduced row-echelon form of \( A \), with \( D = \{ d_1, d_2, d_3, \ldots, d_r \} \) the set of column indices corresponding to the pivot columns of \( B \). Then

1. \( T = \{ \mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \ldots, \mathbf{v}_{d_r} \} \) is a linearly independent set.

2. \( W = \langle T \rangle \).

\[ \square \]

**Proof** To prove that \( T \) is linearly independent, begin with a relation of linear dependence on \( T \),

\[ 0 = \alpha_1\mathbf{v}_{d_1} + \alpha_2\mathbf{v}_{d_2} + \alpha_3\mathbf{v}_{d_3} + \ldots + \alpha_r\mathbf{v}_{d_r} \]

and we will try to conclude that the only possibility for the scalars \( \alpha_i \) is that they are all zero. Denote the non-pivot columns of \( B \) by \( F = \{ f_1, f_2, f_3, \ldots, f_{n-r} \} \). Then we can preserve the equality by adding a big fat zero to the linear combination,

\[ 0 = \alpha_1\mathbf{v}_{d_1} + \alpha_2\mathbf{v}_{d_2} + \alpha_3\mathbf{v}_{d_3} + \ldots + \alpha_r\mathbf{v}_{d_r} + 0\mathbf{v}_{f_1} + 0\mathbf{v}_{f_2} + 0\mathbf{v}_{f_3} + \ldots + 0\mathbf{v}_{f_{n-r}} \]

By [Theorem SLSLC][82], the scalars in this linear combination (suitably reordered) are a solution to the homogeneous system \( \mathcal{L}S(A, 0) \). But notice that this is the solution obtained by setting each free variable to zero. If we consider the description of a solution vector in the conclusion of [Theorem VFSLS][88], in the case of a homogeneous system, then we see that if all the free variables are set to zero the resulting solution vector is trivial (all zeros). So it must be that \( \alpha_i = 0 \), \( 1 \leq i \leq r \). This implies by [Definition LICV][12] that \( T \) is a linearly independent set.

The second conclusion of this theorem is an equality of sets [Definition SE][616]). Since \( T \) is a subset of \( S \), any linear combination of elements of the set \( T \) can also be viewed as a linear combination of elements of the set \( S \). So \( \langle T \rangle \subseteq \langle S \rangle = W \). It remains to prove that \( W = \langle S \rangle \subseteq \langle T \rangle \).

For each \( k \), \( 1 \leq k \leq n-r \), form a solution \( \mathbf{x} \) to \( \mathcal{L}S(A, 0) \) by setting the free variables as follows:

\[ x_{f_1} = 0 \quad x_{f_2} = 0 \quad x_{f_3} = 0 \quad \ldots \quad x_{f_k} = 1 \quad \ldots \quad x_{f_{n-r}} = 0 \]

By [Theorem VFSLS][88], the remainder of this solution vector is given by,

\[ x_{d_1} = -[B]_{1,f_k} \quad x_{d_2} = -[B]_{2,f_k} \quad x_{d_3} = -[B]_{3,f_k} \quad \ldots \quad x_{d_r} = -[B]_{r,f_k} \]

From this solution, we obtain a relation of linear dependence on the columns of \( A \),

\[ -[B]_{1,f_k} \mathbf{v}_{d_1} - [B]_{2,f_k} \mathbf{v}_{d_2} - [B]_{3,f_k} \mathbf{v}_{d_3} - \ldots - [B]_{r,f_k} \mathbf{v}_{d_r} + 1\mathbf{v}_{f_k} = 0 \]

which can be arranged as the equality

\[ \mathbf{v}_{f_k} = [B]_{1,f_k} \mathbf{v}_{d_1} + [B]_{2,f_k} \mathbf{v}_{d_2} + [B]_{3,f_k} \mathbf{v}_{d_3} + \ldots + [B]_{r,f_k} \mathbf{v}_{d_r} \]

Now, suppose we take an arbitrary element, \( \mathbf{w} \), of \( W = \langle S \rangle \) and write it as a linear combination of the elements of \( S \), but with the terms organized according to the indices in \( D \) and \( F \),

\[ \mathbf{w} = \alpha_1\mathbf{v}_{d_1} + \alpha_2\mathbf{v}_{d_2} + \alpha_3\mathbf{v}_{d_3} + \ldots + \alpha_r\mathbf{v}_{d_r} + \beta_1\mathbf{v}_{f_1} + \beta_2\mathbf{v}_{f_2} + \beta_3\mathbf{v}_{f_3} + \ldots + \beta_{n-r}\mathbf{v}_{f_{n-r}} \]

From the above, we can replace each \( \mathbf{v}_{f_j} \) by a linear combination of the \( \mathbf{v}_{d_i} \),

\[ \mathbf{w} = \alpha_1\mathbf{v}_{d_1} + \alpha_2\mathbf{v}_{d_2} + \alpha_3\mathbf{v}_{d_3} + \ldots + \alpha_r\mathbf{v}_{d_r} + \beta_1 ([B]_{1,f_1} \mathbf{v}_{d_1} + [B]_{2,f_1} \mathbf{v}_{d_2} + [B]_{3,f_1} \mathbf{v}_{d_3} + \ldots + [B]_{r,f_1} \mathbf{v}_{d_r}) + \beta_2 ([B]_{1,f_2} \mathbf{v}_{d_1} + [B]_{2,f_2} \mathbf{v}_{d_2} + [B]_{3,f_2} \mathbf{v}_{d_3} + \ldots + [B]_{r,f_2} \mathbf{v}_{d_r}) + \beta_3 ([B]_{1,f_3} \mathbf{v}_{d_1} + [B]_{2,f_3} \mathbf{v}_{d_2} + [B]_{3,f_3} \mathbf{v}_{d_3} + \ldots + [B]_{r,f_3} \mathbf{v}_{d_r}) + \ldots \]

\[ \]
\[ \beta_{n-r}\left( [B]_{1,f_{n-r}} \mathbf{v}_d_1 + [B]_{2,f_{n-r}} \mathbf{v}_d_2 + [B]_{3,f_{n-r}} \mathbf{v}_d_3 + \ldots + [B]_{r,f_{n-r}} \mathbf{v}_d_r \right) \]

With repeated applications of several of the properties of Theorem VSPCV \[75\] we can rearrange this expression as,

\[
\begin{align*}
\mathbf{w} = (\alpha_1 + \beta_1 [B]_{1,f_1} + \beta_2 [B]_{1,f_2} + \beta_3 [B]_{1,f_3} + \ldots + \beta_{n-r} [B]_{1,f_{n-r}}) \mathbf{v}_d_1 + \\
(\alpha_2 + \beta_1 [B]_{2,f_1} + \beta_2 [B]_{2,f_2} + \beta_3 [B]_{2,f_3} + \ldots + \beta_{n-r} [B]_{2,f_{n-r}}) \mathbf{v}_d_2 + \\
(\alpha_3 + \beta_1 [B]_{3,f_1} + \beta_2 [B]_{3,f_2} + \beta_3 [B]_{3,f_3} + \ldots + \beta_{n-r} [B]_{3,f_{n-r}}) \mathbf{v}_d_3 + \\
\vdots \\
(\alpha_r + \beta_1 [B]_{r,f_1} + \beta_2 [B]_{r,f_2} + \beta_3 [B]_{r,f_3} + \ldots + \beta_{n-r} [B]_{r,f_{n-r}}) \mathbf{v}_d_r
\end{align*}
\]

This mess expresses the vector \( \mathbf{w} \) as a linear combination of the vectors in

\[ T = \{ \mathbf{v}_d_1, \mathbf{v}_d_2, \mathbf{v}_d_3, \ldots \mathbf{v}_d_r \} \]

thus saying that \( \mathbf{w} \in \langle T \rangle \). Therefore, \( W = \langle S \rangle \subseteq \langle T \rangle \). \( \blacksquare \)

In Example COV \[141\], we tossed-out vectors one at a time. But in each instance, we rewrote the offending vector as a linear combination of those vectors that corresponded to the pivot columns of the reduced row-echelon form of the matrix of columns. In the proof of Theorem BS \[143\], we accomplish this reduction in one big step. In Example COV \[141\] we arrived at a linearly independent set at exactly the same moment that we ran out of free variables to exploit. This was not a coincidence, it is the substance of our conclusion of linear independence in Theorem BS \[143\].

Here’s a straightforward application of Theorem BS \[143\].

**Example RSSC4**

**Reducing a span in \( \mathbb{C}^4 \)**

Begin with a set of five vectors from \( \mathbb{C}^4 \),

\[
S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}
\]

and let \( W = \langle S \rangle \). To arrive at a (smaller) linearly independent set, follow the procedure described in Theorem BS \[143\]. Place the vectors from \( S \) into a matrix as columns, and row-reduce,

\[
\begin{bmatrix}
1 & 2 & 2 & 7 & 0 \\
1 & 2 & 0 & 1 & 2 \\
2 & 4 & -1 & -1 & 5 \\
1 & 2 & 1 & 4 & 1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 0 & 1 & 2 \\
0 & 0 & 1 & 3 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Columns 1 and 3 are the pivot columns \( (D = \{1, 3\}) \) so the set

\[
T = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}
\]

is linearly independent and \( \langle T \rangle = \langle S \rangle = W \). Boom!

Since the reduced row-echelon form of a matrix is unique (Theorem RREFU \[96\]), the procedure of Theorem BS \[143\] leads us to a unique set \( T \). However, there is a wide variety of possibilities for
sets $T$ that are linearly independent and which can be employed in a span to create $W$. Without proof, we list two other possibilities:

$$T' = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ -1 \\ -9 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -10 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\}$$

$$T^* = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \\ 0 \end{bmatrix} \right\}$$

Can you prove that $T'$ and $T^*$ are linearly independent sets and $W = \langle S \rangle = \langle T' \rangle = \langle T^* \rangle$?

**Example RES**

**Reworking elements of a span**

Begin with a set of five vectors from $C^4$,

$$R = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ -1 \\ -9 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -10 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\}$$

It is easy to create elements of $X = \langle R \rangle$ — we will create one at random,

$$y = 6 \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix} + (-7) \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -8 \\ -1 \\ -9 \\ -4 \end{bmatrix} + 6 \begin{bmatrix} 3 \\ -1 \\ -2 \\ 4 \end{bmatrix} + 2 \begin{bmatrix} -10 \\ -1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 2 \\ -3 \end{bmatrix}$$

We know we can replace $R$ by a smaller set (since it is obviously linearly dependent by Theorem MVSLD [126]) that will create the same span. Here goes,

$$\begin{bmatrix} 2 & -1 & -8 & 3 & -10 \\ 1 & 1 & -1 & 1 & -1 \\ 3 & 0 & -9 & -1 & -1 \\ 2 & 1 & -4 & -2 & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -3 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, if we collect the first, second and fourth vectors from $R$,

$$P = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ -2 \end{bmatrix} \right\}$$

then $P$ is linearly independent and $\langle P \rangle = \langle R \rangle = X$ by Theorem BS [143]. Since we built $y$ as an element of $\langle R \rangle$ it must also be an element of $\langle P \rangle$. Can we write $y$ as a linear combination of just the three vectors in $P$? The answer is, of course, yes. But let’s compute an explicit linear combination just for fun. By Theorem SLSLC [82] we can get such a linear combination by solving a system of equations with the column vectors of $R$ as the columns of a coefficient matrix, and $y$ as the vector of constants. Employing an augmented matrix to solve this system,

$$\begin{bmatrix} 2 & -1 & 3 & 9 \\ 1 & 1 & 1 & 2 \\ 3 & 0 & -1 & 1 \\ 2 & 1 & -2 & -3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
So we see, as expected, that

\[
\begin{bmatrix}
1 \\
1 \\
3 \\
2
\end{bmatrix}
+ (-1) \begin{bmatrix}
-1 \\
0 \\
1 \\
1
\end{bmatrix}
+ 2 \begin{bmatrix}
3 \\
1 \\
-1 \\
-2
\end{bmatrix}
= \begin{bmatrix}
9 \\
2 \\
1 \\
-3
\end{bmatrix}
= y
\]

A key feature of this example is that the linear combination that expresses \( y \) as a linear combination of the vectors in \( P \) is unique. This is a consequence of the linear independence of \( P \). The linearly independent set \( P \) is smaller than \( R \), but still just (barely) big enough to create elements of the set \( X = \langle R \rangle \). There are many, many ways to write \( y \) as a linear combination of the five vectors in \( R \) (the appropriate system of equations to verify this claim has two free variables in the description of the solution set), yet there is precisely one way to write \( y \) as a linear combination of the three vectors in \( P \).

**Subsection READ**

**Reading Questions**

1. Let \( S \) be the linearly dependent set of three vectors below.

\[
S = \left\{ \begin{bmatrix}
1 \\
1 \\
10 \\
100 \\
1000
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix}, \begin{bmatrix}
5 \\
23 \\
203 \\
2003
\end{bmatrix} \right\}
\]

Write one vector from \( S \) as a linear combination of the other two (you should be able to do this on sight, rather than doing some computations). Convert this expression into a relation of linear dependence on \( S \).

2. Explain why the word “dependent” is used in the definition of linear dependence.

3. Suppose that \( Y = \langle P \rangle = \langle Q \rangle \), where \( P \) is a linearly dependent set and \( Q \) is linearly independent. Would you rather use \( P \) or \( Q \) to describe \( Y \)? Why?
Subsection EXC
Exercises

C20 Let $T$ be the set of columns of the matrix $B$ below. Define $W = \langle T \rangle$. Find a set $R$ so that (1) $R$ has 3 vectors, (2) $R$ is a subset of $T$, and (3) $W = \langle R \rangle$.

$$B = \begin{bmatrix} -3 & 1 & -2 & 7 \\ -1 & 2 & 1 & 4 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

Contributed by Robert Beezer  Solution [149]

C40 Verify that the set $R' = \{v_1, v_2, v_4\}$ at the end of Example RSC5 [140] is linearly independent.

Contributed by Robert Beezer

C50 Consider the set of vectors from $\mathbb{C}^3$, $W$, given below. Find a linearly independent set $T$ that contains three vectors from $W$ and such that $\langle W \rangle = \langle T \rangle$.

$$W = \{v_1, v_2, v_3, v_4, v_5\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Contributed by Robert Beezer  Solution [149]

C51 Given the set $S$ below, find a linearly independent set $T$ so that $\langle T \rangle = \langle S \rangle$.

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}$$

Contributed by Robert Beezer  Solution [150]

C55 Let $T$ be the set of vectors $T = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix} \right\}$. Find two different subsets of $T$, named $R$ and $S$, so that $R$ and $S$ each contain three vectors, and so that $\langle R \rangle = \langle T \rangle$ and $\langle S \rangle = \langle T \rangle$. Prove that both $R$ and $S$ are linearly independent.

Contributed by Robert Beezer  Solution [149]

C70 Reprise Example RES [146] by creating a new version of the vector $y$. In other words, form a new, different linear combination of the vectors in $R$ to create a new vector $y$ (but do not simplify the problem too much by choosing any of the five new scalars to be zero). Then express this new $y$ as a combination of the vectors in $P$.

Contributed by Robert Beezer

M10 At the conclusion of Example RSSC4 [145] two alternative solutions, sets $T'$ and $T^*$, are proposed. Verify these claims by proving that $\langle T \rangle = \langle T' \rangle$ and $\langle T \rangle = \langle T^* \rangle$.

Contributed by Robert Beezer

T40 Suppose that $v_1$ and $v_2$ are any two vectors from $\mathbb{C}^m$. Prove the following set equality.

$$\langle \{v_1, v_2\} \rangle = \langle \{v_1 + v_2, v_1 - v_2\} \rangle$$

Contributed by Robert Beezer  Solution [150]
Let \( T = \{ \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4 \} \). The vector
\[
\begin{bmatrix}
2 \\
-1 \\
0 \\
1 \\
\end{bmatrix}
\]
is a solution to the homogeneous system with the matrix \( B \) as the coefficient matrix (check this!). By Theorem SLSLC \[82\] it provides the scalars for a linear combination of the columns of \( B \) (the vectors in \( T \)) that equals the zero vector, a relation of linear dependence on \( T \),
\[
2\mathbf{w}_1 + (-1)\mathbf{w}_2 + (1)\mathbf{w}_4 = \mathbf{0}
\]
We can rearrange this equation by solving for \( \mathbf{w}_4 \),
\[
\mathbf{w}_4 = (-2)\mathbf{w}_1 + \mathbf{w}_2
\]
This equation tells us that the vector \( \mathbf{w}_4 \) is superfluous in the span construction that creates \( W \). So \( W = \langle \{ \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \} \rangle \). The requested set is \( R = \{ \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \} \).

To apply Theorem BS \[143\], we formulate a matrix \( A \) whose columns are \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5 \). Then we row-reduce \( A \). After row-reducing, we obtain
\[
\begin{bmatrix}
1 & 0 & 0 & 2 & -1 \\
0 & 1 & 0 & 1 & -2 \\
0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]
From this we that the pivot columns are \( D = \{ 1, 2, 3 \} \). Thus
\[
\begin{align*}
T &= \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \\
&= \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}
\end{align*}
\]
is a linearly independent set and \( \langle T \rangle = W \). Compare this problem with Exercise LI.M50 \[133\].
this equation can be rewritten with the second vector staying put, and the other three moving to
the other side of the equality,
\[
\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = (2) \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + (1) \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}
\]

We could have chosen other vectors to stay put, but may have then needed to divide by a nonzero scalar. This equation is enough to conclude that the second vector in \( T \) is “surplus” and can be replaced (see the careful argument in Example RSC5 \[140\]). So set

\[
S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} \right\}
\]

and then \( \langle S \rangle = \langle T \rangle \). \( T \) is also a linearly independent set, which we can show directly. Make a matrix \( C \) whose columns are the vectors in \( S \). Row-reduce \( B \) and you will obtain the identity matrix \( I_3 \). By Theorem LIVRN \[125\], the set \( S \) is linearly independent.

C51 Contributed by Robert Beezer Statement \[148\]

Theorem BS \[143\] says we can make a matrix with these four vectors as columns, row-reduce, and just keep the columns with indices in the set \( D \). Here we go, forming the relevant matrix and row-reducing,

\[
\begin{bmatrix} 2 & 3 & 1 & 5 \\ -1 & 0 & 1 & -1 \\ 2 & 1 & -1 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

Analyzing the row-reduced version of this matrix, we see that the first two columns are pivot columns, so \( D = \{1, 2\} \). Theorem BS \[143\] says we need only “keep” the first two columns to create a set with the requisite properties,

\[
T = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}
\]

T40 Contributed by Robert Beezer Statement \[148\]

This is an equality of sets, so Definition SE \[616\] applies.

The “easy” half first. Show that \( X = \langle \{ \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2 \} \rangle \subseteq \langle \{ \mathbf{v}_1, \mathbf{v}_2 \} \rangle = Y \).

Choose \( \mathbf{x} \in X \). Then \( \mathbf{x} = a_1 (\mathbf{v}_1 + \mathbf{v}_2) + a_2 (\mathbf{v}_1 - \mathbf{v}_2) \) for some scalars \( a_1 \) and \( a_2 \). Then,

\[
\mathbf{x} = a_1 (\mathbf{v}_1 + \mathbf{v}_2) + a_2 (\mathbf{v}_1 - \mathbf{v}_2) = a_1 \mathbf{v}_1 + a_1 \mathbf{v}_2 + a_2 \mathbf{v}_1 + (-a_2) \mathbf{v}_2 = (a_1 + a_2) \mathbf{v}_1 + (a_1 - a_2) \mathbf{v}_2
\]

which qualifies \( \mathbf{x} \) for membership in \( Y \), as it is a linear combination of \( \mathbf{v}_1, \mathbf{v}_2 \).

Now show the opposite inclusion, \( Y = \langle \{ \mathbf{v}_1, \mathbf{v}_2 \} \rangle \subseteq \langle \{ \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2 \} \rangle = X \).

Choose \( \mathbf{y} \in Y \). Then there are scalars \( b_1, b_2 \) such that \( \mathbf{y} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 \). Rearranging, we obtain,

\[
\mathbf{y} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 = \frac{b_1}{2} [(\mathbf{v}_1 + \mathbf{v}_2) + (\mathbf{v}_1 - \mathbf{v}_2)] + \frac{b_2}{2} [(\mathbf{v}_1 + \mathbf{v}_2) - (\mathbf{v}_1 - \mathbf{v}_2)]
\]

\[
= \frac{b_1 + b_2}{2} (\mathbf{v}_1 + \mathbf{v}_2) + \frac{b_1 - b_2}{2} (\mathbf{v}_1 - \mathbf{v}_2)
\]

This is an expression for \( \mathbf{y} \) as a linear combination of \( \mathbf{v}_1 + \mathbf{v}_2 \) and \( \mathbf{v}_1 - \mathbf{v}_2 \), earning \( \mathbf{y} \) membership in \( X \). Since \( X \) is a subset of \( Y \), and vice versa, we see that \( X = Y \), as desired.
In this section we define a couple more operations with vectors, and prove a few theorems. These definitions and results are not central to what follows, but we will make use of them frequently throughout the remainder of the course on various occasions. Because we have chosen to use \( \mathbb{C} \) as our set of scalars, this subsection is a bit more, uh, . . . complex than it would be for the real numbers. We'll explain as we go along how things get easier for the real numbers \( \mathbb{R} \). If you haven’t already, now would be a good time to review some of the basic properties of arithmetic with complex numbers described in Section CNO [611]. With that done, we can extend the basics of complex number arithmetic to our study of vectors in \( \mathbb{C}^m \).

**Subsection CAV**

**Complex Arithmetic and Vectors**

We know how the addition and multiplication of complex numbers is employed in defining the operations for vectors in \( \mathbb{C}^m \) ([Definition CVA [73] and Definition CVSM [74]]). We can also extend the idea of the conjugate to vectors.

**Definition CCCV**

**Complex Conjugate of a Column Vector**

Suppose that \( \mathbf{u} \) is a vector from \( \mathbb{C}^m \). Then the conjugate of the vector, \( \overline{\mathbf{u}} \), is defined by

\[
\overline{\mathbf{u}}_i = \overline{\mathbf{u}}_i
\]

\( 1 \leq i \leq m \) (This definition contains Notation CCCV.)

With this definition we can show that the conjugate of a column vector behaves as we would expect with regard to vector addition and scalar multiplication.

**Theorem CRVA**

**Conjugation Respects Vector Addition**

Suppose \( \mathbf{x} \) and \( \mathbf{y} \) are two vectors from \( \mathbb{C}^m \). Then

\[
\overline{\mathbf{x} + \mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}
\]

**Proof**

For each \( 1 \leq i \leq m \),

\[
\overline{\mathbf{x} + \mathbf{y}}_i = \overline{\mathbf{x} + \mathbf{y}}_i = \overline{\mathbf{x}}_i + \overline{\mathbf{y}}_i = \mathbf{x}_i + \mathbf{y}_i
\]

Then by [Definition CVE [73]] we have

\[
\overline{\mathbf{x} + \mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}.
\]

**Theorem CRSM**

**Conjugation Respects Vector Scalar Multiplication**

Suppose \( \mathbf{x} \) is a vector from \( \mathbb{C}^m \), and \( \alpha \in \mathbb{C} \) is a scalar. Then

\[
\overline{\alpha \mathbf{x}} = \overline{\alpha} \overline{\mathbf{x}}
\]
Proof  For $1 \leq i \leq m$,
\[
[\overline{\alpha x}]_i = \overline{[\alpha x]_i} \quad \text{Definition CCCV} \quad [151]
\]
\[
= \alpha [\overline{x}]_i \quad \text{Definition CVSM} \quad [74]
\]
\[
= [\overline{\alpha x}]_i \quad \text{Theorem CCRM} \quad [613]
\]
\[
= [\alpha x]_i \quad \text{Definition CCCV} \quad [151]
\]
\[
= [\alpha x]_i \quad \text{Definition CVSM} \quad [74]
\]

Then by Definition CVE \quad [73] we have $\overline{\alpha x} = \alpha x$.

These two theorems together tell us how we can “push” complex conjugation through linear combinations.

Subsection IP
Inner products

Definition IP
Inner Product
Given the vectors $u, v \in \mathbb{C}^m$ the inner product of $u$ and $v$ is the scalar quantity in $\mathbb{C}$,
\[
\langle u, v \rangle = [u_1 \overline{v}_1] + [u_2 \overline{v}_2] + [u_3 \overline{v}_3] + \cdots + [u_m \overline{v}_m] = \sum_{i=1}^{m} [u_i \overline{v}_i]
\]
(This definition contains Notation IP.)

This operation is a bit different in that we begin with two vectors but produce a scalar. Computing one is straightforward.

Example CSIP
Computing some inner products
The scalar product of
\[
u = \begin{bmatrix} 2 + 3i \\ 5 + 2i \\ -3 + i \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 1 + 2i \\ -4 + 5i \\ 0 + 5i \end{bmatrix}
\]
is
\[
\langle u, v \rangle = (2 + 3i)(1 + 2i) + (5 + 2i)(-4 + 5i) + (3 + i)(0 + 5i)
\]
\[
= (2 + 3i)(1 + 2i) + (5 + 2i)(-4 - 5i) + (3 + i)(0 - 5i)
\]
\[
= (8 - i) + (-10 - 33i) + (5 + 15i)
\]
\[
= 3 - 19i
\]
The scalar product of
\[
w = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 2 \\ 8 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}
\]
The inner product is
\[
\langle w, x \rangle = 2(3) + 4(\overline{1}) + (-3)(\bar{0}) + 2(-1) + 8(-2) = 2(3) + 4(1) + (-3)0 + 2(-1) + 8(-2) = -8.
\]

In the case where the entries of our vectors are all real numbers (as in the second part of Example CSIP [152]), the computation of the inner product may look familiar and be known to you as a dot product or scalar product. So you can view the inner product as a generalization of the scalar product to vectors from \( \mathbb{C}^m \) (rather than \( \mathbb{R}^m \)).

Also, note that we have chosen to conjugate the entries of the second vector listed in the inner product, while many authors choose to conjugate entries from the first component. It really makes no difference which choice is made, it just requires that subsequent definitions and theorems are consistent with the choice. You can study the conclusion of Theorem IPAC [154] as an explanation of the magnitude of the difference that results from this choice. But be careful as you read other treatments of the inner product or its use in applications, and be sure you know ahead of time which choice has been made.

There are several quick theorems we can now prove, and they will each be useful later.

**Theorem IPVA**

**Inner Product and Vector Addition**

Suppose \( u, v, w \in \mathbb{C}^m \). Then
\begin{align*}
1. \quad \langle u + v, w \rangle &= \langle u, w \rangle + \langle v, w \rangle \\
2. \quad \langle u, v + w \rangle &= \langle u, v \rangle + \langle u, w \rangle
\end{align*}

**Proof** The proofs of the two parts are very similar, with the second one requiring just a bit more effort due to the conjugation that occurs. We will prove part 2 and you can prove part 1 (Exercise O.T10 [162]).

\[
\langle u, v + w \rangle = \sum_{i=1}^{m} [u_i |v + w|_i] = \sum_{i=1}^{m} [u_i |v|_i + |w|_i] = \sum_{i=1}^{m} [u_i |v|_i + |u_i |w|_i] = \sum_{i=1}^{m} [u_i |v|_i] + \sum_{i=1}^{m} [u_i |w|_i] = \langle u, v \rangle + \langle u, w \rangle
\]

**Theorem IPSM**

**Inner Product and Scalar Multiplication**

Suppose \( u, v \in \mathbb{C}^m \) and \( \alpha \in \mathbb{C} \). Then
\begin{align*}
1. \quad \langle \alpha u, v \rangle &= \alpha \langle u, v \rangle \\
2. \quad \langle u, \alpha v \rangle &= \bar{\alpha} \langle u, v \rangle
\end{align*}
Proof The proofs of the two parts are very similar, with the second one requiring just a bit more effort due to the conjugation that occurs. We will prove part 2 and you can prove part 1 (Exercise O.T11 [162]).

\[
\langle u, \alpha v \rangle = \sum_{i=1}^{m} |u_i| |\alpha v_i| \quad \text{Definition IP [152]}
\]

\[
= \sum_{i=1}^{m} |u_i| |\alpha| |v_i| \quad \text{Definition CVSM [74]}
\]

\[
= \sum_{i=1}^{m} |u_i| |\bar{\alpha}| |v_i| \quad \text{Theorem CCRM [613]}
\]

\[
= \sum_{i=1}^{m} |\bar{\alpha} u_i| |v_i| \quad \text{Property MCCN [612]}
\]

\[
= \bar{\alpha} \sum_{i=1}^{m} |u_i| |v_i| \quad \text{Property DCN [612]}
\]

\[
= \bar{\alpha} \langle u, v \rangle \quad \text{Definition IP [152]}
\]

\[\square\]

Theorem IPAC
Inner Product is Anti-Commutative
Suppose that \( u \) and \( v \) are vectors in \( \mathbb{C}^m \). Then \( \langle u, v \rangle = \overline{\langle v, u \rangle} \).

\[\square\]

Subsection O.N Norm

If treating linear algebra in a more geometric fashion, the length of a vector occurs naturally, and is what you would expect from its name. With complex numbers, we will define a similar function. Recall that if \( c \) is a complex number, then \( |c| \) denotes its modulus (Definition MCN [614]).
Definition NV
Norm of a Vector
The norm of the vector \( \mathbf{u} \) is the scalar quantity in \( \mathbb{C} \)

\[
\| \mathbf{u} \| = \sqrt{|u_1|^2 + |u_2|^2 + |u_3|^2 + \cdots + |u_m|^2} = \sqrt{\sum_{i=1}^{m} |u_i|^2}
\]

(This definition contains Notation NV.)

Computing a norm is also easy to do.

Example CNSV
Computing the norm of some vectors
The norm of

\[
\mathbf{u} = \begin{bmatrix} 3+2i \\ 1-6i \\ 2+4i \\ 2+i \end{bmatrix}
\]

is

\[
\| \mathbf{u} \| = \sqrt{|3+2i|^2 + |1-6i|^2 + |2+4i|^2 + |2+i|^2} = \sqrt{13 + 37 + 20 + 5} = \sqrt{75} = 5\sqrt{3}.
\]

The norm of

\[
\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ -3 \end{bmatrix}
\]

is

\[
\| \mathbf{v} \| = \sqrt{|3|^2 + |-1|^2 + |2|^2 + |4|^2 + |-3|^2} = \sqrt{3^2 + 1^2 + 2^2 + 4^2 + 3^2} = \sqrt{39}.
\]

Notice how the norm of a vector with real number entries is just the length of the vector. Inner products and norms are related by the following theorem.

Theorem IPN
Inner Products and Norms
Suppose that \( \mathbf{u} \) is a vector in \( \mathbb{C}^m \). Then \( \| \mathbf{u} \|^2 = \langle \mathbf{u}, \mathbf{u} \rangle \).

Proof

\[
\| \mathbf{u} \|^2 = \left( \sum_{i=1}^{m} |u_i|^2 \right)^2 
\]

\[
= \sum_{i=1}^{m} |u_i|^2 
\]

\[
= \sum_{i=1}^{m} u_i \overline{u_i} 
\]

\[
= \langle \mathbf{u}, \mathbf{u} \rangle 
\]

When our vectors have entries only from the real numbers Theorem IPN says that the dot product of a vector with itself is equal to the length of the vector squared.
Theorem PIP
Positive Inner Products
Suppose that \( \mathbf{u} \) is a vector in \( \mathbb{C}^m \). Then \( \langle \mathbf{u}, \mathbf{u} \rangle \geq 0 \) with equality if and only if \( \mathbf{u} = \mathbf{0} \). \( \Box \)

Proof From the proof of Theorem IPN [155] we see that 
\[
\langle \mathbf{u}, \mathbf{u} \rangle = |[\mathbf{u}]_1|^2 + |[\mathbf{u}]_2|^2 + |[\mathbf{u}]_3|^2 + \cdots + |[\mathbf{u}]_m|^2
\]
Since each modulus is squared, every term is positive, and the sum must also be positive. (Notice that in general the inner product is a complex number and cannot be compared with zero, but in the special case of \( \langle \mathbf{u}, \mathbf{u} \rangle \) the result is a real number.) The phrase, “with equality if and only if” means that we want to show that the statement \( \langle \mathbf{u}, \mathbf{u} \rangle = 0 \) (i.e. with equality) is equivalent (“if and only if”) to the statement \( \mathbf{u} = \mathbf{0} \).

If \( \mathbf{u} = \mathbf{0} \), then it is a straightforward computation to see that \( \langle \mathbf{u}, \mathbf{u} \rangle = 0 \). In the other direction, assume that \( \langle \mathbf{u}, \mathbf{u} \rangle = 0 \). As before, \( \langle \mathbf{u}, \mathbf{u} \rangle \) is a sum of moduli. So we have
\[
0 = \langle \mathbf{u}, \mathbf{u} \rangle = |[\mathbf{u}]_1|^2 + |[\mathbf{u}]_2|^2 + |[\mathbf{u}]_3|^2 + \cdots + |[\mathbf{u}]_m|^2
\]
Now we have a sum of squares equaling zero, so each term must be zero. Then by similar logic, \( |[\mathbf{u}]_i| = 0 \) will imply that \( [\mathbf{u}]_i = 0 \), since \( 0 + 0i \) is the only complex number with zero modulus. Thus every entry of \( \mathbf{u} \) is zero and so \( \mathbf{u} = \mathbf{0} \), as desired. \( \Box \)

Notice that Theorem PIP [156] contains three implications: \( \mathbf{u} \) is any vector \Rightarrow \( \langle \mathbf{u}, \mathbf{u} \rangle \geq 0 \), \( \mathbf{u} = \mathbf{0} \Rightarrow \langle \mathbf{u}, \mathbf{u} \rangle = 0 \), and \( \langle \mathbf{u}, \mathbf{u} \rangle = 0 \Rightarrow \mathbf{u} = \mathbf{0} \). The results contained in Theorem PIP [156] are summarized by saying “the inner product is positive definite.”

Subsection OV
Orthogonal Vectors

“Orthogonal” is a generalization of “perpendicular.” You may have used mutually perpendicular vectors in a physics class, or you may recall from a calculus class that perpendicular vectors have a zero dot product. We will now extend these ideas into the realm of higher dimensions and complex scalars.

Definition OV
Orthogonal Vectors
A pair of vectors, \( \mathbf{u} \) and \( \mathbf{v} \), from \( \mathbb{C}^m \) are orthogonal if their inner product is zero, that is, \( \langle \mathbf{u}, \mathbf{v} \rangle = 0 \). \( \triangle \)

Example TOV
Two orthogonal vectors
The vectors
\[
\mathbf{u} = \begin{bmatrix} 2 + 3i \\ 4 - 2i \\ 1 + i \\ 1 + i \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 - i \\ 2 + 3i \\ 4 - 6i \\ 1 \end{bmatrix}
\]
are orthogonal since
\[
\langle \mathbf{u}, \mathbf{v} \rangle = (2 + 3i)(1 + i) + (4 - 2i)(2 - 3i) + (1 + i)(4 + 6i) + (1 + i)(1)
\]
\[
= (-1 + 5i) + (2 - 16i) + (-2 + 10i) + (1 + i)
\]
\[
= 0 + 0i.
\]

We extend this definition to whole sets by requiring vectors to be pairwise orthogonal. Despite using the same word, careful thought about what objects you are using will eliminate any source of confusion.
Definition OSV
Orthogonal Set of Vectors
Suppose that \( S = \{u_1, u_2, u_3, \ldots, u_n\} \) is a set of vectors from \( \mathbb{C}^m \). Then the set \( S \) is orthogonal if every pair of different vectors from \( S \) is orthogonal, that is \( \langle u_i, u_j \rangle = 0 \) whenever \( i \neq j \). \( \triangle \)

The next example is trivial in some respects, but is still worthy of discussion since it is the prototypical orthogonal set.

Example SUVOS
Standard Unit Vectors are an Orthogonal Set
The standard unit vectors are the columns of the identity matrix (Definition SUV [190]). Computing the inner product of two distinct vectors, \( e_i, e_j, i \neq j \), gives,
\[
\langle e_i, e_j \rangle = 0 + 0 + 0 + \cdots + 0 + 0 + 0 + 0 + 0 = 0
\]

Example AOS
An orthogonal set
The set
\[
\{x_1, x_2, x_3, x_4\} = \left\{ \begin{pmatrix} 1 + i \\ 1 \\ -7 - i \\ i \end{pmatrix}, \begin{pmatrix} 1 + 5i \\ 6 + 5i \\ -7 - i \\ 1 - 6i \end{pmatrix}, \begin{pmatrix} -7 + 34i \\ -8 - 23i \\ -10 + 22i \\ 30 + 13i \end{pmatrix}, \begin{pmatrix} -2 - 4i \\ 6 + i \\ 4 + 3i \\ 6 - i \end{pmatrix} \right\}
\]
is an orthogonal set. Since the inner product is anti-commutative (Theorem IPAC [154]) we can test pairs of different vectors in any order. If the result is zero, then it will also be zero if the inner product is computed in the opposite order. This means there are six pairs of different vectors to use in an inner product computation. We’ll do two and you can practice your inner products on the other four.

\[
\langle x_1, x_3 \rangle = (1 + i)(-7 - 34i) + (1)(-8 + 23i) + (1 - i)(-10 - 22i) + (i)(30 - 13i)
\]
\[
= (27 - 41i) + (-8 + 23i) + (-32 - 12i) + (13 + 30i)
\]
\[
= 0 + 0i
\]

and

\[
\langle x_2, x_4 \rangle = (1 + 5i)(-2 + 4i) + (6 + 5i)(6 - i) + (-7 - i)(4 - 3i) + (1 - 6i)(6 + i)
\]
\[
= (-22 - 6i) + (41 + 24i) + (-31 + 17i) + (12 - 35i)
\]
\[
= 0 + 0i
\]

So far, this section has seen lots of definitions, and lots of theorems establishing un-surprising consequences of those definitions. But here is our first theorem that suggests that inner products and orthogonal vectors have some utility. It is also one of our first illustrations of how to arrive at linear independence as the conclusion of a theorem.

Theorem OSLI
Orthogonal Sets are Linearly Independent
Suppose that \( S = \{u_1, u_2, u_3, \ldots, u_n\} \) is an orthogonal set of nonzero vectors. Then \( S \) is linearly independent. \( \square \)

Proof
To prove linear independence of a set of vectors, we can appeal to the definition (Definition LICV [121]) and begin with a relation of linear dependence (Definition RLDCV [121]),
\[
\alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3 + \cdots + \alpha_nu_n = 0.
\]
Then, for every \( 1 \leq i \leq n \), we have

\[
0 = 0 \langle u_i, u_i \rangle = \langle 0u_i, u_i \rangle = \langle \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_n u_n, u_i \rangle = \langle \alpha_1 u_1, u_i \rangle + \alpha_2 \langle u_2, u_i \rangle + \alpha_3 \langle u_3, u_i \rangle + \cdots + \alpha_n \langle u_n, u_i \rangle = \alpha_1 \langle u_1, u_i \rangle + \alpha_2 \langle u_2, u_i \rangle + \alpha_3 \langle u_3, u_i \rangle + \cdots + \alpha_n \langle u_n, u_i \rangle = \alpha_1 (0) + \alpha_2 (0) + \alpha_3 (0) + \cdots + \alpha_i (0) + \cdots + \alpha_n (0) = \alpha_i \langle u_i, u_i \rangle
\]

So we have \( 0 = \alpha_i \langle u_i, u_i \rangle \). However, since \( u_i \neq 0 \) (the hypothesis said our vectors were nonzero), Theorem PIP \(^{156}\) says that \( \langle u_i, u_i \rangle > 0 \). So we must conclude that \( \alpha_i = 0 \) for all \( 1 \leq i \leq n \). But this says that \( S \) is a linearly independent set since the only way to form a relation of linear dependence is the trivial way, with all the scalars zero. Boom!

**Subsection GSP**

**Gram-Schmidt Procedure**

The Gram-Schmidt Procedure is really a theorem. It says that if we begin with a linearly independent set of \( p \) vectors, \( S \), then we can do a number of calculations with these vectors and produce an orthogonal set of \( p \) vectors, \( T \), so that \( \langle S \rangle = \langle T \rangle \). Given the large number of computations involved, it is indeed a procedure to do all the necessary computations, and it is best employed on a computer. However, it also has value in proofs where we may on occasion wish to replace a linearly independent set by an orthogonal set.

This is our first occasion to use the technique of “mathematical induction” for a proof, a technique we will see again several times, especially in Chapter D \(^{333}\). So study the simple example described in Technique I \(^{626}\) first.

**Theorem GSPCV**

**Gram-Schmidt Procedure, Column Vectors**

Suppose that \( S = \{v_1, v_2, v_3, \ldots, v_p\} \) is a linearly independent set of vectors in \( \mathbb{C}^m \). Define the vectors \( u_i, 1 \leq i \leq p \) by

\[
\langle u_i, u_i \rangle = \frac{\langle v_i, u_i \rangle}{\langle u_i, u_i \rangle} u_1 = \frac{\langle v_i, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle v_i, u_3 \rangle}{\langle u_3, u_3 \rangle} u_3 - \cdots - \frac{\langle v_i, u_{i-1} \rangle}{\langle u_{i-1}, u_{i-1} \rangle} u_{i-1}
\]

Then if \( T = \{u_1, u_2, u_3, \ldots, u_p\} \), then \( T \) is an orthogonal set of non-zero vectors, and \( \langle T \rangle = \langle S \rangle \).

**Proof**

We will prove the result by using induction on \( p \) (Technique I \(^{626}\)). To begin, we prove that \( T \) has the desired properties when \( p = 1 \). In this case \( u_1 = v_1 \) and \( T = \{u_1\} = \{v_1\} = S \). Because \( S \) and \( T \) are equal, \( \langle S \rangle = \langle T \rangle \). Equally trivial, \( T \) is an orthogonal set. If \( u_1 = 0 \), then \( S \) would be a linearly dependent set, a contradiction.

Now suppose that the theorem is true for any set of \( p - 1 \) linearly independent vectors. Let \( S = \{v_1, v_2, v_3, \ldots, v_p\} \) be a linearly independent set of \( p \) vectors. Then \( S' = \{v_1, v_2, v_3, \ldots, v_{p-1}\} \) is also linearly independent. So we can apply the theorem to \( S' \) and construct the vectors \( T' = \{u_1, u_2, u_3, \ldots, u_{p-1}\} \). \( T' \) is therefore an orthogonal set of nonzero vectors and \( \langle S' \rangle = \langle T' \rangle \). Define

\[
u_p = v_p - \frac{\langle v_p, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = \frac{\langle v_p, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle v_p, u_3 \rangle}{\langle u_3, u_3 \rangle} u_3 - \cdots - \frac{\langle v_p, u_{p-1} \rangle}{\langle u_{p-1}, u_{p-1} \rangle} u_{p-1}
\]
and let $T = T' \cup \{u_p\}$. We need to now show that $T$ has several properties by building on what we know about $T'$. But first notice that the above equation has no problems with the denominators $(\langle u_i, v \rangle)$ being zero, since the $u_i$ are from $T'$, which is composed of nonzero vectors.

We show that $\langle T \rangle = \langle S \rangle$, by first establishing that $\langle T \rangle \subseteq \langle S \rangle$. Suppose $x \in \langle T \rangle$, so

$$x = a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_pu_p$$

The term $a_pu_p$ is a linear combination of vectors from $T'$ and the vector $v_p$, while the remaining terms are a linear combination of vectors from $T'$. Since $(T') = \langle S' \rangle$, any term that is a multiple of a vector from $T'$ can be rewritten as a linear combination of vectors from $S'$. The remaining term $a_pv_p$ is a multiple of a vector in $S$. So we see that $x$ can be rewritten as a linear combination of vectors from $S$, i.e. $x \in \langle S \rangle$.

To show that $\langle S \rangle \subseteq \langle T \rangle$, begin with $y \in \langle S \rangle$, so

$$y = a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_pv_p$$

Rearrange our defining equation for $u_p$ by solving for $v_p$. Then the term $a_pv_p$ is a multiple of a linear combination of elements of $T$. The remaining terms are a linear combination of $v_1, v_2, v_3, \ldots, v_{p-1}$, hence an element of $(S') = \langle T' \rangle$. Thus these remaining terms can be written as a linear combination of the vectors in $T'$. So $y$ is a linear combination of vectors from $T$, i.e. $y \in \langle T \rangle$.

The elements of $T'$ are nonzero, but what about $u_p$? Suppose to the contrary that $u_p = 0$,

$$0 = u_p = v_p - \sum_{k=1}^{p-1} \frac{\langle v_p, u_k \rangle}{\langle u_k, u_k \rangle} u_k - \frac{\langle v_p, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_p, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle v_p, u_3 \rangle}{\langle u_3, u_3 \rangle} u_3 - \cdots - \frac{\langle v_p, u_{p-1} \rangle}{\langle u_{p-1}, u_{p-1} \rangle} u_{p-1}$$

Since $(S') = \langle T' \rangle$ we can write the vectors $u_1, u_2, u_3, \ldots, u_{p-1}$ on the right side of this equation in terms of the vectors $v_1, v_2, v_3, \ldots, v_{p-1}$ and we then have the vector $v_p$ expressed as a linear combination of the other $p - 1$ vectors in $S$, implying that $S$ is a linearly dependent set (Theorem DLDS [139]), contrary to our lone hypothesis about $S$.

Finally, it is a simple matter to establish that $T$ is an orthogonal set, though it will not appear so simple looking. Think about your objects as you work through the following — what is a vector and what is a scalar. Since $T'$ is an orthogonal set by induction, most pairs of elements in $T$ are orthogonal. We just need to test inner products between $u_p$ and $u_i$, for $1 \leq i \leq p - 1$. Here we go, using summation notation,

$$\langle u_p, u_i \rangle = \langle v_p - \sum_{k=1}^{p-1} \frac{\langle v_p, u_k \rangle}{\langle u_k, u_k \rangle} u_k, u_i \rangle$$

$$\begin{align*}
= \langle v_p, u_i \rangle - \sum_{k=1}^{p-1} \frac{\langle v_p, u_k \rangle}{\langle u_k, u_k \rangle} \langle u_k, u_i \rangle & \quad \text{(Theorem IPVA [153])} \\
= \langle v_p, u_i \rangle - \sum_{k=1}^{p-1} \frac{\langle v_p, u_k \rangle}{\langle u_k, u_k \rangle} \langle u_k, u_i \rangle & \quad \text{(Theorem IPVA [153])} \\
= \langle v_p, u_i \rangle - \sum_{k=1}^{p-1} \frac{\langle v_p, u_k \rangle}{\langle u_k, u_k \rangle} \langle u_k, u_i \rangle & \quad \text{(Theorem IPSM [153])} \\
= \langle v_p, u_i \rangle - \frac{\langle v_p, u_1 \rangle}{\langle u_1, u_1 \rangle} \langle u_1, u_i \rangle - \sum_{k \neq i} \frac{\langle v_p, u_k \rangle}{\langle u_k, u_k \rangle} (0) & \quad \text{Induction Hypothesis} \\
= \langle v_p, u_i \rangle - \langle v_p, u_i \rangle - \sum_{k \neq i} 0 & \\
= 0
\end{align*}$$
Example GSTV
Gram-Schmidt of three vectors
We will illustrate the Gram-Schmidt process with three vectors. Begin with the linearly independent (check this!) set
\[ S = \{v_1, v_2, v_3\} = \begin{bmatrix} 1+i \\ 1+i \\ 1 + i + 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \\ 1 + i \end{bmatrix}, \begin{bmatrix} 0 \\ i \\ i \end{bmatrix} \]
Then
\[ u_1 = v_1 = \begin{bmatrix} 1+i \\ 1+i \\ 1 \end{bmatrix} \]
\[ u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = \frac{1}{4} \begin{bmatrix} -2-3i \\ 1-i \\ 2+5i \end{bmatrix} \]
\[ u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 = \frac{1}{11} \begin{bmatrix} -3-i \\ 1+3i \\ -1-i \end{bmatrix} \]
and
\[ T = \{u_1, u_2, u_3\} = \begin{bmatrix} 1+i \\ 1+i \\ 1 \end{bmatrix}, \begin{bmatrix} -2-3i \\ 1-i \\ 2+5i \end{bmatrix}, \begin{bmatrix} -3-i \\ 1+3i \\ -1-i \end{bmatrix} \]
is an orthogonal set (which you can check) of nonzero vectors and \( \langle T \rangle = \langle S \rangle \) (all by Theorem GSPCV [158]). Of course, as a by-product of orthogonality, the set \( T \) is also linearly independent (Theorem OSLI [157]).

One final definition related to orthogonal vectors.

Definition ONS
OrthoNormal Set
Suppose \( S = \{u_1, u_2, u_3, \ldots, u_n\} \) is an orthogonal set of vectors such that \( \|u_i\| = 1 \) for all \( 1 \leq i \leq n \). Then \( S \) is an orthonormal set of vectors.

Once you have an orthogonal set, it is easy to convert it to an orthonormal set — multiply each vector by the reciprocal of its norm, and the resulting vector will have norm 1. This scaling of each vector will not affect the orthogonality properties (apply Theorem IPSM [153]).

Example ONTV
Orthonormal set, three vectors
The set
\[ T = \{u_1, u_2, u_3\} = \begin{bmatrix} 1+i \\ 1+i \\ 1 \end{bmatrix}, \begin{bmatrix} -2-3i \\ 1-i \\ 2+5i \end{bmatrix}, \begin{bmatrix} -3-i \\ 1+3i \\ -1-i \end{bmatrix} \]
from Example GSTV [160] is an orthogonal set. We compute the norm of each vector,
\[ \|u_1\| = 2 \]
\[ \|u_2\| = \frac{1}{2} \sqrt{11} \]
\[ \|u_3\| = \sqrt{\frac{2}{11}} \]
Converting each vector to a norm of 1, yields an orthonormal set,
\[ w_1 = \frac{1}{2} \begin{bmatrix} 1+i \\ 1+i \\ 1 \end{bmatrix} \]
Example ONFV
Orthonormal set, four vectors
As an exercise convert the linearly independent set

\[
S = \left\{ \begin{bmatrix} 1 + i \\ 1 \\ 1 - i \end{bmatrix}, \begin{bmatrix} i \\ 1 + i \\ -1 \end{bmatrix}, \begin{bmatrix} -i \\ -1 + i \\ 1 \end{bmatrix}, \begin{bmatrix} i \\ 1 \\ 1 \end{bmatrix} \right\}
\]

to an orthogonal set via the Gram-Schmidt Process (Theorem GSPCV \[158\]) and then scale the vectors to norm 1 to create an orthonormal set. You should get the same set you would if you scaled the orthogonal set of Example AOS \([157\) to become an orthonormal set.

It is crazy to do all but the simplest and smallest instances of the Gram-Schmidt procedure by hand. Well, OK, maybe just once or twice to get a good understanding of Theorem GSPCV \[158\]. After that, let a machine do the work for you. That’s what they are for. See: Computation GSP.MMA \[607\].

We will see orthonormal sets again in Subsection MINM.UM \[205\]. They are intimately related to unitary matrices (Definition UM \[205\]) through Theorem CUMOS \[206\]. Some of the utility of orthonormal sets is captured by Theorem COB \[300\] in Subsection B.OBC \[300\]. Orthonormal sets appear once again in Section OD \[540\] where they are key in orthonormal diagonalization.

Subsection READ
Reading Questions

1. Is the set

\[
\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ -2 \end{bmatrix} \right\}
\]

an orthogonal set? Why?

2. What is the distinction between an orthogonal set and an orthonormal set?

3. What is nice about the output of the Gram-Schmidt process?
Subsection EXC
Exercises

C20  Complete Example AOS [157] by verifying that the four remaining inner products are zero.
Contributed by Robert Beezer

C21  Verify that the set $T$ created in Example GSTV [160] by the Gram-Schmidt Procedure is an orthogonal set.
Contributed by Robert Beezer

T10  Prove part 1 of the conclusion of Theorem IPVA [153].
Contributed by Robert Beezer

T11  Prove part 1 of the conclusion of Theorem IPSM [153].
Contributed by Robert Beezer
Chapter M
Matrices

We have made frequent use of matrices for solving systems of equations, and we have begun to investigate a few of their properties, such as the null space and nonsingularity. In this chapter, we will take a more systematic approach to the study of matrices.

Section MO
Matrix Operations

In this section we will back up and start simple. First a definition of a totally general set of matrices.

Definition VSM
Vector Space of $m \times n$ Matrices
The vector space $M_{mn}$ is the set of all $m \times n$ matrices with entries from the set of complex numbers.

(This definition contains Notation VSM.)

Subsection MEASM
Matrix Equality, Addition, Scalar Multiplication

Just as we made, and used, a careful definition of equality for column vectors, so too, we have precise definitions for matrices.

Definition ME
Matrix Equality
The $m \times n$ matrices $A$ and $B$ are equal, written $A = B$ provided $[A]_{ij} = [B]_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$.

(This definition contains Notation ME.)

So equality of matrices translates to the equality of complex numbers, on an entry-by-entry basis. Notice that we now have yet another definition that uses the symbol “=” for shorthand. Whenever a theorem has a conclusion saying two matrices are equal (think about your objects), we will consider appealing to this definition as a way of formulating the top-level structure of the proof. We will now define two operations on the set $M_{mn}$. Again, we will overload a symbol (‘+’) and a convention (juxtaposition for scalar multiplication).

Definition MA
Matrix Addition
Given the $m \times n$ matrices $A$ and $B$, define the sum of $A$ and $B$ as an $m \times n$ matrix, written $A + B$, according to

$$[A + B]_{ij} = [A]_{ij} + [B]_{ij} \quad 1 \leq i \leq m, 1 \leq j \leq n$$
So matrix addition takes two matrices of the same size and combines them (in a natural way!) to create a new matrix of the same size. Perhaps this is the “obvious” thing to do, but it doesn’t relieve us from the obligation to state it carefully.

Example MA
Addition of two matrices in $M_{23}$

If

$$A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -7 \end{bmatrix} \quad B = \begin{bmatrix} 6 & 2 & -4 \\ 3 & 5 & 2 \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -7 \end{bmatrix} + \begin{bmatrix} 6 & 2 & -4 \\ 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 + 6 & -3 + 2 & 4 + (-4) \\ 1 + 3 & 0 + 5 & -7 + 2 \end{bmatrix} = \begin{bmatrix} 8 & -1 & 0 \\ 4 & 5 & -5 \end{bmatrix}$$

Our second operation takes two objects of different types, specifically a number and a matrix, and combines them to create another matrix. As with vectors, in this context we call a number a scalar in order to emphasize that it is not a matrix.

Definition MSM
Matrix Scalar Multiplication

Given the $m \times n$ matrix $A$ and the scalar $\alpha \in \mathbb{C}$, the scalar multiple of $A$ is an $m \times n$ matrix, written $\alpha A$ and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij} \quad 1 \leq i \leq m, 1 \leq j \leq n$$

Our second operation takes two objects of different types, specifically a number and a matrix, and combines them to create another matrix. As with vectors, in this context we call a number a scalar in order to emphasize that it is not a matrix.

Example MSM
Scalar multiplication in $M_{32}$

If

$$A = \begin{bmatrix} 2 & 8 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}$$

and $\alpha = 7$, then

$$\alpha A = 7 \begin{bmatrix} 2 & 8 \\ -3 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7(2) & 7(8) \\ 7(-3) & 7(5) \\ 7(0) & 7(1) \end{bmatrix} = \begin{bmatrix} 14 & 56 \\ -21 & 35 \\ 0 & 7 \end{bmatrix}$$

Subsection VSP
Vector Space Properties

With definitions of matrix addition and scalar multiplication we can now state, and prove, several properties of each operation, and some properties that involve their interplay. We now collect ten of them here for later reference.

Theorem VSPM
Vector Space Properties of Matrices

Suppose that $M_{mn}$ is the set of all $m \times n$ matrices (Definition VSM [163]) with addition and scalar multiplication as defined in Definition MA [163] and Definition MSM [164]. Then
• ACM  Additive Closure, Matrices
  If $A, B \in M_{mn}$, then $A + B \in M_{mn}$.

• SCM  Scalar Closure, Matrices
  If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.

• CM  Commutativity, Matrices
  If $A, B \in M_{mn}$, then $A + B = B + A$.

• AAM  Additive Associativity, Matrices
  If $A, B, C \in M_{mn}$, then $A + (B + C) = (A + B) + C$.

• ZM  Zero Vector, Matrices
  There is a matrix, $O$, called the zero matrix, such that $A + O = A$ for all $A \in M_{mn}$.

• AIM  Additive Inverses, Matrices
  If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = O$.

• SMAM  Scalar Multiplication Associativity, Matrices
  If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha \beta)A$.

• DMAM  Distributivity across Matrix Addition, Matrices
  If $\alpha \in \mathbb{C}$ and $A, B \in M_{mn}$, then $\alpha(A + B) = \alpha A + \alpha B$.

• DSAM  Distributivity across Scalar Addition, Matrices
  If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.

• OM  One, Matrices
  If $A \in M_{mn}$, then $1A = A$.

Proof  While some of these properties seem very obvious, they all require proof. However, the proofs are not very interesting, and border on tedious. We’ll prove one version of distributivity very carefully, and you can test your proof-building skills on some of the others. We’ll give our new notation for matrix entries a workout here. Compare the style of the proofs here with those given for vectors in Theorem VSPCV [75] — while the objects here are more complicated, our notation makes the proofs cleaner.

To prove Property DSAM [165], $(\alpha + \beta)A = \alpha A + \beta A$, we need to establish the equality of two matrices (see Technique GS [621]). Definition ME [163] says we need to establish the equality of their entries, one-by-one. How do we do this, when we do not even know how many entries the two matrices might have? This is where Notation ME [163] comes into play. Ready? Here we go.

For any $i$ and $j$, $1 \leq i \leq m$, $1 \leq j \leq n$,

$$[(\alpha + \beta)A]_{ij} = (\alpha + \beta) [A]_{ij}$$

Definition MSM [164]

$$= \alpha [A]_{ij} + \beta [A]_{ij}$$

Distributivity in $\mathbb{C}$

$$= [\alpha A]_{ij} + [\beta A]_{ij}$$

Definition MSM [164]

$$= [\alpha A + \beta A]_{ij}$$

Definition MA [163]

There are several things to notice here. (1) Each equals sign is an equality of numbers. (2) The two ends of the equation, being true for any $i$ and $j$, allow us to conclude the equality of the matrices by Definition ME [163]. (3) There are several plus signs, and several instances of juxtaposition. Identify each one, and state exactly what operation is being represented by each.

For now, note the similarities between Theorem VSPM [164] about matrices and Theorem VSPCV [75] about vectors.

The zero matrix described in this theorem, $O$, is what you would expect — a matrix full of zeros.

Version 1.04
Definition ZM

Zero Matrix
The \( m \times n \) zero matrix is written as \( \mathcal{O} = \mathcal{O}_{m \times n} \) and defined by \( [\mathcal{O}]_{ij} = 0 \), for all \( 1 \leq i \leq m, 1 \leq j \leq n \).

(This definition contains Notation ZM.) △

Subsection TSM

Transposes and Symmetric Matrices

We describe one more common operation we can perform on matrices. Informally, to transpose a matrix is to build a new matrix by swapping its rows and columns.

Definition TM

Transpose of a Matrix
Given an \( m \times n \) matrix \( A \), its transpose is the \( n \times m \) matrix \( A^t \) given by

\[
[A^t]_{ij} = [A]_{ji}, \quad 1 \leq i \leq n, 1 \leq j \leq m.
\]

(This definition contains Notation TM.) △

Example TM

Transpose of a \( 3 \times 4 \) matrix
Suppose

\[
D = \begin{bmatrix}
3 & 7 & 2 & -3 \\
-1 & 4 & 2 & 8 \\
0 & 3 & -2 & 5
\end{bmatrix}.
\]

We could formulate the transpose, entry-by-entry, using the definition. But it is easier to just systematically rewrite rows as columns (or vice-versa). The form of the definition given will be more useful in proofs. So we have

\[
D^t = \begin{bmatrix}
3 & -1 & 0 \\
7 & 4 & 3 \\
2 & 2 & -2 \\
-3 & 8 & 5
\end{bmatrix}.
\]

It will sometimes happen that a matrix is equal to its transpose. In this case, we will call a matrix symmetric. These matrices occur naturally in certain situations, and also have some nice properties, so it is worth stating the definition carefully. Informally a matrix is symmetric if we can “flip” it about the main diagonal (upper-left corner, running down to the lower-right corner) and have it look unchanged.

Definition SYM

Symmetric Matrix
The matrix \( A \) is symmetric if \( A = A^t \).

Example SYM

A symmetric \( 5 \times 5 \) matrix
The matrix

\[
E = \begin{bmatrix}
2 & 3 & -9 & 5 & 7 \\
3 & 1 & 6 & -2 & -3 \\
-9 & 6 & 0 & -1 & 9 \\
5 & -2 & -1 & 4 & -8 \\
7 & -3 & 9 & -8 & -3
\end{bmatrix}
\]

△
is symmetric.

You might have noticed that Definition SYM \[166\] did not specify the size of the matrix \(A\), as has been our custom. That’s because it wasn’t necessary. An alternative would have been to state the definition just for square matrices, but this is the substance of the next proof. Before reading the next proof, we want to offer you some advice about how to become more proficient at constructing proofs. Perhaps you can apply this advice to the next theorem. Have a peek at Technique P \[627\] now.

**Theorem SMS**

Symmetric Matrices are Square

Suppose that \(A\) is a symmetric matrix. Then \(A\) is square. □

**Proof** We start by specifying \(A\!’s\) size, without assuming it is square, since we are trying to prove that, so we can’t also assume it. Suppose \(A\) is an \(m \times n\) matrix. Because \(A\) is symmetric, we know by Definition SM \[337\] that \(A = A^t\). So, in particular, Definition ME \[163\] requires that \(A\) and \(A^t\) must have the same size. The size of \(A^t\) is \(n \times m\). Because \(A\) has \(m\) rows and \(A^t\) has \(n\) rows, we conclude that \(m = n\), and hence \(A\) must be square by Definition SQM \[61\]. ■

We finish this section with three easy theorems, but they illustrate the interplay of our three new operations, our new notation, and the techniques used to prove matrix equalities.

**Theorem TMA**

Transpose and Matrix Addition

Suppose that \(A\) and \(B\) are \(m \times n\) matrices. Then \((A + B)^t = A^t + B^t\). □

**Proof** The statement to be proved is an equality of matrices, so we work entry-by-entry and use Definition ME \[163\]. Think carefully about the objects involved here, and the many uses of the plus sign.

\[
\left[(A + B)^t\right]_{ij} = [A + B]_{ji} \quad \text{Definition TM} \ [166]
\]

\[
= [A]_{ji} + [B]_{ji} \quad \text{Definition MA} \ [163]
\]

\[
= [A^t]_{ij} + [B^t]_{ij} \quad \text{Definition TM} \ [166]
\]

\[
= [A^t + B^t]_{ij} \quad \text{Definition MA} \ [163]
\]

Since the matrices \((A + B)^t\) and \(A^t + B^t\) agree at each entry, Definition ME \[163\] tells us the two matrices are equal. ■

**Theorem TMSM**

Transpose and Matrix Scalar Multiplication

Suppose that \(\alpha \in \mathbb{C}\) and \(A\) is an \(m \times n\) matrix. Then \((\alpha A)^t = \alpha A^t\). □

**Proof** The statement to be proved is an equality of matrices, so we work entry-by-entry and use Definition ME \[163\]. Think carefully about the objects involved here, the many uses of juxtaposition.

\[
\left[(\alpha A)^t\right]_{ij} = [\alpha A]_{ji} \quad \text{Definition TM} \ [166]
\]

\[
= \alpha [A]_{ji} \quad \text{Definition MSM} \ [164]
\]

\[
= \alpha [A^t]_{ij} \quad \text{Definition TM} \ [166]
\]

\[
= [\alpha A^t]_{ij} \quad \text{Definition MSM} \ [164]
\]

Since the matrices \((\alpha A)^t\) and \(\alpha A^t\) agree at each entry, Definition ME \[163\] tells us the two matrices are equal. ■

**Theorem TT**

Transpose of a Transpose

\[
\left[(A^t)^t\right]_{ij} = [A]_{ji} \quad \text{Definition TM} \ [166]
\]

\[
= [A]_{ji} \quad \text{Definition MSM} \ [164]
\]

\[
= [A^t]_{ij} \quad \text{Definition TM} \ [166]
\]

\[
= [\alpha A^t]_{ij} \quad \text{Definition MSM} \ [164]
\]

Since the matrices \((A^t)^t\) and \(A^t\) agree at each entry, Definition ME \[163\] tells us the two matrices are equal. ■
Suppose that $A$ is an $m \times n$ matrix. Then $\left( A^t \right)^t = A$. □

**Proof** We again want to prove an equality of matrices, so we work entry-by-entry and use Definition ME [163].

\[
\left[ \left( A^t \right)^t \right]_{ij} = [A^t]_{ji} = [A]_{ij}
\]

Definition TM [166]

□

It’s usually straightforward to coax the transpose of a matrix out of a computational device. See: Computation TM.MMA 607 Computation TM.TI86 609.

**Subsection MCC**

**Matrices and Complex Conjugation**

As we did with vectors (Definition CCCV [151]), we can define what it means to take the conjugate of a matrix.

**Definition CCM**

**Complex Conjugate of a Matrix**

Suppose $A$ is an $m \times n$ matrix. Then the conjugate of $A$, written $\overline{A}$ is an $m \times n$ matrix defined by

\[
\left[ \overline{A} \right]_{ij} = \overline{[A]_{ij}}
\]

(This definition contains Notation CCM.) △

**Example CCM**

**Complex conjugate of a matrix**

If

\[
A = \begin{bmatrix}
2 - i & 3 & 5 + 4i \\
-3 + 6i & 2 - 3i & 0
\end{bmatrix}
\]

then

\[
\overline{A} = \begin{bmatrix}
2 + i & 3 & 5 - 4i \\
-3 - 6i & 2 + 3i & 0
\end{bmatrix}
\]

⊗

The interplay between the conjugate of a matrix and the two operations on matrices is what you might expect.

**Theorem CRMA**

**Conjugation Respects Matrix Addition**

Suppose that $A$ and $B$ are $m \times n$ matrices. Then $\overline{A + B} = \overline{A} + \overline{B}$. □

**Proof**

\[
\left[ \overline{A + B} \right]_{ij} = \overline{[A + B]_{ij}}
\]

Definition CCM 168

\[
= [A]_{ij} + [\overline{B}]_{ij}
\]

Definition MA 163

\[
= [\overline{A}]_{ij} + [\overline{B}]_{ij}
\]

Theorem CCRA 613

\[
= [\overline{A}]_{ij} + [B]_{ij}
\]

Definition CCM 168

\[
= [\overline{A + B}]_{ij}
\]

Definition MA 163
Since the matrices $A + B$ and $A + B$ are equal in each entry, \[\text{Definition ME 163}\] says that $A + B = A + B$. □

**Theorem CRMSM**

**Conjugation Respects Matrix Scalar Multiplication**

Suppose that $\alpha \in \mathbb{C}$ and $A$ is an $m \times n$ matrix. Then $\alpha A = \alpha A$.

**Proof**

\[
\begin{align*}
[\alpha A]_{ij} &= \alpha [A]_{ij} \quad \text{Definition CCM 168} \\
&= \alpha [A]_{ij} \quad \text{Definition MSM 164} \\
&= [\alpha A]_{ij} \quad \text{Theorem CCRM 613} \\
&= [\alpha A]_{ij} \quad \text{Definition CCM 168} \\
&= [\alpha A]_{ij} \quad \text{Definition MSM 164}
\end{align*}
\]

Since the matrices $\alpha A$ and $\alpha A$ are equal in each entry, \[\text{Definition ME 163}\] says that $\alpha A = \alpha A$. □

Finally, we will need the following result about matrix conjugation and transposes later.

**Theorem MCT**

**Matrix Conjugation and Transposes**

Suppose that $A$ is an $m \times n$ matrix. Then $(A^t) = \overline{(A)}^t$.

**Proof**

\[
\begin{align*}
{(A^t)}_{ij} &= {A^t}_{ij} \quad \text{Definition CCM 168} \\
&= {A}_{ji} \quad \text{Definition TM 166} \\
&= \overline{(A)}_{ji} \quad \text{Definition CCM 168} \\
&= \overline{{(A^t)}_{ij}} \quad \text{Definition TM 166}
\end{align*}
\]

Since the matrices $(A^t)$ and $\overline{(A)}^t$ are equal in each entry, \[\text{Definition ME 163}\] says that $(A^t) = \overline{(A)}^t$. □

### Subsection READD Reading Questions

1. Perform the following matrix computation.

\[
\begin{pmatrix}
2 & -2 & 8 & 1 \\
4 & 5 & -1 & 3 \\
7 & -3 & 0 & 2
\end{pmatrix}
+ (-2)
\begin{pmatrix}
2 & 7 & 1 & 2 \\
3 & -1 & 0 & 5 \\
1 & 7 & 3 & 3
\end{pmatrix}
\]

2. \[\text{Theorem VSPM 164}\] reminds you of what previous theorem? How strong is the similarity?

3. Compute the transpose of the matrix below.

\[
\begin{pmatrix}
6 & 8 & 4 \\
-2 & 1 & 0 \\
9 & -5 & 6
\end{pmatrix}
\]
In Chapter V we defined the operations of vector addition and vector scalar multiplication in Definition CVA and Definition CVSM. These two operations formed the underpinnings of the remainder of the chapter. We have now defined similar operations for matrices in Definition MA and Definition MSM. You will have noticed the resulting similarities between Theorem VSPCV and Theorem VSPM.

In Exercises M20–M25, you will be asked to extend these similarities to other fundamental definitions and concepts we first saw in Chapter V. This sequence of problems was suggested by Martin Jackson.

**M20** Suppose \( S = \{B_1, B_2, B_3, \ldots, B_p\} \) is a set of matrices from \( M_{mn} \). Formulate appropriate definitions for the following terms and give an example of the use of each.

1. A linear combination of elements of \( S \).
2. A relation of linear dependence on \( S \), both trivial and non-trivial.
3. \( S \) is a linearly independent set.
4. \( \langle S \rangle \).

Contributed by Robert Beezer

**M21** Show that the set \( S \) is linearly independent in \( M_{2,2} \).

\[
S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]

Contributed by Robert Beezer

**M22** Determine if the set

\[
S = \left\{ \begin{bmatrix} -2 & 3 & 4 \\ -1 & 3 & -2 \end{bmatrix}, \begin{bmatrix} 4 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -2 & -2 \\ 2 & 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 2 & -2 \\ 0 & -1 & -2 \end{bmatrix} \right\}
\]

is linearly independent in \( M_{2,3} \).

Contributed by Robert Beezer

**M23** Determine if the matrix \( A \) is in the span of \( S \). In other words, is \( A \in \langle S \rangle \)? If so write \( A \) as a linear combination of the elements of \( S \).

\[
A = \begin{bmatrix} -13 & 24 & 2 \\ -8 & -2 & -20 \end{bmatrix}
\]

\[
S = \left\{ \begin{bmatrix} -2 & 3 & 4 \\ -1 & 3 & -2 \end{bmatrix}, \begin{bmatrix} 4 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -2 & -2 \\ 2 & 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 2 & -2 \\ 0 & -1 & -2 \end{bmatrix} \right\}
\]

Contributed by Robert Beezer

**M24** Suppose \( Y \) is the set of all \( 3 \times 3 \) symmetric matrices (Definition SYM). Find a set \( T \) so that \( T \) is linearly independent and \( \langle T \rangle = Y \).

Contributed by Robert Beezer

**M25** Define a subset of \( M_{3,3} \) by

\[
U_{33} = \left\{ A \in M_{3,3} \mid [A]_{ij} = 0 \text{ whenever } i > j \right\}
\]
Find a set \( R \) so that \( R \) is linearly independent and \( \langle R \rangle = U_{33} \).

Contributed by Robert Beezer

**T13** Prove Property CM 165 of Theorem VSPM 164. Write your proof in the style of the proof of Property DSAM 165 given in this section.

Contributed by Robert Beezer Solution 172

**T17** Prove Property SMAM 165 of Theorem VSPM 164. Write your proof in the style of the proof of Property DSAM 165 given in this section.

Contributed by Robert Beezer

**T18** Prove Property DMAM 165 of Theorem VSPM 164. Write your proof in the style of the proof of Property DSAM 165 given in this section.

Contributed by Robert Beezer
For all $A, B \in M_{mn}$ and for all $1 \leq i \leq m$, $1 \leq j \leq n$, 

$$
[A + B]_{ij} = [A]_{ij} + [B]_{ij} \quad \text{(Definition MA 163)}
$$

$$
= [B]_{ij} + [A]_{ij} \quad \text{(Commutativity in C)}
$$

$$
= [B + A]_{ij} \quad \text{(Definition MA 163)}
$$

With equality of each entry of the matrices $A + B$ and $B + A$ being equal, Definition ME 163 tells us the two matrices are equal.
We know how to add vectors and how to multiply them by scalars. Together, these operations give us the possibility of making linear combinations. Similarly, we know how to add matrices and how to multiply matrices by scalars. In this section we mix all these ideas together and produce an operation known as matrix multiplication. This will lead to some results that are both surprising and central. We begin with a definition of how to multiply a vector by a matrix.

Subsection MVP
Matrix-Vector Product

We have repeatedly seen the importance of forming linear combinations of the columns of a matrix. As one example of this, the oft-used Theorem SLSLC said that every solution to a system of linear equations gives rise to a linear combination of the column vectors of the coefficient matrix that equals the vector of constants. This theorem, and others, motivate the following central definition.

Definition MVP
Matrix-Vector Product
Suppose $A$ is an $m \times n$ matrix with columns $A_1, A_2, A_3, \ldots, A_n$ and $u$ is a vector of size $n$. Then the matrix-vector product of $A$ with $u$ is the linear combination

$$Au = [u]_1 A_1 + [u]_2 A_2 + [u]_3 A_3 + \cdots + [u]_n A_n$$

(This definition contains Notation MVP.)

So, the matrix-vector product is yet another version of “multiplication,” at least in the sense that we have yet again overloaded juxtaposition of two symbols as our notation. Remember your objects, an $m \times n$ matrix times a vector of size $n$ will create a vector of size $m$. So if $A$ is rectangular, then the size of the vector changes. With all the linear combinations we have performed so far, this computation should now seem second nature.

Example MTV
A matrix times a vector
Consider

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 & 4 \\ -3 & 2 & 0 & 1 & -2 \\ 1 & 6 & -3 & -1 & 5 \end{bmatrix}, \quad u = \begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \\ -1 \end{bmatrix}$$

Then

$$Au = 2 \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} + (-2) \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}.$$

This definition now makes it possible to represent systems of linear equations compactly in terms of an operation.

Theorem SLEMM
Systems of Linear Equations as Matrix Multiplication
Solutions to the linear system $LS(A, b)$ are the solutions for $x$ in the vector equation $Ax = b$. □

Proof This theorem says that two sets (of solutions) are equal. So we need to show that one set of solutions is a subset of the other, and vice versa (recall Definition SE). Let
\( A_1, A_2, A_3, \ldots, A_n \) be the columns of \( A \). Both of these set inclusions then follow from the following chain of equivalences,

\[
\begin{align*}
\mathbf{x} \text{ is a solution to } & \mathcal{L}S(A, \mathbf{b}) \\
\iff & |x|_1 A_1 + |x|_2 A_2 + |x|_3 A_3 + \cdots + |x|_n A_n = \mathbf{b} \quad \text{(Theorem SLSLC 82)} \\
\iff & \mathbf{x} \text{ is a solution to } A\mathbf{x} = \mathbf{b} \quad \text{(Definition MVP 173)}
\end{align*}
\]

\[\blacksquare\]

**Example MNSLE**

**Matrix notation for systems of linear equations**

Consider the system of linear equations from [Example NSLE 23]

\[
\begin{align*}
2x_1 + 4x_2 - 3x_3 + 5x_4 + x_5 &= 9 \\
3x_1 + x_2 + x_4 - 3x_5 &= 0 \\
-2x_1 + 7x_2 - 5x_3 + 2x_4 + 2x_5 &= -3
\end{align*}
\]

has coefficient matrix

\[
A = \begin{bmatrix} 2 & 4 & -3 & 5 & 1 \\ 3 & 1 & 0 & 1 & -3 \\ -2 & 7 & -5 & 2 & 2 \end{bmatrix}
\]

and vector of constants

\[
\mathbf{b} = \begin{bmatrix} 9 \\ 0 \\ -3 \end{bmatrix}
\]

and so will be described compactly by the vector equation \( A\mathbf{x} = \mathbf{b} \).

The matrix-vector product is a very natural computation. We have motivated it by its connections with systems of equations, but here is another example.

**Example MBC**

**Money’s best cities**

Every year *Money* magazine selects several cities in the United States as the “best” cities to live in, based on a wide array of statistics about each city. This is an example of how the editors of *Money* might arrive at a single number that consolidates the statistics about a city. We will analyze Los Angeles, Chicago and New York City, based on four criteria: average high temperature in July (Farenheit), number of colleges and universities in a 30-mile radius, number of toxic waste sites in the Superfund clean-up program and a personal crime index based on FBI statistics (average = 100, smaller is safer). It should be apparent how to generalize the example to a greater number of cities and a greater number of statistics.

We begin by building a table of statistics. The rows will be labeled with the cities, and the columns with statistical categories. These values are from *Money’s* website in early 2005.

<table>
<thead>
<tr>
<th>City</th>
<th>Temp</th>
<th>Colleges</th>
<th>Superfund</th>
<th>Crime</th>
</tr>
</thead>
<tbody>
<tr>
<td>Los Angeles</td>
<td>77</td>
<td>28</td>
<td>93</td>
<td>254</td>
</tr>
<tr>
<td>Chicago</td>
<td>84</td>
<td>38</td>
<td>85</td>
<td>363</td>
</tr>
<tr>
<td>New York</td>
<td>84</td>
<td>99</td>
<td>1</td>
<td>193</td>
</tr>
</tbody>
</table>

Conceivably these data might reside in a spreadsheet. Now we must combine the statistics for each city. We could accomplish this by weighting each category, scaling the values and summing them. The sizes of the weights would depend upon the numerical size of each statistic generally, but more importantly, they would reflect the editors opinions or beliefs about which statistics were most important to their readers. Is the crime index more important than the number of colleges and universities? Of course, there is no right answer to this question.
Suppose the editors finally decide on the following weights to employ: temperature, 0.23; colleges, 0.46; Superfund, −0.05; crime, −0.20. Notice how negative weights are used for undesirable statistics. Then, for example, the editors would compute for Los Angeles,

\[
(0.23)(77) + (0.46)(28) + (-0.05)(93) + (-0.20)(254) = -24.86
\]

This computation might remind you of an inner product, but we will produce the computations for all of the cities as a matrix-vector product. Write the table of raw statistics as a matrix

\[
T = \begin{bmatrix}
77 & 28 & 93 & 254 \\
84 & 38 & 85 & 363 \\
84 & 99 & 1 & 193
\end{bmatrix}
\]

and the weights as a vector

\[
w = \begin{bmatrix}
0.23 \\
0.46 \\
-0.05 \\
-0.20
\end{bmatrix}
\]

then the matrix-vector product (Definition MVP [173]) yields

\[
Tw = (0.23) \begin{bmatrix} 77 \\ 84 \\ 84 \end{bmatrix} + (0.46) \begin{bmatrix} 28 \\ 38 \\ 99 \end{bmatrix} + (-0.05) \begin{bmatrix} 93 \\ 85 \\ 1 \end{bmatrix} + (-0.20) \begin{bmatrix} 254 \\ 363 \\ 193 \end{bmatrix} = \begin{bmatrix} -24.86 \\ -40.05 \\ 26.21 \end{bmatrix}
\]

This vector contains a single number for each of the cities being studied, so the editors would rank New York best, Los Angeles next, and Chicago third. Of course, the mayor’s offices in Chicago and Los Angeles are free to counter with a different set of weights that cause their city to be ranked best. These alternative weights would be chosen to play to each cities’ strengths, and minimize their problem areas.

If a spreadsheet were used to make these computations, a row of weights would be entered somewhere near the table of data and the formulas in the spreadsheet would effect a matrix-vector product. This example is meant to illustrate how “linear” computations (addition, multiplication) can be organized as a matrix-vector product.

Another example would be the matrix of numerical scores on examinations and exercises for students in a class. The rows would correspond to students and the columns to exams and assignments. The instructor could then assign weights to the different exams and assignments, and via a matrix-vector product, compute a single score for each student.

Later (much later) we will need the following theorem, which is really a technical lemma (see Technique LC [627]). Since we are in a position to prove it now, we will. But you can safely skip it now, if you promise to come back later to study the proof when the theorem is employed.

**Theorem EMMVP**

**Equal Matrices and Matrix-Vector Products**

Suppose that \(A\) and \(B\) are \(m \times n\) matrices such that \(Ax = Bx\) for every \(x \in \mathbb{C}^n\). Then \(A = B\). □

**Proof** Since \(Ax = Bx\) for all \(x \in \mathbb{C}^n\), choose \(x\) to be a vector of all zeros, with a lone 1 in the \(i\)-th slot. Then

\[
Ax = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}
\]
Subsection MM.MM Matrix Multiplication 176

\[ 0A_1 + 0A_2 + 0A_3 + \cdots + 0A_{i-1} + 1A_i + 0A_{i+1} + \cdots + 0A_n \quad \text{Definition MVP} \ 173 \]

\[ = A_i \]

Similarly, \( Bx = B_i \), so \( A_i = B_i \), \( 1 \leq i \leq n \) and so all the columns of \( A \) and \( B \) are equal. Then our definition of column vector equality (Definition CVE 173) establishes that the individual entries of \( A \) and \( B \) in each column are equal. So by Definition ME 163 the matrices \( A \) and \( B \) are equal.

The hypotheses of this theorem could be weakened to suppose only the equality of the matrix-vector products for just the standard unit vectors (Definition SUV 190) or any other basis (Definition B 294) of \( \mathbb{C}^n \). However, when we apply this theorem we will only need this weaker form.

Subsection MM
Matrix Multiplication

We now define how to multiply two matrices together. Stop for a minute and think about how you might define this new operation.

Many books would present this definition much earlier in the course. However, we have taken great care to delay it as long as possible and to present as many ideas as practical based mostly on the notion of linear combinations. Towards the conclusion of the course, or when you perhaps take a second course in linear algebra, you may be in a position to appreciate the reasons for this. For now, understand that matrix multiplication is a central definition and perhaps you will appreciate its importance more by having saved it for later.

Definition MM
Matrix Multiplication

Suppose \( A \) is an \( m \times n \) matrix and \( B \) is an \( n \times p \) matrix with columns \( B_1, B_2, B_3, \ldots, B_p \). Then the matrix product of \( A \) with \( B \) is the \( m \times p \) matrix where column \( i \) is the matrix-vector product \( A B_i \). Symbolically,

\[
AB = A \begin{bmatrix} B_1 & B_2 & B_3 & \cdots & B_p \end{bmatrix} = \begin{bmatrix} AB_1 & AB_2 & AB_3 & \cdots & AB_p \end{bmatrix}.
\]

\( \triangle \)

Example PTM
Product of two matrices

Set

\[
A = \begin{bmatrix} 1 & 2 & -1 & 4 & 6 \\ 0 & -4 & 1 & 2 & 3 \\ -5 & 1 & 2 & -3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 6 & 2 & 1 \\ -1 & 4 & 3 & 2 \\ 1 & 1 & 2 & 3 \\ 6 & 4 & -1 & 2 \\ 1 & -2 & 3 & 0 \end{bmatrix}
\]

Then

\[
AB = \begin{bmatrix} 1 & 2 & 1 \\ -1 \end{bmatrix} \begin{bmatrix} 6 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ -1 \end{bmatrix} \begin{bmatrix} 6 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ -1 \end{bmatrix} \begin{bmatrix} 6 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ -1 \end{bmatrix} \begin{bmatrix} 6 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 28 & 17 & 20 & 10 \\ 20 & -13 & -3 & -1 \end{bmatrix}
\]

\( \boxtimes \)

Is this the definition of matrix multiplication you expected? Perhaps our previous operations for matrices caused you to think that we might multiply two matrices of the same size, entry-by-entry? Notice that our current definition uses matrices of different sizes (though the number of columns in the first must equal the number of rows in the second), and the result is of a third size. Notice
too in the previous example that we cannot even consider the product $BA$, since the sizes of the two matrices in this order aren’t right.

But it gets weirder than that. Many of your old ideas about “multiplication” won’t apply to matrix multiplication, but some still will. So make no assumptions, and don’t do anything until you have a theorem that says you can. Even if the sizes are right, matrix multiplication is not commutative — order matters.

**Example MMNC**

Matrix multiplication is not commutative

Set

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 \\ 5 & 1 \end{bmatrix}. $$

Then we have two square, $2 \times 2$ matrices, so **Definition MM** [176] allows us to multiply them in either order. We find

$$AB = \begin{bmatrix} 19 & 3 \\ 6 & 2 \end{bmatrix}, \quad BA = \begin{bmatrix} 4 & 12 \\ 4 & 17 \end{bmatrix}$$

and $AB \neq BA$. Not even close. It should not be hard for you to construct other pairs of matrices that do not commute (try a couple of $3 \times 3$’s). Can you find a pair of non-identical matrices that do commute? ☐

Matrix multiplication is fundamental, so it is a natural procedure for any computational device. See: **Computation MM.MMA** [608].

**Subsection MMEE**

Matrix Multiplication, Entry-by-Entry

While certain “natural” properties of multiplication don’t hold, many more do. In the next subsection, we’ll state and prove the relevant theorems. But first, we need a theorem that provides an alternate means of multiplying two matrices. In many texts, this would be given as the definition of matrix multiplication. We prefer to turn it around and have the following formula as a consequence of our definition. It will prove useful for proofs of matrix equality, where we need to examine products of matrices, entry-by-entry.

**Theorem EMP**

**Entries of Matrix Products**

Suppose $A$ is an $m \times n$ matrix and $B = \text{ is an } n \times p$ matrix. Then for $1 \leq i \leq m$, $1 \leq j \leq p$, the individual entries of $AB$ are given by

$$[AB]_{i,j} = [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \cdots + [A]_{in} [B]_{nj}$$

$$= \sum_{k=1}^{n} [A]_{ik} [B]_{kj}.$$ 

**Proof** Denote the columns of $A$ as the vectors $A_1, A_2, A_3, \ldots, A_n$ and the columns of $B$ as the vectors $B_1, B_2, B_3, \ldots, B_p$. Then for $1 \leq i \leq m$, $1 \leq j \leq p$,

$$[AB]_{i,j} = [AB]_{i}$$

$$= [B_1]_{i} A_1 + [B_2]_{i} A_2 + [B_3]_{i} A_3 + \cdots + [B_p]_{i} A_n \quad \text{[Definition MM] [176]}$$

$$= [B_1]_{i} [A_1]_{i} + [B_2]_{i} [A_2]_{i} + [B_3]_{i} [A_3]_{i} + \cdots + [B_p]_{i} [A_n]_{i} \quad \text{[Definition MVP] [173]}$$

$$= [B_1]_{i} [A_1]_{i} + [B_2]_{i} [A_2]_{i} + [B_3]_{i} [A_3]_{i} + \cdots + [B_p]_{i} [A_n]_{i} \quad \text{[Definition CVA] [73]}$$

$$= [B]_{1j} [A]_{i1} + [B]_{2j} [A]_{i2} + [B]_{3j} [A]_{i3} + \cdots + [B]_{nj} [A]_{in} \quad \text{[Definition CVSM] [74]}$$

$$= [B]_{1j} [A]_{i1} + [B]_{2j} [A]_{i2} + [B]_{3j} [A]_{i3} + \cdots + [B]_{nj} [A]_{in} \quad \text{[Notation ME] [163]}$$
\[ [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \cdots + [A]_{in} [B]_{nj} \quad \text{Commutativity in } C \]
\[ = \sum_{k=1}^{n} [A]_{ik} [B]_{kj} \]

**Example PTMEE**

**Product of two matrices, entry-by-entry**

Consider again the two matrices from Example PTM [176]

\[ A = \begin{bmatrix} 1 & 2 & -1 & 4 & 6 \\ 0 & -4 & 1 & 2 & 3 \\ -5 & 1 & 2 & -3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 6 & 2 & 1 \\ -1 & 4 & 3 & 2 \\ 1 & 1 & 2 & 3 \\ 6 & 4 & -1 & 2 \\ 1 & -2 & 3 & 0 \end{bmatrix} \]

Then suppose we just wanted the entry of \( AB \) in the second row, third column:

\[ [AB]_{23} = [A]_{21} [B]_{13} + [A]_{22} [B]_{23} + [A]_{23} [B]_{33} + [A]_{24} [B]_{43} + [A]_{25} [B]_{53} \]
\[ = (0)(2) + (-4)(3) + (1)(2) + (2)(-1) + (3)(3) = -3 \]

Notice how there are 5 terms in the sum, since 5 is the common dimension of the two matrices (column count for \( A \), row count for \( B \)). In the conclusion of **Theorem EMP** [177], it would be the index \( k \) that would run from 1 to 5 in this computation. Here’s a bit more practice.

The entry of third row, first column:

\[ [AB]_{31} = [A]_{31} [B]_{11} + [A]_{32} [B]_{21} + [A]_{33} [B]_{31} + [A]_{34} [B]_{41} + [A]_{35} [B]_{51} \]
\[ = (-5)(1) + (1)(-1) + (2)(1) + (-3)(6) + (4)(1) = -18 \]

To get some more practice on your own, complete the computation of the other 10 entries of this product. Construct some other pairs of matrices (of compatible sizes) and compute their product two ways. First use **Definition MM** [176]. Since linear combinations are straightforward for you now, this should be easy to do and to do correctly. Then do it again, using **Theorem EMP** [177]. Since this process may take some practice, use your first computation to check your work.

**Theorem EMP** [177] is the way most people compute matrix products by hand. It will also be very useful for the theorems we are going to prove shortly. However, the definition (**Definition MM** [176]) is frequently the most useful for its connections with deeper ideas like the null space and the upcoming column space.

---

**Subsection PMM**

**Properties of Matrix Multiplication**

In this subsection, we collect properties of matrix multiplication and its interaction with the zero matrix (**Definition ZM** [166]), the identity matrix (**Definition IM** [62]), matrix addition (**Definition MA** [163]), scalar matrix multiplication (**Definition MSM** [164]), the inner product (**Definition IP** [152]), conjugation (**Theorem MMCC** [181]), and the transpose (**Definition TM** [166]). Whew! Here we go. These are great proofs to practice with, so try to concoct the proofs before reading them, they’ll get progressively more complicated as we go.

**Theorem MMZM**

**Matrix Multiplication and the Zero Matrix**

Suppose \( A \) is an \( m \times n \) matrix. Then
1. \( AO_{n \times p} = O_{m \times p} \)
2. \( O_{p \times m} A = O_{p \times n} \)

**Proof** We’ll prove (1) and leave (2) to you. Entry-by-entry,

\[
[AO_{n \times p}]_{ij} = \sum_{k=1}^{n} [A]_{ik} [O_{n \times p}]_{kj} \quad \text{Theorem EMP 177}
\]

\[
= \sum_{k=1}^{n} [A]_{ik} 0 \quad \text{Definition ZM 166}
\]

\[
= \sum_{k=1}^{n} 0 = 0.
\]

So every entry of the product is the scalar zero, i.e. the result is the zero matrix.  

**Theorem MMIM**

**Matrix Multiplication and Identity Matrix**

Suppose \( A \) is an \( m \times n \) matrix. Then
1. \( AI_{n} = A \)
2. \( I_{m} A = A \)

**Proof** Again, we’ll prove (1) and leave (2) to you. Entry-by-entry,

\[
[AI_{n}]_{ij} = \sum_{k=1}^{n} [A]_{ik} [I_{n}]_{kj} \quad \text{Theorem EMP 177}
\]

\[
= [A]_{ij} [I_{n}]_{jj} + \sum_{k=1, k \neq j}^{n} [A]_{ik} [I_{n}]_{kj} \quad \text{Definition MA 163}
\]

\[
= [A]_{ij} (1) + \sum_{k=1, k \neq j}^{n} [A]_{ik} (0) \quad \text{Definition IM 62}
\]

\[
= [A]_{ij} + \sum_{k=1, k \neq j}^{n} 0 = [A]_{ij}
\]

So the matrices \( A \) and \( AI_{n} \) are equal, entry-by-entry, and by the definition of matrix equality (Definition ME 163) we can say they are equal matrices.

It is this theorem that gives the identity matrix its name. It is a matrix that behaves with matrix multiplication like the scalar 1 does with scalar multiplication. To multiply by the identity matrix is to have no effect on the other matrix.

**Theorem MMDAA**

**Matrix Multiplication Distributes Across Addition**

Suppose \( A \) is an \( m \times n \) matrix and \( B \) and \( C \) are \( n \times p \) matrices and \( D \) is a \( p \times s \) matrix. Then
1. \( A(B + C) = AB + AC \)
2. \( (B + C)D = BD + CD \)

**Proof** We’ll do (1), you do (2). Entry-by-entry,

\[
[A(B + C)]_{ij} = \sum_{k=1}^{n} [A]_{ik} [B + C]_{kj} \quad \text{Theorem EMP 177}
\]

\[
= \sum_{k=1}^{n} [A]_{ik} ([B]_{kj} + [C]_{kj}) \quad \text{Definition MA 163}
\]

\[
= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} + [A]_{ik} [C]_{kj} \quad \text{Distributivity in C}
\]
\[\begin{align*}
&= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} + \sum_{k=1}^{n} [A]_{ik} [C]_{kj} \quad \text{Commutativity in } \mathbb{C} \\
&= [AB]_{ij} + [AC]_{ij} \quad \text{Theorem EMP [177]} \\
&= [AB + AC]_{ij} \quad \text{Definition MA [163]}
\end{align*}\]

So the matrices \(A(B + C)\) and \(AB + AC\) are equal, entry-by-entry, and by the definition of matrix equality (Definition ME [163]) we can say they are equal matrices. \(\blacksquare\)

**Theorem MMSMM**

Matrix Multiplication and Scalar Matrix Multiplication

Suppose \(A\) is an \(m \times n\) matrix and \(B\) is an \(n \times p\) matrix. Let \(\alpha\) be a scalar. Then \(\alpha(AB) = (\alpha A)B = A(\alpha B)\).

**Proof** These are equalities of matrices. We’ll do the first one, the second is similar and will be good practice for you.

\[
[\alpha(AB)]_{ij} = \alpha [AB]_{ij}
\]

\[
= \alpha \sum_{k=1}^{n} [A]_{ik} [B]_{kj} \quad \text{Theorem EMP [177]}
\]

\[
= \sum_{k=1}^{n} \alpha [A]_{ik} [B]_{kj} \quad \text{Distributivity in } \mathbb{C}
\]

\[
= \sum_{k=1}^{n} [\alpha A]_{ik} [B]_{kj} \quad \text{Definition MSM [164]}
\]

\[
= ([\alpha A] B)_{ij} \quad \text{Theorem EMP [177]}
\]

So the matrices \(\alpha(AB)\) and \((\alpha A)B\) are equal, entry-by-entry, and by the definition of matrix equality (Definition ME [163]) we can say they are equal matrices. \(\blacksquare\)

**Theorem MMA**

Matrix Multiplication is Associative

Suppose \(A\) is an \(m \times n\) matrix, \(B\) is an \(n \times p\) matrix and \(D\) is a \(p \times s\) matrix. Then \(A(BD) = (AB)D\).

**Proof** A matrix equality, so we’ll go entry-by-entry, no surprise there.

\[
[A(BD)]_{ij} = \sum_{k=1}^{n} [A]_{ik} [BD]_{kj}
\]

\[
= \sum_{k=1}^{n} [A]_{ik} \left( \sum_{\ell=1}^{p} [B]_{k\ell} [D]_{\ell j} \right) \quad \text{Theorem EMP [177]}
\]

\[
= \sum_{k=1}^{n} \sum_{\ell=1}^{p} [A]_{ik} [B]_{k\ell} [D]_{\ell j} \quad \text{Distributivity in } \mathbb{C}
\]

We can switch the order of the summation since these are finite sums,

\[
= \sum_{\ell=1}^{p} \sum_{k=1}^{n} [A]_{ik} [B]_{k\ell} [D]_{\ell j} \quad \text{Commutativity in } \mathbb{C}
\]

As \([D]_{\ell j}\) does not depend on the index \(k\), we can factor it out of the inner sum,

\[
= \sum_{\ell=1}^{p} [D]_{\ell j} \left( \sum_{k=1}^{n} [A]_{ik} [B]_{k\ell} \right) \quad \text{Distributivity in } \mathbb{C}
\]
\[ \begin{align*}
&\sum_{\ell=1}^{p} [D]_{\ell j} [AB]_{i\ell} \quad \text{Theorem EMP 177} \\
= &\sum_{\ell=1}^{p} [AB]_{i\ell} [D]_{\ell j} \quad \text{Commutativity in } \mathbb{C} \\
= &[(AB)D]_{ij} \quad \text{Theorem EMP 177}
\end{align*} \]

So the matrices \((AB)D\) and \(A(BD)\) are equal, entry-by-entry, and by the definition of matrix equality (Definition ME 163) we can say they are equal matrices. □

**Theorem MMIP**

**Matrix Multiplication and Inner Products**

If we consider the vectors \(u, v \in \mathbb{C}^m\) as \(m \times 1\) matrices then

\[ \langle u, v \rangle = u^t v \]

\[ \square \]

**Proof**

\[ \langle u, v \rangle = \sum_{k=1}^{m} [u]_k [\overline{v}]_k \quad \text{Definition IP 152} \]
\[ = \sum_{k=1}^{m} [u]_{1k} [\overline{v}]_{1k} \quad \text{Column vectors as matrices} \]
\[ = \sum_{k=1}^{m} [u']_{1k} [\overline{v}]_{1k} \quad \text{Definition TM 166} \]
\[ = \sum_{k=1}^{m} [u']_{1k} [\overline{v}]_{k1} \quad \text{Definition CCCV 151} \]
\[ = [u' \overline{v}]_{11} \quad \text{Theorem EMP 177} \]

To finish we just blur the distinction between a \(1 \times 1\) matrix \((u' \overline{v})\) and its lone entry. □

**Theorem MMCC**

**Matrix Multiplication and Complex Conjugation**

Suppose \(A\) is an \(m \times n\) matrix and \(B\) is an \(n \times p\) matrix. Then \(\overline{AB} = \overline{A} \overline{B}\). □

**Proof** To obtain this matrix equality, we will work entry-by-entry,

\[ [AB]_{ij} = [\overline{AB}]_{ij} \quad \text{Definition CM 22} \]
\[ = \sum_{k=1}^{n} [A]_{ik} [B]_{kj} \quad \text{Theorem EMP 177} \]
\[ = \sum_{k=1}^{n} [\overline{A}]_{ik} [\overline{B}]_{kj} \quad \text{Theorem CCRA 613} \]
\[ = \sum_{k=1}^{n} [\overline{A}]_{ik} [\overline{B}]_{kj} \quad \text{Theorem CCRM 613} \]
\[ = \sum_{k=1}^{n} [\overline{A}]_{ik} [\overline{B}]_{kj} \quad \text{Definition CCM 168} \]
\[ = [\overline{A} \overline{B}]_{ij} \quad \text{Theorem EMP 177} \]
So the matrices $\overline{AB}$ and $\overline{BA}$ are equal, entry-by-entry, and by the definition of matrix equality (Definition ME [163]) we can say they are equal matrices.

One more theorem in this style, and its a good one. If you’ve been practicing with the previous proofs you should be able to do this one yourself.

**Theorem MMT**  
Matrix Multiplication and Transposes

Suppose $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix. Then $(AB)^t = B^tA^t$.

**Proof**  
This theorem may be surprising but if we check the sizes of the matrices involved, then maybe it will not seem so far-fetched. First, $AB$ has size $m \times p$, so its transpose has size $p \times m$. The product of $B^t$ with $A^t$ is a $p \times n$ matrix times an $n \times m$ matrix, also resulting in a $p \times m$ matrix. So at least our objects are compatible for equality (and would not be, in general, if we didn’t reverse the order of the operation).

Here we go again, entry-by-entry,

\[
(AB)^t_{ij} = [AB]_{ji} = \sum_{k=1}^{n} [A]_{jk} [B]_{ki} = \sum_{k=1}^{n} [B]_{ki} [A]_{jk} = \sum_{k=1}^{n} [B^t]_{ik} [A^t]_{kj} = [B^tA^t]_{ij}
\]

So the matrices $(AB)^t$ and $B^tA^t$ are equal, entry-by-entry, and by the definition of matrix equality (Definition ME [163]) we can say they are equal matrices.

This theorem seems odd at first glance, since we have to switch the order of $A$ and $B$. But if we simply consider the sizes of the matrices involved, we can see that the switch is necessary for this reason alone. That the individual entries of the products then come along to be equal is a bonus.

Notice how none of these proofs above relied on writing out huge general matrices with lots of ellipses ("...") and trying to formulate the equalities a whole matrix at a time. This messy business is a “proof technique” to be avoided at all costs.

These theorems, along with Theorem VSPM [164], give you the “rules” for how matrices interact with the various operations we have defined. Use them and use them often. But don’t try to do anything with a matrix that you don’t have a rule for. Together, we would informally call all these operations, and the attendant theorems, “the algebra of matrices.” Notice, too, that every column vector is just a $n \times 1$ matrix, so these theorems apply to column vectors also. Finally, these results may make us feel that the definition of matrix multiplication is not so unnatural.

**Subsection READ**  
Reading Questions

1. Form the matrix vector product of

\[
\begin{bmatrix}
2 & 3 & -1 & 0 \\
1 & -2 & 7 & 3 \\
1 & 5 & 3 & 2
\end{bmatrix}
\]

with

\[
\begin{bmatrix}
2 \\
-3 \\
0 \\
5
\end{bmatrix}
\]
2. Multiply together the two matrices below (in the order given).

\[
\begin{bmatrix}
2 & 3 & -1 & 0 \\
1 & -2 & 7 & 3 \\
1 & 5 & 3 & 2 \\
\end{bmatrix}
\begin{bmatrix}
2 & 6 \\
-3 & -4 \\
0 & 2 \\
3 & -1 \\
\end{bmatrix}
\]

3. Rewrite the system of linear equations below as a vector equality and using a matrix-vector product. (This question does not ask for a solution to the system. But it does ask you to express the system of equations in a new form using tools from this section.)

\[
\begin{align*}
2x_1 + 3x_2 - x_3 &= 0 \\
x_1 + 2x_2 + x_3 &= 3 \\
x_1 + 3x_2 + 3x_3 &= 7
\end{align*}
\]
Compute the product of the two matrices below, $AB$. Do this using the definitions of the matrix-vector product (Definition MVP 173) and the definition of matrix multiplication (Definition MM 176).

$$A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 5 & -3 & 4 \\ 2 & 0 & 2 & -3 \end{bmatrix}$$

Contributed by Robert Beezer Solution 186

Suppose that $A$ is a square matrix and there is a vector, $b$, such that $\mathcal{LS}(A, b)$ has a unique solution. Prove that $A$ is nonsingular. Give a direct proof (perhaps appealing to Theorem PSPHS 93) rather than just negating a sentence from the text discussing a similar situation.

Contributed by Robert Beezer Solution 186

Prove the second part of Theorem MMZM 178.

Contributed by Robert Beezer

Prove the second part of Theorem MMIM 179.

Contributed by Robert Beezer

Prove the second part of Theorem MMDAA 179.

Contributed by Robert Beezer

Prove the second part of Theorem MMSMM 180.

Contributed by Robert Beezer Solution 186

Suppose that $A$ is an $m \times n$ matrix and $x, y \in \mathcal{N}(A)$. Prove that $x + y \in \mathcal{N}(A)$.

Contributed by Robert Beezer

Suppose that $A$ is an $m \times n$ matrix, $\alpha \in \mathbb{C}$, and $x \in \mathcal{N}(A)$. Prove that $\alpha x \in \mathcal{N}(A)$.

Contributed by Robert Beezer

Suppose that $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix. Prove that the null space of $B$ is a subset of the null space of $AB$, that is $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$. Provide an example where the opposite is false, in other words give an example where $\mathcal{N}(AB) \nsubseteq \mathcal{N}(B)$.

Contributed by Robert Beezer Solution 186

Suppose that $A$ is an $n \times n$ nonsingular matrix and $B$ is an $n \times p$ matrix. Prove that the null space of $B$ is equal to the null space of $AB$, that is $\mathcal{N}(B) = \mathcal{N}(AB)$. (Compare with Exercise MM.T40 184.)

Contributed by Robert Beezer Solution 187

Suppose $u$ and $v$ are any two solutions of the linear system $\mathcal{LS}(A, b)$. Prove that $u - v$ is an element of the null space of $A$, that is, $u - v \in \mathcal{N}(A)$.

Contributed by Robert Beezer

Give a new proof of Theorem PSPHS 93 replacing applications of Theorem SLSLC 82 with matrix-vector products (Theorem SLEMM 173).

Contributed by Robert Beezer Solution 187

Suppose that $x, y \in \mathbb{C}^n$, $b \in \mathbb{C}^m$ and $A$ is an $m \times n$ matrix. If $x, y$ and $x + y$ are each a solution to the linear system $\mathcal{LS}(A, b)$, what interesting can you say about $b'$? Form an implication with the existence of the three solutions as the hypothesis and an interesting stateme
about \( \mathcal{L}(A, b) \) as the conclusion, and then give a proof.
Contributed by Robert Beezer Solution
Subsection MM.SOL Solutions

C20 Contributed by Robert Beezer  Statement 184
By Definition MM 176.

\[ AB = \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} \]

Repeated applications of Definition MVP 173 give

\[ AB = \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} + 2 \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} \]

T10 Contributed by Robert Beezer  Statement 184
Since \( LS(A, b) \) has at least one solution, we can apply Theorem PSPHS 93. Because the solution is assumed to be unique, the null space of \( A \) must be trivial. Then Theorem NMTNS 64 implies that \( A \) is nonsingular.

The converse of this statement is a trivial application of Theorem NMUS 64. That said, we could extend our NSMxx series of theorems with an added equivalence for nonsingularity, “Given a single vector of constants, \( b \), the system \( LS(A, b) \) has a unique solution.”

T23 Contributed by Robert Beezer  Statement 184
We’ll run the proof entry-by-entry.

\[ \alpha(AB)_{ij} = \alpha[AB]_{ij} \]

Definition MSM 164

\[ = \alpha \sum_{k=1}^{n} [A]_{ik}[B]_{kj} \]

Theorem EMP 177

\[ = \sum_{k=1}^{n} \alpha[A]_{ik}[B]_{kj} \]

Distributivity in \( \mathbb{C} \)

\[ = \sum_{k=1}^{n} [A]_{ik} \alpha[B]_{kj} \]

Commutativity in \( \mathbb{C} \)

\[ = \sum_{k=1}^{n} [A]_{ik} \alpha[B]_{kj} \]

Definition MSM 164

\[ = [A(\alpha B)]_{ij} \]

Theorem EMP 177

So the matrices \( \alpha(AB) \) and \( A(\alpha B) \) are equal, entry-by-entry, and by the definition of matrix equality Definition ME 163, we can say they are equal matrices.

T40 Contributed by Robert Beezer  Statement 184
To prove that one set is a subset of another, we start with an element of the smaller set and see if we can determine that it is a member of the larger set (Definition SSET 615). Suppose \( x \in N(B) \). Then we know that \( Bx = 0 \) by Definition NSM 64. Consider

\[ (AB)x = A(Bx) \]

Theorem MMA 180

Hypothesis
This establishes that $x \in \mathcal{N}(AB)$, so $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$.

To show that the inclusion does not hold in the opposite direction, choose $B$ to be any non-singular matrix of size $n$. Then $\mathcal{N}(B) = \{0\}$ by Theorem NMTNS \[64\]. Let $A$ be the square zero matrix, $O$, of the same size. Then $AB = OB = O$ by Theorem MMZM \[178\] and therefore $\mathcal{N}(AB) = \mathbb{C}^n$, and is not a subset of $\mathcal{N}(B) = \{0\}$.

\textbf{T41} Contributed by Robert Beezer Statement \[184\] 
From the solution to Exercise MM.T40 \[184\] we know that $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$. So to establish the set equality (Definition SE \[616\]) we need to show that $\mathcal{N}(AB) \subseteq \mathcal{N}(B)$.

Suppose $x \in \mathcal{N}(AB)$. Then we know that $ABx = 0$ by Definition NSM \[54\]. Consider
\begin{align*}
Bx &= I_n Bx \\
&= (A^{-1}A) Bx \quad \text{Theorem MMIM \[179\]} \\
&= A^{-1} (AB) x \quad \text{Theorem NI \[204\]} \\
&= 0 \quad \text{Theorem MMZM \[178\]}
\end{align*}
This establishes that $x \in \mathcal{N}(B)$, so $\mathcal{N}(AB) \subseteq \mathcal{N}(B)$ and combined with the solution to Exercise MM.T40 \[184\] we have $\mathcal{N}(B) = \mathcal{N}(AB)$ when $A$ is nonsingular.

\textbf{T51} Contributed by Robert Beezer Statement \[184\] 
We will work with the vector equality representations of the relevant systems of equations, as described by Theorem SLEMM \[173\].

$(\Leftarrow)$ Suppose $y = w + z$ and $z \in \mathcal{N}(A)$. Then
\begin{align*}
Ay &= A(w + z) \\
&= Aw + Az \\
&= b + 0 \quad z \in \mathcal{N}(A) \\
&= b \quad \text{Property ZC \[75\]}
\end{align*}
demonstrating that $y$ is a solution.

$(\Rightarrow)$ Suppose $y$ is a solution to $\mathcal{L}S(A, b)$. Then
\begin{align*}
A(y - w) &= Ay - Aw \\
&= b - b \quad y, w \text{ solutions to } Ax = b \\
&= 0 \quad \text{Property AIC \[75\]}
\end{align*}
which says that $y - w \in \mathcal{N}(A)$. In other words, $y - w = z$ for some vector $z \in \mathcal{N}(A)$. Rewritten, this is $y = w + z$, as desired.

\textbf{T52} Contributed by Robert Beezer Statement \[184\] 
$\mathcal{L}S(A, b)$ must be homogeneous. To see this consider that
\begin{align*}
b &= Ax \\
&= Ax + 0 \quad \text{Property ZC \[75\]} \\
&= Ax + Ay - Ay \quad \text{Property AIC \[75\]} \\
&= A(x + y) - Ay \quad \text{Theorem MMDAA \[179\]} \\
&= b - b \quad \text{Theorem SLEMM \[173\]} \\
&= 0 \quad \text{Property AIC \[75\]}
\end{align*}
By Definition HS \[52\] we see that $\mathcal{L}S(A, b)$ is homogeneous.
We begin with a familiar example, performed in a novel way.

Example SABMI
Solutions to Archetype B with a matrix inverse

Archetype B [638] is the system of \( m = 3 \) linear equations in \( n = 3 \) variables,

\[
\begin{align*}
-7x_1 - 6x_2 - 12x_3 &= -33 \\
5x_1 + 5x_2 + 7x_3 &= 24 \\
x_1 + 4x_3 &= 5
\end{align*}
\]

By Theorem SLEMM [173] we can represent this system of equations as

\[
Ax = b
\]

where

\[
A = \begin{bmatrix}
-7 & -6 & -12 \\
5 & 5 & 7 \\
1 & 0 & 4
\end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}
\]

We’ll pull a rabbit out of our hat and present the \( 3 \times 3 \) matrix \( B \),

\[
B = \begin{bmatrix}
-10 & -12 & -9 \\
\frac{13}{2} & 8 & \frac{11}{2} \\
\frac{3}{2} & \frac{1}{2} & \frac{5}{2}
\end{bmatrix}
\]

and note that

\[
BA = \begin{bmatrix}
\frac{-10}{2} & -12 & -9 \\
\frac{13}{2} & 8 & \frac{11}{2} \\
\frac{3}{2} & \frac{1}{2} & \frac{5}{2}
\end{bmatrix} \begin{bmatrix}
-7 & -6 & -12 \\
5 & 5 & 7 \\
1 & 0 & 4
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Now apply this computation to the problem of solving the system of equations,

\[
x = I_3 x \quad \text{Theorem MMIM [179]} \\
= (BA)x \quad \text{Substitution} \\
= B(Ax) \quad \text{Theorem MMA [180]} \\
= Bb \quad \text{Substitution}
\]

So we have

\[
x = Bb = \begin{bmatrix}
-10 & -12 & -9 \\
\frac{13}{2} & 8 & \frac{11}{2} \\
\frac{3}{2} & \frac{1}{2} & \frac{5}{2}
\end{bmatrix} \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}
\]

So with the help and assistance of \( B \) we have been able to determine a solution to the system represented by \( Ax = b \) through judicious use of matrix multiplication. We know by Theorem NMUS [64] that since the coefficient matrix in this example is nonsingular, there would be a unique solution, no matter what the choice of \( b \). The derivation above amplifies this result, since we were forced to conclude that \( x = Bb \) and the solution couldn’t be anything else. You should notice that this argument would hold for any particular value of \( b \).

The matrix \( B \) of the previous example is called the inverse of \( A \). When \( A \) and \( B \) are combined via matrix multiplication, the result is the identity matrix, which can be inserted “in front” of \( x \)
as the first step in finding the solution. This is entirely analogous to how we might solve a single linear equation like $3x = 12$.

$$x = 1x = \left(\frac{1}{3}(3)\right) x = \frac{1}{3}(3x) = \frac{1}{3}(12) = 4$$

Here we have obtained a solution by employing the “multiplicative inverse” of 3, $3^{-1} = \frac{1}{3}$. This works fine for any scalar multiple of $x$, except for zero, since zero does not have a multiplicative inverse. For matrices, it is more complicated. Some matrices have inverses, some do not. And when a matrix does have an inverse, just how would we compute it? In other words, just where did that matrix $B$ in the last example come from? Are there other matrices that might have worked just as well?

Subsection IM
Inverse of a Matrix

Definition MI
Matrix Inverse
Suppose $A$ and $B$ are square matrices of size $n$ such that $AB = I_n$ and $BA = I_n$. Then $A$ is invertible and $B$ is the inverse of $A$. In this situation, we write $B = A^{-1}$.

(This definition contains Notation MI.)

Notice that if $B$ is the inverse of $A$, then we can just as easily say $A$ is the inverse of $B$, or $A$ and $B$ are inverses of each other.

Not every square matrix has an inverse. In Example SABMI [188] the matrix $B$ is the inverse the coefficient matrix of Archetype B [638]. To see this it only remains to check that $AB = I_3$. What about Archetype A [634]? It is an example of a square matrix without an inverse.

Example MWIAA
A matrix without an inverse, Archetype A
Consider the coefficient matrix from Archetype A [634],

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Suppose that $A$ is invertible and does have an inverse, say $B$. Choose the vector of constants

$$b = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

and consider the system of equations $LS(A, b)$. Just as in Example SABMI [188], this vector equation would have the unique solution $x = Bb$.

However, the system $LS(A, b)$ is inconsistent. Form the augmented matrix $[A | b]$ and row-reduce to

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which allows to recognize the inconsistency by Theorem RCLS [45].

So the assumption of $A$’s inverse leads to a logical inconsistency (the system can’t be both consistent and inconsistent), so our assumption is false. $A$ is not invertible.

It’s possible this example is less than satisfying. Just where did that particular choice of the vector $b$ come from anyway? Stay tuned for an application of the future Theorem CSCS [212] in Example CSAA [215].

Let’s look at one more matrix inverse before we embark on a more systematic study.
Example MI
Matrix inverse
Consider the matrix,

\[
A = \begin{bmatrix}
1 & 2 & 1 & 2 & 1 \\
-2 & -3 & 0 & -5 & -1 \\
1 & 1 & 0 & 2 & 1 \\
-2 & -3 & -1 & -3 & -2 \\
-1 & -3 & -1 & -3 & 1
\end{bmatrix}
\]

And the matrix

\[
B = \begin{bmatrix}
-3 & 3 & 6 & -1 & -2 \\
0 & -2 & -5 & -1 & 1 \\
1 & 2 & 4 & 1 & -1 \\
1 & 0 & 1 & 1 & 0 \\
1 & -1 & -2 & 0 & 1
\end{bmatrix}
\]

Then

\[
AB = \begin{bmatrix}
1 & 2 & 1 & 2 & 1 \\
-2 & -3 & 0 & -5 & -1 \\
1 & 1 & 0 & 2 & 1 \\
-2 & -3 & -1 & -3 & -2 \\
-1 & -3 & -1 & -3 & 1
\end{bmatrix}\begin{bmatrix}
-3 & 3 & 6 & -1 & -2 \\
0 & -2 & -5 & -1 & 1 \\
1 & 2 & 4 & 1 & -1 \\
1 & 0 & 1 & 1 & 0 \\
1 & -1 & -2 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

and

\[
BA = \begin{bmatrix}
-3 & 3 & 6 & -1 & -2 \\
0 & -2 & -5 & -1 & 1 \\
1 & 2 & 4 & 1 & -1 \\
1 & 0 & 1 & 1 & 0 \\
1 & -1 & -2 & 0 & 1
\end{bmatrix}\begin{bmatrix}
1 & 2 & 1 & 2 & 1 \\
-2 & -3 & 0 & -5 & -1 \\
1 & 1 & 0 & 2 & 1 \\
-2 & -3 & -1 & -3 & -2 \\
-1 & -3 & -1 & -3 & 1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

so by Definition MI[189], we can say that A is invertible and write \( B = A^{-1} \).

We will now concern ourselves less with whether or not an inverse of a matrix exists, but instead with how you can find one when it does exist. In Section MINM[202] we will have some theorems that allow us to more quickly and easily determine when a matrix is invertible.

Subsection CIM
Computing the Inverse of a Matrix

We will have occasion in this subsection (and later) to reference the following frequently used vectors, so we will make a useful definition now.

**Definition SUV**
Standard Unit Vectors

Let \( e_j \in \mathbb{C}^m \) denote the column vector that is column \( j \) of the \( m \times m \) identity matrix \( I_m \). Then the set

\[
\{ e_1, e_2, e_3, \ldots, e_m \} = \{ e_j \mid 1 \leq j \leq m \}
\]

is the set of standard unit vectors in \( \mathbb{C}^m \).

We will make reference to these vectors often. Notice that \( e_j \) is a column vector full of zeros, with a lone 1 in the \( j \)-th position, so an alternate definition is

\[
[e_j]_i = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j 
\end{cases}
\]
We’ve seen that the matrices from Archetype B and Archetype K both have inverses, but these inverse matrices have just dropped from the sky. How would we compute an inverse? And just when is a matrix invertible, and when is it not? Writing a putative inverse with \( n^2 \) unknowns and solving the resultant \( n^2 \) equations is one approach. Applying this approach to \( 2 \times 2 \) matrices can get us somewhere, so just for fun, let’s do it.

**Theorem TTMI**

**Two-by-Two Matrix Inverse**

Suppose

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

Then \( A \) is invertible if and only if \( ad - bc \neq 0 \). When \( A \) is invertible, we have

\[
A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]

**Proof**  

\((\Leftarrow)\) If \( ad - bc \neq 0 \) then the displayed formula is legitimate (we are not dividing by zero), and it is a simple matter to actually check that \( A^{-1}A = AA^{-1} = I_2 \).

\((\Rightarrow)\) Assume that \( A \) is invertible, and proceed with a proof by contradiction (Technique CD), by assuming also that \( ad - bc = 0 \). This means that \( ad = bc \). Let

\[
B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}
\]

be a putative inverse of \( A \). This means that

\[
I_2 = AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}
\]

Working on the matrices on both ends of this equation, we will multiply the top row by \( c \) and the bottom row by \( a \).

\[
\begin{bmatrix} c & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} ac + beg & acf + bch \\ ace + adg & acf + adh \end{bmatrix}
\]

We are assuming that \( ad = bc \), so we can replace two occurrences of \( ad \) by \( bc \) in the bottom row of the right matrix.

\[
\begin{bmatrix} c & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} ac + beg & acf + bch \\ ace + beg & acf + bch \end{bmatrix}
\]

The matrix on the right now has two rows that are identical, and therefore the same must be true of the matrix on the left. Given the form of the matrix on the left, identical rows implies that \( a = 0 \) and \( c = 0 \).

With this information, the product \( AB \) becomes

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 = AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} = \begin{bmatrix} bg & bh \\ dg & dh \end{bmatrix}
\]

So \( bg = dh = 1 \) and thus \( b, g, d, h \) are all nonzero. But then \( bh \) and \( dg \) (the “other corners”) must also be nonzero, so this is (finally) a contradiction. So our assumption was false and we see that \( ad - bc \neq 0 \) whenever \( A \) has an inverse.

There are several ways one could try to prove this theorem, but there is a continual temptation to divide by one of the eight entries involved (\( a \) through \( f \)), but we can never be sure if these numbers are zero or not. This could lead to an analysis by cases, which is messy, messy, messy. Note how the above proof never divides, but always multiplies, and how zero/nonzero considerations are handled. Pay attention to the expression \( ad - bc \), we will see it again in a while.

This theorem is cute, and it is nice to have a formula for the inverse, and a condition that tells us when we can use it. However, this approach becomes impractical for larger matrices, even
though it is possible to demonstrate that, in theory, there is a general formula. (Think for a minute
about extending this result to just $3 \times 3$ matrices. For starters, we need 18 letters!) Instead, we
will work column-by-column. Let’s first work an example that will motivate the main theorem and
remove some of the previous mystery.

**Example CMI**
Computing a matrix inverse
Consider the matrix defined in Example MI [190] as,

$$A = \begin{bmatrix}
1 & 2 & 1 & 2 & 1 \\
-2 & -3 & 0 & -5 & -1 \\
1 & 1 & 0 & 2 & 1 \\
-2 & -3 & -1 & -3 & -2 \\
-1 & -3 & -1 & -3 & 1
\end{bmatrix}$$

For its inverse, we desire a matrix $B$ so that $AB = I_5$. Emphasizing the structure of the columns
and employing the definition of matrix multiplication Definition MM [176],

$$AB = I_5$$

$$A|B_1|B_2|B_3|B_4|B_5 = [e_1|e_2|e_3|e_4|e_5]$$

$$[AB_1|AB_2|AB_3|AB_4|AB_5] = [e_1|e_2|e_3|e_4|e_5].$$

Equating the matrices column-by-column we have

$$AB_1 = e_1 \quad AB_2 = e_2 \quad AB_3 = e_3 \quad AB_4 = e_4 \quad AB_5 = e_5.$$

Since the matrix $B$ is what we are trying to compute, we can view each column, $B_i$, as a column
vector of unknowns. Then we have five systems of equations to solve, each with 5 equations in 5
variables. Notice that all 5 of these systems have the same coefficient matrix. We’ll now solve each
system in turn,

Row-reduce the augmented matrix of the linear system $LS(A, e_1),$

$$\begin{bmatrix}
1 & 2 & 1 & 2 & 1 & 1 \\
-2 & -3 & 0 & -5 & -1 & 0 \\
1 & 1 & 0 & 2 & 1 & 0 \\
-2 & -3 & -1 & -3 & -2 & 0 \\
-1 & -3 & -1 & -3 & 1 & 0
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -3 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}$$

so $B_1 = \begin{bmatrix}
-3 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}.$

Row-reduce the augmented matrix of the linear system $LS(A, e_2),$

$$\begin{bmatrix}
1 & 2 & 1 & 2 & 1 & 0 \\
-2 & -3 & 0 & -5 & -1 & 1 \\
1 & 1 & 0 & 2 & 1 & 0 \\
-2 & -3 & -1 & -3 & -2 & 0 \\
-1 & -3 & -1 & -3 & 1 & 0
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 3 \\
0 & 1 & 0 & 0 & 0 & -2 \\
0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}$$

so $B_2 = \begin{bmatrix}
3 \\
-2 \\
2 \\
0 \\
-1
\end{bmatrix}.$

Row-reduce the augmented matrix of the linear system $LS(A, e_3),$

$$\begin{bmatrix}
1 & 2 & 1 & 2 & 1 & 0 \\
-2 & -3 & 0 & -5 & -1 & 0 \\
1 & 1 & 0 & 2 & 1 & 1 \\
-2 & -3 & -1 & -3 & -2 & 0 \\
-1 & -3 & -1 & -3 & 1 & 0
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 6 \\
0 & 1 & 0 & 0 & 0 & -5 \\
0 & 0 & 1 & 0 & 0 & 4 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -2
\end{bmatrix}$$

so $B_3 = \begin{bmatrix}
6 \\
-5 \\
4 \\
1 \\
-2
\end{bmatrix}.$
Row-reduce the augmented matrix of the linear system $LS(A, \mathbf{e}_4)$,

\[
\begin{bmatrix}
1 & 2 & 1 & 2 & 1 & 0 \\
-2 & -3 & 0 & -5 & -1 & 0 \\
1 & 1 & 0 & 2 & 1 & 0 \\
-2 & -3 & -1 & -3 & -2 & 1 \\
-1 & -3 & -1 & -3 & 1 & 0
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

So $B_4 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$.

Row-reduce the augmented matrix of the linear system $LS(A, \mathbf{e}_5)$,

\[
\begin{bmatrix}
1 & 2 & 1 & 2 & 1 & 0 \\
-2 & -3 & 0 & -5 & -1 & 0 \\
1 & 1 & 0 & 2 & 1 & 0 \\
-2 & -3 & -1 & -3 & -2 & 0 \\
-1 & -3 & -1 & -3 & 1 & 1
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -2 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

So $B_5 = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$.

We can now collect our 5 solution vectors into the matrix $B$,

$$B = \begin{bmatrix} B_1 | B_2 | B_3 | B_4 | B_5 \end{bmatrix} = \begin{bmatrix}
-3 & 3 & 6 & -1 & -2 \\
0 & -2 & -5 & -1 & 1 \\
1 & 2 & 4 & 1 & -1 \\
1 & 0 & 1 & 1 & 0 \\
1 & -1 & -2 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
-3 & 3 & 6 & -1 & -2 \\
0 & -2 & -5 & -1 & 1 \\
1 & 2 & 4 & 1 & -1 \\
1 & 0 & 1 & 1 & 0 \\
1 & -1 & -2 & 0 & 1
\end{bmatrix}$$

By this method, we know that $AB = I_5$. Check that $BA = I_5$, and then we will know that we have the inverse of $A$.

Notice how the five systems of equations in the preceding example were all solved by exactly the same sequence of row operations. Wouldn’t it be nice to avoid this obvious duplication of effort? Our main theorem for this section follows, and it mimics this previous example, while also avoiding all the overhead.

**Theorem CINM**

**Computing the Inverse of a Nonsingular Matrix**

Suppose $A$ is a nonsingular square matrix of size $n$. Create the $n \times 2n$ matrix $M$ by placing the $n \times n$ identity matrix $I_n$ to the right of the matrix $A$. Let $N$ be a matrix that is row-equivalent to $M$ and in reduced row-echelon form. Finally, let $J$ be the matrix formed from the final $n$ columns of $N$. Then $AJ = I_n$.

**Proof** $A$ is nonsingular, so by **Theorem NMRRI** there is a sequence of row operations that will convert $A$ into $I_n$. It is this same sequence of row operations that will convert $M$ into $N$, since having the identity matrix in the first $n$ columns of $N$ is sufficient to guarantee that $N$ is in reduced row-echelon form.

If we consider the systems of linear equations, $LS(A, \mathbf{e}_i)$, $1 \leq i \leq n$, we see that the aforementioned sequence of row operations will also bring the augmented matrix of each of these systems into reduced row-echelon form. Furthermore, the unique solution to $LS(A, \mathbf{e}_i)$ appears in column $n+i$ of the row-reduced augmented matrix of the system and is identical to column $n+i$ of $N$. Let $N_1, N_2, N_3, \ldots, N_{2n}$ denote the columns of $N$. So we find,

$$AJ = A|N_{n+1}|N_{n+2}|N_{n+3}|\ldots|N_{n+n}$$
We have to be just a bit careful here about both what this theorem says and what it doesn’t say. If $A$ is a nonsingular matrix, then we are guaranteed a matrix $B$ such that $AB = I_n$, and the proof gives us a process for constructing $B$. However, the definition of the inverse of a matrix (Definition MI [189]) requires that $BA = I_n$ also. So at this juncture we must compute the matrix product in the “opposite” order before we claim $B$ as the inverse of $A$. However, we’ll soon see that this is always the case, in Theorem OSIS [203], so the title of this theorem is not inaccurate.

What if $A$ is singular? At this point we only know that Theorem CINM [193] cannot be applied. The question of $A$’s inverse is still open. (But see Theorem NI [204] in the next section.) We’ll finish by computing the inverse for the coefficient matrix of Archetype B [638], the one we just pulled from a hat in Example SABMI [188]. There are more examples in the Archetypes (Appendix A [630]) to practice with, though notice that it is silly to ask for the inverse of a rectangular matrix (the sizes aren’t right) and not every square matrix has an inverse (remember Example MWIAA [189]?).

Example CMIAB
Computing a matrix inverse, Archetype B

Archetype B [638] has a coefficient matrix given as

$$
B = \begin{bmatrix}
-7 & -6 & -12 \\
5 & 5 & 7 \\
1 & 0 & 4
\end{bmatrix}
$$

Exercising Theorem CINM [193] we set

$$
M = \begin{bmatrix}
-7 & -6 & -12 & 1 & 0 & 0 \\
5 & 5 & 7 & 0 & 1 & 0 \\
1 & 0 & 4 & 0 & 0 & 1
\end{bmatrix}
$$

which row reduces to

$$
N = \begin{bmatrix}
1 & 0 & 0 & -10 & -12 & -9 \\
0 & 1 & 0 & \frac{13}{2} & 8 & \frac{11}{2} \\
0 & 0 & 1 & \frac{5}{2} & 3 & \frac{5}{2}
\end{bmatrix}
$$

So

$$
B^{-1} = \begin{bmatrix}
-10 & -12 & -9 \\
\frac{13}{2} & 8 & \frac{11}{2} \\
\frac{5}{2} & 3 & \frac{5}{2}
\end{bmatrix}
$$

once we check that $B^{-1}B = I_3$ (the product in the opposite order is a consequence of the theorem).

While we can use a row-reducing procedure to compute any needed inverse, most computational devices have a built-in procedure to compute the inverse of a matrix straightaway. See: Computation MI.MMA [608].

Subsection PMI
Properties of Matrix Inverses

The inverse of a matrix enjoys some nice properties. We collect a few here. First, a matrix can have but one inverse.
**Theorem MIU**

**Matrix Inverse is Unique**

Suppose the square matrix $A$ has an inverse. Then $A^{-1}$ is unique. □

**Proof**  As described in Technique U [624], we will assume that $A$ has two inverses. The hypothesis tells there is at least one. Suppose then that $B$ and $C$ are both inverses for $A$. Then, repeated use of Definition MI [189] and Theorem MMIM [179] plus one application of Theorem MMA [180] gives

$$
B = BI_n = B(AC) = (BA)C = I_nC = C
$$

So we conclude that $B$ and $C$ are the same, and cannot be different. So any matrix that acts like an inverse, must be the inverse. ■

When most of us dress in the morning, we put on our socks first, followed by our shoes. In the evening we must then first remove our shoes, followed by our socks. Try to connect the conclusion of the following theorem with this everyday example.

**Theorem SS**

**Socks and Shoes**

Suppose $A$ and $B$ are invertible matrices of size $n$. Then $(AB)^{-1} = B^{-1}A^{-1}$ and $AB$ is an invertible matrix. □

**Proof**  At the risk of carrying our everyday analogies too far, the proof of this theorem is quite easy when we compare it to the workings of a dating service. We have a statement about the inverse of the matrix $AB$, which for all we know right now might not even exist. Suppose $AB$ was to sign up for a dating service with two requirements for a compatible date. Upon multiplication on the left, and on the right, the result should be the identity matrix. In other words, $AB$’s ideal date would be its inverse.

Now along comes the matrix $B^{-1}A^{-1}$ (which we know exists because our hypothesis says both $A$ and $B$ are invertible and we can form the product of these two matrices), also looking for a date. Let’s see if $B^{-1}A^{-1}$ is a good match for $AB$. First they meet at a non-committal neutral location, say a coffee shop, for quiet conversation:

$$
(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n
$$

The first date having gone smoothly, a second, more serious, date is arranged, say dinner and a show:

$$
(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n
$$

So the matrix $B^{-1}A^{-1}$ has met all of the requirements to be $AB$’s inverse (date) and with the ensuing marriage proposal we can announce that $(AB)^{-1} = B^{-1}A^{-1}$. ■

**Theorem MIMI**

Version 1.04
Matrix Inverse of a Matrix Inverse
Suppose $A$ is an invertible matrix. Then $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$.

**Proof** As with the proof of [Theorem SS 195](#), we examine if $A$ is a suitable inverse for $A^{-1}$ (by definition, the opposite is true).

$$AA^{-1} = I_n$$  [Definition MI 189]

and

$$A^{-1}A = I_n$$  [Definition MI 189]

The matrix $A$ has met all the requirements to be the inverse of $A^{-1}$, and so is invertible and we can write $A = (A^{-1})^{-1}$.

**Theorem MIT**
Matrix Inverse of a Transpose
Suppose $A$ is an invertible matrix. Then $A^t$ is invertible and $(A^t)^{-1} = (A^{-1})^t$.

**Proof** As with the proof of [Theorem SS 195](#), we see if $(A^{-1})^t$ is a suitable inverse for $A^t$. Apply [Theorem MMT 182](#) to see that

$$(A^{-1})^t A^t = (AA^{-1})^t = I_n$$  [Theorem MMT 182]

$$I_n$$ is symmetric

and

$$A^t (A^{-1})^t = (A^{-1}A)^t = I_n$$  [Theorem MMT 182]

$I_n$ is symmetric

The matrix $(A^{-1})^t$ has met all the requirements to be the inverse of $A^t$, and so is invertible and we can write $(A^t)^{-1} = (A^{-1})^t$.

**Theorem MISM**
Matrix Inverse of a Scalar Multiple
Suppose $A$ is an invertible matrix and $\alpha$ is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$ and $\alpha A$ is invertible.

**Proof** As with the proof of [Theorem SS 195](#), we see if $\frac{1}{\alpha} A^{-1}$ is a suitable inverse for $\alpha A$.

$$\left( \frac{1}{\alpha} A^{-1} \right) (\alpha A) = \left( \frac{1}{\alpha} \right) (AA^{-1})$$  [Theorem MMSMM 180]

Scalar multiplicative inverses

$$= 1 I_n$$  [Property OM 165]

$$= I_n$$

and

$$\left( \alpha A \right) \left( \frac{1}{\alpha} A^{-1} \right) = \left( \frac{1}{\alpha} \alpha \right) (A^{-1}A)$$  [Theorem MMSMM 180]

Scalar multiplicative inverses

$$= 1 I_n$$  [Property OM 165]

$$= I_n$$

The matrix $\frac{1}{\alpha} A^{-1}$ has met all the requirements to be the inverse of $\alpha A$, so we can write $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$.

Notice that there are some likely theorems that are missing here. For example, it would be tempting to think that $(A + B)^{-1} = A^{-1} + B^{-1}$, but this is false. Can you find a counterexample (see Exercise MISLE.T10 [199])?
1. Compute the inverse of the matrix below.

\[
\begin{bmatrix}
4 & 10 \\
2 & 6
\end{bmatrix}
\]

2. Compute the inverse of the matrix below.

\[
\begin{bmatrix}
2 & 3 & 1 \\
1 & -2 & -3 \\
-2 & 4 & 6
\end{bmatrix}
\]

3. Explain why Theorem SS has the title it does. (Do not just state the theorem, explain the choice of the title making reference to the theorem itself.)
Subsection EXC
Exercises

C21 Verify that $B$ is the inverse of $A$.

\[
A = \begin{bmatrix}
1 & 1 & -1 & 2 \\
-2 & -1 & 2 & -3 \\
1 & 1 & 0 & 2 \\
-1 & 2 & 0 & 2
\end{bmatrix}
\quad B = \begin{bmatrix}
4 & 2 & 0 & -1 \\
8 & 4 & -1 & -1 \\
-1 & 0 & 1 & 0 \\
-6 & -3 & 1 & 1
\end{bmatrix}
\]

Contributed by Robert Beezer Solution 200

C22 Recycle the matrices $A$ and $B$ from Exercise MISLE.C21 and set

\[
c = \begin{bmatrix}
2 \\
1 \\
-3 \\
2
\end{bmatrix}
\quad d = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

Employ the matrix $B$ to solve the two linear systems $LS(A, c)$ and $LS(A, d)$.

Contributed by Robert Beezer Solution 200

C23 If it exists, find the inverse of the $2 \times 2$ matrix

\[
A = \begin{bmatrix}
7 & 3 \\
5 & 2
\end{bmatrix}
\]

and check your answer. (See Theorem TTMI.)

Contributed by Robert Beezer

C24 If it exists, find the inverse of the $2 \times 2$ matrix

\[
A = \begin{bmatrix}
6 & 3 \\
4 & 2
\end{bmatrix}
\]

and check your answer. (See Theorem TTMI.)

Contributed by Robert Beezer

C25 At the conclusion of Example CMI, verify that $BA = I_5$ by computing the matrix product.

Contributed by Robert Beezer

C26 Let

\[
D = \begin{bmatrix}
1 & -1 & 3 & -2 & 1 \\
-2 & 3 & -5 & 3 & 0 \\
1 & -1 & 4 & -2 & 2 \\
-1 & 4 & -1 & 0 & 4 \\
1 & 0 & 5 & -2 & 5
\end{bmatrix}
\]

Compute the inverse of $D$, $D^{-1}$, by forming the $5 \times 10$ matrix $[D | I_5]$ and row-reducing (Theorem CINM). Then use a calculator to compute $D^{-1}$ directly.

Contributed by Robert Beezer Solution 200

C27 Let

\[
E = \begin{bmatrix}
1 & -1 & 3 & -2 & 1 \\
-2 & 3 & -5 & 3 & -1 \\
1 & -1 & 4 & -2 & 2 \\
-1 & 4 & -1 & 0 & 2 \\
1 & 0 & 5 & -2 & 4
\end{bmatrix}
\]
Compute the inverse of \( E \), \( E^{-1} \), by forming the \( 5 \times 10 \) matrix \( [E \mid I_5] \) and row-reducing (Theorem CINM [193]). Then use a calculator to compute \( E^{-1} \) directly.

Contributed by Robert Beezer  
Solution [200]

C28  Let

\[
C = \begin{bmatrix}
1 & 1 & 3 & 1 \\
-2 & -1 & -4 & -1 \\
1 & 4 & 10 & 2 \\
-2 & 0 & -4 & 5 \\
\end{bmatrix}
\]

Compute the inverse of \( C \), \( C^{-1} \), by forming the \( 4 \times 8 \) matrix \( [C \mid I_4] \) and row-reducing (Theorem CINM [193]). Then use a calculator to compute \( C^{-1} \) directly.

Contributed by Robert Beezer  
Solution [200]

C40  Find all solutions to the system of equations below, making use of the matrix inverse found in Exercise MISLE.C28 [199].

\[
\begin{align*}
x_1 + x_2 + 3x_3 + x_4 &= -4 \\
-2x_1 - x_2 - 4x_3 - x_4 &= 4 \\
x_1 + 4x_2 + 10x_3 + 2x_4 &= -20 \\
-2x_1 - 4x_3 + 5x_4 &= 9
\end{align*}
\]

Contributed by Robert Beezer  
Solution [200]

C41  Use the inverse of a matrix to find all the solutions to the following system of equations.

\[
\begin{align*}
x_1 + 2x_2 - x_3 &= -3 \\
2x_1 + 5x_2 - x_3 &= -4 \\
-x_1 - 4x_2 &= 2
\end{align*}
\]

Contributed by Robert Beezer  
Solution [201]

C42  Use a matrix inverse to solve the linear system of equations.

\[
\begin{align*}
x_1 - x_2 + 2x_3 &= 5 \\
x_1 - 2x_3 &= -8 \\
2x_1 - x_2 - x_3 &= -6
\end{align*}
\]

Contributed by Robert Beezer  
Solution [201]

T10  Construct an example to demonstrate that \( (A + B)^{-1} = A^{-1} + B^{-1} \) is not true for all square matrices \( A \) and \( B \) of the same size.

Contributed by Robert Beezer  
Solution [201]
Subsection SOL

Solutions

C21  Contributed by Robert Beezer  Statement 198
Check that both matrix products (Definition MM 176) \(AB\) and \(BA\) equal the \(4 \times 4\) identity matrix \(I_4\) (Definition IM 62).

C22  Contributed by Robert Beezer  Statement 198
Represent each of the two systems by a vector equality, \(Ax = c\) and \(Ay = d\). Then in the spirit of Example SABMI 188, solutions are given by

\[
x = Bc = \begin{bmatrix} 8 \\ 21 \\ -5 \\ -16 \end{bmatrix} \quad y = Bd = \begin{bmatrix} 5 \\ 10 \\ 0 \\ -7 \end{bmatrix}
\]

Notice how we could solve many more systems having \(A\) as the coefficient matrix, and how each such system has a unique solution. You might check your work by substituting the solutions back into the systems of equations, or forming the linear combinations of the columns of \(A\) suggested by Theorem SLSLC 82.

C26  Contributed by Robert Beezer  Statement 198
The inverse of \(D\) is

\[
D^{-1} = \begin{bmatrix} -7 & -6 & -3 & 2 & 1 \\ -7 & -4 & 2 & 2 & -1 \\ -5 & -2 & 3 & 1 & -1 \\ -6 & -3 & 1 & 1 & 0 \\ 4 & 2 & -2 & -1 & 1 \end{bmatrix}
\]

C27  Contributed by Robert Beezer  Statement 198
The matrix \(E\) has no inverse, though we do not yet have a theorem that allows us to reach this conclusion. However, when row-reducing the matrix \([E \mid I_5]\), the first 5 columns will not row-reduce to the \(5 \times 5\) identity matrix, so we are at a loss on how we might compute the inverse. When requesting that your calculator compute \(E^{-1}\), it should give some indication that \(E\) does not have an inverse.

C28  Contributed by Robert Beezer  Statement 199
Employ Theorem CINM 193,

\[
\begin{bmatrix} 1 & 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ -2 & -1 & -4 & -1 & 0 & 1 & 0 & 0 \\ 1 & 4 & 10 & 2 & 0 & 0 & 1 & 0 \\ -2 & 0 & -4 & 5 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 & 38 & 18 & -5 & -2 \\ 0 & 1 & 0 & 0 & 96 & 47 & -12 & -5 \\ 0 & 0 & 1 & 0 & -39 & -19 & 5 & 2 \\ 0 & 0 & 0 & 1 & -16 & -8 & 2 & 1 \end{bmatrix}
\]

And therefore we see that \(C\) is nonsingular (\(C\) row-reduces to the identity matrix, Theorem NMRR1 62) and by Theorem CINM 193,

\[
C^{-1} = \begin{bmatrix} 38 & 18 & -5 & -2 \\ 96 & 47 & -12 & -5 \\ -39 & -19 & 5 & 2 \\ -16 & -8 & 2 & 1 \end{bmatrix}
\]

C40  Contributed by Robert Beezer  Statement 199
View this system as \(\mathcal{L}\mathcal{S}(C, b)\), where \(C\) is the \(4 \times 4\) matrix from Exercise MISLE.C28 199 and
\[ \mathbf{b} = \begin{bmatrix} -4 \\ 4 \\ -20 \\ 9 \end{bmatrix} \]. Since \( C \) was seen to be nonsingular in Exercise MISLE.C28 \[199\] Theorem SNCM \[204\] says the solution, which is unique by Theorem NMUS \[64\], is given by

\[
C^{-1}\mathbf{b} = \begin{bmatrix} 38 & 18 & -5 & -2 \\ 96 & 47 & -12 & -5 \\ -39 & -19 & 5 & 2 \\ -16 & -8 & 2 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 4 \\ -20 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \\ 1 \end{bmatrix}
\]

Notice that this solution can be easily checked in the original system of equations.

C41 Contributed by Robert Beezer Statement \[199\] The coefficient matrix of this system of equations is

\[
A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ -1 & -4 & 0 \end{bmatrix}
\]

and the vector of constants is \( \mathbf{b} = \begin{bmatrix} -3 \\ -4 \\ 2 \end{bmatrix} \). So by Theorem SLEMM \[173\] we can convert the system to the form \( Ax = \mathbf{b} \). Row-reducing this matrix yields the identity matrix so by Theorem NMRRI \[62\] we know \( A \) is nonsingular. This allows us to apply Theorem SNCM \[204\] to find the unique solution as

\[
\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -4 & 4 & 3 \\ 1 & -1 & -1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}
\]

Remember, you can check this solution easily by evaluating the matrix-vector product \( Ax \) (Definition MVP \[173\]).

C42 Contributed by Robert Beezer Statement \[199\] We can reformulate the linear system as a vector equality with a matrix-vector product via Theorem SLEMM \[173\]. The system is then represented by \( Ax = \mathbf{b} \) where

\[
A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 5 \\ -8 \\ -6 \end{bmatrix}
\]

According to Theorem SNCM \[204\], if \( A \) is nonsingular then the (unique) solution will be given by \( A^{-1}\mathbf{b} \). We attempt the computation of \( A^{-1} \) through Theorem CINM \[193\], or with our favorite computational device and obtain,

\[
A^{-1} = \begin{bmatrix} 2 & 3 & -2 \\ 3 & 5 & -4 \\ 1 & 1 & -1 \end{bmatrix}
\]

So by Theorem NI \[204\], we know \( A \) is nonsingular, and so the unique solution is

\[
A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 & -2 \\ 3 & 5 & -4 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -8 \\ -6 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}
\]

T10 Contributed by Robert Beezer Statement \[199\] Let \( D \) be any \( 2 \times 2 \) matrix that has an inverse (Theorem TTMI \[191\] can help you construct such a matrix, \( I_2 \) is a simple choice). Set \( A = D \) and \( B = (-1)D \). While \( A^{-1} \) and \( B^{-1} \) both exist, what is \( (A + B)^{-1} \)? Can the proposed statement be a theorem?
Section MINM
Matrix Inverses and Nonsingular Matrices

We saw in Theorem CINM that if a square matrix $A$ is nonsingular, then there is a matrix $B$ so that $AB = I_n$. In other words, $B$ is halfway to being an inverse of $A$. We will see in this section that $B$ automatically fulfills the second condition $(BA = I_n)$. Example MWIAA showed us that the coefficient matrix from Archetype A had no inverse. Not coincidentally, this coefficient matrix is singular. We’ll make all these connections precise now. Not many examples or definitions in this section, just theorems.

Subsection NMI
Nonsingular Matrices are Invertible

We need a couple of technical results for starters. Some books would call these minor, but essential, results “lemmas.” We’ll just call ’em theorems. See Technique LC for more on the distinction.

The first of these technical results is interesting in that the hypothesis says something about a product of two square matrices and the conclusion then says the same thing about each individual matrix in the product.

Theorem NPNT
Nonsingular Product has Nonsingular Terms
Suppose that $A$ and $B$ are square matrices of size $n$ and the product $AB$ is nonsingular. Then $A$ and $B$ are both nonsingular.

Proof We’ll do the proof in two parts, each as a proof by contradiction. Establishing that $B$ is nonsingular is the easier part, so we will do it first, but in reality, we will need to know that $B$ is nonsingular when we prove that $A$ is nonsingular.

You can also think of this proof as being a study of four possible conclusions in the table below. One of the four rows must happen (the list is exhaustive). In the proof we learn that the first three rows lead to contradictions, and so are impossible. That leaves the fourth row as a certainty, which is our desired conclusion.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>Singular</td>
<td>Singular</td>
<td>1</td>
</tr>
<tr>
<td>Nonsingular</td>
<td>Singular</td>
<td>1</td>
</tr>
<tr>
<td>Singular</td>
<td>Nonsingular</td>
<td>2</td>
</tr>
<tr>
<td>Nonsingular</td>
<td>Nonsingular</td>
<td></td>
</tr>
</tbody>
</table>

Case 1. Suppose $B$ is singular. Then there is a nonzero vector $z$ that is a solution to $LS(B, 0)$. So

$$(AB)z = A(Bz) = A0 = 0$$

Because $z$ is a nonzero solution to $LS(AB, 0)$, we conclude that $AB$ is singular. This is a contradiction, so $B$ is nonsingular, as desired.

Case 2. Suppose $A$ is singular. Then there is a nonzero vector $y$ that is a solution to $LS(A, 0)$. Now use this vector $y$ and consider the linear system $LS(B, y)$. Since we know $B$ is nonsingular (from Case 1), the system has a unique solution, which we will call $w$. We claim $w$ is not the zero vector either. Assuming the opposite, suppose that $w = 0$. Then

$$y = Bw$$
contrary to \( y \) being nonzero. So \( w \neq 0 \). The pieces are in place, so here we go,

\[
(AB)w = A(Bw) = Ay = 0
\]

So \( w \) is a nonzero solution to \( LS(AB, 0) \), and thus we can say that \( AB \) is singular

This is a powerful result, because it allows us to begin with a hypothesis that something complicated (the matrix product \( AB \)) has the property of being nonsingular, and we can then conclude that the simpler constituents (\( A \) and \( B \) individually) then also have the property of being nonsingular. If we had thought that the matrix product was an artificial construction, results like this would make us begin to think twice.

The contrapositive of this result is equally interesting. It says that if either \( A \) or \( B \) (or both) is a singular matrix, then the product \( AB \) is also singular. Notice how the negation of the theorem’s conclusion (\( A \) and \( B \) both nonsingular) becomes the statement “at least one of \( A \) and \( B \) is singular.”

(See Technique CP [623].)

Theorem OSIS

One-Sided Inverse is Sufficient

Suppose \( A \) and \( B \) are square matrices of size \( n \) such that \( AB = I_n \). Then \( BA = I_n \).

Proof The matrix \( I_n \) is nonsingular (since it row-reduces easily to \( I_n \), Theorem NMRRI [62]). So \( A \) and \( B \) are nonsingular by Theorem NPNT [202], so in particular \( B \) is nonsingular. We can therefore apply Theorem CINM [193] to assert the existence of a matrix \( C \) so that \( BC = I_n \). This application of Theorem CINM [193] could be a bit confusing, mostly because of the names of the matrices involved. \( B \) is nonsingular, so there must be a “right-inverse” for \( B \), and we’re calling it \( C \).

Now

\[
BA = (BA)I_n = (BA)(BC) = B(AB)C = BI_nC = BC = I_n
\]

which is the desired conclusion.

So Theorem OSIS [203] tells us that if \( A \) is nonsingular, then the matrix \( B \) guaranteed by Theorem CINM [193] will be both a “right-inverse” and a “left-inverse” for \( A \), so \( A \) is invertible and \( A^{-1} = B \).

So if you have a nonsingular matrix, \( A \), you can use the procedure described in Theorem CINM [193] to find an inverse for \( A \). If \( A \) is singular, then the procedure in Theorem CINM [193] will fail as the first \( n \) columns of \( M \) will not row-reduce to the identity matrix. However, we can say a bit more. When \( A \) is singular, then \( A \) does not have an inverse (which is very different from saying that the procedure in Theorem CINM [193] fails to find an inverse). This may feel like we are splitting hairs, but its important that we do not make unfounded assumptions. These observations motivate the next theorem.
Theorem NI
Nonsingularity is Invertibility
Suppose that $A$ is a square matrix. Then $A$ is nonsingular if and only if $A$ is invertible.

Proof  ($\iff$) Suppose $A$ is invertible, and suppose that $x$ is any solution to the homogeneous system $LS(A, \mathbf{0})$. Then
\[
x = I_n x \\
  = (A^{-1}A)x \\
  = A^{-1}(Ax) \\
  = A^{-1}\mathbf{0} \\
  = \mathbf{0}
\]
So the only solution to $LS(A, \mathbf{0})$ is the zero vector, so by Definition NM, $A$ is nonsingular.

($\Rightarrow$) Suppose now that $A$ is nonsingular. By Theorem CINM, we find $B$ so that $AB = I_n$. Then Theorem OSIS tells us that $BA = I_n$. So $B$ is $A$’s inverse, and by construction, $A$ is invertible.

So for a square matrix, the properties of having an inverse and of having a trivial null space are one and the same. Can’t have one without the other.

Theorem NME3
Nonsingular Matrix Equivalences, Round 3
Suppose that $A$ is a square matrix of size $n$. The following are equivalent.

1. $A$ is nonsingular.
2. $A$ row-reduces to the identity matrix.
3. The null space of $A$ contains only the zero vector, $N(A) = \{\mathbf{0}\}$.
4. The linear system $LS(A, b)$ has a unique solution for every possible choice of $b$.
5. The columns of $A$ are a linearly independent set.
6. $A$ is invertible.

Proof  We can update our list of equivalences for nonsingular matrices with the equivalent condition from Theorem NI.

In the case that $A$ is a nonsingular coefficient matrix of a system of equations, the inverse allows us to very quickly compute the unique solution, for any vector of constants.

Theorem SNCM
Solution with Nonsingular Coefficient Matrix
Suppose that $A$ is nonsingular. Then the unique solution to $LS(A, b)$ is $A^{-1}b$.

Proof  By Theorem NMUS we know already that $LS(A, b)$ has a unique solution for every choice of $b$. We need to show that the expression stated is indeed a solution (the solution). That’s easy, just “plug it in” to the corresponding vector equation representation,
\[
A (A^{-1}b) = (AA^{-1}) b \\
  = I_n b \\
  = b
\]
Since $Ax = b$ is true when we substitute $A^{-1}b$ for $x$, $A^{-1}b$ is a (the!) solution to $LS(A, b)$.
Subsection UM
Unitary Matrices

Definition UM
Unitary Matrices
Suppose that $Q$ is a square matrix of size $n$ such that $(Q)^t Q = I_n$. Then we say $Q$ is unitary. △

This condition may seem rather far-fetched at first glance. Would there be any matrix that behaved this way? Well, yes, here’s one.

Example UM3
Unitary matrix of size $3$

$$Q = \begin{bmatrix} \frac{1+i}{\sqrt{5}} & \frac{3+2i}{\sqrt{5}} & \frac{2+2i}{\sqrt{22}} \\ \frac{1-i}{\sqrt{5}} & \frac{2+2i}{\sqrt{5}} & -\frac{\sqrt{2}}{\sqrt{22}} \\ \frac{\sqrt{5}}{\sqrt{5}} & \frac{3+2i}{\sqrt{55}} & -\frac{2}{\sqrt{22}} \end{bmatrix}$$

The computations get a bit tiresome, but if you work your way through $(Q)^t Q$, you will arrive at the $3 \times 3$ identity matrix $I_3$. ⊠

Unitary matrices do not have to look quite so gruesome. Here’s a larger one that is a bit more pleasing.

Example UPM
Unitary permutation matrix

The matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

is unitary as can be easily checked. Notice that it is just a rearrangement of the columns of the $5 \times 5$ identity matrix, $I_5$ (Definition IM [62]).

An interesting exercise is to build another $5 \times 5$ unitary matrix, $R$, using a different rearrangement of the columns of $I_5$. Then form the product $PR$. This will be another unitary matrix (Exercise MINM.T10 [209]). If you were to build all $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ matrices of this type you would have a set that remains closed under matrix multiplication. It is an example of another algebraic structure known as a group since together the set and the one operation (matrix multiplication here) is closed, associative, has an identity ($I_5$), and inverses (Theorem UMI [205]). Notice though that the operation in this group is not commutative! ⊠

If a matrix $A$ has only real number entries (we say it is a real matrix) then the defining property of being unitary simplifies to $A^t A = I_n$. In this case we, and everybody else, calls the matrix orthogonal, so you may often encounter this term in your other reading.

Unitary matrices have easily computed inverses. They also have columns that form orthonormal sets. Here are the theorems that show us that unitary matrices are not as strange as they might initially appear.

Theorem UMI
Unitary Matrices are Invertible
Suppose that $Q$ is a unitary matrix of size $n$. Then $Q$ is nonsingular, and $Q^{-1} = (Q)^t$. □

Proof By Definition UM [205], we know that $(Q)^t Q = I_n$. The matrix $I_n$ is nonsingular (since it row-reduces easily to $I_n$, Theorem NMRRI [62]). So by Theorem NPNT [202], $Q$ and $(Q)^t$ are both nonsingular matrices.
The equation \((Q)^t Q = I_n\) gets us halfway to an inverse of \(Q\), and Theorem OSIS \(\text{[203]}\) tells us that \(Q(Q)^t = I_n\) also. So \(Q\) and \((Q)^t\) are inverses of each other (Definition MI \(\text{[189]}\)). \(\blacksquare\)

**Theorem CUMOS**

**Columns of Unitary Matrices are Orthonormal Sets**

Suppose that \(A\) is a square matrix of size \(n\) with columns \(S = \{A_1, A_2, A_3, \ldots, A_n\}\). Then \(A\) is a unitary matrix if and only if \(S\) is an orthonormal set. \(\square\)

**Proof**

The proof revolves around recognizing that a typical entry of the product \((A)^t A\) is an inner product of columns of \(A\). Here are the details to support this claim.

\[
\begin{align*}
\left[(\overline{A})^{t} A\right]_{ij} &= \sum_{k=1}^{n} \left[(\overline{A})^t\right]_{ik} [A]_{kj} \quad \text{Theorem EMP \[177\]} \\
&= \sum_{k=1}^{n} [\overline{A}]_{ki} [A]_{kj} \quad \text{Definition TM \[166\]} \\
&= \sum_{k=1}^{n} [A]_{kj} [\overline{A}]_{ki} \quad \text{Definition CCM \[168\]} \\
&= \sum_{k=1}^{n} [A]_{jk} [\overline{A}]_{ki} \quad \text{Commutativity in } \mathbb{C} \\
&= \sum_{k=1}^{n} [A]_{jk} [\overline{A}]_{ki} \quad \text{Notation} \\
&= \langle A_j, A_i \rangle \quad \text{Definition IP \[152\]}
\end{align*}
\]

We now employ this equality in a chain of equivalences,

\[
\begin{align*}
S = \{A_1, A_2, A_3, \ldots, A_n\} &\text{ is an orthonormal set} \\
\iff \langle A_j, A_i \rangle &= \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j
\end{cases} \quad \text{Definition ONS \[160\]} \\
\iff \left[(\overline{A})^{t} A\right]_{ij} &= \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j
\end{cases} \quad \text{Substitution of above} \\
\iff \left[(\overline{A})^{t} A\right]_{ij} &= [I_n]_{ij}, \ 1 \leq i \leq n, \ 1 \leq j \leq n \quad \text{Definition IM \[62\]} \\
\iff (\overline{A})^{t} A &= I_n \quad \text{Definition ME \[163\]} \\
\iff A &\text{ is a unitary matrix} \quad \text{Definition UM \[205\]}
\end{align*}
\]

\(\blacksquare\)

**Example OSMC**

**Orthonormal set from matrix columns**

The matrix

\[
Q = \begin{bmatrix}
\frac{1+i}{\sqrt{2}} & \frac{3+2i}{\sqrt{5}} & \frac{2+2i}{\sqrt{22}} \\
\frac{1-i}{\sqrt{2}} & \frac{3-2i}{\sqrt{5}} & \frac{-2+2i}{\sqrt{22}} \\
\frac{1+2i}{\sqrt{5}} & \frac{2+2i}{\sqrt{5}} & \frac{-2+2i}{\sqrt{22}}
\end{bmatrix}
\]

from Example UM3 \(\text{[205]}\) is a unitary matrix. By Theorem CUMOS \(\text{[206]}\), its columns

\[
\left\{ \begin{bmatrix} \frac{1+i}{\sqrt{2}} \\ \frac{1-i}{\sqrt{2}} \\ \frac{1+2i}{\sqrt{5}} \end{bmatrix}, \quad \begin{bmatrix} \frac{3+2i}{\sqrt{5}} \\ \frac{3-2i}{\sqrt{5}} \\ \frac{2+2i}{\sqrt{5}} \end{bmatrix}, \quad \begin{bmatrix} \frac{2+2i}{\sqrt{22}} \\ \frac{-2+2i}{\sqrt{22}} \\ \frac{-2+2i}{\sqrt{22}} \end{bmatrix} \right\}
\]

form an orthonormal set. You might find checking the six inner products of pairs of these vectors easier than doing the matrix product \((\overline{Q})^{t} Q\). Or, because the inner product is anti-commutative
(Theorem IPAC [154]) you only need check three inner products (see Exercise MINM.T12 [209]).

When using vectors and matrices that only have real number entries, orthogonal matrices are those matrices with inverses that equal their transpose. Similarly, the inner product is the familiar dot product. Keep this special case in mind as you read the next theorem.

**Theorem UMPIP**

**Unitary Matrices Preserve Inner Products**

Suppose that \( Q \) is a unitary matrix of size \( n \) and \( u \) and \( v \) are two vectors from \( \mathbb{C}^n \). Then

\[
\langle Qu, Qv \rangle = \langle u, v \rangle \quad \text{and} \quad \|Qv\| = \|v\|
\]

\[
\square
\]

**Proof**

\[
\langle Qu, Qv \rangle = (Qu)^t Qv \quad \text{Theorem MMIP [181]}
\]

\[
= u^t Q^t Qv \quad \text{Theorem MMT [182]}
\]

\[
= u^t (Q)^t Qv \quad \text{Theorem MMCC [181]}
\]

\[
= u^t (Q)^t Qv \quad \text{Definition CCM [168], Theorem CCT [613]}
\]

\[
= u^t (Q)^t Qv \quad \text{Theorem MCT [169]}
\]

\[
= u^t (Q)^t Qv \quad \text{Theorem MMCC [181]}
\]

\[
= u^t I_n v \quad \text{Definition UM [205]}
\]

\[
= u^t I_n v \quad I_n \text{ has real entries}
\]

\[
= u^t v \quad \text{Theorem MMIM [179]}
\]

\[
= \langle u, v \rangle \quad \text{Theorem MMIP [181]}
\]

The second conclusion is just a specialization of the first conclusion.

\[
\|Qv\| = \sqrt{\|Qv\|^2}
\]

\[
= \sqrt{\langle Qv, Qv \rangle} \quad \text{Theorem IPN [155]}
\]

\[
= \sqrt{\langle v, v \rangle} \quad \text{Previous conclusion}
\]

\[
= \sqrt{\|v\|^2} \quad \text{Theorem IPN [155]}
\]

\[
= \|v\|
\]

\[
\square
\]

We will often have occasion (as above) to conjugate the entries of a matrix, and transpose the result. So, as a convenience, we give this construction a name and some notation. You will see the adjoint written elsewhere variously as \( A^H \), \( A^* \) or \( A^\dagger \).

**Definition A**

**Adjoint**

If \( A \) is a square matrix, then its adjoint is \( A^* = (\overline{A})^t \).

(This definition contains Notation A.)

\[
\triangle
\]

Sometimes a matrix is equal to its adjoint, and these matrices have interesting properties. One of the most common situations where this occurs is when a matrix has only real number entries. Then we are simply talking about symmetric matrices (Definition SYM [166]).

**Definition HM**

**Hermitian Matrix**

The square matrix \( A \) is **Hermitian** (or self-adjoint) if \( A = (\overline{A})^t = A^* \).

\[
\triangle
\]

Again, the set of real matrices that are Hermitian is exactly the set of symmetric matrices. In Section PEE [378] we will uncover some amazing properties of Hermitian matrices, so when you...
get there, run back here to remind yourself of this definition. Further properties will also appear in various sections of the Topics Part T (718).

A final reminder: the terms “dot product,” “orthogonal matrix” and “symmetric matrix” used in reference to vectors or matrices with real number entries correspond to the terms inner product, unitary matrix and Hermitian matrix when we generalize to include complex number entries.

Subsection READ
Reading Questions

1. Show how to use the inverse of a matrix to solve the system of equations below and state the resulting solution.

\[4x_1 + 10x_2 = 12\]
\[2x_1 + 6x_2 = 4\]

2. In the reading questions for Section MISLE 188 you were asked to find the inverse of the 3 × 3 matrix below.

\[
\begin{bmatrix}
2 & 3 & 1 \\
1 & -2 & -3 \\
-2 & 4 & 6
\end{bmatrix}
\]

Because the matrix was not nonsingular, you had no theorems at that point that would allow you to compute the inverse. Explain why you now know that the inverse does not exist (which is different than not being able to compute it) by quoting the relevant theorem’s acronym.

3. Is the matrix \(A\) orthogonal? Why?

\[
A = \begin{bmatrix}
\frac{1}{\sqrt{2}} (4 + 2i) & \frac{1}{\sqrt{374}} (5 + 3i) \\
\frac{1}{\sqrt{2}} (-1 - i) & \frac{1}{\sqrt{374}} (12 + 14i)
\end{bmatrix}
\]
Subsection EXC
Exercises

C40  Solve the system of equations below using the inverse of a matrix.

\[
\begin{align*}
    x_1 + x_2 + 3x_3 + x_4 &= 5 \\
    -2x_1 - x_2 - 4x_3 - x_4 &= -7 \\
    x_1 + 4x_2 + 10x_3 + 2x_4 &= 9 \\
    -2x_1 - 4x_3 + 5x_4 &= 9
\end{align*}
\]

Contributed by Robert Beezer  Solution 210

M20  Construct an example of a \(4 \times 4\) unitary matrix.
Contributed by Robert Beezer  Solution 210

T10  Suppose that \(Q\) and \(P\) are unitary matrices of size \(n\). Prove that \(QP\) is a unitary matrix.
Contributed by Robert Beezer

T11  Prove that Hermitian matrices (Definition HM [208]) have real entries on the diagonal. More precisely, suppose that \(A\) is a Hermitian matrix of size \(n\). Then \(A_{ii} \in \mathbb{R}, 1 \leq i \leq n\).
Contributed by Robert Beezer

T12  Suppose that we are checking if a square matrix of size \(n\) is unitary. Show that a straightforward application of Theorem CUMOS [206] requires the computation of \(n^2\) inner products when the matrix is unitary, and fewer when the matrix is not orthogonal. Then show that this maximum number of inner products can be reduced to \(\frac{1}{2}n(n + 1)\) in light of Theorem IPAC [154].
Contributed by Robert Beezer
The coefficient matrix and vector of constants for the system are

\[
\begin{bmatrix}
1 & 1 & 3 & 1 \\
-2 & -1 & -4 & -1 \\
1 & 4 & 10 & 2 \\
-2 & 0 & -4 & 5
\end{bmatrix}
\quad
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{bmatrix}
\]

\[b = \begin{bmatrix} 5 \\ -7 \\ 9 \\ 9 \end{bmatrix}\]

\(A^{-1}\) can be computed by using a calculator, or by the method of Theorem CINM. Then Theorem SNCM says the unique solution is

\[A^{-1}b = \begin{bmatrix}
38 & 18 & -5 & -2 \\
-96 & 47 & -12 & -5 \\
-39 & -19 & 5 & 2 \\
-16 & -8 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
5 \\
-7 \\
9 \\
9
\end{bmatrix}
= \begin{bmatrix}
1 \\
-2 \\
1 \\
3
\end{bmatrix}\]

The 4 \times 4 identity matrix, \(I_4\), would be one example. Any of the 23 other rearrangements of the columns of \(I_4\) would be a simple, but less trivial, example. See Example UPM.
Section CRS
Column and Row Spaces

Theorem SLSLC \[82\] showed us that there is a natural correspondence between solutions to linear systems and linear combinations of the columns of the coefficient matrix. This idea motivates the following important definition.

Definition CSM
Column Space of a Matrix
Suppose that \(A\) is an \(m \times n\) matrix with columns \(\{A_1, A_2, A_3, \ldots, A_n\}\). Then the column space of \(A\), written \(C(A)\), is the subset of \(\mathbb{C}^m\) containing all linear combinations of the columns of \(A\),
\[C(A) = \langle\{A_1, A_2, A_3, \ldots, A_n\}\rangle\]
(This definition contains Notation CSM.)

Some authors refer to the column space of a matrix as the range, but we will reserve this term for use with linear transformations (Definition RLT \[444\]).

Subsection CSSE
Column Spaces and Systems of Equations

Upon encountering any new set, the first question we ask is what objects are in the set, and which objects are not? Here’s an example of one way to answer this question, and it will motivate a theorem that will then answer the question precisely.

Example CSMCS
Column space of a matrix and consistent systems
Archetype D \[647\] and Archetype E \[651\] are linear systems of equations, with an identical \(3 \times 4\) coefficient matrix, which we call \(A\) here. However, Archetype D \[647\] is consistent, while Archetype E \[651\] is not. We can explain this difference by employing the column space of the matrix \(A\).

The column vector of constants, \(b\), in Archetype D \[647\] is
\[b = \begin{bmatrix} 8 \\ -12 \\ 4 \end{bmatrix}\]

One solution to \(\text{LS}(A, b)\), as listed, is
\[x = \begin{bmatrix} 7 \\ 8 \\ 1 \\ 3 \end{bmatrix}\]

By Theorem SLSLC \[82\], we can summarize this solution as a linear combination of the columns of \(A\) that equals \(b\),
\[7 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + 8 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} -7 \\ -6 \\ -5 \end{bmatrix} = \begin{bmatrix} 8 \\ -12 \\ 4 \end{bmatrix} = b.\]

This equation says that \(b\) is a linear combination of the columns of \(A\), and then by Definition CSM \[211\], we can say that \(b \in C(A)\).
On the other hand, Archetype E \[651\] is the linear system \(LS(A, c)\), where the vector of constants is

\[
c = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}
\]

and this system of equations is inconsistent. This means \(c \not\in \mathcal{C}(A)\), for if it were, then it would equal a linear combination of the columns of \(A\) and Theorem SLSLC \[82\] would lead us to a solution of the system \(LS(A, c)\).

So if we fix the coefficient matrix, and vary the vector of constants, we can sometimes find consistent systems, and sometimes inconsistent systems. The vectors of constants that lead to consistent systems are exactly the elements of the column space. This is the content of the next theorem, and since it is an equivalence, it provides an alternate view of the column space.

**Theorem CSCS**

**Column Spaces and Consistent Systems**

Suppose \(A\) is an \(m \times n\) matrix and \(b\) is a vector of size \(m\). Then \(b \in \mathcal{C}(A)\) if and only if \(LS(A, b)\) is consistent.

**Proof** \((\Rightarrow)\) Suppose \(b \in \mathcal{C}(A)\). Then we can write \(b\) as some linear combination of the columns of \(A\). By Theorem SLSLC \[82\] we can use the scalars from this linear combination to form a solution to \(LS(A, b)\), so this system is consistent.

\((\Leftarrow)\) If \(LS(A, b)\) is consistent, there is a solution that may be used with Theorem SLSLC \[82\] to write \(b\) as a linear combination of the columns of \(A\). This qualifies \(b\) for membership in \(\mathcal{C}(A)\).

This theorem tells us that asking if the system \(LS(A, b)\) is consistent is exactly the same question as asking if \(b\) is in the column space of \(A\). Or equivalently, it tells us that the column space of the matrix \(A\) is precisely those vectors of constants, \(b\), that can be paired with \(A\) to create a system of linear equations \(LS(A, b)\) that is consistent.

Employing Theorem SLEMM \[173\] we can form the chain of equivalences

\[
b \in \mathcal{C}(A) \iff LS(A, b) \text{ is consistent} \iff Ax = b \text{ for some } x
\]

Thus, an alternative (and popular) definition of the column space of an \(m \times n\) matrix \(A\) is

\[
\mathcal{C}(A) = \{ y \in \mathbb{C}^m \mid y = Ax \text{ for some } x \in \mathbb{C}^n \}
\]

We recognize this as saying create all the matrix vector products possible with the matrix \(A\) by letting \(x\) range over all of the possibilities. By Definition MVP \[173\] we see that this means take all possible linear combinations of the columns of \(A\) — precisely the definition of the column space (Definition CSM \[211\]) we have chosen.

Given a vector \(b\) and a matrix \(A\) it is now very mechanical to test if \(b \in \mathcal{C}(A)\). Form the linear system \(LS(A, b)\), row-reduce the augmented matrix, \([A \mid b]\), and test for consistency with Theorem RCLS \[45\]. Here’s an example of this procedure.

**Example MCSM**

**Membership in the column space of a matrix**

Consider the column space of the \(3 \times 4\) matrix \(A\),

\[
A = \begin{bmatrix} 3 & 2 & 1 & -4 \\ -1 & 1 & -2 & 3 \\ 2 & -4 & 6 & -8 \end{bmatrix}
\]

We first show that \(v = \begin{bmatrix} 18 \\ -6 \\ 12 \end{bmatrix}\) is in the column space of \(A\), \(v \in \mathcal{C}(A)\). Theorem CSCS \[212\] says we need only check the consistency of \(LS(A, v)\). Form the augmented matrix and row-reduce,

\[
\begin{bmatrix} 3 & 2 & 1 & -4 & 18 \\ -1 & 1 & -2 & 3 & -6 \\ 2 & -4 & 6 & -8 & 12 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 & -2 & 6 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

This is consistent, so \(v \in \mathcal{C}(A)\).
Without a leading 1 in the final column, Theorem RCLS \[45\] tells us the system is consistent and therefore by Theorem CSCS \[212\], \(v \in C(A)\).

If we wished to demonstrate explicitly that \(v\) is a linear combination of the columns of \(A\), we can find a solution (any solution) of \(LS(A, v)\) and use Theorem SLSLC \[82\] to construct the desired linear combination. For example, set the free variables to \(x_3 = 2\) and \(x_4 = 1\). Then a solution has \(x_2 = 1\) and \(x_1 = 6\). Then by Theorem SLSLC \[82\],
\[v = \begin{bmatrix} 18 \\ -6 \\ 12 \end{bmatrix} = 6 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix} + 1 \begin{bmatrix} -4 \\ 3 \\ -8 \end{bmatrix}.
\]

Now we show that \(w = \begin{bmatrix} 2 \\ -3 \end{bmatrix}\) is not in the column space of \(A\), \(w \notin C(A)\). Theorem CSCS \[212\] says we need only check the consistency of \(LS(A, w)\). Form the augmented matrix and row-reduce,
\[
\begin{bmatrix}
3 & 2 & 1 & -4 & 2 \\
-1 & 1 & -2 & 3 & 1 \\
2 & -4 & 6 & -8 & -3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 & -2 & 0 \\
0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

With a leading 1 in the final column, Theorem RCLS \[45\] tells us the system is inconsistent and therefore by Theorem CSCS \[212\], \(w \notin C(A)\).

Subsection CSSOC
Column Space Spanned by Original Columns

So we have a foolproof, automated procedure for determining membership in \(C(A)\). While this works just fine a vector at a time, we would like to have a more useful description of the set \(C(A)\) as a whole. The next example will preview the first of two fundamental results about the column space of a matrix.

Example CSTW
Column space, two ways
Consider the \(5 \times 7\) matrix \(A\),
\[
\begin{bmatrix}
2 & 4 & 1 & -1 & 1 & 4 & 4 \\
1 & 2 & 1 & 0 & 2 & 4 & 7 \\
0 & 0 & 1 & 4 & 1 & 8 & 7 \\
1 & 2 & -1 & 2 & 1 & 9 & 6 \\
-2 & -4 & 1 & 3 & -1 & -2 & -2
\end{bmatrix}
\]

According to the definition (Definition CSM \[211\]), the column space of \(A\) is
\[C(A) = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\} \]

While this is a concise description of an infinite set, we might be able to describe the span with fewer than seven vectors. This is the substance of Theorem BS \[143\]. So we take these seven vectors and make them the columns of matrix, which is simply the original matrix \(A\) again. Now we row-reduce,
\[
\begin{bmatrix}
2 & 4 & 1 & -1 & 1 & 4 & 4 \\
1 & 2 & 1 & 0 & 2 & 4 & 7 \\
0 & 0 & 1 & 4 & 1 & 8 & 7 \\
1 & 2 & -1 & 2 & 1 & 9 & 6 \\
-2 & -4 & 1 & 3 & -1 & -2 & -2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 0 & 0 & 0 & 3 & 1 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
The pivot columns are \( D = \{1, 3, 4, 5\} \), so we can create the set

\[
T = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \\ 3 \end{bmatrix} \right\}
\]

and know that \( \mathcal{C}(A) = \langle T \rangle \) and \( T \) is a linearly independent set of columns from the set of columns of \( A \).

We will now formalize the previous example, which will make it trivial to determine a linearly independent set of vectors that will span the column space of a matrix, and is constituted of just columns of \( A \).

**Theorem BCS**

**Basis of the Column Space**

Suppose that \( A \) is an \( m \times n \) matrix with columns \( A_1, A_2, A_3, \ldots, A_n \), and \( B \) is a row-equivalent matrix in reduced row-echelon form with \( r \) nonzero rows. Let \( D = \{d_1, d_2, d_3, \ldots, d_r\} \) be the set of column indices where \( B \) has leading 1’s. Let \( T = \{A_{d_1}, A_{d_2}, A_{d_3}, \ldots, A_{d_r}\} \). Then

1. \( T \) is a linearly independent set.
2. \( \mathcal{C}(A) = \langle T \rangle \).

**Proof** Definition CSM [211] describes the column space as the span of the set of columns of \( A \). Theorem BS [143] tells us that we can reduce the set of vectors used in a span. If we apply Theorem BS [143] to \( \mathcal{C}(A) \), we would collect the columns of \( A \) into a matrix (which would just be \( A \) again) and bring the matrix to reduced row-echelon form, which is the matrix \( B \) in the statement of the theorem. In this case, the conclusions of [Theorem BS] [143] applied to \( A, B \) and \( \mathcal{C}(A) \) are exactly the conclusions we desire.

This is a nice result since it gives us a handful of vectors that describe the entire column space (through the span), and we believe this set is as small as possible because we cannot create any more relations of linear dependence to trim it down further. Furthermore, we defined the column space (Definition CSM [211]) as all linear combinations of the columns of the matrix, and the elements of the set \( S \) are still columns of the matrix (we won’t be so lucky in the next two constructions of the column space).

Procedurally this theorem is extremely easy to apply. Row-reduce the original matrix, identify \( r \) columns with leading 1’s in this reduced matrix, and grab the corresponding columns of the original matrix. But it is still important to study the proof of Theorem BS [143] and its motivation in Example COV [141] which lie at the root of this theorem. We’ll trot through an example all the same.

**Example CSOCD**

**Column space, original columns, Archetype D**

Let’s determine a compact expression for the entire column space of the coefficient matrix of the system of equations that is [Archetype D] [647]. Notice that in Example CSMCS [211] we were only determining if individual vectors were in the column space or not, now we are describing the entire column space.

To start with the application of Theorem BCS [214], call the coefficient matrix \( A \)

\[
A = \begin{bmatrix} 2 & 1 & 7 & -7 \\ -3 & 4 & -5 & -6 \\ 1 & 1 & 4 & -5 \end{bmatrix}.
\]
and row-reduce it to reduced row-echelon form,

\[ B = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

There are leading 1’s in columns 1 and 2, so \( D = \{1, 2\} \). To construct a set that spans \( \mathcal{C}(A) \), just grab the columns of \( A \) indicated by the set \( D \), so

\[ \mathcal{C}(A) = \left\langle \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \right\rangle. \]

That’s it.

In Example CSMCS [211] we determined that the vector \( c = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \) was not in the column space of \( A \). Try to write \( c \) as a linear combination of the first two columns of \( A \). What happens?

Also in Example CSMCS [211] we determined that the vector \( b = \begin{bmatrix} 8 \\ -12 \\ 4 \end{bmatrix} \) was in the column space of \( A \). Try to write \( b \) as a linear combination of the first two columns of \( A \). What happens? Did you find a unique solution to this question? Hmmmm.

Subsection CSNM
Column Space of a Nonsingular Matrix

Let’s specialize to square matrices and contrast the column spaces of the coefficient matrices in Archetype A [634] and Archetype B [638].

Example CSAA
Column space of Archetype A
The coefficient matrix in Archetype A [634] is

\[ A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \]

which row-reduces to

\[ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \]

Columns 1 and 2 have leading 1’s, so by Theorem BCS [214] we can write

\[ \mathcal{C}(A) = \langle \{A_1, A_2\} \rangle = \left\langle \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\rangle. \]
So the matrix $B$ is a “right-inverse” for $A$. By Theorem NMRRI \[62\], $I_n$ is a nonsingular matrix, so by Theorem NPNT \[202\] both $A$ and $B$ are nonsingular. Thus, in particular, $A$ is nonsingular. (Travis Osborne contributed to this proof.)

With this equivalence for nonsingular matrices we can update our list. Theorem NME3 \[204\].
3. The null space of $A$ contains only the zero vector, $\mathcal{N}(A) = \{0\}$.

4. The linear system $\mathcal{L}\mathcal{S}(A, b)$ has a unique solution for every possible choice of $b$.

5. The columns of $A$ are a linearly independent set.

6. $A$ is invertible.

7. The column space of $A$ is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

□

**Proof**  Since Theorem [CSNM][216] is an equivalence, we can add it to the list in Theorem [NME3][204].

---

**Subsection RSM  
Row Space of a Matrix**

The rows of a matrix can be viewed as vectors, since they are just lists of numbers, arranged horizontally. So we will transpose a matrix, turning rows into columns, so we can then manipulate rows as column vectors. As a result we will be able to make some new connections between row operations and solutions to systems of equations. OK, here is the second primary definition of this section.

**Definition RSM  
Row Space of a Matrix**

Suppose $A$ is an $m \times n$ matrix. Then the **row space** of $A$, $\mathcal{R}(A)$, is the column space of $A^t$, i.e. $\mathcal{R}(A) = \mathcal{C}(A^t)$.

(This definition contains Notation RSM.)

Informally, the row space is the set of all linear combinations of the rows of $A$. However, we write the rows as column vectors, thus the necessity of using the transpose to make the rows into columns. Additionally, with the row space defined in terms of the column space, all of the previous results of this section can be applied to row spaces.

Notice that if $A$ is a rectangular $m \times n$ matrix, then $\mathcal{C}(A) \subseteq \mathbb{C}^m$, while $\mathcal{R}(A) \subseteq \mathbb{C}^n$ and the two sets are not comparable since they do not even hold objects of the same type. However, when $A$ is square of size $n$, both $\mathcal{C}(A)$ and $\mathcal{R}(A)$ are subsets of $\mathbb{C}^n$, though usually the sets will not be equal (but see Exercise [CRS.M20][225]).

**Example RSAI  
Row space of Archetype I**

The coefficient matrix in [Archetype I][667] is

$$ I = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}. $$

To build the row space, we transpose the matrix,

$$ I^t = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 4 & 8 & 0 & -4 \\ 0 & -1 & 2 & 2 \\ -1 & 3 & -3 & 4 \\ 0 & 9 & -4 & 8 \\ 7 & -13 & 12 & -31 \\ -9 & 7 & -8 & 37 \end{bmatrix}. $$

□
Then the columns of this matrix are used in a span to build the row space,

$$\mathcal{R}(I) = \mathcal{C}(I^t) = \left\{ \begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \\ -1 \\ 3 \\ 9 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ -3 \\ -4 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ 4 \\ 8 \end{pmatrix} \right\}.$$ 

However, we can use Theorem BCS to get a slightly better description. First, row-reduce $I^t$,

$$\begin{pmatrix} 1 & 0 & 0 & -31 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 13 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

Since there are leading 1’s in columns with indices $D = \{1, 2, 3\}$, the column space of $I^t$ can be spanned by just the first three columns of $I^t$,

$$\mathcal{R}(I) = \mathcal{C}(I^t) = \left\{ \begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \\ -1 \\ 3 \\ 9 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ -3 \\ -4 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ 4 \\ 8 \end{pmatrix} \right\}.$$ 

The row space would not be too interesting if it was simply the column space of the transpose. However, when we do row operations on a matrix we have no effect on the many linear combinations that can be formed with the rows of the matrix. This is stated more carefully in the following theorem.

**Theorem REMRS**

**Row-Equivalent Matrices have equal Row Spaces**

Suppose $A$ and $B$ are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$. □

**Proof** Two matrices are row-equivalent (Definition REM) if one can be obtained from another by a sequence of (possibly many) row operations. We will prove the theorem for two matrices that differ by a single row operation, and then this result can be applied repeatedly to get the full statement of the theorem. The row spaces of $A$ and $B$ are spans of the columns of their transposes. For each row operation we perform on a matrix, we can define analogous operations on the columns. Perhaps we should call these column operations. Instead, we will still call them row operations, but we will apply them to the columns of the transposes.

Refer to the columns of $A^t$ and $B^t$ as $A_i$ and $B_i$, $1 \leq i \leq m$. The row operation that switches rows will just switch columns of the transposed matrices. This will have no effect on the possible linear combinations formed by the columns.

Suppose that $B^t$ is formed from $A^t$ by multiplying column $A_i$ by $\alpha \neq 0$. In other words, $B_i = \alpha A_i$, and $B_i = A_i$ for all $i \neq t$. We need to establish that two sets are equal, $\mathcal{C}(A^t) = \mathcal{C}(B^t)$. We will take a generic element of one and show that it is contained in the other.

$$\beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3 + \ldots + \beta_t B_t + \ldots + \beta_m B_m$$
Row spaces of two row-equivalent matrices

In Example TREM 25 we saw that the matrices

\[
A = \begin{bmatrix}
2 & -1 & 3 & 4 \\
5 & 2 & -2 & 3 \\
1 & 1 & 0 & 6
\end{bmatrix}
\quad B = \begin{bmatrix}
1 & 1 & 0 & 6 \\
3 & 0 & -2 & -9 \\
2 & -1 & 3 & 4
\end{bmatrix}
\]

are row-equivalent by demonstrating a sequence of two row operations that converted \(A\) into \(B\). Applying Theorem REMRS 218 we can say

\[
\mathcal{R}(A) = \begin{bmatrix}
\begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 6 \end{bmatrix}\end{bmatrix} = \begin{bmatrix}
\begin{bmatrix} 1 \\ 0 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 3 \\ 4 \end{bmatrix}\end{bmatrix} = \mathcal{R}(B)
\]

Theorem REMRS 218 is at its best when one of the row-equivalent matrices is in reduced row-echelon form. The vectors that correspond to the zero rows can be ignored (who needs the...
zero vector when building a span?, see Exercise LI.T10 [133]). The echelon pattern insures that the nonzero rows yield vectors that are linearly independent. Here’s the theorem.

**Theorem BRS**

**Basis for the Row Space**

Suppose that $A$ is a matrix and $B$ is a row-equivalent matrix in reduced row-echelon form. Let $S$ be the set of nonzero columns of $B^t$. Then

1. $\mathcal{R}(A) = \langle S \rangle$.

2. $S$ is a linearly independent set.

**Proof** From Theorem REMRS [218] we know that $\mathcal{R}(A) = \mathcal{R}(B)$. If $B$ has any zero rows, these correspond to columns of $B^t$ that are the zero vector. We can safely toss out the zero vector in the span construction, since it can be recreated from the nonzero vectors by a linear combination where all the scalars are zero. So $\mathcal{R}(A) = \langle S \rangle$.

Suppose $B$ has $r$ nonzero rows and let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ denote the column indices of $B$ that have a leading one in them. Denote the $r$ column vectors of $B^t$, the vectors in $S$, as $B_1, B_2, B_3, \ldots, B_r$. To show that $S$ is linearly independent, start with a relation of linear dependence

$$\alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 + \cdots + \alpha_r B_r = 0$$

Now consider this vector equality in location $d_i$. Since $B$ is in reduced row-echelon form, the entries of column $d_i$ of $B$ are all zero, except for a (leading) 1 in row $i$. Thus, in $B^t$, row $d_i$ is all zeros, excepting a 1 in column $i$. So, for $1 \leq i \leq r$,

$$0 = [0]_{d_i} = [\alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 + \cdots + \alpha_r B_r]_{d_i} = [\alpha_1 B_1]_{d_i} + [\alpha_2 B_2]_{d_i} + [\alpha_3 B_3]_{d_i} + \cdots + [\alpha_r B_r]_{d_i} = \alpha_1 [B_1]_{d_i} + \alpha_2 [B_2]_{d_i} + \alpha_3 [B_3]_{d_i} + \cdots + \alpha_r [B_r]_{d_i} = \alpha_1 (0) + \alpha_2 (0) + \alpha_3 (0) + \cdots + \alpha_i (1) + \cdots + \alpha_r (0)$$

So we conclude that $\alpha_i = 0$ for all $1 \leq i \leq r$, establishing the linear independence of $S$ (Definition LICV [121]).

**Example IAS**

**Improving a span**

Suppose in the course of analyzing a matrix (its column space, its null space, its...) we encounter the following set of vectors, described by a span

$$X = \langle \begin{bmatrix} 1 \\ 2 \\ 1 \\ 6 \\ 6 \\ -3 \\ 2 \\ -1 \\ 0 \\ -2 \\ 6 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ -3 \\ 6 \\ -10 \end{bmatrix} \rangle$$

Let $A$ be the matrix whose rows are the vectors in $X$, so by design $X = \mathcal{R}(A)$,
Row-reduce $A$ to form a row-equivalent matrix in reduced row-echelon form,

$$B = \begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then Theorem BRS 220 says we can grab the nonzero columns of $B^t$ and write

$$X = \mathcal{R}(A) = \mathcal{R}(B) = \langle \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 5 \end{bmatrix} \rangle$$

These three vectors provide a much-improved description of $X$. There are fewer vectors, and the pattern of zeros and ones in the first three entries makes it easier to determine membership in $X$. And all we had to do was row-reduce the right matrix and toss out a zero row. Next to row operations themselves, this is probably the most powerful computational technique at your disposal as it quickly provides a much improved description of a span, any span.

Theorem BRS 220 and the techniques of Example IAS 220 will provide yet another description of the column space of a matrix. First we state a triviality as a theorem, so we can reference it later.

Theorem CSRST
Column Space, Row Space, Transpose
Suppose $A$ is a matrix. Then $\mathcal{C}(A) = \mathcal{R}(A^t)$.

Proof

$$\mathcal{C}(A) = \mathcal{C}( (A^t)^t ) = \mathcal{R}(A^t)$$

Theorem TT 167
Definition RSM 217

So to find another expression for the column space of a matrix, build its transpose, row-reduce it, toss out the zero rows, and convert the nonzero rows to column vectors to yield an improved set for the span construction. We’ll do Archetype I 667, then you do Archetype J 671.

Example CSROI
Column space from row operations, Archetype I
To find the column space of the coefficient matrix of Archetype I 667, we proceed as follows. The matrix is

$$I = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}$$

The transpose is

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 4 \\ 8 \\ 0 \\ -4 \\ 0 \\ -1 \\ 2 \\ 2 \\ -1 \\ 3 \\ -3 \\ 4 \\ 0 \\ 9 \\ -4 \\ 8 \\ 7 \\ -13 \\ 12 \\ -31 \\ -9 \\ 7 \\ -8 \\ 37 \end{bmatrix}$$
Row-reduced this becomes,
\[
\begin{bmatrix}
1 & 0 & 0 & -\frac{31}{7} \\
0 & 1 & 0 & \frac{12}{7} \\
0 & 0 & 1 & \frac{13}{7} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\].

Now, using Theorem CSRST \[221\] and Theorem BRS \[220\]
\[
C(I) = R(I^t) = \langle \begin{bmatrix}
1 \\
0 \\
-\frac{31}{7}
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
\frac{12}{7}
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
\frac{13}{7}
\end{bmatrix}\rangle.
\]

This is a very nice description of the column space. Fewer vectors than the 7 involved in the definition, and the pattern of the zeros and ones in the first 3 slots can be used to advantage. For example, Archetype I \[667\] is presented as a consistent system of equations with a vector of constants
\[
b = \begin{bmatrix}
3 \\
9 \\
1
\end{bmatrix}.
\]

Since \(LS(I, b)\) is consistent, Theorem CSCS \[212\] tells us that \(b \in C(I)\). But we could see this quickly with the following computation, which really only involves any work in the 4th entry of the vectors as the scalars in the linear combination are \textit{dictated} by the first three entries of \(b\).
\[
b = \begin{bmatrix}
3 \\
9 \\
1
\end{bmatrix} = 3 \begin{bmatrix}
1 \\
0 \\
-\frac{31}{7}
\end{bmatrix} + 9 \begin{bmatrix}
0 \\
1 \\
\frac{12}{7}
\end{bmatrix} + 1 \begin{bmatrix}
0 \\
0 \\
\frac{13}{7}
\end{bmatrix}
\]

Can you now rapidly construct several vectors, \(b\), so that \(LS(I, b)\) is consistent, and several more so that the system is inconsistent?

Subsection READ

Reading Questions

1. Write the column space of the matrix below as the span of a set of three vectors and explain your choice of method.
\[
\begin{bmatrix}
1 & 3 & 1 & 3 \\
2 & 0 & 1 & 1 \\
-1 & 2 & 1 & 0
\end{bmatrix}
\]

2. Suppose that \(A\) is an \(n \times n\) nonsingular matrix. What can you say about its column space?

3. Is the vector \[
\begin{bmatrix}
0 \\
5 \\
2 \\
3
\end{bmatrix}
\]
in the row space of the following matrix? Why or why not?
\[
\begin{bmatrix}
1 & 3 & 1 & 3 \\
2 & 0 & 1 & 1 \\
-1 & 2 & 1 & 0
\end{bmatrix}
\]
Subsection EXC
Exercises

C30  Example CSOCD 214 expresses the column space of the coefficient matrix from Archetype D 647 (call the matrix \( A \) here) as the span of the first two columns of \( A \). In Example CSMCS 211 we determined that the vector

\[
c = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}
\]

was not in the column space of \( A \) and that the vector

\[
b = \begin{bmatrix} 8 \\ -12 \\ 4 \end{bmatrix}
\]

was in the column space of \( A \). Attempt to write \( c \) and \( b \) as linear combinations of the two vectors in the span construction for the column space in Example CSOCD 214 and record your observations.

Contributed by Robert Beezer  Solution 227

C31  For the matrix \( A \) below find a set of vectors \( T \) meeting the following requirements: (1) the span of \( T \) is the column space of \( A \), that is, \( \langle T \rangle = \text{C}(A) \), (2) \( T \) is linearly independent, and (3) the elements of \( T \) are columns of \( A \).

\[
A = \begin{bmatrix}
2 & 1 & 4 & -1 & 2 \\
1 & -1 & 5 & 1 & 1 \\
-1 & 2 & -7 & 0 & 1 \\
2 & -1 & 8 & -1 & 2
\end{bmatrix}
\]

Contributed by Robert Beezer  Solution 227

C32  In Example CSAA 215, verify that the vector \( b \) is not in the column space of the coefficient matrix.

Contributed by Robert Beezer

C33  Find a linearly independent set \( S \) so that the span of \( S \), \( \langle S \rangle \), is row space of the matrix \( B \), and \( S \) is linearly independent.

\[
B = \begin{bmatrix}
2 & 3 & 1 & 1 \\
1 & 1 & 0 & 1 \\
-1 & 2 & 3 & -4
\end{bmatrix}
\]

Contributed by Robert Beezer  Solution 227

C34  For the \( 3 \times 4 \) matrix \( A \) and the column vector \( y \in \mathbb{C}^4 \) given below, determine if \( y \) is in the row space of \( A \). In other words, answer the question: \( y \in \text{R}(A) \)? (15 points)

\[
A = \begin{bmatrix}
-2 & 6 & 7 & -1 \\
7 & -3 & 0 & -3 \\
8 & 0 & 7 & 6
\end{bmatrix}
\quad y = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}
\]

Contributed by Robert Beezer  Solution 227

C35  For the matrix \( A \) below, find two different linearly independent sets whose spans equal the column space of \( A \), \( \text{C}(A) \), such that

(a) the elements are each columns of \( A \).
(b) the set is obtained by a procedure that is substantially different from the procedure you use in part (a).

\[
A = \begin{bmatrix}
3 & 5 & 1 & -2 \\
1 & 2 & 3 & 3 \\
-3 & -4 & 7 & 13
\end{bmatrix}
\]

Contributed by Robert Beezer

C40 The following archetypes are systems of equations. For each system, write the vector of constants as a linear combination of the vectors in the span construction for the column space provided by Theorem BCS [214] (these vectors are listed for each of these archetypes).

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671

Contributed by Robert Beezer

C42 The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute a set of column vectors such that (1) the vectors are columns of the matrix, (2) the set is linearly independent, and (3) the span of the set is the column space of the matrix. See Theorem BCS [214].

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671
Archetype K 676
Archetype L 680

Contributed by Robert Beezer

C50 The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute a set of column vectors such that (1) the set is linearly independent, and (2) the span of the set is the row space of the matrix. See Theorem BRS [220].

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671
Archetype K 676
C51 The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute the column space as the span of a linearly independent set as follows: transpose the matrix, row-reduce, toss out zero rows, convert rows into column vectors. See Example CSROI 221.

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671
Archetype K 676
Archetype L 680

C52 The following archetypes are systems of equations. For each different coefficient matrix build two new vectors of constants. The first should lead to a consistent system and the second should lead to an inconsistent system. Descriptions of the column space as spans of linearly independent sets of vectors with “nice patterns” of zeros and ones might be most useful and instructive in connection with this exercise. (See the end of Example CSROI 221.)

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671

M10 For the matrix $E$ below, find vectors $b$ and $c$ so that the system $LS(E, b)$ is consistent and $LS(E, c)$ is inconsistent.

$$E = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 3 & -1 & 0 & 2 \\ 4 & 1 & 1 & 6 \end{bmatrix}$$

M20 Usually the column space and null space of a matrix contain vectors of different sizes. For a square matrix, though, the vectors in these two sets are the same size. Usually the two sets will be different. Construct an example of a square matrix where the column space and null space are equal.

M21 We have a variety of theorems about how to create column spaces and row spaces and they frequently involve row-reducing a matrix. Here is a procedure that some try to use to get a column space. Begin with an $m \times n$ matrix $A$ and row-reduce to a matrix $B$ with columns
\(B_1, B_2, B_3, \ldots, B_n\). Then form the column space of \(A\) as

\[C(A) = \{B_1, B_2, B_3, \ldots, B_n\} = C(B)\]

This is not a legitimate procedure, and therefore is not a theorem. Construct an example to show that the procedure will not in general create the column space of \(A\).

Contributed by Robert Beezer

Solution [228]

**T40** Suppose that \(A\) is an \(m \times n\) matrix and \(B\) is an \(n \times p\) matrix. Prove that the column space of \(AB\) is a subset of the column space of \(A\), that is \(C(AB) \subseteq C(A)\). Provide an example where the opposite is false, in other words give an example where \(C(A) \nsubseteq C(AB)\). (Compare with Exercise [MM.T40](184).)

Contributed by Robert Beezer

Solution [229]

**T41** Suppose that \(A\) is an \(m \times n\) matrix and \(B\) is an \(n \times n\) nonsingular matrix. Prove that the column space of \(A\) is equal to the column space of \(AB\), that is \(C(A) = C(AB)\). (Compare with Exercise [MM.T41](184) and Exercise [CRS.T40](226).)

Contributed by Robert Beezer

Solution [229]

**T45** Suppose that \(A\) is an \(m \times n\) matrix and \(B\) is an \(n \times m\) matrix where \(AB\) is a nonsingular matrix. Prove that

1. \(N(B) = \{0\}\)
2. \(C(B) \cap N(A) = \{0\}\)

Discuss the case when \(m = n\) in connection with [Theorem NPNT](202).

Contributed by Robert Beezer

Solution [229]
Subsection SOL
Solutions

C30  Contributed by Robert Beezer  Statement 223
In each case, begin with a vector equation where one side contains a linear combination of the two vectors from the span construction that gives the column space of $A$ with unknowns for scalars, and then use Theorem SLSLC [82] to set up a system of equations. For $c$, the corresponding system has no solution, as we would expect.

For $b$ there is a solution, as we would expect. What is interesting is that the solution is unique. This is a consequence of the linear independence of the set of two vectors in the span construction. If we wrote $b$ as a linear combination of all four columns of $A$, then there would be infinitely many ways to do this.

C31  Contributed by Robert Beezer  Statement 223
Theorem BCS [214] is the right tool for this problem. Row-reduce this matrix, identify the pivot columns and then grab the corresponding columns of $A$ for the set $T$. The matrix $A$ row-reduces to

$$
\begin{bmatrix}
1 & 0 & 3 & 0 & 0 \\
0 & 1 & -2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

So $D = \{1, 2, 4, 5\}$ and then

$$
T = \{A_1, A_2, A_4, A_5\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}
$$

has the requested properties.

C33  Contributed by Robert Beezer  Statement 223
Theorem BRS [220] is the most direct route to a set with these properties. Row-reduce, toss zero rows, keep the others. You could also transpose the matrix, then look for the column space by row-reducing the transpose and applying Theorem BCS [214]. We’ll do the former,

$$
B \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & -1 & 2 \\
0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

So the set $S$ is

$$
S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}
$$

C34  Contributed by Robert Beezer  Statement 223
$$
y \in \mathcal{R}(A) \iff y \in \mathcal{C}(A^t) \\
\iff \mathcal{L}S(A^t, y) \text{ is consistent}
$$

The augmented matrix $[A^t | y]$ row reduces to

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$
and with a leading 1 in the final column \( \text{Theorem RCLS} \) tells us the linear system is inconsistent and so \( y \not\in \mathbb{R}(A) \).

**C35** Contributed by Robert Beezer Statement 223

(a) By \( \text{Theorem BCS} \) we can row-reduce \( A \), identify pivot columns with the set \( D \), and “keep” those columns of \( A \) and we will have a set with the desired properties.

\[
A \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & -13 & -19 \\
0 & 1 & 8 & 11 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

So we have the set of pivot columns \( D = \{1, 2\} \) and we “keep” the first two columns of \( A \),

\[
\left\{ \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} \right\}
\]

(b) We can view the column space as the row space of the transpose \( \text{Theorem CSRST} \). We can get a basis of the row space of a matrix quickly by bringing the matrix to reduced row-echelon form and keeping the nonzero rows as column vectors \( \text{Theorem BRS} \). Here goes.

\[
A^t \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & -2 \\
0 & 1 & 3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Taking the nonzero rows and tilting them up as columns gives us

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}
\]

An approach based on the matrix \( L \) from extended echelon form \( \text{Definition EEF} \) and \( \text{Theorem FS} \) will work as well as an alternative approach.

**M10** Contributed by Robert Beezer Statement 225

Any vector from \( \mathbb{C}^3 \) will lead to a consistent system, and therefore there is no vector that will lead to an inconsistent system.

How do we convince ourselves of this? First, row-reduce \( E \),

\[
E \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

If we augment \( E \) with any vector of constants, and row-reduce the augmented matrix, we will never find a leading 1 in the final column, so by \( \text{Theorem RCLS} \) the system will always be consistent.

Said another way, the column space of \( E \) is all of \( \mathbb{C}^3 \), \( \mathbb{C}(E) = \mathbb{C}^3 \). So by \( \text{Theorem CSCS} \) any vector of constants will create a consistent system (and none will create an inconsistent system).

**M20** Contributed by Robert Beezer Statement 225

The \( 2 \times 2 \) matrix

\[
\begin{bmatrix}
1 & 1 \\
-1 & -1
\end{bmatrix}
\]

has \( \mathcal{C}(A) = \mathcal{N}(A) = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \).

**M21** Contributed by Robert Beezer Statement 225

Begin with a matrix \( A \) (of any size) that does not have any zero rows, but which when row-reduced...
to \( B \) yields at least one row of zeros. Such a matrix should be easy to construct (or find, like say from Archetype A [634]).

\( C(A) \) will contain some vectors whose final slot (entry \( m \)) is non-zero, however, every column vector from the matrix \( B \) will have a zero in slot \( m \) and so every vector in \( C(B) \) will also contain a zero in the final slot. This means that \( C(A) \neq C(B) \), since we have vectors in \( C(A) \) that cannot be elements of \( C(B) \).

**T40** Contributed by Robert Beezer Statement [226]
Choose \( x \in C(AB) \). Then by Theorem CSCS [212] there is a vector \( w \) that is a solution to \( LS(AB, x) \). Define the vector \( y \) by \( y = Bw \). We’re set,

\[
Ay = A(Bw) \quad \text{Definition of } y
\]

\[
= (AB)w \quad \text{Theorem MMA [180]}
\]

\[
= x \quad w \text{ solution to } LS(AB, x)
\]

This says that \( LS(A, x) \) is a consistent system, and by Theorem CSCS [212], we see that \( x \in C(A) \) and therefore \( C(AB) \subseteq C(A) \).

For an example where \( C(A) \nsubseteq C(AB) \) choose \( A \) to be any nonzero matrix and choose \( B \) to be a zero matrix. Then \( C(A) \neq \{0\} \) and \( C(AB) = C(O) = \{0\} \).

**T41** Contributed by Robert Beezer Statement [226]
From the solution to Exercise CRS.T40 [226] we know that \( C(AB) \subseteq C(A) \). So to establish the set equality (Definition SE [616]) we need to show that \( C(A) \subseteq C(AB) \).

Choose \( x \in C(A) \). By Theorem CSCS [212] the linear system \( LS(A, x) \) is consistent, so let \( y \) be one such solution. Because \( B \) is nonsingular, and linear system using \( B \) as a coefficient matrix will have a solution (Theorem NMUS [64]). Let \( w \) be the unique solution to the linear system \( LS(B, y) \). All set, here we go,

\[
(AB)w = A(Bw) \quad \text{Theorem MMA [180]}
\]

\[
= Ay \quad w \text{ solution to } LS(B, y)
\]

\[
= x \quad y \text{ solution to } LS(A, x)
\]

This says that the linear system \( LS(AB, x) \) is consistent, so by Theorem CSCS [212], \( x \in C(AB) \). So \( C(A) \subseteq C(AB) \).

**T45** Contributed by Robert Beezer Statement [226]
First, \( 0 \in N(B) \) trivially. Now suppose that \( x \in N(B) \). Then

\[
ABx = A(Bx) \quad \text{Theorem MMA [180]}
\]

\[
= A0 \quad x \in N(B)
\]

\[
= 0 \quad \text{Theorem MMZM [178]}
\]

Since we have assumed \( AB \) is nonsingular, Definition NM [61] implies that \( x = 0 \).

Second, \( 0 \in C(B) \) and \( 0 \in N(A) \) trivially, and so the zero vector is in the intersection as well (Definition SI [617]). Now suppose that \( y \in C(B) \cap N(A) \). Because \( y \in C(B) \), Theorem CSCS [212] says the system \( LS(B, y) \) is consistent. Let \( x \in \mathbb{C}^n \) be one solution to this system. Then

\[
ABx = A(Bx) \quad \text{Theorem MMA [180]}
\]

\[
= Ay \quad x \text{ solution to } LS(B, y)
\]

\[
= 0 \quad y \in N(A)
\]

Since we have assumed \( AB \) is nonsingular, Definition NM [61] implies that \( x = 0 \). Then \( y = Bx = B0 = 0 \).
When $AB$ is nonsingular and $m = n$ we know that the first condition, $\mathcal{N}(B) = \{0\}$, means that $B$ is nonsingular (Theorem NMTNS [64]). Because $B$ is nonsingular Theorem CSNM [216] implies that $\mathcal{C}(B) = \mathbb{C}^m$. In order to have the second condition fulfilled, $\mathcal{C}(B) \cap \mathcal{N}(A) = \{0\}$, we must realize that $\mathcal{N}(A) = \{0\}$. However, a second application of Theorem NMTNS [64] shows that $A$ must be nonsingular. This reproduces Theorem NPNT [202].
Section FS  
Four Subsets

There are four natural subsets associated with a matrix. We have met three already: the null space, the column space and the row space. In this section we will introduce a fourth, the left null space. The objective of this section is to describe one procedure that will allow us to find linearly independent sets that span each of these four sets of column vectors. Along the way, we will make a connection with the inverse of a matrix, so Theorem FS 237 will tie together most all of this chapter (and the entire course so far).

Subsection LNS  
Left Null Space

Definition LNS  
Left Null Space
Suppose $A$ is an $m \times n$ matrix. Then the left null space is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$.  
(This definition contains Notation LNS.)

The left null space will not feature prominently in the sequel, but we can explain its name and connect it to row operations. Suppose $y \in \mathcal{L}(A)$. Then by Definition LNS 231, $A^ty = 0$. We can then write

$$0^t = (A^ty)^t$$
$$\quad = y^t(A^t)^t$$
$$\quad = y^tA$$  
(The product $y^tA$ can be viewed as the components of $y$ acting as the scalars in a linear combination of the rows of $A$. And the result is a “row vector”, $0^t$ that is totally zeros. When we apply a sequence of row operations to a matrix, each row of the resulting matrix is some linear combination of the rows. These observations tell us that the vectors in the left null space are scalars that record a sequence of row operations that result in a row of zeros in the row-reduced version of the matrix. We will see this idea more explicitly in the course of proving Theorem FS 237.

Example LNS  
Left null space
We will find the left null space of

$$A = \begin{bmatrix}
1 & -3 & 1 \\
-2 & 1 & 1 \\
1 & 5 & 1 \\
9 & -4 & 0
\end{bmatrix}$$

We transpose $A$ and row-reduce,

$$A^t = \begin{bmatrix}
1 & -2 & 1 & 9 \\
-3 & 1 & 5 & -4 \\
1 & 1 & 1 & 0
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & 1
\end{bmatrix}$$

Applying Definition LNS 231 and Theorem BNS 128 we have

$$\mathcal{L}(A) = \mathcal{N}(A^t) = \left\{ \begin{bmatrix}
-2 \\
3 \\
-1 \\
1
\end{bmatrix} \right\}$$
If you row-reduce $A$ you will discover one zero row in the reduced row-echelon form. This zero row is created by a sequence of row operations, which in total amounts to a linear combination, with scalars $a_1 = -2$, $a_2 = 3$, $a_3 = -1$ and $a_4 = 1$, on the rows of $A$ and which results in the zero vector (check this!). So the components of the vector describing the left null space of $A$ provide a relation of linear dependence on the rows of $A$. 

Subsection CRS
Computing Column Spaces

We have three ways to build the column space of a matrix. First, we can use just the definition, Definition CSM [211], and express the column space as a span of the columns of the matrix. A second approach gives us the column space as the span of some of the columns of the matrix, but this set is linearly independent (Theorem BCS [214]). Finally, we can transpose the matrix, row-reduce the transpose, kick out zero rows, and transpose the remaining rows back into column vectors. Theorem CSRST [221] and Theorem BRS [220] tell us that the resulting vectors are linearly independent and their span is the column space of the original matrix.

We will now demonstrate a fourth method by way of a rather complicated example. Study this example carefully, but realize that its main purpose is to motivate a theorem that simplifies much of the apparent complexity. So other than an instructive exercise or two, the procedure we are about to describe will not be a usual approach to computing a column space.

Example CSANS
Column space as null space

Lets find the column space of the matrix $A$ below with a new approach.

\[
A = \begin{bmatrix}
10 & 0 & 3 & 8 & 7 \\
-16 & -1 & -4 & -10 & -13 \\
-6 & 1 & -3 & -6 & -6 \\
0 & 2 & -2 & -3 & -2 \\
3 & 0 & 1 & 2 & 3 \\
-1 & -1 & 1 & 1 & 0
\end{bmatrix}
\]

By Theorem CSCS [212] we know that the column vector $b$ is in the column space of $A$ if and only if the linear system $\mathcal{L}(A, b)$ is consistent. So let’s try to solve this system in full generality, using a vector of variables for the vector of constants. In other words, which vectors $b$ lead to consistent systems? Begin by forming the augmented matrix $[A | b]$ with a general version of $b$,

\[
[A | b] = \begin{bmatrix}
10 & 0 & 3 & 8 & 7 & b_1 \\
-16 & -1 & -4 & -10 & -13 & b_2 \\
-6 & 1 & -3 & -6 & -6 & b_3 \\
0 & 2 & -2 & -3 & -2 & b_4 \\
3 & 0 & 1 & 2 & 3 & b_5 \\
-1 & -1 & 1 & 1 & 0 & b_6
\end{bmatrix}
\]

To identify solutions we will row-reduce this matrix and bring it to reduced row-echelon form. Despite the presence of variables in the last column, there is nothing to stop us from doing this. Except our numerical routines on calculators can’t be used, and even some of the symbolic algebra routines do some unexpected maneuvers with this computation. So do it by hand. Yes, it is a bit of work. But worth it. We’ll still be here when you get back. Notice along the way that the row operations are exactly the same ones you would do if you were just row-reducing the coefficient matrix alone, say in connection with a homogeneous system of equations. The column with the $b_i$ acts as a sort of bookkeeping device. There are many different possibilities for the result, depending on what order you choose to perform the row operations, but shortly we’ll all be on the same page.
Here’s one possibility (you can find this same result by doing additional row operations with the fifth and sixth rows to remove any occurrences of $b_1$ and $b_2$ from the first four rows of your result):

$$
\begin{bmatrix}
1 & 0 & 0 & 2 & b_3 - b_4 + 2b_5 - b_6 \\
0 & 1 & 0 & 0 & -3 & -2b_3 + 3b_4 - 3b_5 + 3b_6 \\
0 & 0 & 1 & 0 & 1 & b_3 + b_4 + 3b_5 + 3b_6 \\
0 & 0 & 0 & 1 & -2 & -2b_3 + b_4 - 4b_5 \\
0 & 0 & 0 & 0 & 0 & b_1 + b_3 - b_4 + 3b_5 + b_6 \\
0 & 0 & 0 & 0 & 0 & b_2 - 2b_3 + b_4 + b_5 - b_6
\end{bmatrix}
$$

Our goal is to identify those vectors $b$ which make $LS(A, b)$ consistent. By Theorem RCLS we know that the consistent systems are precisely those without a leading 1 in the last column. Are the expressions in the last column of rows 5 and 6 equal to zero, or are they leading 1’s? The answer is: maybe. It depends on $b$. With a nonzero value for either of these expressions, we would scale the row and produce a leading 1. So we get a consistent system, and $b$ is in the column space, if and only if these two expressions are both simultaneously zero. In other words, members of the column space of $A$ are exactly those vectors $b$ that satisfy

$$
b_1 + 3b_3 - b_4 + 3b_5 + b_6 = 0 \quad \text{and} \quad b_2 - 2b_3 + b_4 + b_5 - b_6 = 0
$$

Hmmm. Looks suspiciously like a homogeneous system of two equations with six variables. If you’ve been playing along (and we hope you have) then you may have a slightly different system, but you should have just two equations. Form the coefficient matrix and row-reduce (notice that the system above has a coefficient matrix that is already in reduced row-echelon form). We should all be together now with the same matrix,

$$
L = \begin{bmatrix}
1 & 0 & 3 & -1 & 3 & 1 \\
0 & 1 & -2 & 1 & 1 & -1
\end{bmatrix}
$$

So, $C(A) = N(L)$ and we can apply Theorem BNS to obtain a linearly independent set to use in a span construction,

$$
C(A) = N(L) = \left\{ \begin{bmatrix}
-3 \\
2 \\
1 \\
0 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
1 \\
-1 \\
0 \\
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
-3 \\
-1 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
0 \\
0 \\
0 \\
1
\end{bmatrix} \right\}
$$

Whew! As a postscript to this central example, you may wish to convince yourself that the four vectors above really are elements of the column space? Do they create consistent systems with $A$ as coefficient matrix? Can you recognize the constant vector in your description of these solution sets?

OK, that was so much fun, let’s do it again. But simpler this time. And we’ll all get the same results all the way through. Doing row operations by hand with variables can be a bit error prone, so let’s see if we can improve the process some. Rather than row-reduce a column vector $b$ full of variables, let’s write $b = I_6b$ and we will row-reduce the matrix $I_6$ and when we finish row-reducing, then we will compute the matrix-vector product. You should first convince yourself that we can operate like this (this is the subject of a future homework exercise). Rather than augmenting $A$ with $b$, we will instead augment it with $I_6$ (does this feel familiar?),

$$
M = \begin{bmatrix}
10 & 0 & 3 & 8 & 7 & 1 & 0 & 0 & 0 & 0 & 0 \\
-16 & -1 & -4 & -10 & -13 & 0 & 1 & 0 & 0 & 0 & 0 \\
-6 & 1 & -3 & -6 & -6 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & -2 & -3 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\
3 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
$$
We want to row-reduce the left-hand side of this matrix, but we will apply the same row operations to the right-hand side as well. And once we get the left-hand side in reduced row-echelon form, we will continue on to put leading 1’s in the final two rows, as well as clearing out the columns containing those two additional leading 1’s. It is these additional row operations that will ensure that we all get to the same place, since the reduced row-echelon form is unique (Theorem RREFU)

\[
N = \begin{bmatrix}
1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & -1 & 2 & -1 \\
0 & 1 & 0 & 0 & -3 & 0 & 0 & -2 & 3 & -3 & 3 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 3 & 3 \\
0 & 0 & 0 & 1 & -2 & 0 & 0 & -2 & 1 & -4 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 3 & -1 & 3 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 1 & -1
\end{bmatrix}
\]

We are after the final six columns of this matrix, which we will multiply by \( b \)

\[
J = \begin{bmatrix}
0 & 0 & 1 & -1 & 2 & -1 \\
0 & 0 & -2 & 3 & -3 & 3 \\
0 & 0 & 1 & 1 & 3 & 3 \\
0 & 0 & -2 & 1 & -4 & 0 \\
1 & 0 & 3 & -1 & 3 & 1 \\
0 & 1 & -2 & 1 & 1 & -1
\end{bmatrix}
\]

so

\[
Jb = \begin{bmatrix}
0 & 0 & 1 & -1 & 2 & -1 \\
0 & 0 & -2 & 3 & -3 & 3 \\
0 & 0 & 1 & 1 & 3 & 3 \\
0 & 0 & -2 & 1 & -4 & 0 \\
1 & 0 & 3 & -1 & 3 & 1 \\
0 & 1 & -2 & 1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
b_5 \\
b_6
\end{bmatrix} = \begin{bmatrix}
b_3 - b_4 + 2b_5 - b_6 \\
-2b_3 + 3b_4 - 3b_5 + 3b_6 \\
b_3 + b_4 + 3b_5 + 3b_6 \\
-2b_3 + b_4 - 4b_5 \\
b_1 + 3b_3 - b_4 + 3b_5 + b_6 \\
b_2 - 2b_3 + b_4 + b_5 - b_6
\end{bmatrix}
\]

So by applying to the identity matrix the same row operations that row-reduce \( A \) (which we could do with a calculator once \( I_6 \) is placed alongside of \( A \)), we can then arrive at the result of row-reducing a column of symbols where the vector of constants usually resides. Since the row-reduced version of \( A \) has two zero rows, for a consistent system we require that

\[
b_1 + 3b_3 - b_4 + 3b_5 + b_6 = 0 \\
b_2 - 2b_3 + b_4 + b_5 - b_6 = 0
\]

Now we are exactly back where we were on the first go-round. Notice that we obtain the matrix \( L \) as simply the last two rows and last six columns of \( N \).

This example motivates the remainder of this section, so it is worth careful study. You might attempt to mimic the second approach with the coefficient matrices of \( \text{Archetype I} \) and \( \text{Archetype J} \). We will see shortly that the matrix \( L \) contains more information about \( A \) than just the column space.

Subsection EEF
Extended echelon form

The final matrix that we row-reduced in \( \text{Example CSANS} \) should look familiar in most respects to the procedure we used to compute the inverse of a nonsingular matrix, \( \text{Theorem CINM} \). We will now generalize that procedure to matrices that are not necessarily nonsingular, or even square. First a definition.

Definition EEF
Extended Echelon Form
Suppose \( A \) is an \( m \times n \) matrix. Add \( m \) new columns to \( A \) that together equal an \( m \times m \) identity
matrix to form an \( m \times (n + m) \) matrix \( M \). Use row operations to bring \( M \) to reduced row-echelon form and call the result \( N \). \( N \) is the extended reduced row-echelon form of \( A \), and we will standardize on names for five submatrices (\( B, C, J, K, L \)) of \( N \).

Let \( B \) denote the \( m \times n \) matrix formed from the first \( n \) columns of \( N \) and let \( J \) denote the \( m \times m \) matrix formed from the last \( m \) columns of \( N \). Suppose that \( B \) has \( r \) nonzero rows. Further partition \( N \) by letting \( C \) denote the \( r \times n \) matrix formed from all of the non-zero rows of \( B \). Let \( K \) be the \( r \times m \) matrix formed from the first \( r \) rows of \( J \), while \( L \) will be the \((m - r) \times m \) matrix formed from the bottom \( m - r \) rows of \( J \). Pictorially,

\[
M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix} C & K \\ 0 & L \end{bmatrix}
\]

\( \triangle \)

**Example SEEF**

Submatrices of extended echelon form

We illustrate Definition EEF with the matrix \( A \),

\[
A = \begin{bmatrix} 1 & -1 & -2 & 7 & 1 & 6 \\ -6 & 2 & -4 & -18 & -3 & -26 \\ 4 & -1 & 4 & 10 & 2 & 17 \\ 3 & -1 & 2 & 9 & 1 & 12 \end{bmatrix}
\]

Augmenting with the \( 4 \times 4 \) identity matrix, \( M = \)

\[
\begin{bmatrix} 1 & -1 & -2 & 7 & 1 & 6 & 1 & 0 & 0 & 0 \\ -6 & 2 & -4 & -18 & -3 & -26 & 0 & 1 & 0 & 0 \\ 4 & -1 & 4 & 10 & 2 & 17 & 0 & 0 & 1 & 0 \\ 3 & -1 & 2 & 9 & 1 & 12 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

and row-reducing, we obtain

\[
N = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 3 & 0 & 1 & 1 & 1 \\ 0 & 1 & 4 & -6 & 0 & -1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 \end{bmatrix}
\]

So we then obtain

\[
B = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 3 \\ 0 & 1 & 4 & -6 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 3 \\ 0 & 1 & 4 & -6 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 2 & 2 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}
\]
You can observe (or verify) the properties of the following theorem with this example.

**Theorem PEEF**

**Properties of Extended Echelon Form**

Suppose that $A$ is an $m \times n$ matrix and that $N$ is its extended echelon form. Then

1. $J$ is nonsingular.
2. $B = JA$.
3. If $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$, then $Ax = y$ if and only if $Bx = Jy$.
4. $C$ is in reduced row-echelon form, has no zero rows and has $r$ pivot columns.
5. $L$ is in reduced row-echelon form, has no zero rows and has $m - r$ pivot columns.

**Proof**  

$J$ is the result of applying a sequence of row operations to $I_m$, as such $J$ and $I_m$ are row-equivalent. $\mathcal{L}S(I_m, 0)$ has only the zero solution, since $I_m$ is nonsingular (Theorem NMRRI \[62\]). Thus, $\mathcal{L}S(J, 0)$ also has only the zero solution (Theorem REMES \[25\], Definition ESYS \[11\]) and $J$ is therefore nonsingular (Definition NSM \[54\]).

To prove the second part of this conclusion, first convince yourself that row operations and the matrix-vector are commutative operations. By this we mean the following. Suppose that $F$ is an $m \times n$ matrix that is row-equivalent to the matrix $G$. Apply to the column vector $Fw$ the same sequence of row operations that converts $F$ to $G$. Then the result is $Gw$. So we can do row operations on the matrix, then do a matrix-vector product, or do a matrix-vector product and then do row operations on a column vector, and the result will be the same either way. Since matrix multiplication is defined by a collection of matrix-vector products (), if we apply to the matrix product $FH$ the same sequence of row operations that converts $F$ to $G$ then the result will equal $GH$. Now apply these observations to $A$.

Write $AI_n = I_mA$ and apply the row operations that convert $M$ to $N$. $A$ is converted to $B$, while $I_m$ is converted to $J$, so we have $BI_n = JA$. Simplifying the left side gives the desired conclusion.

For the third conclusion, we now establish the two equivalences

$$Ax = y \iff JAx = Jy \iff Bx = Jy$$

The forward direction of the first equivalence is accomplished by multiplying both sides of the matrix equality by $J$, while the backward direction is accomplished by multiplying by the inverse of $J$ (which we know exists by Theorem NI \[204\] since $J$ is nonsingular). The second equivalence is obtained simply by the substitutions given by $JA = B$.

The first $r$ rows of $N$ are in reduced row-echelon form, since any contiguous collection of rows taken from a matrix in reduced row-echelon form will form a matrix that is again in reduced row-echelon form. Since the matrix $C$ is formed by removing the last $n$ entries of each these rows, the remainder is still in reduced row-echelon form. By its construction, $C$ has no zero rows. $C$ has $r$ rows and each contains a leading 1, so there are $r$ pivot columns in $C$.

The final $m - r$ rows of $N$ are in reduced row-echelon form, since any contiguous collection of rows taken from a matrix in reduced row-echelon form will form a matrix that is again in reduced row-echelon form. Since the matrix $L$ is formed by removing the first $n$ entries of each of these rows, and these entries are all zero (they form the zero rows of $B$), the remainder is still in reduced row-echelon form. $L$ is the final $m - r$ rows of the nonsingular matrix $J$, so none of these rows can be totally zero, or $J$ would not row-reduce to the identity matrix. $L$ has $m - r$ rows and each contains a leading 1, so there are $m - r$ pivot columns in $L$.

Notice that in the case where $A$ is a nonsingular matrix we know that the reduced row-echelon form of $A$ is the identity matrix (Theorem NMRRI \[62\]), so $B = I_n$. Then the second conclusion
above says \( JA = B = I_n \), so \( J \) is the inverse of \( A \). Thus this theorem generalizes Theorem CINM 193, though the result is a “left-inverse” of \( A \) rather than a “right-inverse.”

The third conclusion of Theorem PEEF 236 is the most telling. It says that \( x \) is a solution to the linear system \( L{\mathcal S}(A, y) \) if and only if \( x \) is a solution to the linear system \( L{\mathcal S}(B, Jy) \). Or said differently, if we row-reduce the augmented matrix \([A | x]\) we will get the augmented matrix \([B | Jy]\). The matrix \( J \) tracks the cumulative effect of the row operations that converts \( A \) to reduced row-echelon form, here effectively applying them to the vector of constants in a system of equations having \( A \) as a coefficient matrix. When \( A \) row-reduces to a matrix with zero rows, then \( Jy \) should also have zero entries in the same rows if the system is to be consistent.

Subsection FS. Four Subsets

With all the preliminaries in place we can state our main result for this section. In essence this result will allow us to say that we can find linearly independent sets to use in span constructions for all four subsets (null space, column space, row space, left null space) by analyzing only the extended echelon form of the matrix, and specifically, just the two submatrices \( C \) and \( L \), which will be ripe for analysis since they are already in reduced row-echelon form (Theorem PEEF 236).

Theorem FS

Four Subsets

Suppose \( A \) is an \( m \times n \) matrix with extended echelon form \( N \). Suppose the reduced row-echelon form of \( A \) has \( r \) nonzero rows. Then \( C \) is the submatrix of \( N \) formed from the first \( r \) rows and the first \( n \) columns and \( L \) is the submatrix of \( N \) formed from the last \( m \) columns and the last \( m - r \) rows. Then

1. The null space of \( A \) is the null space of \( C \), \( \mathcal{N}(A) = \mathcal{N}(C) \).
2. The row space of \( A \) is the row space of \( C \), \( \mathcal{R}(A) = \mathcal{R}(C) \).
3. The column space of \( A \) is the null space of \( L \), \( \mathcal{C}(A) = \mathcal{N}(L) \).
4. The left null space of \( A \) is the row space of \( L \), \( \mathcal{L}(A) = \mathcal{R}(L) \).

□

Proof  First, \( \mathcal{N}(A) = \mathcal{N}(B) \) since \( B \) is row-equivalent to \( A \) (Theorem REMES 25). The zero rows of \( B \) represent equations that are always true in the homogeneous system \( L{\mathcal S}(B, \mathbf{0}) \), so the removal of these equations will not change the solution set. Thus, in turn, \( \mathcal{N}(B) = \mathcal{N}(C) \).

Second, \( \mathcal{R}(A) = \mathcal{R}(B) \) since \( B \) is row-equivalent to \( A \) (Theorem REMRS 218). The zero rows of \( B \) contribute nothing to the span that is the row space of \( B \), so the removal of these rows will not change the row space. Thus, in turn, \( \mathcal{R}(B) = \mathcal{R}(C) \).

Third, we prove the set equality \( \mathcal{C}(A) = \mathcal{N}(L) \) with Definition SE 616. Begin by showing that \( \mathcal{C}(A) \subseteq \mathcal{N}(L) \). Choose \( y \in \mathcal{C}(A) \subseteq \mathbb{C}^m \). Then there exists a vector \( x \in \mathbb{C}^n \) such that \( Ax = y \) (Theorem CSCS 212). Then for \( 1 \leq k \leq m - r \),

\[
[Ly]_k = [Jy]_{r+k} = [Bx]_{r+k} = [Cx]_k = [0]_k
\]

So, for all \( 1 \leq k \leq m - r \), \( [Ly]_k = [0]_k \). So by Definition CVE 73 we have \( Ly = \mathbf{0} \) and thus \( y \in \mathcal{N}(L) \).

Now, show that \( \mathcal{N}(L) \subseteq \mathcal{C}(A) \). Choose \( y \in \mathcal{N}(L) \subseteq \mathbb{C}^m \). Form the vector \( Ky \in \mathbb{C}^r \). The linear system \( L{\mathcal S}(C, Ky) \) is consistent since \( C \) is in reduced row-echelon form and has no zero rows (Theorem PEEF 236). Let \( x \in \mathbb{C}^n \) denote a solution to \( L{\mathcal S}(C, Ky) \).
Then for \(1 \leq j \leq r\),
\[
[Bx]_j = [Cx]_j
\]
\[
= [Ky]_j
\]
\[
= [Jy]_j
\]
\(C\) a submatrix of \(B\)
x a solution to \(\mathcal{L}S(C, Ky)\)
\(K\) a submatrix of \(J\)

And for \(r + 1 \leq k \leq m\),
\[
[Bx]_k = [Ox]_{k-r}
\]
\[
= [0]_{k-r}
\]
\[
= [Ly]_{k-r}
\]
\[
= [Jy]_k
\]
Zero matrix a submatrix of \(B\)

So for all \(1 \leq i \leq m\), \([Bx]_i = [Jy]_i\), and by Definition CVE [73] we have \(Bx = Jy\). From Theorem PEEF [236] we know then that \(Ax = y\), and therefore \(y \in \mathcal{C}(A)\) (Theorem CSCS [212]). By Definition SE [616] we have \(\mathcal{C}(A) = \mathcal{N}(L)\).

Fourth, we prove the set equality \(\mathcal{L}(A) = \mathcal{R}(L)\) with Definition SE [616]. Begin by showing that \(\mathcal{R}(L) \subseteq \mathcal{L}(A)\). Choose \(y \in \mathcal{R}(L) \subseteq \mathbb{C}^m\). Then there exists a vector \(w \in \mathbb{C}^{m-r}\) such that \(y = L'w\) (Definition RSM [217], Theorem CSCS [212]). Then for \(1 \leq i \leq n\),
\[
[A'y]_i = \sum_{k=1}^{m} \left[ A^T \right]_{ik} [y]_k
\]
\[
= \sum_{k=1}^{m} \left[ A^T \right]_{ik} [L'w]_k
\]
\[
= \sum_{k=1}^{m} \left[ A^T \right]_{ik} \sum_{\ell=1}^{m-r} [L']_{k\ell} [w]_{\ell}
\]
\[
= \sum_{\ell=1}^{m-r} \left( \sum_{k=1}^{m} \left[ A^T \right]_{ik} [L']_{k\ell} \right) [w]_{\ell}
\]
\[
= \sum_{\ell=1}^{m-r} \left( \sum_{k=1}^{m} \left[ A^T \right]_{ik} [J']_{k,r+\ell} \right) [w]_{\ell}
\]
\(L\) a submatrix of \(J\)

Since \([A'y]_i = [0]_i\) for \(1 \leq i \leq n\), Definition CVE [73] implies that \(A'y = 0\). This means that \(y \in \mathcal{N}(A')\).

Now, show that \(\mathcal{L}(A) \subseteq \mathcal{R}(L)\). Choose \(y \in \mathcal{L}(A) \subseteq \mathbb{C}^m\). The matrix \(J\) is nonsingular (Theorem PEEF [236]), so \(J'\) is also nonsingular (Theorem MIT [196]) and therefore the linear system...
\( \mathcal{L}(J^t, y) \) has a unique solution. Denote this solution as \( x \in \mathbb{C}^m \). We will need to work with two “halves” of \( x \), which we will denote as \( z \) and \( w \) with formal definitions given by

\[
[z]_j = [x]_i, \quad 1 \leq j \leq r, \quad [w]_k = [x]_{r+k}, \quad 1 \leq k \leq m-r
\]

Now, for \( 1 \leq j \leq r \),

\[
[C^t z]_j = \sum_{k=1}^{r} [C^t]_{jk} [z]_k \quad \text{Theorem EMP 177}
\]

\[
= \sum_{k=1}^{r} [C^t]_{jk} [z]_k + \sum_{\ell=1}^{m-r} [O]_{j\ell} [w]_\ell \quad \text{Definition ZM 166}
\]

\[
= \sum_{k=1}^{r} [B^t]_{jk} [z]_k + \sum_{\ell=1}^{m-r} [B^t]_{j,r+\ell} [w]_\ell \quad C, O \text{ submatrices of } B
\]

\[
= \sum_{k=1}^{r} [B^t]_{jk} [x]_k + \sum_{\ell=1}^{m-r} [B^t]_{j,r+\ell} [x]_{r+\ell} \quad \text{Definitions of } z \text{ and } w
\]

\[
= \sum_{k=1}^{r} [B^t]_{jk} [x]_k + \sum_{k=r+1}^{m} [B^t]_{jk} [x]_k \quad \text{Re-index second sum}
\]

\[
= \sum_{k=1}^{m} [B^t]_{jk} [x]_k \quad \text{Combine sums}
\]

\[
= \sum_{k=1}^{m} [(J A)^t]_{jk} [x]_k \quad \text{Theorem PEEF 236}
\]

\[
= \sum_{k=1}^{m} [A^t J^t]_{jk} [x]_k \quad \text{Theorem MMT 182}
\]

\[
= \sum_{k=1}^{m} \sum_{\ell=1}^{m} [A^t]_{j\ell} [J^t]_{\ell k} [x]_k \quad \text{Theorem EMP 177}
\]

\[
= \sum_{\ell=1}^{m} [A^t]_{j\ell} \left( \sum_{k=1}^{m} [J^t]_{\ell k} [x]_k \right) \quad \text{Commutativity, Distributivity in } \mathbb{C}
\]

\[
= \sum_{\ell=1}^{m} [A^t]_{j\ell} [J^t x]_\ell \quad \text{Theorem EMP 177}
\]

\[
= \sum_{\ell=1}^{m} [A^t]_{j\ell} [y]_\ell \quad \text{Definition of } x
\]

\[
= [A^t y]_j \quad \text{Theorem EMP 177}
\]

\[
= [0]_j \quad y \in \mathcal{L}(A)
\]

So, by Definition CVE 73, \( C^t z = 0 \) and the vector \( z \) gives us a linear combination of the columns of \( C^t \) that equals the zero vector. In other words, \( z \) gives a relation of linear dependence on the the rows of \( C \). However, the rows of \( C \) are a linearly independent set by Theorem BRS 220. According to Definition LICV 121, we must conclude that the entries of \( z \) are all zero, i.e. \( z = 0 \).

Now, for \( 1 \leq i \leq m \), we have

\[
[y]_i = [J^t x]_i \quad \text{Definition of } x
\]

\[
= \sum_{k=1}^{m} [J^t]_{ik} [x]_k \quad \text{Theorem EMP 177}
\]

\[
= \sum_{k=1}^{r} [J^t]_{ik} [x]_k + \sum_{k=r+1}^{m} [J^t]_{ik} [x]_k \quad \text{Break apart sum}
\]

\[ \]
\[ \begin{align*}
\sum_{k=1}^{r} [J^T]_{ik} [z]_{k} + \sum_{k=r+1}^{m} [J^T]_{ik} [w]_{k-r} &= \text{Definition of } z \text{ and } w \\
\sum_{k=1}^{r} [J^T]_{ik} 0 + \sum_{\ell=1}^{m-r} [J^T]_{i,r+\ell} [w]_{\ell} &= \text{z = 0, re-index } \\
0 + \sum_{\ell=1}^{m-r} [L^T]_{i,\ell} [w]_{\ell} &= L \text{ a submatrix of } J \text{ (Theorem EMP 177)}
\end{align*} \]

So by Definition CVE 73, \( y = L^t w \). The existence of \( w \) implies that \( y \in \mathcal{R}(L) \), and therefore \( \mathcal{L}(A) \subseteq \mathcal{R}(L) \). So by Definition SE 616 we have \( \mathcal{L}(A) = \mathcal{R}(L) \). \( \square \)

The first two conclusions of this theorem are nearly trivial. But they set up a pattern of results for \( C \) that is reflected in the latter two conclusions about \( L \). In total, they tell us that we can compute all four subsets just by finding null spaces and row spaces. This theorem does not tell us exactly how to compute these subsets, but instead simply expresses them as null spaces and row spaces of matrices in reduced row-echelon form without any zero rows (\( C \) and \( L \)). A linearly independent set that spans the null space of a matrix in reduced row-echelon form can be found easily with Theorem BNS 128. It is an even easier matter to find a linearly independent set that spans the row space of a matrix in reduced row-echelon form with Theorem BRS 220. 

The situation when \( r = m \) deserves comment, since now the matrix \( L \) has no rows. What is \( \mathcal{C}(A) \) when we try to apply Theorem FS 237 and encounter \( \mathcal{N}(L) \)? One interpretation of this situation is that \( L \) is the coefficient matrix of a homogeneous system that has no equations. How hard is it to find a solution vector to this system? Some thought will convince you that any proposed vector will qualify as a solution, since it makes all of the equations true. So every possible vector is in the null space of \( L \) and therefore \( \mathcal{C}(A) = \mathcal{N}(L) = \mathbb{C}^m \). OK, perhaps this sounds like some twisted argument from Alice in Wonderland. Let us try another argument that might solidly convince you of this logic.

If \( r = m \), when we row-reduce the augmented matrix of \( \mathcal{L}S(A, b) \) the result will have no zero rows, and all the leading 1’s will occur in first \( n \) columns, so by Theorem RCLS 45 the system will be consistent. By Theorem CSCS 212, \( b \in \mathcal{C}(A) \). Since \( b \) was arbitrary, every possible vector is in the column space of \( A \), so we again have \( \mathcal{C}(A) = \mathbb{C}^m \). The situation when a matrix has \( r = m \) is known by the term full rank, and in the case of a square matrix coincides with nonsingularity (see Exercise FS.M50 246). 

The properties of the matrix \( L \) described by this theorem can be explained informally as follows. A column vector \( y \in \mathbb{C}^m \) is in the column space of \( A \) if the linear system \( \mathcal{L}S(A, y) \) is consistent (Theorem CSCS 212). By Theorem RCLS 45, the reduced row-echelon form of the augmented matrix \( [A \mid y] \) of a consistent system will have zeros in the bottom \( m - r \) locations of the last column. By Theorem PEEF 236 this final column is the vector \( Jy \) and so should then have zeros in the final \( m - r \) locations. But since \( L \) comprises the final \( m - r \) rows of \( J \), this condition is expressed by saying \( y \in \mathcal{N}(L) \).

Additionally, the rows of \( J \) are the scalars in linear combinations of the rows of \( A \) that create the rows of \( B \). That is, the rows of \( J \) record the net effect of the sequence of row operations that takes \( A \) to its reduced row-echelon form, \( B \). This can be seen in the equation \( JA = B \) (Theorem PEEF 236). As such, the rows of \( L \) are scalars for linear combinations of the rows of \( A \) that yield zero rows. But such linear combinations are precisely the elements of the left null space. So any element of the row space of \( L \) is also an element of the left null space of \( A \). We will now illustrate Theorem FS 237 with a few examples.

Example FS1
**Four subsets, #1**

In Example SEEF \[235\] we found the five relevant submatrices of the matrix

\[
A = \begin{bmatrix}
1 & -1 & -2 & 7 & 1 & 6 \\
-6 & 2 & -4 & -18 & -3 & -26 \\
4 & -1 & 4 & 10 & 2 & 17 \\
3 & -1 & 2 & 9 & 1 & 12
\end{bmatrix}
\]

To apply Theorem FS \[237\] we only need \(C\) and \(L\),

\[
C = \begin{bmatrix}
1 & 0 & 2 & 1 & 0 & 3 \\
0 & 1 & 4 & -6 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 2
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
1 & 2 & 2 & 1
\end{bmatrix}
\]

Then we use Theorem FS \[237\] to obtain

\[\mathcal{N}(A) = \mathcal{N}(C) = \langle \begin{bmatrix}
-2 \\
-4 \\
-1 \\
0 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
6 \\
1 \\
0 \\
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
-3 \\
1 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}\rangle \quad \text{Theorem BNS} \[128\]

\[\mathcal{R}(A) = \mathcal{R}(C) = \langle \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
2
\end{bmatrix}\rangle \quad \text{Theorem BRS} \[220\]

\[\mathcal{C}(A) = \mathcal{N}(L) = \langle \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
-2 \\
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}\rangle \quad \text{Theorem BNS} \[128\]

\[\mathcal{L}(A) = \mathcal{R}(L) = \langle \begin{bmatrix}
1 \\
2 \\
2
\end{bmatrix}\rangle \quad \text{Theorem BRS} \[220\]

Boom!

**Example FS2**

**Four subsets, #2**

Now let’s return to the matrix \(A\) that we used to motivate this section in Example CSANS \[232\],

\[
A = \begin{bmatrix}
10 & 0 & 3 & 8 & 7 \\
-16 & -1 & -4 & -10 & -13 \\
-6 & 1 & -3 & -6 & -6 \\
0 & 2 & -2 & -3 & -2 \\
3 & 0 & 1 & 2 & 3 \\
-1 & -1 & 1 & 1 & 0
\end{bmatrix}
\]

We form the matrix \(M\) by adjoining the \(6 \times 6\) identity matrix \(I_6\),

\[
M = \begin{bmatrix}
10 & 0 & 3 & 8 & 7 & 1 & 0 & 0 & 0 & 0 & 0 \\
-16 & -1 & -4 & -10 & -13 & 0 & 1 & 0 & 0 & 0 & 0 \\
-6 & 1 & -3 & -6 & -6 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & -2 & -3 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\
3 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
and row-reduce to obtain $N$

$$N = \begin{bmatrix}
1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & -1 & 2 & -1 \\
0 & 1 & 0 & 0 & -3 & 0 & 0 & -2 & 3 & -3 & 3 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 3 & 3 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & -2 & 1 & -4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 1 & -1
\end{bmatrix}$$

To find the four subsets for $A$, we only need identify the $4 \times 5$ matrix $C$ and the $2 \times 6$ matrix $L$,

$$C = \begin{bmatrix}
1 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & -3 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & -2
\end{bmatrix} \quad \quad L = \begin{bmatrix}
1 & 0 & 3 & -1 & 3 & 1 \\
0 & 1 & -2 & 1 & 1 & -1
\end{bmatrix}$$

Then we apply Theorem FS [237],

$$\mathcal{N}(A) = \mathcal{N}(C) = \span\begin{bmatrix}
-2 \\
-1 \\
2 \\
1
\end{bmatrix} \tag*{Theorem BNS [128]}$$

$$\mathcal{R}(A) = \mathcal{R}(C) = \span\begin{bmatrix}
1 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & -3 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & -2
\end{bmatrix} \tag*{Theorem BRS [220]}$$

$$\mathcal{C}(A) = \mathcal{N}(L) = \span\begin{bmatrix}
1 & 0 & 0 & 0 & 2 \\
1 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -2
\end{bmatrix} \tag*{Theorem BNS [128]}$$

$$\mathcal{L}(A) = \mathcal{R}(L) = \span\begin{bmatrix}
1 & 0 & 0 \\
3 & -1 & 0 \\
-1 & 1 & 1 \\
3 & 1 & -1
\end{bmatrix} \tag*{Theorem BRS [220]}$$

The next example is just a bit different since the matrix has more rows than columns, and a trivial null space.

**Example FSAG**

**Four subsets, Archetype G**

Archetype G [659] and Archetype H [663] are both systems of $m = 5$ equations in $n = 2$ variables. They have identical coefficient matrices, which we will denote here as the matrix $G$,

$$G = \begin{bmatrix}
2 & 3 \\
-1 & 4 \\
3 & 10 \\
3 & -1 \\
6 & 9
\end{bmatrix}$$

Version 1.04
Adjoin the $5 \times 5$ identity matrix, $I_5$, to form

$$M = \begin{bmatrix}
2 & 3 & 1 & 0 & 0 & 0 \\
-1 & 4 & 0 & 1 & 0 & 0 \\
3 & 10 & 0 & 0 & 1 & 0 \\
3 & -1 & 0 & 0 & 0 & 1 \\
6 & 9 & 0 & 0 & 0 & 1
\end{bmatrix}$$

This row-reduces to

$$N = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \frac{3}{\pi} & \frac{1}{\pi} \\
0 & 1 & 0 & 0 & 0 & -\frac{1}{\pi} & \frac{1}{\pi} \\
0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{\pi} \\
0 & 0 & 0 & 1 & 0 & 1 & -\frac{1}{\pi} \\
0 & 0 & 0 & 0 & 1 & 1 & -1
\end{bmatrix}$$

The first $n = 2$ columns contain $r = 2$ leading 1’s, so we obtain $C$ as the $2 \times 2$ identity matrix and extract $L$ from the final $m - r = 3$ rows in the final $m = 5$ columns.

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{\pi} \\
0 & 1 & 0 & 1 & -\frac{1}{\pi} \\
0 & 0 & 1 & 1 & -1
\end{bmatrix}$$

Then we apply Theorem FS [237],

$\mathcal{N}(G) = \mathcal{N}(C) = \langle \rangle = \{0\}$ \hspace{1cm} \text{Theorem BNS} 128

$\mathcal{R}(G) = \mathcal{R}(C) = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle = \mathbb{C}^2$ \hspace{1cm} \text{Theorem BRS} 220

$\mathcal{C}(G) = \mathcal{N}(L) = \langle \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \end{bmatrix} \rangle$ \hspace{1cm} \text{Theorem BNS} 128

$\mathcal{L}(G) = \mathcal{R}(L) = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\frac{1}{3} \\ 0 \\ \frac{1}{3} \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} \rangle$ \hspace{1cm} \text{Theorem BRS} 220

As mentioned earlier, Archetype G [659] is consistent, while Archetype H [663] is inconsistent. See if you can write the two different vectors of constants from these two archetypes as linear combinations of the two vectors in $\mathcal{C}(G)$. How about the two columns of $G$, can you write each individually as a linear combination of the two vectors in $\mathcal{C}(G)$? They must be in the column space of $G$ also. Are your answers unique? Do you notice anything about the scalars that appear in the linear combinations you are forming?

Example COV [141] and Example CSROI [221] each describes the column space of the coefficient matrix from Archetype I [667] as the span of a set of $r = 3$ linearly independent vectors. It is no
accident that these two different sets both have the same size. If we (you?) were to calculate the column space of this matrix using the null space of the matrix $L$ from Theorem FS \[237\] then we would again find a set of 3 linearly independent vectors that span the range. More on this later.

So we have three different methods to obtain a description of the column space of a matrix as the span of a linearly independent set. Theorem BCS \[214\] is sometimes useful since the vectors it specifies are equal to actual columns of the matrix. Theorem BRS \[220\] and Theorem CSRST \[221\] combine to create vectors with lots of zeros, and strategically placed 1’s near the top of the vector. Theorem FS \[237\] and the matrix $L$ from the extended echelon form gives us a third method, which tends to create vectors with lots of zeros, and strategically placed 1’s near the bottom of the vector. If we don’t care about linear independence we can also appeal to Definition CSM \[211\] and simply express the column space as the span of all the columns of the matrix, giving us a fourth description.

Although we have many ways to describe a column space, notice that one tempting strategy will usually fail. It is not possible to simply row-reduce a matrix directly and then use the columns of the row-reduced matrix as a set whose span equals the column space. In other words, row operations do not preserve column spaces (however row operations do preserve row spaces, Theorem REMRS \[218\]). See Exercise CRS.M21 \[225\].

Subsection READ  
Reading Questions

1. Find a nontrivial element of the left null space of $A$.

\[
A = \begin{bmatrix}
2 & 1 & -3 & 4 \\
-1 & -1 & 2 & -1 \\
0 & -1 & 1 & 2
\end{bmatrix}
\]

2. Find the matrices $C$ and $L$ in the extended echelon form of $A$.

\[
A = \begin{bmatrix}
-9 & 5 & -3 \\
2 & -1 & 1 \\
-5 & 3 & -1
\end{bmatrix}
\]

3. Why is Theorem FS \[237\] a great way to conclude Chapter M \[163\]?
Subsection EXC
Exercises

C20 Example FSAG concludes with several questions. Perform the analysis suggested by these questions. Contributed by Robert Beezer

C25 Given the matrix $A$ below, use the extended echelon form of $A$ to answer each part of this problem. In each part, find a linearly independent set of vectors, $S$, so that the span of $S$, $\langle S \rangle$, equals the specified set of vectors.

$$A = \begin{bmatrix}
-5 & 3 & -1 \\
-1 & 1 & 1 \\
-8 & 5 & -1 \\
3 & -2 & 0
\end{bmatrix}$$

(a) The row space of $A$, $\mathcal{R}(A)$.
(b) The column space of $A$, $\mathcal{C}(A)$.
(c) The null space of $A$, $\mathcal{N}(A)$.
(d) The left null space of $A$, $\mathcal{L}(A)$.

Contributed by Robert Beezer Solution

C26 For the matrix $D$ below use the extended echelon form to find

(a) a linearly independent set whose span is the column space of $D$.
(b) a linearly independent set whose span is the left null space of $D$.

$$D = \begin{bmatrix}
-7 & -11 & -19 & -15 \\
6 & 10 & 18 & 14 \\
3 & 5 & 9 & 7 \\
-1 & -2 & -4 & -3
\end{bmatrix}$$

Contributed by Robert Beezer Solution

C41 The following archetypes are systems of equations. For each system, write the vector of constants as a linear combination of the vectors in the span construction for the column space provided by Theorem FS and Theorem BNS (these vectors are listed for each of these archetypes).

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671

Contributed by Robert Beezer

C43 The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute the extended echelon form $N$ and identify the matrices $C$ and $L$. Using Theorem FS, Theorem BNS, and Theorem BRS express the null space, the row space, the column space and left null space of each coefficient matrix as a span of a linearly
independent set.

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647
Archetype E 651
Archetype F 654
Archetype G 659
Archetype H 663
Archetype I 667
Archetype J 671
Archetype K 676
Archetype L 680

Contributed by Robert Beezer

C60 For the matrix $B$ below, find sets of vectors whose span equals the column space of $B$ ($C(B)$) and which individually meet the following extra requirements.

(a) The set illustrates the definition of the column space.
(b) The set is linearly independent and the members of the set are columns of $B$.
(c) The set is linearly independent with a “nice pattern of zeros and ones” at the top of each vector.
(d) The set is linearly independent with a “nice pattern of zeros and ones” at the bottom of each vector.

$$B = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 2 & 3 & -4 \end{bmatrix}$$

Contributed by Robert Beezer Solution 248

C61 Let $A$ be the matrix below, and find the indicated sets with the requested properties.

$$A = \begin{bmatrix} 2 & -1 & 5 & -3 \\ -5 & 3 & -12 & 7 \\ 1 & 1 & 4 & -3 \end{bmatrix}$$

(a) A linearly independent set $S$ so that $C(A) = \langle S \rangle$ and $S$ is composed of columns of $A$.
(b) A linearly independent set $S$ so that $C(A) = \langle S \rangle$ and the vectors in $S$ have a nice pattern of zeros and ones at the top of the vectors.
(c) A linearly independent set $S$ so that $C(A) = \langle S \rangle$ and the vectors in $S$ have a nice pattern of zeros and ones at the bottom of the vectors.
(d) A linearly independent set $S$ so that $R(A) = \langle S \rangle$.

Contributed by Robert Beezer Solution 249

M50 Suppose that $A$ is a nonsingular matrix. Extend the four conclusions of Theorem FS 237 in this special case and discuss connections with previous results (such as Theorem NME4 216).

Contributed by Robert Beezer

M51 Suppose that $A$ is a singular matrix. Extend the four conclusions of Theorem FS 237 in this special case and discuss connections with previous results (such as Theorem NME4 216).

Contributed by Robert Beezer
Add a $4 \times 4$ identity matrix to the right of $A$ to form the matrix $M$ and then row-reduce to the matrix $N$,

$$M = \begin{bmatrix}
-5 & 3 & -1 & 1 & 0 & 0 & 0 \\
-1 & 1 & 1 & 0 & 1 & 0 & 0 \\
-8 & 5 & -1 & 0 & 0 & 1 & 0 \\
3 & -2 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \text{ RREF } \begin{bmatrix}
1 & 0 & 2 & 0 & 0 & -2 & -5 \\
0 & 1 & 3 & 0 & 0 & -3 & -8 \\
0 & 0 & 0 & 1 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 1 & 1 & 3
\end{bmatrix} = N$$

To apply Theorem FS in each of these four parts, we need the two matrices,

$$C = \begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 3
\end{bmatrix} \quad L = \begin{bmatrix}
1 & 0 & -1 & -1 \\
0 & 1 & 1 & 3
\end{bmatrix}$$

(a) $\mathcal{R}(A) = \mathcal{R}(C)$

$$= \left\langle \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\rangle$$

(b) $\mathcal{C}(A) = \mathcal{N}(L)$

$$= \left\langle \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

(c) $\mathcal{N}(A) = \mathcal{N}(C)$

$$= \left\langle \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} \right\rangle$$

(d) $\mathcal{L}(A) = \mathcal{R}(L)$

$$= \left\langle \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\rangle$$

For both parts, we need the extended echelon form of the matrix.

$$\begin{bmatrix}
-7 & -11 & -19 & -15 & 1 & 0 & 0 & 0 \\
6 & 10 & 18 & 14 & 0 & 1 & 0 & 0 \\
3 & 5 & 9 & 7 & 0 & 0 & 1 & 0 \\
-1 & -2 & -4 & -3 & 0 & 0 & 0 & 1
\end{bmatrix} \text{ RREF } \begin{bmatrix}
1 & 0 & -2 & -1 & 0 & 0 & 2 & 5 \\
0 & 1 & 3 & 2 & 0 & 0 & -1 & -3 \\
0 & 0 & 0 & 0 & 1 & 0 & 3 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 0
\end{bmatrix}$$
From this matrix we extract the last two rows, in the last four columns to form the matrix $L$,

$$L = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

(a) By [Theorem FS 237](#) and [Theorem BNS 128](#) we have

$$C(D) = N(L) = \left\langle \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

(b) By [Theorem FS 237](#) and [Theorem BRS 220](#) we have

$$L(D) = R(L) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix} \right\rangle$$

**C60** Contributed by Robert Beezer  Statement 246

(a) The definition of the column space is the span of the set of columns ([Definition CSM 211](#)). So the desired set is just the four columns of $B$,

$$S = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} \right\}$$

(b) [Theorem BCS 214](#) suggests row-reducing the matrix and using the columns of $B$ that correspond to the pivot columns.

$$B \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the pivot columns are numbered by elements of $D = \{1, 2\}$, so the requested set is

$$S = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}$$

(c) We can find this set by row-reducing the transpose of $B$, deleting the zero rows, and using the nonzero rows as column vectors in the set. This is an application of [Theorem CSRST 221](#) followed by [Theorem BRS 220](#).

$$B^t \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

So the requested set is

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -7 \end{bmatrix} \right\}$$

(d) With the column space expressed as a null space, the vectors obtained via [Theorem BNS 128](#) will be of the desired shape. So we first proceed with [Theorem FS 237](#) and create the extended echelon form,

$$[B | I_3] = \begin{bmatrix} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ -1 & 2 & 3 & -4 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 & 2 & 0 & \frac{3}{5} \\ 0 & 1 & 1 & -1 & 0 & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
So, employing Theorem \textit{FS} 237, we have $\mathcal{C}(B) = \mathcal{N}(L)$, where

\[ L = \begin{bmatrix} 1 & -\frac{7}{3} & -\frac{1}{3} \end{bmatrix} \]

We can find the desired set of vectors from Theorem \textit{BNS} 128 as

\[ S = \left\{ \begin{bmatrix} \frac{7}{3} & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 0 & 1 \end{bmatrix} \right\} \]

\textbf{C61} Contributed by \textit{Robert Beezer} Statement 246

(a) First find a matrix $B$ that is row-equivalent to $A$ and in reduced row-echelon form

\[ B = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

By Theorem \textit{BCS} 214 we can choose the columns of $A$ that correspond to dependent variables ($D = \{1, 2\}$) as the elements of $S$ and obtain the desired properties. So

\[ S = \left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\} \]

(b) We can write the column space of $A$ as the row space of the transpose (Theorem \textit{CSRST} 221). So we row-reduce the transpose of $A$ to obtain the row-equivalent matrix $C$ in reduced row-echelon form

\[ C = \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

The nonzero rows (written as columns) will be a linearly independent set that spans the row space of $A'$, by Theorem \textit{BRS} 220, and the zeros and ones will be at the top of the vectors,

\[ S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\} \]

(c) In preparation for Theorem \textit{FS} 237, augment $A$ with the $3 \times 3$ identity matrix $I_3$ and row-reduce to obtain the extended echelon form,

\[ \begin{bmatrix} 1 & 0 & 3 & -2 & 0 & -\frac{1}{3} \\ 0 & 1 & 1 & -1 & 0 & \frac{3}{8} \\ 0 & 0 & 0 & 1 & \frac{3}{8} & -\frac{1}{8} \end{bmatrix} \]

Then since the first four columns of row 3 are all zeros, we extract

\[ L = \begin{bmatrix} 1 & \frac{3}{8} & -\frac{1}{8} \end{bmatrix} \]

Theorem \textit{FS} 237 says that $\mathcal{C}(A) = \mathcal{N}(L)$. We can then use Theorem \textit{BNS} 128 to construct the desired set $S$, based on the free variables with indices in $F = \{2, 3\}$ for the homogeneous system $\mathcal{L}S(L, 0)$, so

\[ S = \left\{ \begin{bmatrix} -\frac{2}{8} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{8} \\ 0 \\ 1 \end{bmatrix} \right\} \]
Notice that the zeros and ones are at the bottom of the vectors.

(d) This is a straightforward application of Theorem BRS \[220\]. Use the row-reduced matrix $B$ from part (a), grab the nonzero rows, and write them as column vectors,

$$S = \begin{Bmatrix}
\begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}
\end{Bmatrix}$$
Chapter VS
Vector Spaces

We now have a computational toolkit in place and so we can begin our study of linear algebra in a more theoretical style.

Linear algebra is the study of two fundamental objects, vector spaces and linear transformations (see Chapter LT [405]). This chapter will focus on the former. The power of mathematics is often derived from generalizing many different situations into one abstract formulation, and that is exactly what we will be doing throughout this chapter.

Section VS
Vector Spaces

In this section we present a formal definition of a vector space, which will lead to an extra increment of abstraction. Once defined, we study its most basic properties.

Subsection VS
Vector Spaces

Here is one of our two most important definitions in the entire course.

Definition VS
Vector Space
Suppose that $V$ is a set upon which we have defined two operations: (1) vector addition, which combines two elements of $V$ and is denoted by “$+$”, and (2) scalar multiplication, which combines a complex number with an element of $V$ and is denoted by juxtaposition. Then $V$, along with the two operations, is a vector space if the following ten properties hold.

- **AC** Additive Closure
  If $u, v \in V$, then $u + v \in V$.

- **SC** Scalar Closure
  If $\alpha \in \mathbb{C}$ and $u \in V$, then $\alpha u \in V$.

- **C** Commutativity
  If $u, v \in V$, then $u + v = v + u$.

- **AA** Additive Associativity
  If $u, v, w \in V$, then $u + (v + w) = (u + v) + w$.

- **Z** Zero Vector
  There is a vector, $0$, called the zero vector, such that $u + 0 = u$ for all $u \in V$. 

251
• **AI Additive Inverses**
  If \( u \in V \), then there exists a vector \(-u \in V\) so that \( u + (-u) = 0 \).

• **SMA Scalar Multiplication Associativity**
  If \( \alpha, \beta \in \mathbb{C} \) and \( u \in V \), then \( \alpha(\beta u) = (\alpha\beta)u \).

• **DVA Distributivity across Vector Addition**
  If \( \alpha \in \mathbb{C} \) and \( u, v \in V \), then \( \alpha(u + v) = \alpha u + \alpha v \).

• **DSA Distributivity across Scalar Addition**
  If \( \alpha, \beta \in \mathbb{C} \) and \( u \in V \), then \( (\alpha + \beta)u = \alpha u + \beta u \).

• **O One**
  If \( u \in V \), then \( 1u = u \).

The objects in \( V \) are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space.

Now, there are several important observations to make. Many of these will be easier to understand on a second or third reading, and especially after carefully studying the examples in Subsection VS.EVS [252].

An **axiom** is often a “self-evident” truth. Something so fundamental that we all agree it is true and accept it without proof. Typically, it would be the logical underpinning that we would begin to build theorems upon. Some might refer to the ten properties of Definition VS [251] as axioms, implying that a vector space is a very natural object and the ten properties are the essence of a vector space. We will instead emphasize that we will begin with a definition of a vector space. After studying the remainder of this chapter, you might return here and remind yourself how all our forthcoming theorems and definitions rest on this foundation.

As we will see shortly, the objects in \( V \) can be anything, even though we will call them vectors. We have been working with vectors frequently, but we should stress here that these have so far just been **column** vectors — scalars arranged in a columnar list of fixed length. In a similar vein, you have used the symbol “+” for many years to represent the addition of numbers (scalars). We have extended its use to the addition of column vectors and to the addition of matrices, and now we are going to recycle it even further and let it denote vector addition in any possible vector space. So when describing a new vector space, we will have to **define** exactly what “+” is. Similar comments apply to scalar multiplication. Conversely, we can define our operations any way we like, so long as the ten properties are fulfilled (see Example CVS [255]).

A vector space is composed of three objects, a set and two operations. However, we usually use the same symbol for both the set and the vector space itself. Do not let this convenience fool you into thinking the operations are secondary!

This discussion has either convinced you that we are really embarking on a new level of abstraction, or they have seemed cryptic, mysterious or nonsensical. You might want to return to this section in a few days and give it another read then. In any case, let’s look at some concrete examples now.

**Subsection EVS Examples of Vector Spaces**

Our aim in this subsection is to give you a storehouse of examples to work with, to become comfortable with the ten vector space properties and to convince you that the multitude of examples justifies (at least initially) making such a broad definition as Definition VS [251]. Some of our claims will be justified by reference to previous theorems, we will prove some facts from scratch, and we will do one non-trivial example completely. In other places, our usual thoroughness will be neglected, so grab paper and pencil and play along.
Example VSCV

The vector space $\mathbb{C}^m$

Set: $\mathbb{C}^m$, all column vectors of size $m$, Definition VSCV [72].

Equality: Entry-wise, Definition CVE [73].

Vector Addition: The “usual” addition, given in Definition CVA [73].

Scalar Multiplication: The “usual” scalar multiplication, given in Definition CVSM [74].

Does this set with these operations fulfill the ten properties? Yes. And by design all we need to do is quote Theorem VSPCV [75]. That was easy.

Example VSM

The vector space of matrices, $M_{mn}$

Set: $M_{mn}$, the set of all matrices of size $m \times n$ and entries from $\mathbb{C}$, Example VSM [253].

Equality: Entry-wise, Definition ME [163].

Vector Addition: The “usual” addition, given in Definition MA [163].

Scalar Multiplication: The “usual” scalar multiplication, given in Definition MSM [164].

Does this set with these operations fulfill the ten properties? Yes. And all we need to do is quote Theorem VSPM [164]. Another easy one (by design).

So, the set of all matrices of a fixed size forms a vector space. That entitles us to call a matrix a vector, since a matrix is an element of a vector space. For example, if $A, B \in M_{3,4}$ then we call $A$ and $B$ “vectors,” and we even use our previous notation for column vectors to refer to $A$ and $B$. So we could legitimately write expressions like

$$u + v = A + B = B + A = v + u$$

This could lead to some confusion, but it is not too great a danger. But it is worth comment.

The previous two examples may be less than satisfying. We made all the relevant definitions long ago. And the required verifications were all handled by quoting old theorems. However, it is important to consider these two examples first. We have been studying vectors and matrices carefully (Chapter V [72], Chapter M [163]), and both objects, along with their operations, have certain properties in common, as you may have noticed in comparing Theorem VSPCV [75] with Theorem VSPM [164]. Indeed, it is these two theorems that motivate us to formulate the abstract definition of a vector space, Definition VS [251]. Now, should we prove some general theorems about vector spaces (as we will shortly in Subsection VS.VSP [257]), we can instantly apply the conclusions to both $\mathbb{C}^m$ and $M_{mn}$. Notice too how we have taken six definitions and two theorems and reduced them down to two examples. With greater generalization and abstraction our old ideas get downgraded in stature.

Let us look at some more examples, now considering some new vector spaces.

Example VSP

The vector space of polynomials, $P_n$

Set: $P_n$, the set of all polynomials of degree $n$ or less in the variable $x$ with coefficients from $\mathbb{C}$.

Equality:

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$$

if and only if $a_i = b_i$ for $0 \leq i \leq n$

Vector Addition:

$$(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n) =$$

$$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n$$

Scalar Multiplication:

$$\alpha(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + \cdots + (\alpha a_n)x^n$$

This set, with these operations, will fulfill the ten properties, though we will not work all the details here. However, we will make a few comments and prove one of the properties. First, the
zero vector (Property Z [251]) is what you might expect, and you can check that it has the required property.

\[ 0 = 0 + 0x + 0x^2 + \cdots + 0x^n \]

The additive inverse (Property AI [252]) is also no surprise, though consider how we have chosen to write it.

\[ -(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = (-a_0) + (-a_1)x + (-a_2)x^2 + \cdots + (-a_n)x^n \]

Now let’s prove the associativity of vector addition (Property AA [251]). This is a bit tedious, though necessary. Throughout, the plus sign (“+”) does triple-duty. You might ask yourself what each plus sign represents as you work through this proof.

\[
\begin{align*}
    u + (v + w) &= (a_0 + a_1x + \cdots + a_nx^n) + ((b_0 + b_1x + \cdots + b_nx^n) + (c_0 + c_1x + \cdots + c_nx^n)) \\
                 &= (a_0 + a_1x + \cdots + a_nx^n) + ((b_0 + c_0) + (b_1 + c_1)x + \cdots + (b_n + c_n)x^n) \\
                 &= (a_0 + (b_0 + c_0)) + ((a_1 + b_1 + c_1)x + \cdots + (a_n + b_n + c_n)x^n) \\
                 &= ((a_0 + b_0) + c_0) + ((a_1 + b_1 + c_1)x + \cdots + (a_n + b_n + c_n)x^n) \\
                 &= ((a_0 + b_0) + (a_1 + b_1 + c_1)x + \cdots + (a_n + b_n + c_n)x^n) \\
                 &\quad + (c_0 + c_1x + \cdots + c_nx^n) \\
                 &= ((a_0 + b_0)x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n) + (c_0 + c_1x + \cdots + c_nx^n) \\
                 &= (u + v) + w
\end{align*}
\]

Notice how it is the application of the associativity of the (old) addition of complex numbers in the middle of this chain of equalities that makes the whole proof happen. The remainder is successive applications of our (new) definition of vector (polynomial) addition. Proving the remainder of the ten properties is similar in style and tedium. You might try proving the commutativity of vector addition (Property AA [251]), or one of the distributivity properties (Property DVA [252], Property DSA [252]).

Example VSIS

The vector space of infinite sequences

Set: \( \mathbb{C}^\infty = \{ (c_0, c_1, c_2, \ldots) \mid c_i \in \mathbb{C}, \ i \in \mathbb{N} \} \).

Equality:

\( (c_0, c_1, c_2, \ldots) = (d_0, d_1, d_2, \ldots) \) if and only if \( c_i = d_i \) for all \( i \geq 0 \)

Vector Addition:

\( (c_0, c_1, c_2, \ldots) + (d_0, d_1, d_2, \ldots) = (c_0 + d_0, c_1 + d_1, c_2 + d_2, \ldots) \)

Scalar Multiplication:

\( \alpha(c_0, c_1, c_2, \ldots) = (\alpha c_0, \alpha c_1, \alpha c_2, \alpha c_3, \ldots) \)

This should remind you of the vector space \( \mathbb{C}^m \), though now our lists of scalars are written horizontally with commas as delimiters and they are allowed to be infinite in length. What does the zero vector look like (Property Z [251])? Additive inverses (Property AI [252])? Can you prove the associativity of vector addition (Property AA [251])?

Example VSF

The vector space of functions

Set: \( F = \{ f : \mathbb{C} \to \mathbb{C} \} \).

Equality: \( f = g \) if and only if \( f(x) = g(x) \) for all \( x \in \mathbb{C} \).

Vector Addition: \( f + g \) is the function with outputs defined by \( (f + g)(x) = f(x) + g(x) \).

Scalar Multiplication: \( \alpha f \) is the function with outputs defined by \( (\alpha f)(x) = \alpha f(x) \).

So this is the set of all functions of one variable that take a complex number to a complex number. You might have studied functions of one variable that take a real number to a real number.
number, and that might be a more natural set to study. But since we are allowing our scalars to be complex numbers, we need to expand the domain and range of our functions also. Study carefully how the definitions of the operation are made, and think about the different uses of “+” and juxtaposition. As an example of what is required when verifying that this is a vector space, consider that the zero vector \( z \) is the function \( z(x) = 0 \) for every input \( x \).

While vector spaces of functions are very important in mathematics and physics, we will not devote them much more attention.

Here’s a unique example.

**Example VSS**

**The singleton vector space**

**Set:** \( Z = \{ z \} \).

**Equality:** Huh?

**Vector Addition:** \( z + z = z \).

**Scalar Multiplication:** \( \alpha z = z \).

This should look pretty wild. First, just what is \( z \)? Column vector, matrix, polynomial, sequence, function? Mineral, plant, or animal? We aren’t saying! \( z \) just is. And we have definitions of vector addition and scalar multiplication that are sufficient for an occurrence of either that may come along.

Our only concern is if this set, along with the definitions of two operations, fulfills the ten properties of Definition VS. Let’s check associativity of vector addition (Property AA). For all \( u, v, w \in Z \),

\[
\begin{align*}
\text{u + (v + w)} &= \text{z + (z + z)} \\
&= \text{z + z} \\
&= \text{(z + z) + z} \\
&= \text{(u + v) + w}
\end{align*}
\]

What is the zero vector in this vector space (Property Z)? With only one element in the set, we do not have much choice. Is \( z = 0 \)? It appears that \( z \) behaves like the zero vector should, so it gets the title. Maybe now the definition of this vector space does not seem so bizarre. It is a set whose only element is the element that behaves like the zero vector, so that lone element is the zero vector.

Perhaps some of the above definitions and verifications seem obvious or like splitting hairs, but the next example should convince you that they are necessary. We will study this one carefully. Ready? Check your preconceptions at the door.

**Example CVS**

**The crazy vector space**

**Set:** \( C = \{ (x_1, x_2) \mid x_1, x_2 \in \mathbb{C} \} \).

**Vector Addition:** \( (x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1) \).

**Scalar Multiplication:** \( \alpha(x_1, x_2) = (\alpha x_1 + \alpha - 1, \alpha x_2 + \alpha - 1) \).

Now, the first thing I hear you say is “You can’t do that!” And my response is, “Oh yes, I can!” I am free to define my set and my operations any way I please. They may not look natural, or even useful, but we will now verify that they provide us with another example of a vector space. And that is enough. If you are adventurous, you might try first checking some of the properties yourself. What is the zero vector? Additive inverses? Can you prove associativity? Ready, here we go.

**Property AC, Property SC:** The result of each operation is a pair of complex numbers, so these two closure properties are fulfilled.

**Property C:**

\[
\text{u + v} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)
\]
= (y_1 + x_1 + 1, y_2 + x_2 + 1) = (y_1, y_2) + (x_1, x_2)
= v + u

Property AA [251]:

u + (v + w) = (x_1, x_2) + ((y_1, y_2) + (z_1, z_2))
= (x_1, x_2) + (y_1 + z_1 + 1, y_2 + z_2 + 1)
= (x_1 + (y_1 + z_1 + 1), x_2 + (y_2 + z_2 + 1))
= (x_1 + y_1 + z_1 + 2, x_2 + y_2 + z_2 + 2)
= ((x_1 + y_1 + 1) + z_1 + 1, (x_2 + y_2 + 1) + z_2 + 1)
= ((x_1, x_2) + (y_1, y_2)) + (z_1, z_2)
= (u + v) + w

Property Z [251]: The zero vector is . . . 0 = (−1, −1). Now I hear you say, “No, no, that can’t be, it must be (0, 0)!” Indulge me for a moment and let us check my proposal.

u + 0 = (x_1, x_2) + (−1, −1) = (x_1 + (−1) + 1, x_2 + (−1) + 1) = (x_1, x_2) = u

Feeling better? Or worse?

Property AI [252]: For each vector, u, we must locate an additive inverse, −u. Here it is, −(x_1, x_2) = (−x_1 − 2, −x_2 − 2). As odd as it may look, I hope you are withholding judgment. Check:

u + (−u) = (x_1, x_2) + (−x_1 − 2, −x_2 − 2) = (x_1 + (−x_1 − 2) + 1, −x_2 + (x_2 − 2) + 1) = (−1, −1) = 0

Property SMA [252]:

α(βu) = α(β(x_1, x_2))
= α(βx_1 + βx_2 + β – 1)
= (αβx_1 + αβx_2 + αβ – α – 1)
= (αβx_1 + αβx_2 + αβ – α + α – 1)
= (αβ)(x_1, x_2)
= (αβ)u

Property DVA [252]: If you have hung on so far, here’s where it gets even wilder. In the next two properties we mix and mash the two operations.

α(u + v) = α((x_1, x_2) + (y_1, y_2))
= α(x_1 + y_1 + 1, x_2 + y_2 + 1)
= (αx_1 + α + α – 1, αx_2 + αy_2 + α + α – 1)
= (αx_1 + α – 1 + αy_1 + α + α – 1, αx_2 + α – 1 + αy_2 + α + α – 1)
= ((αx_1 + α – 1) + (αy_1 + α + α – 1), (αx_2 + α – 1) + (αy_2 + α + α – 1) + 1)
= (αx_1 + α – 1, αx_2 + α – 1) + (αy_1 + α – 1, αy_2 + α – 1)
= α(x_1, x_2) + α(y_1, y_2)
= αu + αv

Property DSA [252]:

(α + β)u = (α + β)(x_1, x_2)
Subsection VS.VSP Vector Space Properties

\[ = ((\alpha + \beta)x_1 + (\alpha + \beta) - 1, (\alpha + \beta)x_2 + (\alpha + \beta) - 1) \]
\[ = (\alpha x_1 + \beta x_1 + \alpha + \beta - 1, \alpha x_2 + \beta x_2 + \alpha + \beta - 1) \]
\[ = (\alpha x_1 + \alpha - 1 + \beta x_1 + \beta - 1 + 1, \alpha x_2 + \alpha - 1 + \beta x_2 + \beta - 1 + 1) \]
\[ = (\alpha x_1 + \alpha - 1) + (\beta x_1 + \beta - 1 + 1, (\alpha x_2 + \alpha - 1) + (\beta x_2 + \beta - 1 + 1) \]
\[ = (\alpha x_1 + \alpha - 1, \alpha x_2 + \alpha - 1) + (\beta x_1 + \beta - 1, \beta x_2 + \beta - 1) \]
\[ = \alpha(x_1, x_2) + \beta(x_1, x_2) \]
\[ = \alpha u + \beta u \]

Property O \[252\]: After all that, this one is easy, but no less pleasing.

\[ 1u = 1(x_1, x_2) = (x_1 + 1 - 1, x_2 + 1 - 1) = (x_1, x_2) = u \]

That’s it, \( C \) is a vector space, as crazy as that may seem.

Notice that in the case of the zero vector and additive inverses, we only had to propose possibilities and then verify that they were the correct choices. You might try to discover how you would arrive at these choices, though you should understand why the process of discovering them is not a necessary component of the proof itself.

Subsection VSP
Vector Space Properties

Subsection VS.EVS \[252\] has provided us with an abundance of examples of vector spaces, most of them containing useful and interesting mathematical objects along with natural operations. In this subsection we will prove some general properties of vector spaces. Some of these results will again seem obvious, but it is important to understand why it is necessary to state and prove them. A typical hypothesis will be “Let \( V \) be a vector space.” From this we may assume the ten properties of Definition VS \[251\], and nothing more. Its like starting over, as we learn about what can happen in this new algebra we are learning. But the power of this careful approach is that we can apply these theorems to any vector space we encounter — those in the previous examples, or new ones we have not yet contemplated. Or perhaps new ones that nobody has ever contemplated. We will illustrate some of these results with examples from the crazy vector space (Example CVS \[255\]), but mostly we are stating theorems and doing proofs. These proofs do not get too involved, but are not trivial either, so these are good theorems to try proving yourself before you study the proof given here. (See Technique P \[627\].)

First we show that there is just one zero vector. Notice that the properties only require there to be at least one, and say nothing about there possibly being more. That is because we can use the ten properties of a vector space (Definition VS \[251\]) to learn that there can never be more than one. To require that this extra condition be stated as an eleventh property would make the definition of a vector space more complicated than it needs to be.

Theorem ZVU
Zero Vector is Unique
Suppose that \( V \) is a vector space. The zero vector, \( 0 \), is unique.

Proof To prove uniqueness, a standard technique is to suppose the existence of two objects (Technique U \[624\]). So let \( 0_1 \) and \( 0_2 \) be two zero vectors in \( V \). Then

\[ 0_1 = 0_1 + 0_2 \quad \text{Property Z} \[251\] for \( 0_2 \]
\[ = 0_2 + 0_1 \quad \text{Property C} \[251\] \]
\[ = 0_2 \quad \text{Property Z} \[251\] for \( 0_1 \]
This proves the uniqueness since the two zero vectors are really the same. □

**Theorem AIU**

**Additive Inverses are Unique**

Suppose that \( V \) is a vector space. For each \( u \in V \), the additive inverse, \(-u\), is unique. □

**Proof** To prove uniqueness, a standard technique is to suppose the existence of two objects (Technique U [624]). So let \(-u_1\) and \(-u_2\) be two additive inverses for \( u \). Then

\[
\begin{align*}
-u_1 &= -u_1 + 0 & \text{Property Z 251} \\
&= -u_1 + (u + -u_2) & \text{Property AI 252} \\
&= (-u_1 + u) + -u_2 & \text{Property AA 251} \\
&= 0 + -u_2 & \text{Property AI 252} \\
&= -u_2 & \text{Property Z 251}
\end{align*}
\]

So the two additive inverses are really the same. □

As obvious as the next three theorems appear, nowhere have we guaranteed that the zero scalar, scalar multiplication and the zero vector all interact this way. Until we have proved it, anyway.

**Theorem ZSSM**

**Zero Scalar in Scalar Multiplication**

Suppose that \( V \) is a vector space and \( u \in V \). Then \( 0u = 0 \).

**Proof** Notice that 0 is a scalar, \( u \) is a vector, so Property SC 251 says 0\( u \) is again a vector. As such, 0\( u \) has an additive inverse, \(-(0u)\) by Property AI 252.

\[
\begin{align*}
0u &= 0 + 0u & \text{Property Z 251} \\
&= (-0u) + 0u & \text{Property AI 252} \\
&= -0u + (0u + 0u) & \text{Property AA 251} \\
&= -(0u) + (0 + 0)u & \text{Property DSA 252} \\
&= -(0u) + 0u & 0 \text{ in } \mathbb{C} \\
&= 0 & \text{Property AI 252}
\end{align*}
\]

**Theorem ZVSM**

**Zero Vector in Scalar Multiplication**

Suppose that \( V \) is a vector space and \( \alpha \in \mathbb{C} \). Then \( \alpha 0 = 0 \).

**Proof** Notice that \( \alpha \) is a scalar, \( 0 \) is a vector, so Property SC 251 means \( \alpha 0 \) is again a vector. As such, \( \alpha 0 \) has an additive inverse, \(-(\alpha 0)\) by Property AI 252.

\[
\begin{align*}
\alpha 0 &= 0 + \alpha 0 & \text{Property Z 251} \\
&= (-\alpha 0) + \alpha 0 & \text{Property AI 252} \\
&= -(\alpha 0) + (\alpha 0 + \alpha 0) & \text{Property AA 251} \\
&= -(\alpha 0) + \alpha (0 + 0) & \text{Property DVA 252} \\
&= -(\alpha 0) + \alpha 0 & \text{Property Z 251} \\
&= 0 & \text{Property AI 252}
\end{align*}
\]

Here’s another one that sure looks obvious. But understand that we have chosen to use certain notation because it makes the theorem’s conclusion look so nice. The theorem is not true because the notation looks so good, it still needs a proof. If we had really wanted to make this point,
we might have defined the additive inverse of \( u \) as \( u^\sharp \). Then we would have written the defining property, Property AI \[252\], as \( u + u^\sharp = 0 \). This theorem would become \( u^\sharp = (-1)u \). Not really quite as pretty, is it?

**Theorem AISM**  
**Additive Inverses from Scalar Multiplication**  
Suppose that \( V \) is a vector space and \( u \in V \). Then \( -u = (-1)u \). □

**Proof**

\[
\begin{align*}
-u &= -u + 0 & \text{Property Z } \[251]\n&= -u + 0u & \text{Theorem ZSM } \[258]\n&= -u + (1 + (-1))u & \text{Property DSA } \[252]\n&= -u + (1u + (-1)u) & \text{Property O } \[252]\n&= -u + (u + (-1)u) & \text{Property AA } \[251]\n&= 0 + (-1)u & \text{Property AI } \[251]\n&= (-1)u & \text{Property Z } \[251]\n\end{align*}
\]

Because of this theorem, we can now write linear combinations like \( 6u_1 + (-4)u_2 \) as \( 6u_1 - 4u_2 \), even though we have not formally defined an operation called vector subtraction.

**Example PCVS**  
**Properties for the Crazy Vector Space**
Several of the above theorems have interesting demonstrations when applied to the crazy vector space, \( C \) (Example CVS \[255\]). We are not proving anything new here, or learning anything we did not know already about \( C \). It is just plain fun to see how these general theorems apply in a specific instance. For most of our examples, the applications are obvious or trivial, but not with \( C \).

Suppose \( u \in C \).
Then, as given by Theorem ZSSM \[258\],

\[
0u = 0(x_1, x_2) = (0x_1 + 0 - 1, 0x_2 + 0 - 1) = (-1, -1) = 0
\]

And as given by Theorem ZVSM \[258\],

\[
\alpha 0 = \alpha(-1, -1) = (\alpha(-1) + \alpha - 1, \alpha(-1) + \alpha - 1) = (-\alpha + \alpha - 1, -\alpha + \alpha - 1) = (-1, -1) = 0
\]

Finally, as given by Theorem AISM \[259\],

\[
(-1)u = (-1)(x_1, x_2) = ((-1)x_1 + (-1) - 1, (-1)x_2 + (-1) - 1) = (-x_1 - 2, -x_2 - 2) = -u
\]

Our next theorem is a bit different from several of the others in the list. Rather than making a declaration (“the zero vector is unique”) it is an implication (“if..., then...”) and so can be used in proofs to move from one statement to another.

**Theorem SMEZV**  
**Scalar Multiplication Equals the Zero Vector**
Suppose that \( V \) is a vector space and \( \alpha \in \mathbb{C} \). If \( \alpha u = 0 \), then either \( \alpha = 0 \) or \( u = 0 \). □

**Proof** We prove this theorem by breaking up the analysis into two cases. The first seems too trivial, and it is, but the logic of the argument is still legitimate.
Case 1. Suppose $\alpha = 0$. In this case our conclusion is true (the first part of the either/or is true) and we are done. That was easy.

Case 2. Suppose $\alpha \neq 0$.

$$u = 1u$$  \hspace{5cm} \text{Property O} \; 252
$$= \left( \frac{1}{\alpha} \right) u$$  \hspace{5cm} \alpha \neq 0
$$= \frac{1}{\alpha} (\alpha u)$$  \hspace{5cm} \text{Property SMA} \; 252
$$= \frac{1}{\alpha} (0)$$  \hspace{5cm} \text{Hypothesis}
$$= 0$$  \hspace{5cm} \text{Theorem ZVSM} \; 258

So in this case, the conclusion is true (the second part of the either/or is true) and we are done since the conclusion was true in each of the two cases.

The next three theorems give us cancellation properties. The two concerned with scalar multiplication are intimately connected with Theorem SMEZV \; 259. All three are implications. So we will prove each once, here and now, and then we can apply them at will in the future, saving several steps in a proof whenever we do.

**Theorem VAC**

**Vector Addition Cancellation**

Suppose that $V$ is a vector space, and $u, v, w \in V$. If $w + u = w + v$, then $u = v$. □

**Proof**

$$u = 0 + u$$  \hspace{5cm} \text{Property Z} \; 251
$$= (-w + w) + u$$  \hspace{5cm} \text{Property AI} \; 252
$$= -w + (w + u)$$  \hspace{5cm} \text{Property AA} \; 251
$$= -w + (w + v)$$  \hspace{5cm} \text{Hypothesis}
$$= (-w + w) + v$$  \hspace{5cm} \text{Property AA} \; 251
$$= 0 + v$$  \hspace{5cm} \text{Property AI} \; 252
$$= v$$  \hspace{5cm} \text{Property Z} \; 251

**Theorem CSSM**

**Canceling Scalars in Scalar Multiplication**

Suppose $V$ is a vector space, $u, v \in V$ and $\alpha$ is a nonzero scalar from $\mathbb{C}$. If $\alpha u = \alpha v$, then $u = v$. □

**Proof**

$$u = 1u$$  \hspace{5cm} \text{Property O} \; 252
$$= \left( \frac{1}{\alpha} \right) u$$  \hspace{5cm} \alpha \neq 0
$$= \frac{1}{\alpha} (\alpha u)$$  \hspace{5cm} \text{Property SMA} \; 252
$$= \frac{1}{\alpha} (\alpha v)$$  \hspace{5cm} \text{Hypothesis}
$$= \left( \frac{1}{\alpha} \right) v$$  \hspace{5cm} \text{Property SMA} \; 252
$$= 1v$$
$$= v$$  \hspace{5cm} \text{Property O} \; 252
Theorem CVSM
Canceling Vectors in Scalar Multiplication
Suppose \( V \) is a vector space, \( \mathbf{u} \neq \mathbf{0} \) is a vector in \( V \) and \( \alpha, \beta \in \mathbb{C} \). If \( \alpha \mathbf{u} = \beta \mathbf{u} \), then \( \alpha = \beta \). □

Proof

\[
\begin{align*}
0 &= \alpha \mathbf{u} + - (\alpha \mathbf{u}) & \text{Property AI } 252 \\
&= \beta \mathbf{u} + - (\alpha \mathbf{u}) & \text{Hypothesis} \\
&= \beta \mathbf{u} + (-1) (\alpha \mathbf{u}) & \text{Theorem AISM } 259 \\
&= \beta \mathbf{u} + ((-1)\alpha) \mathbf{u} & \text{Property SMA } 252 \\
&= \beta \mathbf{u} + (-\alpha) \mathbf{u} \\
&= (\beta - \alpha) \mathbf{u} & \text{Property DSA } 252
\end{align*}
\]

By hypothesis, \( \mathbf{u} \neq \mathbf{0} \), so Theorem SMEZV 259 implies

\[
\begin{align*}
0 &= \beta - \alpha \\
\alpha &= \beta
\end{align*}
\]

So with these three theorems in hand, we can return to our practice of “slashing” out parts of an equation, so long as we are careful about not canceling a scalar that might possibly be zero, or canceling a vector in a scalar multiplication that might be the zero vector.

Subsection RD
Recycling Definitions

When we say that \( V \) is a vector space, we then know we have a set of objects (the “vectors”), but we also know we have been provided with two operations ("vector addition" and “scalar multiplication”) and these operations behave with these objects according to the ten properties of Definition VS 251. One combines two vectors and produces a vector, the other takes a scalar and a vector, producing a vector as the result. So if \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in V \) then an expression like

\[
5\mathbf{u}_1 + 7\mathbf{u}_2 - 13\mathbf{u}_3
\]

would be unambiguous in any of the vector spaces we have discussed in this section. And the resulting object would be another vector in the vector space. If you were tempted to call the above expression a linear combination, you would be right. Four of the definitions that were central to our discussions in Chapter V 72 were stated in the context of vectors being column vectors, but were purposely kept broad enough that they could be applied in the context of any vector space. They only rely on the presence of scalars, vectors, vector addition and scalar multiplication to make sense. We will restate them shortly, unchanged, except that their titles and acronyms no longer refer to column vectors, and the hypothesis of being in a vector space has been added. Take the time now to look forward and review each one, and begin to form some connections to what we have done earlier and what we will be doing in subsequent sections and chapters. Specifically, compare the following pairs of definitions:

- Definition LCCV 79 and Definition LC 269
- Definition SSCV 102 and Definition SS 270
- Definition RLDCV 121 and Definition RLD 280
- Definition LICV 121 and Definition LI 280

Version 1.04
1. Comment on how the vector space \( \mathbb{C}^m \) went from a theorem (Theorem VSPCV 75) to an example (Example VSCV 253).

2. In the crazy vector space, \( C \), compute the linear combination \( 2(3, 4) + (-6)(1, 2) \).

3. Suppose that \( \alpha \) is a scalar and \( 0 \) is the zero vector. Why should we prove anything as obvious as \( \alpha 0 = 0 \) such as we did in Theorem ZVSM 258?
T10  Prove each of the ten properties of Definition VS for each of the following examples of a vector space:

Example VSP
Example VSIS
Example VSF
Example VSS

Contributed by Robert Beezer

M10  Define a possibly new vector space by beginning with the set and vector addition from $\mathbb{C}^2$ (Example VSCV) but change the definition of scalar multiplication to

$$\alpha x = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \alpha \in \mathbb{C}, \ x \in \mathbb{C}^2$$

Prove that the first nine properties required for a vector space hold, but Property O does not hold.

This example shows us that we cannot expect to be able to derive Property O as a consequence of assuming the first nine properties. In other words, we cannot slim down our list of properties by jettisoning the last one, and still have the same collection of objects qualify as vector spaces.

Contributed by Robert Beezer
Section S
Subspaces

A subspace is a vector space that is contained within another vector space. So every subspace is a vector space in its own right, but it is also defined relative to some other (larger) vector space. We will discover shortly that we are already familiar with a wide variety of subspaces from previous sections. Here’s the definition.

Definition S Subspace
Suppose that $V$ and $W$ are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that $W$ is a subset of $V$, $W \subseteq V$. Then $W$ is a subspace of $V$. △

Let’s look at an example of a vector space inside another vector space.

Example SC3
A subspace of $\mathbb{C}^3$
We know that $\mathbb{C}^3$ is a vector space (Example VSCV [253]). Consider the subset,

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid 2x_1 - 5x_2 + 7x_3 = 0 \right\}$$

It is clear that $W \subseteq \mathbb{C}^3$, since the objects in $W$ are column vectors of size 3. But is $W$ a vector space? Does it satisfy the ten properties of Definition VS [251] when we use the same operations?

That is the main question. Suppose $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are vectors from $W$. Then we know that these vectors cannot be totally arbitrary, they must have gained membership in $W$ by virtue of meeting the membership test. For example, we know that $x$ must satisfy $2x_1 - 5x_2 + 7x_3 = 0$ while $y$ must satisfy $2y_1 - 5y_2 + 7y_3 = 0$. Our first property (Property AC [251]) asks the question, is $x + y \in W$? When our set of vectors was $\mathbb{C}^3$, this was an easy question to answer. Now it is not so obvious. Notice first that

$$x + y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

and we can test this vector for membership in $W$ as follows,

$$2(x_1 + y_1) - 5(x_2 + y_2) + 7(x_3 + y_3) = 2x_1 + 2y_1 - 5x_2 - 5y_2 + 7x_3 + 7y_3$$

$$= (2x_1 - 5x_2 + 7x_3) + (2y_1 - 5y_2 + 7y_3)$$

$$= 0 + 0$$

$$= 0$$

and by this computation we see that $x + y \in W$. One property down, nine to go.

If $\alpha$ is a scalar and $x \in W$, is it always true that $\alpha x \in W$? This is what we need to establish Property SC [251]. Again, the answer is not as obvious as it was when our set of vectors was all of $\mathbb{C}^3$. Let’s see.

$$\alpha x = \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}$$

and we can test this vector for membership in $W$ with

$$2(\alpha x_1) - 5(\alpha x_2) + 7(\alpha x_3) = \alpha(2x_1 - 5x_2 + 7x_3)$$
\[ x \in W \]
\[
\begin{align*}
\alpha x &= 0 \\
\alpha &= 0
\end{align*}
\]

and we see that indeed \( \alpha x \in W \). Always.

If \( W \) has a zero vector, it will be unique (Theorem ZVU [257]). The zero vector for \( \mathbb{C}^3 \) should also perform the required duties when added to elements of \( W \). So the likely candidate for a zero vector in \( W \) is the same zero vector that we know \( \mathbb{C}^3 \) has. You can check that \( 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \) is a zero vector in \( W \) too (Property Z [251]).

With a zero vector, we can now ask about additive inverses (Property AI [252]). As you might suspect, the natural candidate for an additive inverse in \( W \) is the same as the additive inverse from \( \mathbb{C}^3 \). However, we must insure that these additive inverses actually are elements of \( W \). Given \( x \in W \), is \(-x \in W\)?

\[ -x = \begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \end{bmatrix} \]

and we can test this vector for membership in \( W \) with

\[
2(-x_1) - 5(-x_2) + 7(-x_3) = -(2x_1 - 5x_2 + 7x_3) \\
= -0 \\
= 0
\]

and we now believe that \(-x \in W\).

Is the vector addition in \( W \) commutative (Property C [251])? Is \( x + y = y + x \)? Of course! Nothing about restricting the scope of our set of vectors will prevent the operation from still being commutative. Indeed, the remaining five properties are unaffected by the transition to a smaller set of vectors, and so remain true. That was convenient.

So \( W \) satisfies all ten properties, is therefore a vector space, and thus earns the title of being a subspace of \( \mathbb{C}^3 \).

Subsection TS
Testing Subspaces

In Example SC3 [264] we proceeded through all ten of the vector space properties before believing that a subset was a subspace. But six of the properties were easy to prove, and we can lean on some of the properties of the vector space (the superset) to make the other four easier. Here is a theorem that will make it easier to test if a subset is a vector space. A shortcut if there ever was one.

Theorem TSS
Testing Subsets for Subspaces

Suppose that \( V \) is a vector space and \( W \) is a subset of \( V \), \( W \subseteq V \). Endow \( W \) with the same operations as \( V \). Then \( W \) is a subspace if and only if three conditions are met

1. \( W \) is non-empty, \( W \neq \emptyset \).
2. If \( x \in W \) and \( y \in W \), then \( x + y \in W \).
3. If \( \alpha \in \mathbb{C} \) and \( x \in W \), then \( \alpha x \in W \).

\[ \square \]

Proof  \((\Rightarrow)\) We have the hypothesis that \( W \) is a subspace, so by Definition VS [251] we know that \( W \) contains a zero vector. This is enough to show that \( W \neq \emptyset \). Also, since \( W \) is a vector space it
satisfies the additive and scalar multiplication closure properties, and so exactly meets the second and third conditions. If that was easy, the other direction might require a bit more work.

\((\Leftarrow)\) We have three properties for our hypothesis, and from this we should conclude that \(W\) has the ten defining properties of a vector space. The second and third conditions of our hypothesis are exactly Property AC \([251]\) and Property SC \([251]\). Our hypothesis that \(V\) is a vector space implies that Property C \([251]\), Property AA \([251]\), Property SMA \([252]\), Property DVA \([252]\), Property DSA \([252]\) and Property O \([252]\) all hold. They continue to be true for vectors from \(W\) since passing to a subset, and keeping the operation the same, leaves their statements unchanged. Eight down, two to go.

Suppose \(x \in W\). Then by the third part of our hypothesis (scalar closure), we know that \((-1)x \in W\). By Theorem AISM \([259]\) \((-1)x = -x\), so together these statements show us that \(-x \in W\). \(-x\) is the additive inverse of \(x\) in \(V\), but will continue in this role when viewed as element of the subset \(W\). So every element of \(W\) has an additive inverse that is an element of \(W\) and Property AI \([252]\) is completed. Just one property left.

While we have implicitly discussed the zero vector in the previous paragraph, we need to be certain that the zero vector (of \(V\)) really lives in \(W\). Since \(W\) is non-empty, we can choose some vector \(z \in W\). Then by the argument in the previous paragraph, we know \(-z \in W\). Now by Property AI \([252]\) for \(V\) and then by the second part of our hypothesis (additive closure) we see that

\[
0 = z + (-z) \in W
\]

So \(W\) contain the zero vector from \(V\). Since this vector performs the required duties of a zero vector in \(V\), it will continue in that role as an element of \(W\). This gives us, Property Z \([251]\), the final property of the ten required. (Sarah Fellez contributed to this proof.)

Three conditions, plus being a subset of a known vector space, gets us all ten properties. Fabulous!

This theorem can be paraphrased by saying that a subspace is “a non-empty subset (of a vector space) that is closed under vector addition and scalar multiplication.”

You might want to go back and rework Example SC3 \([264]\) in light of this result, perhaps seeing where we can now economize or where the work done in the example mirrored the proof and where it did not. We will press on and apply this theorem in a slightly more abstract setting.

Example SP4
A subspace of \(P_4\)

\(P_4\) is the vector space of polynomials with degree at most 4 \([Example VSP 253]\). Define a subset \(W\) as

\[
W = \{ p(x) \mid p \in P_4, p(2) = 0 \}
\]

so \(W\) is the collection of those polynomials (with degree 4 or less) whose graphs cross the \(x\)-axis at \(x = 2\). Whenever we encounter a new set it is a good idea to gain a better understanding of the set by finding a few elements in the set, and a few outside it. For example \(x^2 - x - 2 \in W\), while \(x^4 + x^3 - 7 \notin W\).

Is \(W\) nonempty? Yes, \(x - 2 \in W\).

Additive closure? Suppose \(p \in W\) and \(q \in W\). Is \(p + q \in W\)? \(p\) and \(q\) are not totally arbitrary, we know that \(p(2) = 0\) and \(q(2) = 0\). Then we can check \(p + q\) for membership in \(W\),

\[
(p + q)(2) = p(2) + q(2) = 0 + 0 = 0
\]

so we see that \(p + q\) qualifies for membership in \(W\).

Scalar multiplication closure? Suppose that \(\alpha \in \mathbb{C}\) and \(p \in W\). Then we know that \(p(2) = 0\). Testing \(\alpha p\) for membership,

\[
(\alpha p)(2) = \alpha p(2)
\]

Scalar multiplication in \(P_4\)
\[ = \alpha 0 \quad p \in W \]

so \( \alpha p \in W \).

We have shown that \( W \) meets the three conditions of \( \text{Theorem TSS} \) and so qualifies as a subspace of \( P_4 \). Notice that by \( \text{Definition S} \) we now know that \( W \) is also a vector space. So all the properties of a vector space (\( \text{Definition VS} \)) and the theorems of \( \text{Section VS} \) apply in full.

Much of the power of \( \text{Theorem TSS} \) is that we can easily establish new vector spaces if we can locate them as subsets of other vector spaces, such as the ones presented in \( \text{Subsection VS.EVS} \).

It can be as instructive to consider some subsets that are not subspaces. Since \( \text{Theorem TSS} \) is an equivalence (see \( \text{Technique E} \)) we can be assured that a subset is not a subspace if it violates one of the three conditions, and in any example of interest this will not be the “non-empty” condition. However, since a subspace has to be a vector space in its own right, we can also search for a violation of any one of the ten defining properties in \( \text{Definition VS} \) or any inherent property of a vector space, such as those given by the basic theorems of \( \text{Subsection VS.VSP} \). Notice also that a violation need only be for a specific vector or pair of vectors.

**Example NSC2Z**

A non-subspace in \( \mathbb{C}^2 \), zero vector

Consider the subset \( W \) below as a candidate for being a subspace of \( \mathbb{C}^2 \)

\[
W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right| 3x_1 - 5x_2 = 12 \right\}
\]

The zero vector of \( \mathbb{C}^2 \), \( 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) will need to be the zero vector in \( W \) also. However, \( 0 \notin W \) since \( 3(0) - 5(0) = 0 \neq 12 \). So \( W \) has no zero vector and fails Property Z of \( \text{Definition VS} \). This subspace also fails to be closed under addition and scalar multiplication. Can you find examples of this?

**Example NSC2A**

A non-subspace in \( \mathbb{C}^2 \), additive closure

Consider the subset \( X \) below as a candidate for being a subspace of \( \mathbb{C}^2 \)

\[
X = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right| x_1x_2 = 0 \right\}
\]

You can check that \( 0 \in X \), so the approach of the last example will not get us anywhere. However, notice that \( x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in X \) and \( y = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in X \). Yet

\[
x + y = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin X
\]

So \( X \) fails the additive closure requirement of either Property AC or \( \text{Theorem TSS} \), and is therefore not a subspace.

**Example NSC2S**

A non-subspace in \( \mathbb{C}^2 \), scalar multiplication closure

Consider the subset \( Y \) below as a candidate for being a subspace of \( \mathbb{C}^2 \)

\[
Y = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right| x_1 \in \mathbb{Z}, \ x_2 \in \mathbb{Z} \right\}
\]
$\mathbb{Z}$ is the set of integers, so we are only allowing "whole numbers" as the constituents of our vectors. Now, $0 \in Y$, and additive closure also holds (can you prove these claims?). So we will have to try something different. Note that $\alpha = \frac{1}{2} \in \mathbb{C}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \in Y$, but 

$$\alpha x = \frac{1}{2} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \notin Y$$

So $Y$ fails the scalar multiplication closure requirement of either Property SC [251] or Theorem TSS [265], and is therefore not a subspace.

There are two examples of subspaces that are trivial. Suppose that $V$ is any vector space. Then $V$ is a subset of itself and is a vector space. By Definition S [264], $V$ qualifies as a subspace of itself. The set containing just the zero vector $Z = \{0\}$ is also a subspace as can be seen by applying Theorem TSS [265] or by simple modifications of the techniques hinted at in Example VSS [255]. Since these subspaces are so obvious (and therefore not too interesting) we will refer to them as being trivial.

**Definition TS**

**Trivial Subspaces**

Given the vector space $V$, the subspaces $V$ and $\{0\}$ are each called a trivial subspace. △

We can also use Theorem TSS [265] to prove more general statements about subspaces, as illustrated in the next theorem.

**Theorem NSMS**

**Null Space of a Matrix is a Subspace**

Suppose that $A$ is an $m \times n$ matrix. Then the null space of $A$, $\mathcal{N}(A)$, is a subspace of $\mathbb{C}^n$. □

**Proof**

We will examine the three requirements of Theorem TSS [265]. Recall that $\mathcal{N}(A) = \{x \in \mathbb{C}^n \mid Ax = 0\}$.

First, $0 \in \mathcal{N}(A)$, which can be inferred as a consequence of Theorem HSC [52]. So $\mathcal{N}(A) \neq \emptyset$.

Second, check additive closure by supposing that $x \in \mathcal{N}(A)$ and $y \in \mathcal{N}(A)$. So we know a little something about $x$ and $y$: $Ax = 0$ and $Ay = 0$, and that is all we know. Question: Is $x + y \in \mathcal{N}(A)$? Let’s check.

$$A(x + y) = Ax + Ay$$

Theorem MMDAA [179]

$$= 0 + 0$$

$x \in \mathcal{N}(A)$, $y \in \mathcal{N}(A)$

Theorem VSPCV [75]

$$= 0$$

So, yes, $x + y$ qualifies for membership in $\mathcal{N}(A)$.

Third, check scalar multiplication closure by supposing that $\alpha \in \mathbb{C}$ and $x \in \mathcal{N}(A)$. So we know a little something about $x$: $Ax = 0$, and that is all we know. Question: Is $\alpha x \in \mathcal{N}(A)$? Let’s check.

$$A(\alpha x) = \alpha (Ax)$$

Theorem MMSMM [180]

$$= \alpha 0$$

$x \in \mathcal{N}(A)$

Theorem ZVSM [258]

$$= 0$$

So, yes, $\alpha x$ qualifies for membership in $\mathcal{N}(A)$.

Having met the three conditions in Theorem TSS [265] we can now say that the null space of a matrix is a subspace (and hence a vector space in its own right!). □

Here is an example where we can exercise Theorem NSMS [268].

**Example RSNS**

**Recasting a subspace as a null space**
Consider the subset of \( \mathbb{C}^5 \) defined as
\[
W = \begin{cases}
  \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 
  \end{bmatrix} & \begin{align*}
  3x_1 + x_2 - 5x_3 + 7x_4 + x_5 &= 0, \\
  4x_1 + 6x_2 + 3x_3 - 6x_4 - 5x_5 &= 0, \\
  -2x_1 + 4x_2 + 7x_4 + x_5 &= 0.
  \end{align*}
\end{cases}
\]

It is possible to show that \( W \) is a subspace of \( \mathbb{C}^5 \) by checking the three conditions of Theorem TSS directly, but it will get tedious rather quickly. Instead, give \( W \) a fresh look and notice that it is a set of solutions to a homogeneous system of equations. Define the matrix
\[
A = \begin{bmatrix}
  3 & 1 & -5 & 7 & 1 \\
  4 & 6 & 3 & -6 & -5 \\
  -2 & 4 & 0 & 7 & 1
\end{bmatrix}
\]
and then recognize that \( W = \mathcal{N}(A) \). By Theorem NSMS we can immediately see that \( W \) is a subspace. Boom!

### Subsection TSS
The Span of a Set

The span of a set of column vectors got a heavy workout in Chapter V and Chapter M. The definition of the span depended only on being able to formulate linear combinations. In any of our more general vector spaces we always have a definition of vector addition and of scalar multiplication. So we can build linear combinations and manufacture spans. This subsection contains two definitions that are just mild variants of definitions we have seen earlier for column vectors. If you haven’t already, compare them with Definition LCCV and Definition SSCV.

**Definition LC**
Linear Combination
Suppose that \( V \) is a vector space. Given \( n \) vectors \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n \) and \( n \) scalars \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \), their linear combination is the vector
\[
\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n.
\]

**Example LCM**
A linear combination of matrices
In the vector space \( M_{23} \) of \( 2 \times 3 \) matrices, we have the vectors
\[
\mathbf{x} = \begin{bmatrix}
  1 & 3 & -2 \\
  2 & 0 & 7
\end{bmatrix}, \quad
\mathbf{y} = \begin{bmatrix}
  3 & -1 & 2 \\
  5 & 5 & 1
\end{bmatrix}, \quad
\mathbf{z} = \begin{bmatrix}
  4 & 2 & -4 \\
  1 & 1 & 1
\end{bmatrix}
\]
and we can form linear combinations such as
\[
2\mathbf{x} + 4\mathbf{y} + (-1)\mathbf{z} = 2 \begin{bmatrix}
  1 & 3 & -2 \\
  2 & 0 & 7
\end{bmatrix} + 4 \begin{bmatrix}
  3 & -1 & 2 \\
  5 & 5 & 1
\end{bmatrix} + (-1) \begin{bmatrix}
  4 & 2 & -4 \\
  1 & 1 & 1
\end{bmatrix}
\]
\[
= \begin{bmatrix}
  2 & 6 & -4 \\
  4 & 0 & 14
\end{bmatrix} + \begin{bmatrix}
  12 & -4 & 8 \\
  20 & 20 & 4
\end{bmatrix} + \begin{bmatrix}
  -4 & -2 & 4 \\
  -1 & -1 & -1
\end{bmatrix}
\]
\[
= \begin{bmatrix}
  10 & 0 & 8 \\
  23 & 19 & 17
\end{bmatrix}
\]
or,
\[
4x - 2y + 3z = 4 \begin{bmatrix} 1 & 3 & -2 \\ 2 & 0 & 7 \\ \end{bmatrix} - 2 \begin{bmatrix} 3 & -1 & 2 \\ 5 & 5 & 1 \\ \end{bmatrix} + 3 \begin{bmatrix} 4 & 2 & -4 \\ 1 & 1 & 1 \\ \end{bmatrix}
\]
\[
= \begin{bmatrix} 4 & 12 & -8 \\ 8 & 0 & 28 \\ \end{bmatrix} + \begin{bmatrix} -6 & 2 & -4 \\ -10 & -10 & -2 \\ \end{bmatrix} + \begin{bmatrix} 12 & 6 & -12 \\ 3 & 3 & 3 \\ \end{bmatrix}
\]
\[
= \begin{bmatrix} 10 & 20 & -24 \\ 1 & -7 & 29 \\ \end{bmatrix}
\]

When we realize that we can form linear combinations in any vector space, then it is natural to revisit our definition of the span of a set, since it is the set of all possible linear combinations of a set of vectors.

**Definition SS**

**Span of a Set**

Suppose that \(V\) is a vector space. Given a set of vectors \(S = \{u_1, u_2, u_3, \ldots, u_t\}\), their span, \(\langle S \rangle\), is the set of all possible linear combinations of \(u_1, u_2, u_3, \ldots, u_t\). Symbolically,

\[
\langle S \rangle = \{ \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_t u_t \mid \alpha_i \in \mathbb{C}, \ 1 \leq i \leq t \}
\]

\[
= \left\{ \sum_{i=1}^{t} \alpha_i u_i \mid \alpha_i \in \mathbb{C}, \ 1 \leq i \leq t \right\}
\]

**Theorem SSS**

**Span of a Set is a Subspace**

Suppose \(V\) is a vector space. Given a set of vectors \(S = \{u_1, u_2, u_3, \ldots, u_t\} \subseteq V\), their span, \(\langle S \rangle\), is a subspace.

**Proof** We will verify the three conditions of [Theorem TSS](#) [265]. First,

\[
0 = 0 + 0 + 0 + \ldots + 0 \quad \text{Property Z [251] for } V
\]

\[
= 0u_1 + 0u_2 + 0u_3 + \cdots + 0u_t \quad \text{Theorem ZSSM [258]}
\]

So we have written \(0\) as a linear combination of the vectors in \(S\) and by [Definition SS](#) [270], \(0 \in \langle S \rangle\) and therefore \(S \neq \emptyset\).

Second, suppose \(x \in \langle S \rangle\) and \(y \in \langle S \rangle\). Can we conclude that \(x + y \in \langle S \rangle\)? What do we know about \(x\) and \(y\) by virtue of their membership in \(\langle S \rangle\)? There must be scalars from \(\mathbb{C}\), \(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_t\) and \(\beta_1, \beta_2, \beta_3, \ldots, \beta_t\) so that

\[
x = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_t u_t
\]

\[
y = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \cdots + \beta_t u_t
\]

Then

\[
x + y = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_t u_t
\]

\[
+ \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \cdots + \beta_t u_t
\]

\[
= \alpha_1 u_1 + \beta_1 u_1 + \alpha_2 u_2 + \beta_2 u_2
\]

\[
+ \alpha_3 u_3 + \beta_3 u_3 + \cdots + \alpha_t u_t + \beta_t u_t \quad \text{Property AA [251], Property C [251]}
\]

\[
= (\alpha_1 + \beta_1) u_1 + (\alpha_2 + \beta_2) u_2
\]

\[
+ (\alpha_3 + \beta_3) u_3 + \cdots + (\alpha_t + \beta_t) u_t \quad \text{Property DSA [252]}
\]

Since each \(\alpha_i + \beta_i\) is again a scalar from \(\mathbb{C}\) we have expressed the vector sum \(x + y\) as a linear combination of the vectors from \(S\), and therefore by [Definition SS](#) [270] we can say that \(x + y \in \langle S \rangle\).
Third, suppose \( \alpha \in \mathbb{C} \) and \( x \in \langle S \rangle \). Can we conclude that \( \alpha x \in \langle S \rangle \)? What do we know about \( x \) by virtue of its membership in \( \langle S \rangle \)? There must be scalars from \( \mathbb{C} \), \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_t \) so that

\[
x = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_t u_t
\]

Then

\[
\alpha x = \alpha (\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_t u_t) = \alpha (\alpha_1 u_1 + \alpha (\alpha_2 u_2) + \alpha (\alpha_3 u_3) + \cdots + \alpha (\alpha_t u_t))
\]

Therefore, \( \alpha x \) is a linear combination of the vectors from \( S \), and thereby by Definition SS \( 270 \) we can say that \( \alpha x \in \langle S \rangle \).

With the three conditions of Theorem TSS \( 265 \) met, we can say that \( \langle S \rangle \) is a subspace (and so is also vector space, Definition VS \( 251 \)). (See Exercise SS.T20 \( 114 \), Exercise SS.T21 \( 114 \), Exercise SS.T22 \( 114 \).)

**Example SSP**

**Span of a set of polynomials**

In Example SP4 \( 266 \) we proved that

\[
W = \{ p(x) \mid p \in P_4, \, p(2) = 0 \}
\]

is a subspace of \( P_4 \), the vector space of polynomials of degree at most 4. Since \( W \) is a vector space itself, let’s construct a span within \( W \). First let

\[
S = \{ x^4 - 4x^3 + 5x^2 - x - 2, 2x^4 - 3x^3 - 6x^2 + 6x + 4 \}
\]

and verify that \( S \) is a subset of \( W \) by checking that each of these two polynomials has \( x = 2 \) as a root. Now, if we define \( U = \langle S \rangle \), then Theorem SSS \( 270 \) tells us that \( U \) is a subspace of \( W \). So quite quickly we have built a chain of subspaces, \( U \) inside \( W \), and \( W \) inside \( P_4 \).

Rather than dwell on how quickly we can build subspaces, let’s try to gain a better understanding of just how the span construction creates subspaces, in the context of this example. We can quickly build representative elements of \( U \):

\[
3(x^4 - 4x^3 + 5x^2 - x - 2) + 5(2x^4 - 3x^3 - 6x^2 + 6x + 4) = 13x^4 - 27x^3 - 15x^2 + 27x + 14
\]

and

\[
(-2)(x^4 - 4x^3 + 5x^2 - x - 2) + 8(2x^4 - 3x^3 - 6x^2 + 6x + 4) = 14x^4 - 16x^3 - 58x^2 + 50x + 36
\]

and each of these polynomials must be in \( W \) since it is closed under addition and scalar multiplication. But you might check for yourself that both of these polynomials have \( x = 2 \) as a root.

I can tell you that \( y = 3x^4 - 7x^3 - x^2 + 7x - 2 \) is not in \( U \), but would you believe me? A first check shows that \( y \) does have \( x = 2 \) as a root, but that only shows that \( y \in W \). What does \( y \) have to do to gain membership in \( U = \langle S \rangle \)? It must be a linear combination of the vectors in \( S \), \( x^4 - 4x^3 + 5x^2 - x - 2 \) and \( 2x^4 - 3x^3 - 6x^2 + 6x + 4 \). So let’s suppose that \( y \) is such a linear combination,

\[
y = 3x^4 - 7x^3 - x^2 + 7x - 2
\]

\[
= \alpha_1 (x^4 - 4x^3 + 5x^2 - x - 2) + \alpha_2 (2x^4 - 3x^3 - 6x^2 + 6x + 4)
\]

\[
= (\alpha_1 + 2\alpha_2)x^4 + (-4\alpha_1 - 3\alpha_2)x^3 + (5\alpha_1 - 6\alpha_2)x^2 + (-\alpha_1 + 6\alpha_2)x - (-2\alpha_1 + 4\alpha_2)
\]
Notice that operations above are done in accordance with the definition of the vector space of polynomials (Example VSP 253). Now, if we equate coefficients, which is the definition of equality for polynomials, then we obtain the system of five linear equations in two variables

\[
\begin{align*}
\alpha_1 + 2\alpha_2 &= 3 \\
-4\alpha_1 - 3\alpha_2 &= -7 \\
5\alpha_1 - 6\alpha_2 &= -1 \\
-\alpha_1 + 6\alpha_2 &= 7 \\
-2\alpha_1 + 4\alpha_2 &= -2
\end{align*}
\]

Build an augmented matrix from the system and row-reduce,

\[
\begin{bmatrix}
1 & 2 & 3 \\
-4 & -3 & -7 \\
5 & -6 & -1 \\
-1 & 6 & 7 \\
-2 & 4 & -2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

With a leading 1 in the final column of the row-reduced augmented matrix, Theorem RCLS 45 tells us the system of equations is inconsistent. Therefore, there are no scalars, \(\alpha_1\) and \(\alpha_2\), to establish \(y\) as a linear combination of the elements in \(U\). So \(y \notin U\).

Let’s again examine membership in a span.

**Example SM32**

**A subspace of \(M_{32}\)**

The set of all \(3 \times 2\) matrices forms a vector space when we use the operations of matrix addition (Definition MA 163) and scalar matrix multiplication (Definition MSM 164), as was show in Example VSM 253. Consider the subset

\[
S = \left\{ \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 14 & -1 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ -1 & 2 \\ -19 & -11 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 1 & -2 \\ 14 & -2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ -4 & 0 \\ -17 & 7 \end{bmatrix} \right\}
\]

and define a new subset of vectors \(W\) in \(M_{32}\) using the span (Definition SS 270), \(W = \langle S \rangle\). So by Theorem SSS 270 we know that \(W\) is a subspace of \(M_{32}\). While \(W\) is an infinite set, and this is a precise description, it would still be worthwhile to investigate whether or not \(W\) contains certain elements.

First, is \(y\) in \(W\)? To answer this, we want to determine if \(y\) can be written as a linear combination of the five matrices in \(S\). Can we find scalars, \(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\) so that

\[
\begin{bmatrix}
9 \\
7 \\
10
\end{bmatrix}
= \alpha_1 \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & -5 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 14 & -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 & -1 \\ -1 & 2 \\ -19 & -11 \end{bmatrix} + \alpha_4 \begin{bmatrix} 4 & 2 \\ 1 & -2 \\ 14 & -2 \end{bmatrix} + \alpha_5 \begin{bmatrix} 3 & 1 \\ -4 & 0 \\ -17 & 7 \end{bmatrix}
\]

Using our definition of matrix equality (Definition ME 163) we can translate this statement into six equations in the five unknowns,

\[
3\alpha_1 + \alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 = 9
\]
\[
\begin{align*}
\alpha_1 + \alpha_2 - \alpha_3 + 2\alpha_4 + \alpha_5 &= 3 \\
4\alpha_1 + 2\alpha_2 - \alpha_3 + \alpha_4 - 4\alpha_5 &= 7 \\
2\alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4 &= 3 \\
5\alpha_1 + 14\alpha_2 - 19\alpha_3 + 14\alpha_4 - 17\alpha_5 &= 10 \\
-5\alpha_1 - \alpha_2 - 11\alpha_3 - 2\alpha_4 + 7\alpha_5 &= -11 \\
\end{align*}
\]

This is a linear system of equations, which we can represent with an augmented matrix and row-reduce in search of solutions. The matrix that is row-equivalent to the augmented matrix is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & \frac{5}{4} & 2 \\
0 & 1 & 0 & 0 & -\frac{19}{4} & -1 \\
0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 1 & \frac{1}{2} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

So we recognize that the system is consistent since there is no leading 1 in the final column (Theorem RCLS [45]), and compute \( n - r = 5 - 4 = 1 \) free variables (Theorem FVCS [46]). While there are infinitely many solutions, we are only in pursuit of a single solution, so let's choose the free variable \( \alpha_5 = 0 \) for simplicity's sake. Then we easily see that \( \alpha_1 = 2, \alpha_2 = -1, \alpha_3 = 0, \alpha_4 = 1 \). So the scalars \( \alpha_1 = 2, \alpha_2 = -1, \alpha_3 = 0, \alpha_4 = 1, \alpha_5 = 0 \) will provide a linear combination of the elements of \( S \) that equals \( y \), as we can verify by checking,

\[
\begin{bmatrix}
9 & 3 \\
7 & 3 \\
10 & -11 \\
\end{bmatrix}
= 2 \begin{bmatrix}
3 & 1 \\
4 & 2 \\
5 & -5 \\
\end{bmatrix}
+ (-1) \begin{bmatrix}
1 & 1 \\
2 & -1 \\
14 & -1 \\
\end{bmatrix}
+ (1) \begin{bmatrix}
4 & 2 \\
1 & -2 \\
14 & -2 \\
\end{bmatrix}
\]

So with one particular linear combination in hand, we are convinced that \( y \) deserves to be a member of \( W = \langle S \rangle \). Second, is \( x \in W \)? To answer this, we want to determine if \( x \) can be written as a linear combination of the five matrices in \( S \). Can we find scalars, \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \) so that

\[
\begin{bmatrix}
2 & 1 \\
3 & 1 \\
4 & -2 \\
\end{bmatrix}
= \begin{bmatrix}
3 & 1 \\
1 & 1 \\
5 & -5 \\
\end{bmatrix}
+ \alpha_2 \begin{bmatrix}
1 & 1 \\
2 & -1 \\
14 & -1 \\
\end{bmatrix}
+ \alpha_3 \begin{bmatrix}
3 & -1 \\
-1 & 2 \\
-19 & -11 \\
\end{bmatrix}
+ \alpha_4 \begin{bmatrix}
4 & 2 \\
1 & -2 \\
14 & -2 \\
\end{bmatrix}
+ \alpha_5 \begin{bmatrix}
3 & 1 \\
-4 & 0 \\
-17 & 7 \\
\end{bmatrix}
\]

Using our definition of matrix equality (Definition ME [163]) we can translate this statement into six equations in the five unknowns,

\[
\begin{align*}
3\alpha_1 + \alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 &= 2 \\
\alpha_1 + \alpha_2 - \alpha_3 + 2\alpha_4 + \alpha_5 &= 1 \\
4\alpha_1 + 2\alpha_2 - \alpha_3 + \alpha_4 - 4\alpha_5 &= 3 \\
2\alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4 &= 1 \\
5\alpha_1 + 14\alpha_2 - 19\alpha_3 + 14\alpha_4 - 17\alpha_5 &= 4 \\
-5\alpha_1 - \alpha_2 - 11\alpha_3 - 2\alpha_4 + 7\alpha_5 &= -2 \\
\end{align*}
\]
This is a linear system of equations, which we can represent with an augmented matrix and row-reduce in search of solutions. The matrix that is row-equivalent to the augmented matrix is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & & & -
\begin{array}{c}
38 \\
17 \\
7 \\
3 \\
2 \\
0
\end{array}
\\
0 & 1 & 0 & 0 & & & -
\begin{array}{c}
-8 \\
-4 \\
-2 \\
0 \\
0 \\
0
\end{array}
\\
0 & 0 & 1 & 0 & & & -
\begin{array}{c}
-8 \\
-4 \\
-2 \\
0 \\
0 \\
0
\end{array}
\\
0 & 0 & 0 & 1 & & & 1
\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

With a leading 1 in the last column, Theorem RCLS tells us that the system is inconsistent. Therefore, there are no values for the scalars that will place \( x \) in \( W \), and so we conclude that \( x \not\in W \).

Notice how Example SSP and Example SM32 contained questions about membership in a span, but these questions quickly became questions about solutions to a system of linear equations. This will be a common theme going forward.

Subsection SC
Subspace Constructions

Several of the subsets of vectors spaces that we worked with in Chapter M are also subspaces— they are closed under vector addition and scalar multiplication in \( \mathbb{C}^m \).

**Theorem CSMS**
**Column Space of a Matrix is a Subspace**
Suppose that \( A \) is an \( m \times n \) matrix. Then \( \mathcal{C}(A) \) is a subspace of \( \mathbb{C}^m \).

**Proof** Definition CSM shows us that \( \mathcal{C}(A) \) is a subset of \( \mathbb{C}^m \), and that it is defined as the span of a set of vectors from \( \mathbb{C}^m \) (the columns of the matrix). Since \( \mathcal{C}(A) \) is a span, Theorem SSS says it is a subspace.

That was easy! Notice that we could have used this same approach to prove that the null space is a subspace, since Theorem SSNS provided a description of the null space of a matrix as the span of a set of vectors. However, I much prefer the current proof of Theorem NSMS. Speaking of easy, here is a very easy theorem that exposes another of our constructions as creating subspaces.

**Theorem RSMS**
**Row Space of a Matrix is a Subspace**
Suppose that \( A \) is an \( m \times n \) matrix. Then \( \mathcal{R}(A) \) is a subspace of \( \mathbb{C}^n \).

**Proof** Definition RSM says \( \mathcal{R}(A) = \mathcal{C}(A^t) \), so the row space of a matrix is a column space, and every column space is a subspace by Theorem CSMS. That’s enough.

One more.

**Theorem LNSMS**
**Left Null Space of a Matrix is a Subspace**
Suppose that \( A \) is an \( m \times n \) matrix. Then \( \mathcal{L}(A) \) is a subspace of \( \mathbb{C}^m \).

**Proof** Definition LNS says \( \mathcal{L}(A) = \mathcal{N}(A^t) \), so the left null space is a null space, and every null space is a subspace by Theorem NSMS. Done.

So the span of a set of vectors, and the null space, column space, row space and left null space of a matrix are all subspaces, and hence are all vector spaces, meaning they have all the properties detailed in Definition VS and in the basic theorems presented in Section VS. We have worked with these objects as just sets in Chapter V and Chapter M, but now we understand that they have much more structure. In particular, being closed under vector addition and scalar multiplication means a subspace is also closed under linear combinations.
1. Summarize the three conditions that allow us to quickly test if a set is a subspace.

2. Consider the set of vectors
\[
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix} \\
3a - 2b + c = 5
\]

Is this set a subspace of \( \mathbb{C}^3 \)? Explain your answer.

3. Name five general constructions of sets of column vectors (subsets of \( \mathbb{C}^m \)) that we now know as subspaces.
Subsection EXC
Exercises

C20 Working within the vector space \( P_3 \) of polynomials of degree 3 or less, determine if \( p(x) = x^3 + 6x + 4 \) is in the subspace \( W \) below.

\[
W = \langle \{ x^3 + x^2 + x, x^3 + 2x - 6, x^2 - 5 \} \rangle
\]

Contributed by Robert Beezer  Solution 277

C21 Consider the subspace

\[
W = \langle \left\{ \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} \right\} \rangle
\]

of the vector space of \( 2 \times 2 \) matrices, \( M_{22} \). Is \( C = \begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix} \) an element of \( W \)?

Contributed by Robert Beezer  Solution 277

C25 Show that the set \( W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \bigg| 3x_1 - 5x_2 = 12 \right\} \) from Example NSC2Z 267 fails Property AC 251 and Property SC 251.

Contributed by Robert Beezer

C26 Show that the set \( Y = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \bigg| x_1 \in \mathbb{Z}, x_2 \in \mathbb{Z} \right\} \) from Example NSC2S 267 has Property AC 251.

Contributed by Robert Beezer

M20 In \( \mathbb{C}^3 \), the vector space of column vectors of size 3, prove that the set \( Z \) is a subspace.

\[
Z = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \bigg| 4x_1 - x_2 + 5x_3 = 0 \right\}
\]

Contributed by Robert Beezer  Solution 277

T20 A square matrix \( A \) of size \( n \) is upper triangular if \( [A]_{ij} = 0 \) whenever \( i > j \). Let \( UT_n \) be the set of all upper triangular matrices of size \( n \). Prove that \( UT_n \) is a subspace of the vector space of all square matrices of size \( n \), \( M_{nn} \).

Contributed by Robert Beezer  Solution 278
The question is if $p$ can be written as a linear combination of the vectors in $W$. To check this, we set $p$ equal to a linear combination and massage with the definitions of vector addition and scalar multiplication that we get with $P_3$ (Example VSP 253):

$$p(x) = a_1(x^3 + x^2 + x) + a_2(x^3 + 2x - 6) + a_3(x^2 - 5)$$

$$x^3 + 6x + 4 = (a_1 + a_2)x^3 + (a_1 + a_3)x^2 + (a_1 + 2a_2)x + (-6a_2 - 5a_3)$$

Equating coefficients of equal powers of $x$, we get the system of equations,

$$a_1 + a_2 = 1$$
$$a_1 + a_3 = 0$$
$$a_1 + 2a_2 = 6$$
$$-6a_2 - 5a_3 = 4$$

The augmented matrix of this system of equations row-reduces to

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

There is a leading 1 in the last column, so Theorem RCLS 45 implies that the system is inconsistent. So there is no way for $p$ to gain membership in $W$, so $p \not\in W$.

In order to belong to $W$, we must be able to express $C$ as a linear combination of the elements in the spanning set of $W$. So we begin with such an expression, using the unknowns $a$, $b$, and $c$ for the scalars in the linear combination.

$$C = \begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix} = a \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} + b \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix} + c \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}$$

Massaging the right-hand side, according to the definition of the vector space operations in $M_{22}$ (Example VSM 253), we find the matrix equality,

$$\begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} 2a + 4b - 3c & a + c \\ 3a + 2b + 2c & -a + 3b + c \end{bmatrix}$$

Matrix equality allows us to form a system of four equations in three variables, whose augmented matrix row-reduces as follows,

$$\begin{bmatrix} 2 & 4 & -3 & -3 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & 2 & 6 \\ -1 & 3 & 1 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this system of equations is consistent (Theorem RCLS 45), a solution will provide values for $a$, $b$, and $c$ that allow us to recognize $C$ as an element of $W$. 

The membership criteria for $Z$ is a single linear equation, which comprises a homogeneous system.
of equations. As such, we can recognize \( Z \) as the solutions to this system, and therefore \( Z \) is a null space. Specifically, \( Z = N([4 \ -1 \ 5]) \). Every null space is a subspace by Theorem NSMS \ref{thm:NSMS}.

A less direct solution appeals to Theorem TSS \ref{thm:TSS}.

First, we want to be certain \( Z \) is non-empty. The zero vector of \( \mathbb{C}^3 \), \( \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \), is a good candidate, since if it fails to be in \( Z \), we will know that \( Z \) is not a vector space. Check that

\[
4(0) - (0) + 5(0) = 0
\]

so that \( \mathbf{0} \in Z \).

Suppose \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) and \( \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \) are vectors from \( Z \). Then we know that these vectors cannot be totally arbitrary, they must have gained membership in \( Z \) by virtue of meeting the membership test. For example, we know that \( \mathbf{x} \) must satisfy \( 4x_1 - x_2 + 5x_3 = 0 \) while \( \mathbf{y} \) must satisfy \( 4y_1 - y_2 + 5y_3 = 0 \). Our second criteria asks the question, is \( \mathbf{x} + \mathbf{y} \in Z \)? Notice first that

\[
\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}
\]

and we can test this vector for membership in \( Z \) as follows,

\[
4(x_1 + y_1) - 1(x_2 + y_2) + 5(x_3 + y_3) = 4x_1 + 4y_1 - x_2 - y_2 + 5x_3 + 5y_3 = (4x_1 - x_2 + 5x_3) + (4y_1 - y_2 + 5y_3) = 0 + 0 = 0
\]

and by this computation we see that \( \mathbf{x} + \mathbf{y} \in Z \).

If \( \alpha \) is a scalar and \( \mathbf{x} \in Z \), is it always true that \( \alpha \mathbf{x} \in Z \)? To check our third criteria, we examine

\[
\alpha \mathbf{x} = \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}
\]

and we can test this vector for membership in \( Z \) with

\[
4(\alpha x_1) - (\alpha x_2) + 5(\alpha x_3) = \alpha (4x_1 - x_2 + 5x_3) = 0 \quad \text{\( \alpha \in \mathbb{C} \)}
\]

and we see that indeed \( \alpha \mathbf{x} \in Z \). With the three conditions of Theorem TSS \ref{thm:TSS} fulfilled, we can conclude that \( Z \) is a subspace of \( \mathbb{C}^3 \).

**T20** Contributed by Robert Beezer Statement \ref{stat:T20} Apply Theorem TSS \ref{thm:TSS}.

First, the zero vector of \( M_{nn} \) is the zero matrix, \( \mathbf{0} \), whose entries are all zero (Definition ZM \ref{def:ZM}). This matrix then meets the condition that \( [\mathbf{0}]_{ij} = 0 \) for \( i > j \) and so is an element of \( UT_n \).

Suppose \( A, B \in UT_n \). Is \( A + B \in UT_n \)? We examine the entries of \( A + B \) “below” the diagonal. That is, in the following, assume that \( i > j \).

\[
[A + B]_{ij} = [A]_{ij} + [B]_{ij} = 0 + 0 \quad \text{\( A, B \in UT_n \)}
\]
= 0

which qualifies \( A + B \) for membership in \( UT_n \).

Suppose \( \alpha \in \mathbb{C} \) and \( A \in UT_n \). Is \( \alpha A \in UT_n \)? We examine the entries of \( \alpha A \) “below” the diagonal. That is, in the following, assume that \( i > j \).

\[
[\alpha A]_{ij} = \alpha [A]_{ij} \\
= \alpha 0 \\
= 0
\]

which qualifies \( \alpha A \) for membership in \( UT_n \).

Having fulfilled the three conditions of Theorem TSS \( 265 \) we see that \( UT_n \) is a subspace of \( M_{nn} \).
Section LISS
Linear Independence and Spanning Sets

A vector space is defined as a set with two operations, meeting ten properties (Definition VS [251]). Just as the definition of span of a set of vectors only required knowing how to add vectors and how to multiply vectors by scalars, so it is with linear independence. A definition of a linear independent set of vectors in an arbitrary vector space only requires knowing how to form linear combinations and equating these with the zero vector. Since every vector space must have a zero vector (Property Z [251]), we always have a zero vector at our disposal.

In this section we will also put a twist on the notion of the span of a set of vectors. Rather than beginning with a set of vectors and creating a subspace that is the span, we will instead begin with a subspace and look for a set of vectors whose span equals the subspace.

The combination of linear independence and spanning will be very important going forward.

Subsection LI
Linear Independence

Our previous definition of linear independence (Definition LI [280]) employed a relation of linear dependence that was a linear combination on one side of an equality and a zero vector on the other side. As a linear combination in a vector space (Definition LC [269]) depends only on vector addition and scalar multiplication, and every vector space must have a zero vector (Property Z [251]), we can extend our definition of linear independence from the setting of \( \mathbb{C}^n \) to the setting of a general vector space \( V \) with almost no changes. Compare these next two definitions with Definition RLDCV [121] and Definition LICV [121].

Definition RLD
Relation of Linear Dependence
Suppose that \( V \) is a vector space. Given a set of vectors \( S = \{ u_1, u_2, u_3, \ldots, u_n \} \), an equation of the form

\[ \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_n u_n = 0 \]

is a relation of linear dependence on \( S \). If this equation is formed in a trivial fashion, i.e. \( \alpha_i = 0 \), \( 1 \leq i \leq n \), then we say it is a trivial relation of linear dependence on \( S \).

Definition LI
Linear Independence
Suppose that \( V \) is a vector space. The set of vectors \( S = \{ u_1, u_2, u_3, \ldots, u_n \} \) from \( V \) is linearly dependent if there is a relation of linear dependence on \( S \) that is not trivial. In the case where the only relation of linear dependence on \( S \) is the trivial one, then \( S \) is a linearly independent set of vectors.

Notice the emphasis on the word “only.” This might remind you of the definition of a nonsingular matrix, where if the matrix is employed as the coefficient matrix of a homogeneous system then the only solution is the trivial one.

Example LIP4
Linear independence in \( P_4 \)
In the vector space of polynomials with degree 4 or less, \( P_4 \) (Example VSP [253]) consider the set

\[ S = \{ 2x^4 + 3x^3 + 2x^2 - x + 10, -x^4 - 2x^3 + x^2 + 5x - 8, 2x^4 + x^3 + 10x^2 + 17x - 2 \} \].

Is this set of vectors linearly independent or dependent? Consider that

\[ 3 \left( 2x^4 + 3x^3 + 2x^2 - x + 10 \right) + 4 \left( -x^4 - 2x^3 + x^2 + 5x - 8 \right) \]
We form the coefficient matrix of this homogeneous system of equations and row-reduce to find

\[\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}\]

This is a nontrivial relation of linear dependence (Definition RLD 280) on the set \(S\) and so convinces us that \(S\) is linearly dependent (Definition LI 280).

Now, I hear you say, “Where did those scalars come from?” Do not worry about that right now, just be sure you understand why the above explanation is sufficient to prove that \(S\) is linearly dependent. The remainder of the example will demonstrate how we might find these scalars if they had not been provided so readily. Let’s look at another set of vectors (polynomials) from \(P_4\). Let

\[T = \{3x^4 - 2x^3 + 4x^2 + 6x - 1, -3x^4 + 1x^3 + 0x^2 + 4x + 2, 4x^4 + 5x^3 - 2x^2 + 3x + 1, 2x^4 - 7x^3 + 4x^2 + 2x + 1\}\]

Suppose we have a relation of linear dependence on this set,

\[0 = 0x^4 + 0x^3 + 0x^2 + 0x + 0\]
\[= \alpha_1(3x^4 - 2x^3 + 4x^2 + 6x - 1) + \alpha_2(-3x^4 + 1x^3 + 0x^2 + 4x + 2)\]
\[+ \alpha_3(4x^4 + 5x^3 - 2x^2 + 3x + 1) + \alpha_4(2x^4 - 7x^3 + 4x^2 + 2x + 1)\]

Using our definitions of vector addition and scalar multiplication in \(P_4\) (Example VSP 253), we arrive at,

\[0x^4 + 0x^3 + 0x^2 + 0x + 0 = (3\alpha_1 - 3\alpha_2 + 4\alpha_3 + 2\alpha_4)x^4 + (-2\alpha_1 + \alpha_2 + 5\alpha_3 - 7\alpha_4)x^3\]
\[+ (4\alpha_1 - 2\alpha_3 + 4\alpha_4)x^2 + (6\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4)x\]
\[+ (-\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)\, .\]

Equating coefficients, we arrive at the homogeneous system of equations,

\[3\alpha_1 - 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = 0\]
\[-2\alpha_1 + \alpha_2 + 5\alpha_3 - 7\alpha_4 = 0\]
\[4\alpha_1 - 2\alpha_3 + 4\alpha_4 = 0\]
\[6\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 = 0\]
\[-\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 0\]

We form the coefficient matrix of this homogeneous system of equations and row-reduce to find

We expected the system to be consistent (Theorem HSC 52) and so can compute \(n - r = 4 - 4 = 0\) and Theorem CSRN 46 tells us that the solution is unique. Since this is a homogeneous system, this unique solution is the trivial solution (Definition TSHSE 52), \(\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \alpha_4 = 0\). So by Definition LI 280 the set \(T\) is linearly independent.

A few observations. If we had discovered infinitely many solutions, then we could have used one of the non-trivial ones to provide a linear combination in the manner we used to show that \(S\) was linearly dependent. It is important to realize that it is not interesting that we can create a relation of linear dependence with zero scalars — we can always do that — but that for \(T\), this is the only way to create a relation of linear dependence. It was no accident that we arrived at a homogeneous system of equations in this example, it is related to our use of the zero vector in defining a relation of linear dependence. It is easy to present a convincing statement that a set is linearly dependent (just exhibit a nontrivial relation of linear dependence) but a convincing statement of linear independence requires demonstrating that there is no relation of linear dependence other than the trivial one.
Notice how we relied on theorems from Chapter SLE to provide this demonstration. Whew! There’s a lot going on in this example. Spend some time with it, we’ll be waiting patiently right here when you get back.

Example LIM32
Linear independence in $M_{32}$
Consider the two sets of vectors $R$ and $S$ from the vector space of all $3 \times 2$ matrices, $M_{32}$ (Example VSM).

$$R = \left\{ \begin{bmatrix} 3 & -1 \\ 1 & 4 \\ 6 & -6 \end{bmatrix}, \begin{bmatrix} -2 & 3 \\ 1 & -3 \\ -2 & -6 \end{bmatrix}, \begin{bmatrix} 6 & -6 \\ -1 & 0 \\ 7 & -9 \end{bmatrix}, \begin{bmatrix} 7 & 9 \\ -4 & -5 \end{bmatrix} \right\}$$

$$S = \left\{ \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} -4 & 0 \\ -2 & 2 \\ -2 & -6 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} -5 & 3 \\ -10 & 7 \end{bmatrix} \right\}$$

One set is linearly independent, the other is not. Which is which? Let’s examine $R$ first. Build a generic relation of linear dependence (Definition RLD),

$$\alpha_1 \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 6 \\ -1 \\ 7 \end{bmatrix} + \alpha_4 \begin{bmatrix} 7 \\ -4 \\ 2 \end{bmatrix} = 0$$

Massaging the left-hand side with our definitions of vector addition and scalar multiplication in $M_{32}$ (Example VSM) we obtain,

$$\begin{bmatrix} 3\alpha_1 - 2\alpha_2 + 6\alpha_3 + 7\alpha_4 \\ -\alpha_1 + 3\alpha_2 - 6\alpha_3 + 9\alpha_4 \\ -\alpha_1 + 3\alpha_2 - 6\alpha_3 + 9\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using our definition of matrix equality (Definition ME) and equating corresponding entries we get the homogeneous system of six equations in four variables,

$$3\alpha_1 - 2\alpha_2 + 6\alpha_3 + 7\alpha_4 = 0$$
$$-\alpha_1 + 3\alpha_2 - 6\alpha_3 + 9\alpha_4 = 0$$
$$-\alpha_1 + 3\alpha_2 - 6\alpha_3 + 9\alpha_4 = 0$$
$$-\alpha_1 + 3\alpha_2 - 6\alpha_3 + 9\alpha_4 = 0$$
$$-\alpha_1 + 3\alpha_2 - 6\alpha_3 + 9\alpha_4 = 0$$
$$-\alpha_1 + 3\alpha_2 - 6\alpha_3 + 9\alpha_4 = 0$$

Form the coefficient matrix of this homogeneous system and row-reduce to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Analyzing this matrix we are led to conclude that $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 = 0$. This means there is only a trivial relation of linear dependence on the vectors of $R$ and so we call $R$ a linearly independent set (Definition LI).

So it must be that $S$ is linearly dependent. Let’s see if we can find a non-trivial relation of linear dependence on $S$. We will begin as with $R$, by constructing a relation of linear dependence (Definition RLD) with unknown scalars,

$$\alpha_1 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -4 \\ -2 \\ -2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} + \alpha_4 \begin{bmatrix} 3 \\ -5 \\ -10 \end{bmatrix} = 0$$
Massaging the left-hand side with our definitions of vector addition and scalar multiplication in $M_{32}$ (Example VSM [253]) we obtain,

$$
\begin{bmatrix}
2\alpha_1 - 4\alpha_2 + \alpha_3 - 5\alpha_4 & \alpha_3 + 3\alpha_4 \\
\alpha_1 - 2\alpha_2 - 2\alpha_3 - 10\alpha_4 & -\alpha_1 + 2\alpha_2 + \alpha_3 + 7\alpha_4 \\
\alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\alpha_4 & 3\alpha_1 - 6\alpha_2 + 4\alpha_3
\end{bmatrix} =
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
$$

Using our definition of matrix equality (Definition ME [163]) and equating corresponding entries we get the homogeneous system of six equations in four variables,

$$
\begin{align*}
2\alpha_1 - 4\alpha_2 + \alpha_3 - 5\alpha_4 &= 0 \\
+\alpha_3 + 3\alpha_4 &= 0 \\
\alpha_1 - 2\alpha_2 - 2\alpha_3 - 10\alpha_4 &= 0 \\
-\alpha_1 + 2\alpha_2 + \alpha_3 + 7\alpha_4 &= 0 \\
\alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\alpha_4 &= 0 \\
3\alpha_1 - 6\alpha_2 + 4\alpha_3 &= 0
\end{align*}
$$

Form the coefficient matrix of this homogeneous system and row-reduce to obtain

$$
\begin{bmatrix}
1 & -2 & 0 & -4 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

Analyzing this we see that the system is consistent (we expected this since the system is homogeneous, Theorem HSC [52]) and has $n - r = 4 - 2 = 2$ free variables, namely $\alpha_2$ and $\alpha_4$. This means there are infinitely many solutions, and in particular, we can find a non-trivial solution, so long as we do not pick all of our free variables to be zero. The mere presence of a nontrivial solution for these scalars is enough to conclude that $S$ is a linearly dependent set (Definition LI [280]). But let’s go ahead and explicitly construct a non-trivial relation of linear dependence.

Choose $\alpha_2 = 1$ and $\alpha_4 = -1$. There is nothing special about this choice, there are infinitely many possibilities, some “easier” than this one, just avoid picking both variables to be zero. Then we find the corresponding dependent variables to be $\alpha_1 = -2$ and $\alpha_3 = 3$. So the relation of linear dependence,

$$
(-2)\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} + (1)\begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix} + (3)\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} + (-1)\begin{bmatrix} -5 & 3 \\ -10 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

is an iron-clad demonstration that $S$ is linearly dependent. Can you construct another such demonstration?

Example LIC
Linearly independent set in the crazy vector space

Is the set $R = \{(1, 0), (6, 3)\}$ linearly independent in the crazy vector space $C$ (Example CVS [255])? We begin with a relation of linear independence and massage it to a point where we can apply the definition of equality in $C$. Recall the definitions of vector addition and scalar multiplication in $C$.

$$
0 = a_1(1, 0) + a_2(6, 3) \quad \text{Definition RLD [280]}
$$

$$
(-1, -1) = (1a_1 + a_1 - 1, 0a_1 + a_1 - 1) + (6a_2 + a_2 - 1, 3a_2 + a_2 - 1) \quad \text{Scalar mult in } C
$$


\begin{align*}
&= (2a_1 - 1, a_1 - 1) + (7a_2 - 1, 4a_2 - 1) \\
&= (2a_1 - 1 + 7a_2 - 1 + 1, a_1 - 1 + 4a_2 - 1 + 1) \\
&= (2a_1 + 7a_2 - 1, a_1 + 4a_2 - 1)
\end{align*}

Equality in $C$ then yields the two equations,

\begin{align*}
2a_1 + 7a_2 - 1 &= -1 \\
a_1 + 4a_2 - 1 &= -1
\end{align*}

which becomes the homogeneous system

\begin{align*}
2a_1 + 7a_2 &= 0 \\
a_1 + 4a_2 &= 0
\end{align*}

Since the coefficient matrix of this system is nonsingular (check this!) the system has only the trivial solution $a_1 = a_2 = 0$. By Definition L SIS 280 the set $R$ is linearly independent. Notice that even though the zero vector of $C$ is not what we might first suspected, a question about linear independence still concludes with a question about a homogeneous system of equations. ☒

### Subsection SS
#### Spanning Sets

In a vector space $V$, suppose we are given a set of vectors $S \subseteq V$. Then we can immediately construct a subspace, $\langle S \rangle$, using Definition SS 270 and then be assured by Theorem SSS 270 that the construction does provide a subspace. We now turn the situation upside-down. Suppose we are first given a subspace $W \subseteq V$. Can we find a set $S$ so that $\langle S \rangle = W$? Typically $W$ is infinite and we are searching for a finite set of vectors $S$ that we can combine in linear combinations and “build” all of $W$.

I like to think of $S$ as the raw materials that are sufficient for the construction of $W$. If you have nails, lumber, wire, copper pipe, drywall, plywood, carpet, shingles, paint (and a few other things), then you can combine them in many different ways to create a house (or infinitely many different houses for that matter). A fast-food restaurant may have beef, chicken, beans, cheese, tortillas, taco shells and hot sauce and from this small list of ingredients build a wide variety of items for sale. Or maybe a better analogy comes from Ben Cordes — the additive primary colors (red, green and blue) can be combined to create many different colors by varying the intensity of each. The intensity is like a scalar multiple, and the combination of the three intensities is like vector addition. The three individual colors, red, yellow and blue, are the elements of the spanning set.

Because we will use terms like “spanned by” and “spanning set,” there is the potential for confusion with “the span.” Come back and reread the first paragraph of this subsection whenever you are uncertain about the difference. Here’s the working definition.

**Definition TSVS**

**To Span a Vector Space**

Suppose $V$ is a vector space. A subset $S$ of $V$ is a **spanning set** for $V$ if $\langle S \rangle = V$. In this case, we also say $S$ spans $V$. ☛

The definition of a spanning set requires that two sets (subspaces actually) be equal. If $S$ is a subset of $V$, then $\langle S \rangle \subseteq V$, always. Thus it is usually only necessary to prove that $V \subseteq \langle S \rangle$. Now would be a good time to review Definition SE 616.

**Example SSP4**

**Spanning set in $P_4$**

In Example SP4 266 we showed that

\[ W = \{ p(x) \mid p \in P_4, \ p(2) = 0 \} \]
is a subspace of $P_4$, the vector space of polynomials with degree at most 4 (Example VSP 253). In this example, we will show that the set

$$S = \{ x - 2, x^2 - 4x + 4, x^3 - 6x^2 + 12x - 8, x^4 - 8x^3 + 24x^2 - 32x + 16 \}$$

is a spanning set for $W$. To do this, we require that $W = \langle S \rangle$. This is an equality of sets. We can check that every polynomial in $S$ has $x = 2$ as a root and therefore $S \subseteq W$. Since $W$ is closed under addition and scalar multiplication, $\langle S \rangle \subseteq W$ also.

So it remains to show that $W \subseteq \langle S \rangle$ (Definition STE 616). To do this, begin by choosing an arbitrary polynomial in $W$, say $r(x) = ax^4 + bx^3 + cx^2 + dx + e \in W$. This polynomial is not as arbitrary as it would appear, since we also know it must have $x = 2$ as a root. This translates to

$$0 = a(2)^4 + b(2)^3 + c(2)^2 + d(2) + e = 16a + 8b + 4c + 2d + e$$

as a condition on $r$.

We wish to show that $r$ is a polynomial in $\langle S \rangle$, that is, we want to show that $r$ can be written as a linear combination of the vectors (polynomials) in $S$. So let’s try.

$$r(x) = ax^4 + bx^3 + cx^2 + dx + e$$

$$= \alpha_1 (x - 2) + \alpha_2 (x^2 - 4x + 4) + \alpha_3 (x^3 - 6x^2 + 12x - 8) + \alpha_4 (x^4 - 8x^3 + 24x^2 - 32x + 16)$$

$$= \alpha_4 x^4 + (\alpha_3 - 8\alpha_4) x^3 + (\alpha_2 - 6\alpha_3 + 24\alpha_4) x^2 + (\alpha_1 - 4\alpha_2 + 12\alpha_3 - 32\alpha_4) x + (-2\alpha_1 + 4\alpha_2 - 8\alpha_3 + 16\alpha_4)$$

Equating coefficients (vector equality in $P_4$) gives the system of five equations in four variables,

$$\begin{align*}
\alpha_4 &= a \\
\alpha_3 - 8\alpha_4 &= b \\
\alpha_2 - 6\alpha_3 + 24\alpha_4 &= c \\
\alpha_1 - 4\alpha_2 + 12\alpha_3 - 32\alpha_4 &= d \\
-2\alpha_1 + 4\alpha_2 - 8\alpha_3 + 16\alpha_4 &= e
\end{align*}$$

Any solution to this system of equations will provide the linear combination we need to determine if $r \in \langle S \rangle$, but we need to be convinced there is a solution for any values of $a$, $b$, $c$, $d$, $e$ that qualify $r$ to be a member of $W$. So the question is: is this system of equations consistent? We will form the augmented matrix, and row-reduce. (We probably need to do this by hand, since the matrix is symbolic — reversing the order of the first four rows is the best way to start). We obtain a matrix in reduced row-echelon form

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 32a + 12b + 4c + d \\
0 & 1 & 0 & 0 & 24a + 6b + c \\
0 & 0 & 1 & 0 & 8a + b \\
0 & 0 & 0 & 1 & 16a + 8b + 4c + 2d + e
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 32a + 12b + 4c + d \\
0 & 1 & 0 & 0 & 24a + 6b + c \\
0 & 0 & 1 & 0 & 8a + b \\
0 & 0 & 0 & 1 & a
\end{bmatrix}$$

For your results to match our first matrix, you may find it necessary to multiply the final row of your row-reduced matrix by the appropriate scalar, and/or add multiples of this row to some of the other rows. To obtain the second version of the matrix, the last entry of the last column has been simplified to zero according to the one condition we were able to impose on an arbitrary polynomial from $W$. So with no leading 1’s in the last column, Theorem RCLS 45 tells us this system is consistent. Therefore, any polynomial from $W$ can be written as a linear combination of the polynomials in $S$, so $W \subseteq \langle S \rangle$. Therefore, $W = \langle S \rangle$ and $S$ is a spanning set for $W$ by Definition TSVS 284.
Notice that an alternative to row-reducing the augmented matrix by hand would be to appeal to Theorem FS \[237\] by expressing the column space of the coefficient matrix as a null space, and then verifying that the condition on \( r \) guarantees that \( r \) is in the column space, thus implying that the system is always consistent. Give it a try, we’ll wait. This has been a complicated example, but worth studying carefully.

Given a subspace and a set of vectors, as in Example SSP4 \[284\] it can take some work to determine that the set actually is a spanning set. An even harder problem is to be confronted with a subspace and required to construct a spanning set with no guidance. We will now work an example of this flavor, but some of the steps will be unmotivated. Fortunately, we will have some better tools for this type of problem later on.

**Example SSM22**

**Spanning set in** \( M_{22} \)

In the space of all \( 2 \times 2 \) matrices, \( M_{22} \) consider the subspace

\[
Z = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + 3b - c - 5d = 0, \ -2a - 6b + 3c + 14d = 0 \right\}
\]

and find a spanning set for \( Z \).

We need to construct a limited number of matrices in \( Z \) so that every matrix in \( Z \) can be expressed as a linear combination of this limited number of matrices. Suppose that \( B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is a matrix in \( Z \). Then we can form a column vector with the entries of \( B \) and write

\[
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathcal{N}\left( \begin{bmatrix} 1 & 3 & -1 & -5 \\ -2 & -6 & 3 & 14 \end{bmatrix} \right)
\]

Row-reducing this matrix and applying Theorem REMES \[25\] we obtain the equivalent statement,

\[
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathcal{N}\left( \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \right)
\]

We can then express the subspace \( Z \) in the following equal forms,

\[
Z = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + 3b - c - 5d = 0, \ -2a - 6b + 3c + 14d = 0 \right\}
= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a + 3b - d = 0, \ c + 4d = 0 \right\}
= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a = -3b + d, \ c = -4d \right\}
= \left\{ \begin{bmatrix} -3b + d & b \\ -4d & d \end{bmatrix} \mid b, d \in \mathbb{C} \right\}
= \left\{ \begin{bmatrix} -3b & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} d & 0 \\ -4d & d \end{bmatrix} \mid b, d \in \mathbb{C} \right\}
= \left\{ \begin{bmatrix} b & -3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix} \mid b, d \in \mathbb{C} \right\}
= \langle \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \rangle
\]

So the set

\[
Q = \left\{ \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \right\}
\]
spans $Z$ by [Definition TSVS 284].

**Example SSC**

**Spanning set in the crazy vector space**

In [Example LIC 283], we determined that the set $R = \{(1, 0), (6, 3)\}$ is linearly independent in the crazy vector space $C$ [Example CVS 255]. We now show that $R$ is a spanning set for $C$.

Given an arbitrary vector $(x, y) \in C$ we desire to show that it can be written as a linear combination of the elements of $R$. In other words, are there scalars $a_1$ and $a_2$ so that

$$(x, y) = a_1(1, 0) + a_2(6, 3)$$

We will act as if this equation is true and try to determine just what $a_1$ and $a_2$ would be (as functions of $x$ and $y$).

$$(x, y) = a_1(1, 0) + a_2(6, 3)$$

$$= (a_1 \cdot 1 + a_1 \cdot 1 - 0a_1 + a_1 - 1) + (6a_2 + a_2 - 1, 3a_2 + a_2 - 1)$$

$$= (2a_1 - 1, a_1 - 1) + (7a_2 - 1, 4a_2 - 1)$$

$$= (2a_1 - 1 + 7a_2 - 1 + 1, a_1 - 1 + 4a_2 - 1 + 1)$$

Equality in $C$ then yields the two equations,

$$2a_1 + 7a_2 - 1 = x$$

$$a_1 + 4a_2 - 1 = y$$

which becomes the linear system with a matrix representation

$$\begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} x + 1 \\ y + 1 \end{bmatrix}$$

The coefficient matrix of this system is nonsingular, hence invertible [Theorem NI 204], and we can employ its inverse to find a solution [Theorem TTMII 191, Theorem SNCM 204],

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} x + 1 \\ y + 1 \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x + 1 \\ y + 1 \end{bmatrix} = \begin{bmatrix} 4x - 7y - 3 \\ -x + 2y + 1 \end{bmatrix}$$

We could chase through the above implications backwards and take the existence of these solutions as sufficient evidence for $R$ being a spanning set for $C$. Instead, let us view the above as simply scratchwork and now get serious with a simple direct proof that $R$ is a spanning set. Ready? Suppose $(x, y)$ is any vector from $C$, then compute the following linear combination using the definitions of the operations in $C$,

$$(4x - 7y - 3)(1, 0) + (-x + 2y + 1)(6, 3)$$

$$= (4x - 7y - 3, 4x - 7y - 3 - 1, 0(4x - 7y - 3) + (4x - 7y - 3 - 1))$$

$$+ (6(-x + 2y + 1) + (-x + 2y + 1) - 1, 3(-x + 2y + 1) + (-x + 2y + 1) - 1)$$

$$= (8x - 14y - 7, 4x - 7y - 4 + (-7x + 14y + 6, -4x + 8y + 3)$$

$$= ((8x - 14y - 7) + (-7x + 14y + 6) + 1, (4x - 7y - 4) + (-4x + 8y + 3) + 1)$$

$$= (x, y)$$

This final sequence of computations in $C$ is sufficient to demonstrate that any element of $C$ can be written (or expressed) as a linear combination of the two vectors in $R$, so $C \subseteq \langle R \rangle$. Since the reverse inclusion $\langle R \rangle \subseteq C$ is trivially true, $C = \langle R \rangle$ and we say $R$ spans $C$. [Definition TSVS 284]. Notice that this demonstration is no more or less valid if we hide from the reader our scratchwork that suggested $a_1 = 4x - 7y - 3$ and $a_2 = -x + 2y + 1$. ☑
Subsection VR  
Vector Representation

In Chapter R 473 we will take up the matter of representations fully, where Theorem VRRB 288 will be critical for Definition VR 473. We will now motivate and prove a critical theorem that tells us how to “represent” a vector. This theorem could wait, but working with it now will provide some extra insight into the nature of linearly independent spanning sets. First an example, then the theorem.

Example AVR  
A vector representation

Consider the set

$$S = \left\{ \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix} \right\}$$

from the vector space \( \mathbb{C}^3 \). Let \( A \) be the matrix whose columns are the set \( S \), and verify that \( A \) is nonsingular. By Theorem NMLIC 126 the elements of \( S \) form a linearly independent set. Suppose that \( b \in \mathbb{C}^3 \). Then \( \mathcal{L}(A, b) \) has a (unique) solution (Theorem NMUS 64) and hence is consistent. By Theorem SLSLC 82, \( b \in (S) \). Since \( b \) is arbitrary, this is enough to show that \( (S) = \mathbb{C}^3 \), and therefore \( S \) is a spanning set for \( \mathbb{C}^3 \) (Definition TSVS 284). (This set comes from the columns of the coefficient matrix of Archetype B 638.)

Now examine the situation for a particular choice of \( b \), say \( b = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix} \). Because \( S \) is a spanning set for \( \mathbb{C}^3 \), we know we can write \( b \) as a linear combination of the vectors in \( S \),

$$\begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix} = (-3) \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix} + (5) \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix}.$$

The nonsingularity of the matrix \( A \) tells that the scalars in this linear combination are unique. More precisely, it is the linear independence of \( S \) that provides the uniqueness. We will refer to the scalars \( a_1 = -3, a_2 = 5, a_3 = 2 \) as a “representation of \( b \) relative to \( S \).” In other words, once we settle on \( S \) as a linearly independent set that spans \( \mathbb{C}^3 \), the vector \( b \) is recoverable just by knowing the scalars \( a_1 = -3, a_2 = 5, a_3 = 2 \) (use these scalars in a linear combination of the vectors in \( S \)). This is all an illustration of the following important theorem, which we prove in the setting of a general vector space.

Theorem VRRB  
Vector Representation Relative to a Basis

Suppose that \( V \) is a vector space and \( B = \{v_1, v_2, v_3, \ldots, v_m\} \) is a linearly independent set that spans \( V \). Let \( w \) be any vector in \( V \). Then there exist unique scalars \( a_1, a_2, a_3, \ldots, a_m \) such that

$$w = a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_m v_m.$$

Proof  That \( w \) can be written as a linear combination of the vectors in \( B \) follows from the spanning property of the set (Definition TSVS 284). This is good, but not the meat of this theorem. We now know that for any choice of the vector \( w \) there exist some scalars that will create \( w \) as a linear combination of the basis vectors. The real question is: Is there more than one way to write \( w \) as a linear combination of \( \{v_1, v_2, v_3, \ldots, v_m\} \)? Are the scalars \( a_1, a_2, a_3, \ldots, a_m \) unique? (Technique U 624)

Assume there are two ways to express \( w \) as a linear combination of \( \{v_1, v_2, v_3, \ldots, v_m\} \). In other words there exist scalars \( a_1, a_2, a_3, \ldots, a_m \) and \( b_1, b_2, b_3, \ldots, b_m \) so that

$$w = a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_m v_m.$$

$$w = b_1 v_1 + b_2 v_2 + b_3 v_3 + \cdots + b_m v_m.$$
Then notice that
\[ 0 = w + (-w) \]
\[ = w + (1)w \]
\[ = (a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_mv_m) + \]
\[ (-1)(b_1v_1 + b_2v_2 + b_3v_3 + \cdots + b_mv_m) \]
\[ = (a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_mv_m) + \]
\[ (-b_1v_1 - b_2v_2 - b_3v_3 - \cdots - b_mv_m) \]
\[ = (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + (a_3 - b_3)v_3 + \]
\[ \cdots + (a_m - b_m)v_m \]

But this is a relation of linear dependence on a linearly independent set of vectors \( \text{Definition RLD} \) \[280\]! Now we are using the other assumption about \( B \), that \( \{v_1, v_2, v_3, \ldots, v_m\} \) is a linearly independent set. So by \( \text{Definition LI} \) \[280\] it \textit{must} happen that the scalars are all zero. That is,
\[
(a_1 - b_1) = 0 \quad (a_2 - b_2) = 0 \quad (a_3 - b_3) = 0 \quad \ldots \quad (a_m - b_m) = 0
\]
\[
a_1 = b_1 \quad a_2 = b_2 \quad a_3 = b_3 \quad \ldots \quad a_m = b_m.
\]

And so we find that the scalars are unique. \( \blacksquare \)

This is a very typical use of the hypothesis that a set is linearly independent — obtain a relation of linear dependence and then conclude that the scalars \textit{must} all be zero. The result of this theorem tells us that we can write any vector in a vector space as a linear combination of the vectors in a linearly independent spanning set, but only just. There is only enough raw material in the spanning set to write each vector one way as a linear combination. So in this sense, we could call a linearly independent spanning set a “minimal spanning set.” These sets are so important that we will give them a simpler name (“basis”) and explore their properties further in the next section.

Subsection READ
Reading Questions

1. Is the set of matrices below linearly independent or linearly dependent in the vector space \( M_{22} \)? Why or why not?
\[
\begin{bmatrix}
1 & 2 \\
4 & 3 \\
3 & 2
\end{bmatrix},
\begin{bmatrix}
-2 & 3 \\
4 & -5 \\
-1 & 3
\end{bmatrix},
\begin{bmatrix}
0 & 9 \\
-6 & 2 \\
2 & 1
\end{bmatrix}
\]

2. Explain the difference between the following two uses of the term “span”:
(a) \( S \) is a subset of the vector space \( V \) and the span of \( S \) is a subspace of \( V \).
(b) \( W \) is subspace of the vector space \( Y \) and \( T \) spans \( W \).

3. The set
\[
S = \begin{bmatrix}
6 & 4 & 5 \\
2 & -3 & 8 \\
1 & 1 & 2
\end{bmatrix}
\]
is linearly independent and spans \( \mathbb{C}^3 \). Write the vector \( x = \begin{bmatrix}
-6 \\
2 \\
2
\end{bmatrix} \) a linear combination of the elements of \( S \). Quote the relevant theorem that tells you how many ways are there to answer this question.
### Subsection EXC Exercises

**C20** In the vector space of $2 \times 2$ matrices, $M_{22}$, determine if the set $S$ below is linearly independent.

$$S = \left\{ \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \right\}$$

Contributed by Robert Beezer  
Solution [291]

**C21** In the crazy vector space $C$ (Example CVS [255]), is the set $S = \{ (0, 2), (2, 8) \}$ linearly independent?

Contributed by Robert Beezer  
Solution [291]

**C22** In the vector space of polynomials $P_3$, determine if the set $S$ is linearly independent or linearly dependent.

$$S = \{ 2 + x - 3x^2 - 8x^3, 1 + x + x^2 + 5x^3, 3 - 4x^2 - 7x^3 \}$$

Contributed by Robert Beezer  
Solution [291]

**C23** Determine if the set $S = \{ (3, 1), (7, 3) \}$ is linearly independent in the crazy vector space $C$ (Example CVS [255]).

Contributed by Robert Beezer  
Solution [292]

**C30** In Example LIM32 [282], find another nontrivial relation of linear dependence on the linearly dependent set of $3 \times 2$ matrices, $S$.

Contributed by Robert Beezer

**C40** Determine if the set $T = \{ x^2 - x + 5, 4x^3 - x^2 + 5x, 3x + 2 \}$ spans the vector space of polynomials with degree 4 or less, $P_4$.

Contributed by Robert Beezer  
Solution [292]

**C41** The set $W$ is a subspace of $M_{22}$, the vector space of all $2 \times 2$ matrices. Prove that $S$ is a spanning set for $W$.

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| 2a - 3b + 4c - d = 0 \right\} \quad S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix} \right\}$$

Contributed by Robert Beezer  
Solution [292]

**C42** Determine if the set $S = \{ (3, 1), (7, 3) \}$ spans the crazy vector space $C$ (Example CVS [255]).

Contributed by Robert Beezer  
Solution [292]

**M10** Halfway through Example SSP4 [284], we need to show that the system of equations

$$LS \left( \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -8 \\ 0 & 1 & -6 & 24 \\ 1 & -4 & 12 & -32 \\ -2 & 4 & -8 & 16 \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} \right)$$

is consistent for every choice of the vector of constants for which $16a + 8b + 4c + 2d + e = 0$.

Express the column space of the coefficient matrix of this system as a null space, using Theorem FS [237]. From this use Theorem CSCS [212] to establish that the system is always consistent. Notice that this approach removes from Example SSP4 [284] the need to row-reduce a symbolic matrix.

Contributed by Robert Beezer  
Solution [293]
Subsection SOL
Solutions

C20 Contributed by Robert Beezer Statement 290

Begin with a relation of linear dependence on the vectors in $S$ and massage it according to the definitions of vector addition and scalar multiplication in $M_{22}$,

$$O = a_1 \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix} + a_3 \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

By our definition of matrix equality (Definition ME 163) we arrive at a homogeneous system of linear equations,

$$2a_1 + 4a_3 = 0$$
$$-a_1 + 4a_2 + 2a_3 = 0$$
$$a_1 - a_2 + a_3 = 0$$
$$3a_1 + 2a_2 + 3a_3 = 0$$

The coefficient matrix of this system row-reduces to the matrix,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and from this we conclude that the only solution is $a_1 = a_2 = a_3 = 0$. Since the relation of linear dependence (Definition RLD 280) is trivial, the set $S$ is linearly independent (Definition LI 280).

C21 Contributed by Robert Beezer Statement 290

We begin with a relation of linear dependence using unknown scalars $a$ and $b$. We wish to know if these scalars must both be zero. Recall that the zero vector in $C$ is $(-1, -1)$ and that the definitions of vector addition and scalar multiplication are not what we might expect.

$$0 = (-1, -1) = a(0, 2) + b(2, 8) = (0a + a - 1, 2a + a - 1) + (2b + b - 1, 8b + b - 1) = (a - 1, 3a - 1) + (3b - 1, 9b - 1) = (a - 1 + 3b - 1 + 1, 3a - 1 + 9b - 1 + 1) = (a + 3b - 1, 3a + 9b - 1)$$

From this we obtain two equalities, which can be converted to a homogeneous system of equations,

$$-1 = a + 3b - 1$$
$$-1 = 3a + 9b - 1$$

This homogeneous system has a singular coefficient matrix (Theorem SMZD 351), and so has more than just the trivial solution (Definition NM 61). Any nontrivial solution will give us a nontrivial relation of linear dependence on $S$. So $S$ is linearly dependent (Definition LI 280).

C22 Contributed by Robert Beezer Statement 290

Begin with a relation of linear dependence (Definition RLD 280),

$$a_1 \left(2 + x - 3x^2 - 8x^3\right) + a_2 \left(1 + x + x^2 + 5x^3\right) + a_3 \left(3 - 4x^2 - 7x^3\right) = 0$$
Massage according to the definitions of scalar multiplication and vector addition in the definition of $P_3$ (Example VSP [253]) and use the zero vector dro this vector space,

$$(2a_1 + a_2 + 3a_3) + (a_1 + a_2) x + (-3a_1 + a_2 - 4a_3) x^2 + (-8a_1 + 5a_2 - 7a_3) x^3 = 0 + 0x + 0x^2 + 0x^3$$

The definition of the equality of polynomials allows us to deduce the following four equations,

$$2a_1 + a_2 + 3a_3 = 0$$
$$a_1 + a_2 = 0$$
$$-3a_1 + a_2 - 4a_3 = 0$$
$$-8a_1 + 5a_2 - 7a_3 = 0$$

Row-reducing the coefficient matrix of this homogeneous system leads to the unique solution $a_1 = a_2 = a_3 = 0$. So the only relation of linear dependence on $S$ is the trivial one, and this is linear independence for $S$ (Definition LI [280]).

C23 Contributed by Robert Beezer Statement [290]

Notice, or discover, that the following gives a nontrivial relation of linear dependence on $S$ in $C$,

$$2(3, 1) + (-1)(7, 3) = (7, 3) + (-9, -5) = (-1, -1) = 0$$

C40 Contributed by Robert Beezer Statement [290]
The vector space $P_4$ has dimension 5 by Theorem DP [311]. Since $T$ contains only 3 vectors, and $3 < 5$, Theorem G [320] tells us that $T$ does not span $P_5$.

C41 Contributed by Robert Beezer Statement [290]
We want to show that $W = \langle S \rangle$ (Definition TSVS [284]), which is an equality of sets (Definition SE [616]).

First, show that $\langle S \rangle \subseteq W$. Begin by checking that each of the three matrices in $S$ is a member of the set $W$. Then, since $W$ is a vector space, the closure properties (Property AC [251], Property SC [251]) guarantee that every linear combination of elements of $S$ remains in $W$.

Second, show that $W \subseteq \langle S \rangle$. We want to convince ourselves that an arbitrary element of $W$ is a linear combination of elements of $S$. Choose

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$$

The values of $a$, $b$, $c$, $d$ are not totally arbitrary, since membership in $W$ requires that $2a - 3b + 4c - d = 0$. Now, rewrite as follows,

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$= \begin{bmatrix} a & b \\ c & 2a - 3b + 4c \end{bmatrix}$$
$$= \begin{bmatrix} a & 0 \\ 0 & 2a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & -3b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 4c \end{bmatrix}$$
$$= a \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix}$$
$$\in \langle S \rangle$$

C42 Contributed by Robert Beezer Statement [290]
We will try to show that $S$ spans $C$. Let $(x, y)$ be an arbitrary element of $C$ and search for scalars $a_1$ and $a_2$ such that

$$(x, y) = a_1(3, 1) + a_2(7, 3)$$
Equality in $C$ leads to the system

$$
4a_1 + 8a_2 = x + 1 \\
2a_1 + 4a_2 = y + 1
$$

This system has a singular coefficient matrix whose column space is simply $\langle \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rangle$. So any choice of $x$ and $y$ that causes the column vector $\begin{bmatrix} x + 1 \\ y + 1 \end{bmatrix}$ to lie outside the column space will lead to an inconsistent system, and hence create an element $(x, y)$ that is not in the span of $S$. So $S$ does not span $C$.

For example, choose $x = 0$ and $y = 5$, and then we can see that $\begin{bmatrix} 1 \\ 6 \end{bmatrix} \not\in \langle \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rangle$ and we know that $(0, 5)$ cannot be written as a linear combination of the vectors in $S$. A shorter solution might begin by asserting that $(0, 5)$ is not in $\langle S \rangle$ and then establishing this claim alone.

M10 Contributed by Robert Beezer Statement 290 Theorem FS 237 provides the matrix

$$
L = \begin{bmatrix} 1 & 1/2 & 1/4 & 1/8 & 1/16 \\
0 & 1/2 & 1/4 & 1/8 & 1/16 \\
\end{bmatrix}
$$

and so if $A$ denotes the coefficient matrix of the system, then $\mathcal{C}(A) = \mathcal{N}(L)$. The single homogeneous equation in $\mathcal{LS}(L, 0)$ is equivalent to the condition on the vector of constants (use $a, b, c, d, e$ as variables and then multiply by 16).
A basis of a vector space is one of the most useful concepts in linear algebra. It often provides a finite description of an infinite vector space.

We now have all the tools in place to define a basis of a vector space.

**Definition B**

**Basis**

Suppose $V$ is a vector space. Then a subset $S \subseteq V$ is a basis of $V$ if it is linearly independent and spans $V$.

So, a basis is a linearly independent spanning set for a vector space. The requirement that the set spans $V$ insures that $S$ has enough raw material to build $V$, while the linear independence requirement insures that we do not have any more raw material than we need. As we shall see soon in Section D, a basis is a minimal spanning set.

You may have noticed that we used the term basis for some of the titles of previous theorems (e.g. Theorem BNS, Theorem BCS, Theorem BRS) and if you review each of these theorems you will see that their conclusions provide linearly independent spanning sets for sets that we now recognize as subspaces of $\mathbb{C}^m$. Examples associated with these theorems include Example NSLIL, Example CSOCD and Example IAS. As we will see, these three theorems will continue to be powerful tools, even in the setting of more general vector spaces.

Furthermore, the archetypes contain an abundance of bases. For each coefficient matrix of a system of equations, and for each archetype defined simply as a matrix, there is a basis for the null space, three bases for the column space, and a basis for the row space. For this reason, our subsequent examples will concentrate on bases for vector spaces other than $\mathbb{C}^m$. Notice that Definition B does not preclude a vector space from having many bases, and this is the case, as hinted above by the statement that the archetypes contain three bases for the column space of a matrix. More generally, we can grab any basis for a vector space, multiply any one basis vector by a non-zero scalar and create a slightly different set that is still a basis. For “important” vector spaces, it will be convenient to have a collection of “nice” bases. When a vector space has a single particularly nice basis, it is sometimes called the standard basis though there is nothing precise enough about this term to allow us to define it formally — it is a question of style. Here are some nice bases for important vector spaces.

**Theorem SUVB**

**Standard Unit Vectors are a Basis**

The set of standard unit vectors for $\mathbb{C}^m$ (Definition SUV), $B = \{e_1, e_2, e_3, \ldots, e_m\} = \{e_i \mid 1 \leq i \leq m\}$ is a basis for the vector space $\mathbb{C}^m$.

**Proof** We must show that the set $B$ is both linearly independent and a spanning set for $\mathbb{C}^m$. First, the vectors in $B$ are, by Definition SUV, the columns of the identity matrix, which we know is nonsingular (since it row-reduces to the identity matrix, Theorem NMRRI). And the columns of a nonsingular matrix are linearly independent by Theorem NMLIC.
Suppose we grab an arbitrary vector from \( \mathbb{C}^m \), say
\[
\mathbf{v} = \begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
\vdots \\
v_m
\end{bmatrix}.
\]
Can we write \( \mathbf{v} \) as a linear combination of the vectors in \( B \)? Yes, and quite simply.
\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
\vdots \\
v_m
\end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + v_m \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
\]
\[
v = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3 + \cdots + v_m \mathbf{e}_m
\]
this shows that \( \mathbb{C}^m \subseteq \langle \mathbf{B} \rangle \), which is sufficient to show that \( B \) is a spanning set for \( \mathbb{C}^m \).

Example BP

Bases for \( P_n \)
The vector space of polynomials with degree at most \( n \), \( P_n \), has the basis
\[
B = \{ 1, x, x^2, \ldots, x^n \}.
\]
Another nice basis for \( P_n \) is
\[
C = \{ 1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3, \ldots, 1 + x + x^2 + x^3 + \cdots + x^n \}.
\]
Checking that each of \( B \) and \( C \) is a linearly independent spanning set are good exercises.

Example BM

A basis for the vector space of matrices
In the vector space \( M_{mn} \) of matrices (Example VSM [253]) define the matrices \( B_{k\ell} \), \( 1 \leq k \leq m \), \( 1 \leq \ell \leq n \) by
\[
[B_{k\ell}]_{ij} = \begin{cases} 
1 & \text{if } k = i, \ell = j \\
0 & \text{otherwise}
\end{cases}
\]
So these matrices have entries that are all zeros, with the exception of a lone entry that is one. The set of all \( mn \) of them,
\[
B = \{ B_{k\ell} \mid 1 \leq k \leq m, \ 1 \leq \ell \leq n \}
\]
forms a basis for \( M_{mn} \).

The bases described above will often be convenient ones to work with. However a basis doesn’t have to obviously look like a basis.

Example BSP4

A basis for a subspace of \( P_4 \)
In Example SSP4 [284] we showed that
\[
S = \{ x - 2, x^2 - 4x + 4, x^3 - 6x^2 + 12x - 8, x^4 - 8x^3 + 24x^2 - 32x + 16 \}
\]
is a spanning set for \( W = \{ p(x) \mid p \in P_4, \ p(2) = 0 \} \). We will now show that \( S \) is also linearly independent in \( W \). Begin with a relation of linear dependence,
\[
0 + 0x + 0x^2 + 0x^3 + 0x^4 = \alpha_1 (x - 2) + \alpha_2 (x^2 - 4x + 4) + \alpha_3 (x^3 - 6x^2 + 12x - 8) + \alpha_4 (x^4 - 8x^3 + 24x^2 - 32x + 16)
\]
\[
0 + 0 + 0 + 0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4
\]
which shows that \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \). Hence \( S \) is linearly independent in \( W \).
\[ = \alpha_4 x^4 + (\alpha_3 - 8\alpha_4) x^3 + (\alpha_2 - 6\alpha_3 + 24\alpha_4) x^2 + (\alpha_1 - 4\alpha_2 + 12\alpha_3 - 32\alpha_4) x + (-2\alpha_1 + 4\alpha_2 - 8\alpha_3 + 16\alpha_4) \]

Equating coefficients (vector equality in \( P_4 \)) gives the homogeneous system of five equations in four variables,

\[
\begin{align*}
\alpha_4 &= 0 \\
\alpha_3 - 8\alpha_4 &= 0 \\
\alpha_2 - 6\alpha_3 + 24\alpha_4 &= 0 \\
\alpha_1 - 4\alpha_2 + 12\alpha_3 - 32\alpha_4 &= 0 \\
-2\alpha_1 + 4\alpha_2 - 8\alpha_3 + 16\alpha_4 &= 0
\end{align*}
\]

We form the coefficient matrix, and row-reduce to obtain a matrix in reduced row-echelon form

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

With only the trivial solution to this homogeneous system, we conclude that only scalars that will form a relation of linear dependence are the trivial ones, and therefore the set \( S \) is linearly independent (Definition LI \[280\]). Finally, \( S \) has earned the right to be called a basis for \( W \) (Definition B \[294\]).

**Example BSM22**

A basis for a subspace of \( M_{22} \)

In Example SSM22 \[286\] we discovered that

\[
Q = \left\{ \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \right\}
\]

is a spanning set for the subspace

\[
Z = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \bigg| a + 3b - c - 5d = 0, -2a - 6b + 3c + 14d = 0 \right\}
\]

of the vector space of all \( 2 \times 2 \) matrices, \( M_{22} \). If we can also determine that \( Q \) is linearly independent in \( Z \) (or in \( M_{22} \)), then it will qualify as a basis for \( Z \). Let’s begin with a relation of linear dependence.

\[
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} -3\alpha_1 + \alpha_2 & \alpha_1 \\ -4\alpha_2 & \alpha_2 \end{bmatrix}
\]

Using our definition of matrix equality (Definition ME \[163\]) we equate corresponding entries and get a homogeneous system of four equations in two variables,

\[
\begin{align*}
-3\alpha_1 + \alpha_2 &= 0 \\
\alpha_1 &= 0 \\
-4\alpha_2 &= 0 \\
\alpha_2 &= 0
\end{align*}
\]

We could row-reduce the coefficient matrix of this homogeneous system, but it is not necessary. The second and fourth equations tell us that \( \alpha_1 = 0, \alpha_2 = 0 \) is the only solution to this homogeneous
system. This qualifies the set $Q$ as being linearly independent, since the only relation of linear dependence is trivial (Definition LI [280]). Therefore $Q$ is a basis for $Z$ (Definition B [294]).

**Example BC**

**Basis for the crazy vector space**

In Example LIC [283] and Example SSC [287] we determined that the set $R = \{(1, 0), (6, 3)\}$ from the crazy vector space, $C$ (Example CVS [255]), is linearly independent and is a spanning set for $C$. By Definition B [294] we see that $R$ is a basis for $C$.

We have seen that several of the sets associated with a matrix are subspaces of vector spaces of column vectors. Specifically these are the null space (Theorem NSMS [268]), column space (Theorem CSMS [274]), row space (Theorem RSMS [274]) and left null space (Theorem LNSMS [274]). As subspaces they are vector spaces (Definition S [264]) and it is natural to ask about bases for these vector spaces. Theorem BNS [128], Theorem BCS [214], Theorem BRS [220] each have conclusions that provide linearly independent spanning sets for (respectively) the null space, column space, and row space. Notice that each of these theorems contains the word “basis” in its title, even though we did not know the precise meaning of the word at the time. To find a basis for a left null space we can use the definition of this subspace as a null space (Definition LNS [231]) and apply Theorem BNS [128]. Or Theorem FS [237] tells us that the left null space can be expressed as a row space and we can then use Theorem BRS [220].

Theorem BS [143] is another early result that provides a linearly independent spanning set (i.e. a basis) as its conclusion. If a vector space of column vectors can be expressed as a span of a set of column vectors, then Theorem BS [143] can be employed in a straightforward manner to quickly yield a basis.

**Subsection BSCV**

**Bases for Spans of Column Vectors**

We have seen several examples of bases in different vector spaces. In this subsection, and the next (Subsection B.BNM [299]), we will consider building bases for $\mathbb{C}^m$ and its subspaces.

Suppose we have a subspace of $\mathbb{C}^m$ that is expressed as the span of a set of vectors, $S$, and $S$ is not necessarily linearly independent, or perhaps not very attractive. Theorem REMRS [218] says that row-equivalent matrices have identical row spaces, while Theorem BRS [220] says the nonzero rows of a matrix in reduced row-echelon form are a basis for the row space. These theorems together give us a great computational tool for quickly finding a basis for a subspace that is expressed originally as a span.

**Example RSB**

**Row space basis**

When we first defined the span of a set of column vectors, in Example SCAD [110] we looked at the set

$$W = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ -5 \end{bmatrix} \right\}$$

with an eye towards realizing $W$ as the span of a smaller set. By building relations of linear dependence (though we did not know them by that name then) we were able to remove two vectors and write $W$ as the span of the other two vectors. These two remaining vectors formed a linearly independent set, even though we did not know that at the time.

Now we know that $W$ is a subspace and must have a basis. Consider the matrix, $C$, whose rows are the vectors in the spanning set for $W$,

$$C = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 4 & 1 \\ 7 & -5 & 4 \\ -7 & -6 & -5 \end{bmatrix}$$
Subsection B.BSCV  Bases for Spans of Column Vectors  298

Then, by Definition RSM [217], the row space of \( C \) will be \( W, \mathcal{R}(C) = W \). Theorem BRS [220] tells us that if we row-reduce \( C \), the nonzero rows of the row-equivalent matrix in reduced row-echelon form will be a basis for \( \mathcal{R}(C) \), and hence a basis for \( W \). Let’s do it — \( C \) row-reduces to

\[
\begin{bmatrix}
1 & 0 & 7 \\
0 & 1 & 11 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

If we convert the two nonzero rows to column vectors then we have a basis,

\[
B = \left\{ \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix} \right\}
\]

and

\[
W = \left\langle \begin{bmatrix} 1 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix} \right\}
\]

For aesthetic reasons, we might wish to multiply each vector in \( B \) by 11, which will not change the spanning or linear independence properties of \( B \) as a basis. Then we can also write

\[
W = \left\langle \begin{bmatrix} 11 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 11 \\ 1 \end{bmatrix} \right\}
\]

Example IAS [220] provides another example of this flavor, though now we can notice that \( X \) is a subspace, and that the resulting set of three vectors is a basis. This is such a powerful technique that we should do one more example.

Example RS
Reducing a span
In Example RSC5 [140] we began with a set of \( n = 4 \) vectors from \( \mathbb{C}^5 \),

\[
R = \{ v_1, v_2, v_3, v_4 \} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 2 \\ -7 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -7 \\ 6 \\ -11 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \\ 6 \\ -2 \\ 6 \end{bmatrix} \right\}
\]

and defined \( V = \langle R \rangle \). Our goal in that problem was to find a relation of linear dependence on the vectors in \( R \), solve the resulting equation for one of the vectors, and re-express \( V \) as the span of a set of three vectors.

Here is another way to accomplish something similar. The row space of the matrix

\[
A = \begin{bmatrix}
1 & 2 & -1 & 3 & 2 \\
2 & 1 & 3 & 1 & 2 \\
0 & -7 & 6 & -11 & -2 \\
4 & 1 & 2 & 1 & 6
\end{bmatrix}
\]

is equal to \( \langle R \rangle \). By Theorem BRS [220] we can row-reduce this matrix, ignore any zero rows, and use the non-zero rows as column vectors that are a basis for the row space of \( A \). Row-reducing \( A \) creates the matrix

\[
\begin{bmatrix}
1 & 0 & 0 & -\frac{1}{17} & \frac{30}{17} \\
0 & 1 & 0 & \frac{2}{17} & \frac{17}{17} \\
0 & 0 & 1 & -\frac{12}{17} & -\frac{8}{17} \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
is a basis for $V$. Our theorem tells us this is a basis, there is no need to verify that the subspace spanned by three vectors (rather than four) is the identical subspace, and there is no need to verify that we have reached the limit in reducing the set, since the set of three vectors is guaranteed to be linearly independent.

Subsection B.NM

Bases and Nonsingular Matrices

A quick source of diverse bases for $\mathbb{C}^m$ is the set of columns of a nonsingular matrix.

**Theorem CNMB**

**Columns of Nonsingular Matrix are a Basis**

Suppose that $A$ is a square matrix of size $m$. Then the columns of $A$ are a basis of $\mathbb{C}^m$ if and only if $A$ is nonsingular. □

**Proof**  
$(\Rightarrow)$ Suppose that the columns of $A$ are a basis for $\mathbb{C}^m$. Then Definition B [294] says the set of columns is linearly independent. Theorem NMLIC [126] then says that $A$ is nonsingular.

$(\Leftarrow)$ Suppose that $A$ is nonsingular. Then by Theorem NMLIC [126] this set of columns is linearly independent. Theorem CSNM [216] says that for a nonsingular matrix, $\mathbb{C}(A) = \mathbb{C}^m$. This is equivalent to saying that the columns of $A$ are a spanning set for the vector space $\mathbb{C}^m$. As a linearly independent spanning set, the columns of $A$ qualify as a basis for $\mathbb{C}^m$ (Definition B [294]). ■

**Example CABAK**

**Columns as Basis, Archetype K**

Archetype K [676] is the $5 \times 5$ matrix

\[
K = \begin{bmatrix}
10 & 18 & 24 & 24 & -12 \\
12 & 2 & -6 & 0 & -18 \\
-30 & -21 & -23 & -30 & 39 \\
27 & 30 & 36 & 37 & -30 \\
18 & 24 & 30 & 30 & -20
\end{bmatrix}
\]

which is row-equivalent to the $5 \times 5$ identity matrix $I_5$. So by Theorem NMRRI [62], $K$ is nonsingular. Then Theorem CNMB [299] says the set

\[
\left\{ \begin{bmatrix} 10 \\ 12 \\ -30 \\ 18 \\ 18 \end{bmatrix}, \begin{bmatrix} 18 \\ -2 \\ -21 \\ 30 \\ 24 \end{bmatrix}, \begin{bmatrix} 24 \\ -6 \\ -23 \\ 36 \\ 24 \end{bmatrix}, \begin{bmatrix} 24 \\ 0 \\ -30 \\ 30 \\ 30 \end{bmatrix}, \begin{bmatrix} -12 \\ 0 \\ 39 \\ 37 \\ -20 \end{bmatrix} \right\}
\]

is a (novel) basis of $\mathbb{C}^5$.

Perhaps we should view the fact that the standard unit vectors are a basis (Theorem SUVB [294]) as just a simple corollary of Theorem CNMB [299]? (See Technique LC [627].)

With a new equivalence for a nonsingular matrix, we can update our list of equivalences.

**Theorem NME5**

**Nonsingular Matrix Equivalences, Round 5**

Suppose that $A$ is a square matrix of size $n$. The following are equivalent.
1. \( A \) is nonsingular.
2. \( A \) row-reduces to the identity matrix.
3. The null space of \( A \) contains only the zero vector, \( \mathcal{N}(A) = \{0\} \).
4. The linear system \( \mathcal{L}(A, b) \) has a unique solution for every possible choice of \( b \).
5. The columns of \( A \) are a linearly independent set.
6. \( A \) is invertible.
7. The column space of \( A \) is \( \mathbb{C}^n \), \( \mathcal{C}(A) = \mathbb{C}^n \).
8. The columns of \( A \) are a basis for \( \mathbb{C}^n \).

\[ \square \]

**Proof** With a new equivalence for a nonsingular matrix in Theorem CNMB [299] we can expand Theorem NME4 [216].

### Subsection OBC
**Orthonormal Bases and Coordinates**

We learned about orthogonal sets of vectors in \( \mathbb{C}^m \) back in Section O [151], and we also learned that orthogonal sets are automatically linearly independent (Theorem OSLI [157]). When an orthogonal set also spans a subspace of \( \mathbb{C}^m \), then the set is a basis. And when the set is orthonormal, then the set is an incredibly nice basis. We will back up this claim with a theorem, but first consider how you might manufacture such a set.

Suppose that \( W \) is a subspace of \( \mathbb{C}^m \) with basis \( B \). Then \( B \) spans \( W \) and is a linearly independent set of nonzero vectors. We can apply the Gram-Schmidt Procedure (Theorem GSPCV [158]) and obtain a linearly independent set \( T \) such that \( \langle T \rangle = \langle B \rangle = W \) and \( T \) is orthogonal. In other words, \( T \) is a basis for \( W \), and is an orthogonal set. By scaling each vector of \( T \) to norm 1, we can convert \( T \) into an orthonormal basis. This process is illustrated in Example GSTV [160], followed by Example ONTV [160].

Unitary matrices (Definition UM [205]) are another good source of orthonormal bases (and vice versa). Suppose that \( Q \) is a unitary matrix of size \( n \). Then the \( n \) columns of \( Q \) form an orthonormal set (Theorem CUMOS [206]) that is therefore linearly independent (Theorem OSLI [157]). Since \( Q \) is invertible (Theorem UMI [205]), we know \( Q \) is nonsingular (Theorem NI [204]), and then the columns of \( Q \) span \( \mathbb{C}^n \) (Theorem CSNM [216]). So the columns of a unitary matrix of size \( n \) are an orthonormal basis for \( \mathbb{C}^n \).

Why all the fuss about orthonormal bases? Theorem VRRB [288] told us that any vector in a vector space could be written, uniquely, as a linear combination of basis vectors. For an orthonormal basis, finding the scalars for this linear combination is extremely easy, and this is the content of the next theorem. Furthermore, with vectors written this way (as linear combinations of the elements of an orthonormal set) certain computations and analysis become much easier. Here’s the promised theorem.

**Theorem COB**
**Coordinates and Orthonormal Bases**

Suppose that \( B = \{v_1, v_2, v_3, \ldots, v_p\} \) is an orthonormal basis of the subspace \( W \) of \( \mathbb{C}^m \). For any \( w \in W \),

\[
\begin{align*}
  w &= \langle w, v_1 \rangle v_1 + \langle w, v_2 \rangle v_2 + \langle w, v_3 \rangle v_3 + \cdots + \langle w, v_p \rangle v_p
\end{align*}
\]

\[ \square \]

**Proof** Because \( B \) is a basis of \( W \), Theorem VRRB [288] tells us that we can write \( w \) uniquely as a linear combination of the vectors in \( B \). So it is not this aspect of the conclusion that makes
this theorem interesting. What is interesting is that the particular scalars are so easy to compute. No need to solve big systems of equations — just do an inner product of \( w \) with \( v_i \) to arrive at the coefficient of \( v_i \) in the linear combination.

So begin the proof by writing \( w \) as a linear combination of the vectors in \( B \), using unknown scalars,

\[
 w = a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_pv_p
\]

and compute,

\[
\langle w, v_i \rangle = \sum_{k=1}^{p} \langle a_kv_k, v_i \rangle = \sum_{k=1}^{p} a_k \langle v_k, v_i \rangle
\]

by Theorem IPVA \[153\] and Theorem IPSM \[153\]. Isolate term with \( k = i \)

\[
= a_i \langle v_i, v_i \rangle + \sum_{k \neq i} a_k \langle v_k, v_i \rangle
\]

So the (unique) scalars for the linear combination are indeed the inner products advertised in the conclusion of the theorem’s statement.

\[\Box\]

Example CROB4

Coordinatization relative to an orthonormal basis, \( \mathbb{C}^4 \)

The set

\[
\{ x_1, x_2, x_3, x_4 \} = \left\{ \begin{bmatrix} 1 + i \\ 1 \\ 1 - i \\ i \end{bmatrix}, \begin{bmatrix} 1 + 5i \\ 6 + 5i \\ -7 - i \\ 1 - 6i \end{bmatrix}, \begin{bmatrix} -7 + 34i \\ -8 - 23i \\ -10 + 22i \\ 30 + 13i \end{bmatrix}, \begin{bmatrix} -2 - 4i \\ 6 + i \\ 4 + 3i \\ 6 - i \end{bmatrix} \right\}
\]

was proposed, and partially verified, as an orthogonal set in Example AOS \[157\]. Let’s scale each vector to norm 1, so as to form an orthonormal basis of \( \mathbb{C}^4 \). (Notice that by Theorem OSLI \[157\] the set is linearly independent. Since we know the dimension of \( \mathbb{C}^4 \) is 4, Theorem G \[320\] tells us the set is just the right size to be a basis of \( \mathbb{C}^4 \).) The norms of these vectors are,

\[
\| x_1 \| = \sqrt{6}, \quad \| x_2 \| = \sqrt{174}, \quad \| x_3 \| = \sqrt{3451}, \quad \| x_4 \| = \sqrt{119}
\]

So an orthonormal basis is

\[
B = \{ v_1, v_2, v_3, v_4 \} = \left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 1 + i \\ 1 \\ 1 - i \\ i \end{bmatrix}, \frac{1}{\sqrt{174}} \begin{bmatrix} 1 + 5i \\ 6 + 5i \\ -7 - i \\ 1 - 6i \end{bmatrix}, \frac{1}{\sqrt{3451}} \begin{bmatrix} -7 + 34i \\ -8 - 23i \\ -10 + 22i \\ 30 + 13i \end{bmatrix}, \frac{1}{\sqrt{119}} \begin{bmatrix} -2 - 4i \\ 6 + i \\ 4 + 3i \\ 6 - i \end{bmatrix} \right\}
\]

Now, choose any vector from \( \mathbb{C}^4 \), say \( w = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 4 \end{bmatrix} \), and compute

\[
\langle w, v_1 \rangle = \frac{-5i}{\sqrt{6}}, \quad \langle w, v_2 \rangle = \frac{-19 + 30i}{\sqrt{174}}, \quad \langle w, v_3 \rangle = \frac{120 - 211i}{\sqrt{3451}}, \quad \langle w, v_4 \rangle = \frac{6 + 12i}{\sqrt{119}}
\]
then \textbf{Theorem COB 300} guarantees that

\[
\begin{bmatrix}
2 \\
-3 \\
1 \\
4
\end{bmatrix} = \frac{-5i}{\sqrt{6}} \begin{bmatrix}
1 + i \\
1 \\
i
\end{bmatrix} + \frac{-19 + 30i}{\sqrt{174}} \begin{bmatrix}
1 + 5i \\
6 + 5i \\
1 - 6i
\end{bmatrix}
\]

\[
+ \frac{120 - 211i}{\sqrt{3451}} \begin{bmatrix}
-7 + 34i \\
-8 - 23i \\
30 + 13i
\end{bmatrix} + \frac{6 + 12i}{\sqrt{119}} \begin{bmatrix}
-2 - 4i \\
6 + i \\
6 - i
\end{bmatrix}
\]

as you might want to check (if you have unlimited patience).

A slightly less intimidating example follows, in three dimensions and with just real numbers.

\textbf{Example CROB3}

\textbf{Coordinatization relative to an orthonormal basis, }\mathbb{C}^3

The set

\[
\{x_1, x_2, x_3\} = \left\{ \begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}, \begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
2 \\
1 \\
1
\end{bmatrix}\right\}
\]

is a linearly independent set, which the Gram-Schmidt Process (\textbf{Theorem GSPCV 158}) converts to an orthogonal set, and which can then be converted to the orthonormal set,

\[
B = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}\right\}
\]

which is therefore an orthonormal basis of \mathbb{C}^3. With three vectors in \mathbb{C}^3, all with real number entries, the inner product (\textbf{Definition IP 152}) reduces to the usual “dot product” (or scalar product) and the orthogonal pairs of vectors can be interpreted as perpendicular pairs of directions. So the vectors in \(B\) serve as replacements for our usual 3-D axes, or the usual 3-D unit vectors \(\vec{i}, \vec{j}\) and \(\vec{k}\).

We would like to decompose arbitrary vectors into “components” in the directions of each of these basis vectors. It is \textbf{Theorem COB 300} that tells us how to do this.

Suppose that we choose \(w = \begin{bmatrix}
2 \\
-1 \\
5
\end{bmatrix}\). Compute

\[
\langle w, v_1 \rangle = \frac{5}{\sqrt{6}}, \quad \langle w, v_2 \rangle = \frac{3}{\sqrt{2}}, \quad \langle w, v_3 \rangle = \frac{8}{\sqrt{3}}
\]

then \textbf{Theorem COB 300} guarantees that

\[
\begin{bmatrix}
2 \\
-1 \\
5
\end{bmatrix} = \frac{5}{\sqrt{6}} \begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix} + \frac{3}{\sqrt{2}} \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix} + \frac{8}{\sqrt{3}} \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

which you should be able to check easily, even if you do not have much patience.

\textbf{Subsection READ}

\textbf{Reading Questions}

1. The matrix below is nonsingular. What can you now say about its columns?

\[
A = \begin{bmatrix}
-3 & 0 & 1 \\
1 & 2 & 1 \\
5 & 1 & 6
\end{bmatrix}
\]
2. Write the vector $\mathbf{w} = \begin{bmatrix} 6 \\ 6 \\ 15 \end{bmatrix}$ as a linear combination of the columns of the matrix $A$ above.

How many ways are there to answer this question?

3. Why is an orthonormal basis desirable?
Subsection EXC
Exercises

C40  From Example RSB 297, form an arbitrary (and nontrivial) linear combination of the four vectors in the original spanning set for $W$. So the result of this computation is of course an element of $W$. As such, this vector should be a linear combination of the basis vectors in $B$. Find the (unique) scalars that provide this linear combination. Repeat with another linear combination of the original four vectors.
Contributed by Robert Beezer  Solution 306

C80  Prove that $\{(1, 2), (2, 3)\}$ is a basis for the crazy vector space $C$ (Example CVS 255).
Contributed by Robert Beezer

M20  In Example BM 295 provide the verifications (linear independence and spanning) to show that $B$ is a basis of $M_{mn}$.
Contributed by Robert Beezer  Solution 305
Subsection SOL
Solutions

M20 Contributed by Robert Beezer Statement 304

We need to establish the linear independence and spanning properties of the set

\[ B = \{ B_{kℓ} | 1 ≤ k ≤ m, \ 1 ≤ ℓ ≤ n \} \]

relative to the vector space \( M_{mn} \).

This proof is more transparent if you write out individual matrices in the basis with lots of zeros and dots and a lone one. But we don’t have room for that here, so we will use summation notation. Think carefully about each step, especially when the double summations seem to “disappear.” Begin with a relation of linear dependence, using double subscripts on the scalars to align with the basis elements.

\[ \mathcal{O} = \sum_{k=1}^{m} \sum_{ℓ=1}^{n} \alpha_{kℓ} B_{kℓ} \]

Now consider the entry in row \( i \) and column \( j \) for these equal matrices,

\[ 0 = [\mathcal{O}]_{ij} \]

\[ = \left[ \sum_{k=1}^{m} \sum_{ℓ=1}^{n} \alpha_{kℓ} B_{kℓ} \right] _{ij} \quad \text{Definition ZM} \quad 166 \]

\[ = \sum_{k=1}^{m} \sum_{ℓ=1}^{n} [\alpha_{kℓ} B_{kℓ}]_{ij} \quad \text{Definition ME} \quad 163 \]

\[ = \sum_{k=1}^{m} \sum_{ℓ=1}^{n} α_{kℓ} [B_{kℓ}]_{ij} \quad \text{Definition MA} \quad 163 \]

\[ = α_{ij} \quad \text{Definition MSM} \quad 164 \]

\[ = α_{ij} [B_{ij}]_{ij} \]

\[ = α_{ij} \quad \text{since } [B_{ij}]_{ij} = 1 \]

Since \( i \) and \( j \) were arbitrary, we find that each scalar is zero and so \( B \) is linearly independent (Definition LI 280).

To establish the spanning property of \( B \) we need only show that an arbitrary matrix \( A \) can be written as a linear combination of the elements of \( B \). So suppose that \( A \) is an arbitrary \( m \times n \) matrix and consider the matrix \( C \) defined as a linear combination of the elements of \( B \) by

\[ C = \sum_{k=1}^{m} \sum_{ℓ=1}^{n} [A]_{kℓ} B_{kℓ} \]

Then,

\[ [C]_{ij} = \left[ \sum_{k=1}^{m} \sum_{ℓ=1}^{n} [A]_{kℓ} B_{kℓ} \right] _{ij} \quad \text{Definition ME} \quad 163 \]

\[ = \sum_{k=1}^{m} \sum_{ℓ=1}^{n} ([A]_{kℓ} B_{kℓ})_{ij} \quad \text{Definition MA} \quad 163 \]

\[ = \sum_{k=1}^{m} \sum_{ℓ=1}^{n} [A]_{kℓ} [B_{kℓ}]_{ij} \quad \text{Definition MSM} \quad 164 \]

\[ = [A]_{ij} [B_{ij}]_{ij} \]

\[ = [A]_{ij} (1) \quad [B_{ij}]_{ij} = 1 \]

\[ [B_{kℓ}]_{ij} = 0 \text{ when } (k, ℓ) \neq (i, j) \]

\[ [B_{ij}]_{ij} = 1 \]

Version 1.04
So by Definition ME 163, $A = C$, and therefore $A \in \langle B \rangle$. By Definition B 294, the set $B$ is a basis of the vector space $M_{mn}$.

Contributed by Robert Beezer Statement 304

An arbitrary linear combination is

$$y = 3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix} + (-2) \begin{bmatrix} -7 \\ -6 \\ -5 \end{bmatrix} = \begin{bmatrix} 25 \\ -10 \end{bmatrix}$$

(You probably used a different collection of scalars.) We want to write $y$ as a linear combination of

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{7}{11} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{1}{11} \end{bmatrix} \right\}$$

We could set this up as vector equation with variables as scalars in a linear combination of the vectors in $B$, but since the first two slots of $B$ have such a nice pattern of zeros and ones, we can determine the necessary scalars easily and then double-check our answer with a computation in the third slot,

$$25 \begin{bmatrix} 1 \\ 0 \\ \frac{7}{11} \end{bmatrix} + (-10) \begin{bmatrix} 0 \\ 1 \\ \frac{1}{11} \end{bmatrix} = \begin{bmatrix} 25 \\ -10 \end{bmatrix} \frac{7}{11} + (-10) \frac{1}{11} = \begin{bmatrix} 25 \\ -10 \end{bmatrix} = y$$

Notice how the uniqueness of these scalars arises. They are forced to be 25 and $-10$. 
Section D  
Dimension  

Almost every vector space we have encountered has been infinite in size (an exception is Example VSS 255). But some are bigger and richer than others. Dimension, once suitably defined, will be a measure of the size of a vector space, and a useful tool for studying its properties. You probably already have a rough notion of what a mathematical definition of dimension might be — try to forget these imprecise ideas and go with the new ones given here.

Subsection D  
Dimension  

Definition D  
Dimension  
Suppose that \( V \) is a vector space and \( \{v_1, v_2, v_3, \ldots, v_t\} \) is a basis of \( V \). Then the dimension of \( V \) is defined by \( \dim(V) = t \). If \( V \) has no finite bases, we say \( V \) has infinite dimension. (This definition contains Notation D.) △

This is a very simple definition, which belies its power. Grab a basis, any basis, and count up the number of vectors it contains. That’s the dimension. However, this simplicity causes a problem. Given a vector space, you and I could each construct different bases — remember that a vector space might have many bases. And what if your basis and my basis had different sizes? Applying Definition D 307 we would arrive at different numbers! With our current knowledge about vector spaces, we would have to say that dimension is not “well-defined.” Fortunately, there is a theorem that will correct this problem.

In a strictly logical progression, the next two theorems would precede the definition of dimension. Many subsequent theorems will trace their lineage back to the following fundamental result.

Theorem SSLD  
Spanning Sets and Linear Dependence  
Suppose that \( S = \{v_1, v_2, v_3, \ldots, v_t\} \) is a finite set of vectors which spans the vector space \( V \). Then any set of \( t + 1 \) or more vectors from \( V \) is linearly dependent. □

Proof  We want to prove that any set of \( t + 1 \) or more vectors from \( V \) is linearly dependent. So we will begin with a totally arbitrary set of vectors from \( V \), \( R = \{u_1, u_2, u_3, \ldots, u_m\} \), where \( m > t \). We will now construct a nontrivial relation of linear dependence on \( R \).

Each vector \( u_1, u_2, u_3, \ldots, u_m \) can be written as a linear combination of \( v_1, v_2, v_3, \ldots, v_t \) since \( S \) is a spanning set of \( V \). This means there exist scalars \( a_{ij} \), \( 1 \leq i \leq t \), \( 1 \leq j \leq m \), so that

\[
\begin{align*}
    u_1 &= a_{11}v_1 + a_{21}v_2 + a_{31}v_3 + \cdots + a_{t1}v_t \\
    u_2 &= a_{12}v_1 + a_{22}v_2 + a_{32}v_3 + \cdots + a_{t2}v_t \\
    u_3 &= a_{13}v_1 + a_{23}v_2 + a_{33}v_3 + \cdots + a_{t3}v_t \\
    & \vdots \\
    u_m &= a_{1m}v_1 + a_{2m}v_2 + a_{3m}v_3 + \cdots + a_{tm}v_t
\end{align*}
\]

Now we form, unmotivated, the homogeneous system of \( t \) equations in the \( m \) variables, \( x_1, x_2, x_3, \ldots, x_m \), where the coefficients are the just-discovered scalars \( a_{ij} \),

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1m}x_m &= 0 \\
    a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2m}x_m &= 0 \\
    a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3m}x_m &= 0
\end{align*}
\]
This is a homogeneous system with more variables than equations (our hypothesis is expressed as \( m > t \)), so by Theorem HMVEI \[54\] there are infinitely many solutions. Choose a nontrivial solution and denote it by \( x_1 = c_1, x_2 = c_2, x_3 = c_3, \ldots, x_m = c_m \). As a solution to the homogeneous system, we then have

\[
\begin{align*}
a_{11}c_1 + a_{12}c_2 + a_{13}c_3 + \cdots + a_{1m}c_m &= 0 \\
a_{21}c_1 + a_{22}c_2 + a_{23}c_3 + \cdots + a_{2m}c_m &= 0 \\
a_{31}c_1 + a_{32}c_2 + a_{33}c_3 + \cdots + a_{3m}c_m &= 0 \\
&\vdots \\
a_{t1}c_1 + a_{t2}c_2 + a_{t3}c_3 + \cdots + a_{tm}c_m &= 0
\end{align*}
\]

As a collection of nontrivial scalars, \( c_1, c_2, c_3, \ldots, c_m \) will provide the nontrivial relation of linear dependence we desire,

\[
c_1u_1 + c_2u_2 + c_3u_3 + \cdots + c_mu_m = S \text{ spans } V
\]

\[
= c_1 \left( a_{11}v_1 + a_{21}v_2 + a_{31}v_3 + \cdots + a_{t1}v_t \right) \\
+ c_2 \left( a_{12}v_1 + a_{22}v_2 + a_{32}v_3 + \cdots + a_{t2}v_t \right) \\
+ c_3 \left( a_{13}v_1 + a_{23}v_2 + a_{33}v_3 + \cdots + a_{t3}v_t \right) \\
&\vdots \\
+ c_m \left( a_{1m}v_1 + a_{2m}v_2 + a_{3m}v_3 + \cdots + a_{tm}v_t \right)
\]

\[
= (c_1a_{11} + c_2a_{12} + c_3a_{13} + \cdots + c_ma_{1m})v_1 \\
+ (c_1a_{21} + c_2a_{22} + c_3a_{23} + \cdots + c_ma_{2m})v_2 \\
+ (c_1a_{31} + c_2a_{32} + c_3a_{33} + \cdots + c_ma_{3m})v_3 \\
&\vdots \\
+ (c_1a_{t1} + c_2a_{t2} + c_3a_{t3} + \cdots + c_ma_{tm})v_t
\]

\[
= (a_{11}c_1 + a_{12}c_2 + a_{13}c_3 + \cdots + a_{1m}c_m)v_1 \\
+ (a_{21}c_1 + a_{22}c_2 + a_{23}c_3 + \cdots + a_{2m}c_m)v_2 \\
+ (a_{31}c_1 + a_{32}c_2 + a_{33}c_3 + \cdots + a_{3m}c_m)v_3 \\
&\vdots \\
+ (a_{t1}c_1 + a_{t2}c_2 + a_{t3}c_3 + \cdots + a_{tm}c_m)v_t
\]

That does it. \( R \) has been undeniably shown to be a linearly dependent set.

The proof just given has some rather monstrous expressions in it, mostly owing to the double subscripts present. Now is a great opportunity to show the value of a more compact notation.
We will rewrite the key steps of the previous proof using summation notation, resulting in a more economical presentation, and even greater insight into the key aspects of the proof. **Proof (Alternate Proof of Theorem SSLD)** We want to prove that any set of \( t + 1 \) or more vectors from \( V \) is linearly dependent. So we will begin with a totally arbitrary set of vectors from \( V \), \( R = \{ u_j \mid 1 \leq j \leq m \} \), where \( m > t \). We will now construct a nontrivial relation of linear dependence on \( R \).

Each vector \( u_j \), \( 1 \leq j \leq m \) can be written as a linear combination of \( v_i \), \( 1 \leq i \leq t \) since \( S \) is a spanning set of \( V \). This means there are scalars \( a_{ij} \), \( 1 \leq i \leq t \), \( 1 \leq j \leq m \), so that

\[
    u_j = \sum_{i=1}^{t} a_{ij} v_i \quad 1 \leq j \leq m
\]

Now we form, unmotivated, the homogeneous system of \( t \) equations in the \( m \) variables, \( x_j \), \( 1 \leq j \leq m \), where the coefficients are the just-discovered scalars \( a_{ij} \),

\[
    \sum_{j=1}^{m} a_{ij} x_j = 0 \quad 1 \leq i \leq t
\]

This is a homogeneous system with more variables than equations (our hypothesis is expressed as \( m > t \)), so by **Theorem HMVEI** \[54\] there are infinitely many solutions. Choose one of these solutions that is not trivial and denote it by \( x_j = c_j \), \( 1 \leq j \leq m \). As a solution to the homogeneous system, we then have \( \sum_{j=1}^{m} a_{ij} c_j = 0 \) for \( 1 \leq i \leq t \). As a collection of nontrivial scalars, \( c_j \), \( 1 \leq j \leq m \), will provide the nontrivial relation of linear dependence we desire,

\[
    \sum_{j=1}^{m} c_j u_j = \sum_{j=1}^{m} c_j \left( \sum_{i=1}^{t} a_{ij} v_i \right) = \sum_{j=1}^{m} \sum_{i=1}^{t} c_j a_{ij} v_i = t \sum_{i=1}^{m} \sum_{j=1}^{t} a_{ij} c_j v_i = \sum_{i=1}^{t} \left( \sum_{j=1}^{m} a_{ij} c_j \right) v_i = \sum_{i=1}^{t} 0 v_i = 0
\]

That does it. \( R \) has been undeniably shown to be a linearly dependent set. ■

Notice how the swap of the two summations is so much easier in the third step above, as opposed to all the rearranging and regrouping that takes place in the previous proof. In about half the space. And there are no ellipses ( . . . ).

**Theorem SSLD** \[307\] can be viewed as a generalization of **Theorem MVSLD** \[126\]. We know that \( \mathbb{C}^m \) has a basis with \( m \) vectors in it (**Theorem SUVB** \[294\]), so it is a set of \( m \) vectors that spans \( \mathbb{C}^m \). By **Theorem SSLD** \[307\], any set of more than \( m \) vectors from \( \mathbb{C}^m \) will be linearly dependent. But this is exactly the conclusion we have in **Theorem MVSLD** \[126\]. Maybe this is
not a total shock, as the proofs of both theorems rely heavily on Theorem HMVEI [54]. The beauty of Theorem SSLD [307] is that it applies in any vector space. We illustrate the generality of this theorem, and hint at its power, in the next example.

**Example LDP4**

Linearly dependent set in $P_4$

In Example SSP4 [284] we showed that $S = \{ x - 2, x^2 - 4x + 4, x^3 - 6x^2 + 12x - 8, x^4 - 8x^3 + 24x^2 - 32x + 16 \}$ is a spanning set for $W = \{ p(x) \mid p \in P_4, p(2) = 0 \}$. So we can apply Theorem SSLD [307] to $W$ with $t = 4$. Here is a set of five vectors from $W$, as you may check by verifying that each is a polynomial of degree 4 or less and has $x = 2$ as a root,

$$
T = \{ p_1, p_2, p_3, p_4, p_5 \} \subseteq W
$$

$$
p_1 = x^4 - 2x^3 + 2x^2 - 8x + 8 \\
p_2 = -x^3 + 6x^2 - 5x - 6 \\
p_3 = 2x^4 - 5x^3 + 5x^2 - 7x + 2 \\
p_4 = -x^4 + 4x^3 - 7x^2 + 6x \\
p_5 = 4x^3 - 9x^2 + 5x - 6
$$

By Theorem SSLD [307] we conclude that $T$ is linearly dependent, with no further computations.

Theorem SSLD [307] is indeed powerful, but our main purpose in proving it right now was to make sure that our definition of dimension (Definition D [307]) is well-defined. Here’s the theorem.

**Theorem BIS**

**Bases have Identical Sizes**

Suppose that $V$ is a vector space with a finite basis $B$ and a second basis $C$. Then $B$ and $C$ have the same size. □

**Proof** Suppose that $C$ has more vectors than $B$. (Allowing for the possibility that $C$ is infinite, we can replace $C$ by a subset that has more vectors than $B$.) As a basis, $B$ is a spanning set for $V$ (Definition B [294]), so Theorem SSLD [307] says that $C$ is linearly dependent. However, this contradicts the fact that as a basis $C$ is linearly independent (Definition B [294]). So $C$ must also be a finite set, with size less than, or equal to, that of $B$.

Suppose that $B$ has more vectors than $C$. As a basis, $C$ is a spanning set for $V$ (Definition B [294]), so Theorem SSLD [307] says that $B$ is linearly dependent. However, this contradicts the fact that as a basis $B$ is linearly independent (Definition B [294]). So $C$ cannot be strictly smaller than $B$.

The only possibility left for the sizes of $B$ and $C$ is for them to be equal. □

Theorem BIS [310] tells us that if we find one finite basis in a vector space, then they all have the same size. This (finally) makes Definition D [307] unambiguous.

**Subsection DVS**

**Dimension of Vector Spaces**

We can now collect the dimension of some common, and not so common, vector spaces.

**Theorem DCM**

**Dimension of $C^m$**
The dimension of $\mathbb{C}^m$ (Example VSCV 253) is $m$. □

**Proof** Theorem SUVB 294 provides a basis with $m$ vectors. ■

**Theorem DP**
**Dimension of $P_n$**
The dimension of $P_n$ (Example VSP 253) is $n + 1$. □

**Proof** Example BP 295 provides two bases with $n + 1$ vectors. Take your pick. ■

**Theorem DM**
**Dimension of $M_{mn}$**
The dimension of $M_{mn}$ (Example VSM 253) is $mn$. □

**Proof** Example BM 295 provides a basis with $mn$ vectors. ■

**Example DSM22**
**Dimension of a subspace of $M_{22}$**
It should now be plausible that

$$Z = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left| \begin{array}{c} 2a + b + 3c + 4d = 0 \\ -a + 3b - 5c - d = 0 \end{array} \right. \right\}$$

is a subspace of the vector space $M_{22}$ (Example VSM 253). (It is.) To find the dimension of $Z$ we must first find a basis, though any old basis will do.

First concentrate on the conditions relating $a$, $b$, $c$ and $d$. They form a homogeneous system of two equations in four variables with coefficient matrix

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ -1 & 3 & -5 & -1 \end{bmatrix}$$

We can row-reduce this matrix to obtain

$$\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

Rewrite the two equations represented by each row of this matrix, expressing the dependent variables ($a$ and $b$) in terms of the free variables ($c$ and $d$), and we obtain,

$$a = -2c - 2d$$
$$b = c$$

We can now write a typical entry of $Z$ strictly in terms of $c$ and $d$, and we can decompose the result,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2c - 2d & c \\ c & d \end{bmatrix} = \begin{bmatrix} -2c & c \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -2d & 0 \\ 0 & 0 \end{bmatrix} = c \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

this equation says that an arbitrary matrix in $Z$ can be written as a linear combination of the two vectors in

$$S = \left\{ \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

so we know that

$$Z = \langle S \rangle = \left\langle \left\{ \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \right\} \right\rangle$$

Are these two matrices (vectors) also linearly independent? Begin with a relation of linear dependence on $S$,

$$a_1 \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} + a_2 \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{0}$$
From the equality of the two entries in the last row, we conclude that \( a_1 = 0, a_2 = 0. \) Thus the only possible relation of linear dependence is the trivial one, and therefore \( S \) is linearly independent (Definition LI [280]). So \( S \) is a basis for \( V \) (Definition B [294]). Finally, we can conclude that \( \dim(Z) = 2 \) (Definition D [307]) since \( S \) has two elements.

**Example DSP4**

**Dimension of a subspace of \( P_4 \)**

In Example BSP4 [295] we showed that

\[
S = \{ x - 2, x^2 - 4x + 4, x^3 - 6x^2 + 12x - 8, x^4 - 8x^3 + 24x^2 - 32x + 16 \}
\]

is a basis for \( W = \{ p(x) \mid p \in P_4, \ p(2) = 0 \} \). Thus, the dimension of \( W \) is four, \( \dim(W) = 4. \)

**Example DC**

**Dimension of the crazy vector space**

In Example BC [297] we determined that the set \( R = \{ (1, 0), (6, 3) \} \) from the crazy vector space, \( C \) (Example CVS [255]), is a basis for \( C \). By Definition D [307] we see that \( C \) has dimension 2, \( \dim(C) = 2. \)

It is possible for a vector space to have no finite bases, in which case we say it has infinite dimension. Many of the best examples of this are vector spaces of functions, which lead to constructions like Hilbert spaces. We will focus exclusively on finite-dimensional vector spaces. OK, one infinite-dimensional example, and then we will focus exclusively on finite-dimensional vector spaces.

**Example VSPUD**

**Vector space of polynomials with unbounded degree**

Define the set \( P \) by

\[
P = \{ p \mid p(x) \text{ is a polynomial in } x \}
\]

Our operations will be the same as those defined for \( P_n \) (Example VSP [253]).

With no restrictions on the possible degrees of our polynomials, any finite set that is a candidate for spanning \( P \) will come up short. We will give a proof by contradiction (Technique CD [623]). To this end, suppose that the dimension of \( P \) is finite, say \( \dim(P) = n \).

The set \( T = \{ 1, x, x^2, \ldots, x^n \} \) is a linearly independent set (check this!) containing \( n + 1 \) polynomials from \( P \). However, a basis of \( P \) will be a spanning set of \( P \) containing \( n \) vectors. This situation is a contradiction of Theorem SSLD [307], so our assumption that \( P \) has finite dimension is false. Thus, we say \( \dim(P) = \infty \).

**Subsection RNM**

**Rank and Nullity of a Matrix**

For any matrix, we have seen that we can associate several subspaces — the null space (Theorem NSMS [268]), the column space (Theorem CSMS [274]), row space (Theorem RSMS [274]) and the left null space (Theorem LNSMS [274]). As vector spaces, each of these has a dimension, and for the null space and column space, they are important enough to warrant names.

**Definition NOM**

**Nullity Of a Matrix**

Suppose that \( A \) is an \( m \times n \) matrix. Then the **nullity** of \( A \) is the dimension of the null space of \( A \), \( n(A) = \dim(N(A)) \).
Definition ROM
Rank Of a Matrix
Suppose that $A$ is an $m \times n$ matrix. Then the rank of $A$ is the dimension of the column space of $A$, $r(A) = \dim(C(A))$.

Example RNM
Rank and nullity of a matrix
Let’s compute the rank and nullity of

$$A = \begin{bmatrix}
2 & -4 & -1 & 3 & 2 & 1 & -4 \\
1 & -2 & 0 & 0 & 4 & 0 & 1 \\
-2 & 4 & 1 & 0 & -5 & -4 & -8 \\
1 & -2 & 1 & 1 & 6 & 1 & -3 \\
2 & -4 & -1 & 1 & 4 & -2 & -1 \\
-1 & 2 & 3 & -1 & 6 & 3 & -1
\end{bmatrix}$$

To do this, we will first row-reduce the matrix since that will help us determine bases for the null space and column space.

$$\begin{bmatrix}
1 & -2 & 0 & 0 & 4 & 0 & 1 \\
0 & 0 & 1 & 0 & 3 & 0 & -2 \\
0 & 0 & 0 & 1 & -1 & 0 & -3 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

From this row-equivalent matrix in reduced row-echelon form we record $D = \{1, 3, 4, 6\}$ and $F = \{2, 5, 7\}$.

For each index in $D$, Theorem BCS [214] creates a single basis vector. In total the basis will have 4 vectors, so the column space of $A$ will have dimension 4 and we write $r(A) = 4$.

For each index in $F$, Theorem BNS [128] creates a single basis vector. In total the basis will have 3 vectors, so the null space of $A$ will have dimension 3 and we write $n(A) = 3$.

There were no accidents or coincidences in the previous example — with the row-reduced version of a matrix in hand, the rank and nullity are easy to compute.

Theorem CRN
Computing Rank and Nullity
Suppose that $A$ is an $m \times n$ matrix and $B$ is a row-equivalent matrix in reduced row-echelon form with $r$ nonzero rows. Then $r(A) = r$ and $n(A) = n - r$.

Proof Theorem BCS [214] provides a basis for the column space by choosing columns of $A$ that correspond to the dependent variables in a description of the solutions to $\mathcal{L}(A, 0)$. In the analysis of $B$, there is one dependent variable for each leading 1, one per nonzero row, or one per pivot column. So there are $r$ column vectors in a basis for $C(A)$.

Theorem BNS [128] provide a basis for the null space by creating basis vectors of the null space of $A$ from entries of $B$, one for each independent variable, one per column with out a leading 1. So there are $n - r$ column vectors in a basis for $n(A)$.

Every archetype (Appendix A [630]) that involves a matrix lists its rank and nullity. You may have noticed as you studied the archetypes that the larger the column space is the smaller the null space is. A simple corollary states this trade-off succinctly. (See Technique LC [627].)
Theorem RPNC
Rank Plus Nullity is Columns
Suppose that $A$ is an $m \times n$ matrix. Then $r(A) + n(A) = n$. □

Proof Let $r$ be the number of nonzero rows in a row-equivalent matrix in reduced row-echelon form. By Theorem CRN 313,

$$r(A) + n(A) = r + (n - r) = n$$

When we first introduced $r$ as our standard notation for the number of nonzero rows in a matrix in reduced row-echelon form you might have thought $r$ stood for “rows.” Not really — it stands for “rank”!

Subsection RNNM
Rank and Nullity of a Nonsingular Matrix

Let’s take a look at the rank and nullity of a square matrix.

Example RNSM
Rank and nullity of a square matrix
The matrix

$$E = \begin{bmatrix} 0 & 4 & -1 & 2 & 2 & 3 & 1 \\ 2 & -2 & 1 & -1 & 0 & -4 & -3 \\ -2 & -3 & 9 & -3 & 9 & -1 & 9 \\ -3 & -4 & 9 & 4 & -1 & 6 & -2 \\ -3 & -4 & 6 & -2 & 5 & 9 & -4 \\ 9 & -3 & 8 & -2 & -4 & 2 & 4 \\ 8 & 2 & 2 & 9 & 3 & 0 & 9 \end{bmatrix}$$

is row-equivalent to the matrix in reduced row-echelon form,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

With $n = 7$ columns and $r = 7$ nonzero rows Theorem CRN 313 tells us the rank is $r(E) = 7$ and the nullity is $n(E) = 7 - 7 = 0$. □

The value of either the nullity or the rank are enough to characterize a nonsingular matrix.

Theorem RNNM
Rank and Nullity of a Nonsingular Matrix
Suppose that $A$ is a square matrix of size $n$. The following are equivalent.

1. $A$ is nonsingular.
2. The rank of $A$ is $n$, $r(A) = n$.
3. The nullity of $A$ is zero, $n(A) = 0$.
Proof (1 $\Rightarrow$ 2) Theorem CSNM [216] says that if $A$ is nonsingular then $\mathcal{C}(A) = \mathbb{C}^n$. If $\mathcal{C}(A) = \mathbb{C}^n$, then the column space has dimension $n$ by Theorem DCM [310], so the rank of $A$ is $n$.

(2 $\Rightarrow$ 3) Suppose $r(A) = n$. Then Theorem RPNC [314] gives

\[
 n(A) = n - r(A) = n - n = 0
\]

(3 $\Rightarrow$ 1) Suppose $n(A) = 0$, so a basis for the null space of $A$ is the empty set. This implies that $\mathcal{N}(A) = \{0\}$ and Theorem NMTNS [64] says $A$ is nonsingular.

With a new equivalence for a nonsingular matrix, we can update our list of equivalences (Theorem NME5 [299]) which now becomes a list requiring double digits to number.

Theorem NME6
Nonsingular Matrix Equivalences, Round 6
Suppose that $A$ is a square matrix of size $n$. The following are equivalent.

1. $A$ is nonsingular.
2. $A$ row-reduces to the identity matrix.
3. The null space of $A$ contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
4. The linear system $\mathcal{L}(A, b)$ has a unique solution for every possible choice of $b$.
5. The columns of $A$ are a linearly independent set.
6. $A$ is invertible.
7. The column space of $A$ is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.
8. The columns of $A$ are a basis for $\mathbb{C}^n$.
9. The rank of $A$ is $n$, $r(A) = n$.
10. The nullity of $A$ is zero, $n(A) = 0$.

Proof Building on Theorem NME5 [299] we can add two of the statements from Theorem RNNM [314].

Subsection READ
Reading Questions

1. What is the dimension of the vector space $P_6$, the set of all polynomials of degree 6 or less?
2. How are the rank and nullity of a matrix related?
3. Explain why we might say that a nonsingular matrix has “full rank.”
Subsection EXC
Exercises

C20 The archetypes listed below are matrices, or systems of equations with coefficient matrices. For each, compute the nullity and rank of the matrix. This information is listed for each archetype (along with the number of columns in the matrix, so as to illustrate Theorem RPNC[314]), and notice how it could have been computed immediately after the determination of the sets D and F associated with the reduced row-echelon form of the matrix.

Archetype A 634
Archetype B 638
Archetype C 643
Archetype D 647 / Archetype E 651
Archetype F 654
Archetype G 659 / Archetype H 663
Archetype I 667
Archetype J 671
Archetype K 676
Archetype L 680
Contributed by Robert Beezer

C30 For the matrix A below, compute the dimension of the null space of A, \( \dim(N(A)) \).

\[
A = \begin{bmatrix}
2 & -1 & -3 & 11 & 9 \\
1 & 2 & 1 & -7 & -3 \\
3 & 1 & -3 & 6 & 8 \\
2 & 1 & 2 & -5 & -3
\end{bmatrix}
\]

Contributed by Robert Beezer Solution 318

C31 The set W below is a subspace of \( \mathbb{C}^4 \). Find the dimension of W.

\[ W = \left\langle \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 2 \\ 5 \end{bmatrix} \right\rangle \]

Contributed by Robert Beezer Solution 318

C40 In Example LDP4 310 we determined that the set of five polynomials, \( T \), is linearly dependent by a simple invocation of Theorem SSLD 307. Prove that \( T \) is linearly dependent from scratch, beginning with Definition LI 280.

Contributed by Robert Beezer

M20 \( M_{22} \) is the vector space of \( 2 \times 2 \) matrices. Let \( S_{22} \) denote the set of all \( 2 \times 2 \) symmetric matrices. That is

\[ S_{22} = \{ A \in M_{22} \mid A^t = A \} \]

(a) Show that \( S_{22} \) is a subspace of \( M_{22} \).
(b) Exhibit a basis for \( S_{22} \) and prove that it has the required properties.
(c) What is the dimension of \( S_{22} \)?

Contributed by Robert Beezer Solution 318

M21 A \( 2 \times 2 \) matrix \( B \) is upper triangular if \( |B|_{21} = 0 \). Let \( UT_2 \) be the set of all \( 2 \times 2 \) upper triangular matrices. Then \( UT_2 \) is a subspace of the vector space of all \( 2 \times 2 \) matrices, \( M_{22} \) (you may assume this). Determine the dimension of \( UT_2 \) providing all of the necessary justifications for
your answer.
Contributed by Robert Beezer  Solution 319
Subsection SOL
Solutions

C30 Contributed by Robert Beezer Statement 316
Row reduce $A$,

\[
A \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & -3 & -1 \\
0 & 0 & 1 & -2 & -2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

So $r = 3$ for this matrix. Then

\[
\dim(N(A)) = n(A) = (n(A) + r(A)) - r(A) = 5 - r(A) = 5 - 3 = 2
\]

We could also use Theorem BNS and create a basis for $N(A)$ with $n - r = 5 - 3 = 2$ vectors (because the solutions are described with 2 free variables) and arrive at the dimension as the size of this basis.

C31 Contributed by Robert Beezer Statement 316
We will appeal to Theorem BS (or you could consider this an appeal to Theorem BCS).
Put the three columnn vectors of this spanning set into a matrix as columns and row-reduce.

\[
\begin{bmatrix}
2 & 3 & -4 \\
-3 & 0 & -3 \\
4 & 1 & 2 \\
1 & -2 & 5
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

The pivot columns are $D = \{1, 2\}$ so we can “keep” the vectors corresponding to the pivot columns and set

\[
T = \left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}
\]

and conclude that $W = \langle T \rangle$ and $T$ is linearly independent. In other words, $T$ is a basis with two vectors, so $W$ has dimension 2.

M20 Contributed by Robert Beezer Statement 316
(a) We will use the three criteria of Theorem TSS. The zero vector of $M_{22}$ is the zero matrix, $O$ (Definition ZM), which is a symmetric matrix. So $S_{22}$ is not empty, since $O \in S_{22}$.

Suppose that $A$ and $B$ are two matrices in $S_{22}$. Then we know that $A^t = A$ and $B^t = B$. We want to know if $A + B \in S_{22}$, so test $A + B$ for membership,

\[
(A + B)^t = A^t + B^t
\]

\[
= A + B
\]

\[
A, B \in S_{22}
\]

So $A + B$ is symmetric and qualifies for membership in $S_{22}$.

Suppose that $A \in S_{22}$ and $\alpha \in \mathbb{C}$. Is $\alpha A \in S_{22}$? We know that $A^t = A$. Now check that,

\[
\alpha A^t = \alpha A^t
\]

\[
= \alpha A
\]

\[
A \in S_{22}
\]
So $\alpha A$ is also symmetric and qualifies for membership in $S_{22}$.

With the three criteria of Theorem [TSS 265] fulfilled, we see that $S_{22}$ is a subspace of $M_{22}$.

(b) An arbitrary matrix from $S_{22}$ can be written as $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$. We can express this matrix as

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

$$= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

this equation says that the set

$$T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

spans $S_{22}$. Is it also linearly independent?

Write a relation of linear dependence on $S$, 

$$\mathcal{O} = a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}$$

The equality of these two matrices (Definition [ME 163]) tells us that $a_1 = a_2 = a_3 = 0$, and the only relation of linear dependence on $T$ is trivial. So $T$ is linearly independent, and hence is a basis of $S_{22}$.

(c) The basis $T$ found in part (b) has size 3. So by Definition [D 307], $\dim (S_{22}) = 3$.  

M21 Contributed by Robert Beezer Statement 316

A typical matrix from $UT_2$ looks like 

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

where $a, b, c \in \mathbb{C}$ are arbitrary scalars. Observing this we can then write

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

which says that

$$R = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a spanning set for $UT_2$ (Definition [TSVS 284]). Is $R$ is linearly independent? If so, it is a basis for $UT_2$. So consider a relation of linear dependence on $R$, 

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

From this equation, one rapidly arrives at the conclusion that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. So $R$ is a linearly independent set (Definition [LI 280]), and hence is a basis (Definition [B 294]) for $UT_2$. Now, we simply count up the size of the set $R$ to see that the dimension of $UT_2$ is $\dim (UT_2) = 3$. 

Version 1.04
Section PD
Properties of Dimension

Once the dimension of a vector space is known, then the determination of whether or not a set of vectors is linearly independent, or if it spans the vector space, can often be much easier. In this section we will state a workhorse theorem and then apply it to the column space and row space of a matrix. It will also help us describe a super-basis for $\mathbb{C}^m$.

Subsection GT
Goldilocks’ Theorem

We begin with a useful theorem that we will need later, and in the proof of the main theorem in this subsection. This theorem says that we can extend linearly independent sets, one vector at a time, by adding vectors from outside the span of the linearly independent set, all the while preserving the linear independence of the set.

**Theorem ELIS**
Extending Linearly Independent Sets

Suppose $V$ is a vector space and $S$ is a linearly independent set of vectors from $V$. Suppose $w$ is a vector such that $w \not\in \langle S \rangle$. Then the set $S' = S \cup \{w\}$ is linearly independent.

**Proof**
Suppose $S = \{v_1, v_2, v_3, \ldots, v_m\}$ and begin with a relation of linear dependence on $S'$,

$$a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_mv_m + a_{m+1}w = 0.$$

There are two cases to consider. First suppose that $a_{m+1} = 0$. Then the relation of linear dependence on $S'$ becomes

$$a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_mv_m = 0,$$

and by the linear independence of the set $S$, we conclude that $a_1 = a_2 = a_3 = \cdots = a_m = 0$. So all of the scalars in the relation of linear dependence on $S'$ are zero.

In the second case, suppose that $a_{m+1} \neq 0$. Then the relation of linear dependence on $S'$ becomes

$$a_{m+1}w = -a_1v_1 - a_2v_2 - a_3v_3 - \cdots - a_mv_m.$$

This equation expresses $w$ as a linear combination of the vectors in $S$, contrary to the assumption that $w \not\in \langle S \rangle$, so this case leads to a contradiction.

The first case yielded only a trivial relation of linear dependence on $S'$ and the second case led to a contradiction. So $S'$ is a linearly independent set since any relation of linear dependence is trivial.

In the story *Goldilocks and the Three Bears*, the young girl Goldilocks visits the empty house of the three bears while out walking in the woods. One bowl of porridge is too hot, the other too cold, the third is just right. One chair is too hard, one too soft, the third is just right. So it is with sets of vectors — some are too big (linearly dependent), some are too small (they don’t span), and some are just right (bases). Here’s Goldilocks’ Theorem.

**Theorem G**
Goldilocks

Suppose that $V$ is a vector space of dimension $t$. Let $S = \{v_1, v_2, v_3, \ldots, v_m\}$ be a set of vectors from $V$. Then
1. If \( m > t \), then \( S \) is linearly dependent.

2. If \( m < t \), then \( S \) does not span \( V \).

3. If \( m = t \) and \( S \) is linearly independent, then \( S \) spans \( V \).

4. If \( m = t \) and \( S \) spans \( V \), then \( S \) is linearly independent.

**Proof** Let \( B \) be a basis of \( V \). Since \( \dim (V) = t \), Definition B \[294\] and Theorem BIS \[310\] imply that \( B \) is a linearly independent set of \( t \) vectors that spans \( V \).

1. Suppose to the contrary that \( S \) is linearly independent. Then \( B \) is a smaller set of vectors that spans \( V \). This contradicts Theorem SSLD \[307\].

2. Suppose to the contrary that \( S \) does span \( V \). Then \( B \) is a larger set of vectors that is linearly independent. This contradicts Theorem SSLD \[307\].

3. Suppose to the contrary that \( S \) does not span \( V \). Then we can choose a vector \( w \) such that \( w \in V \) and \( w \not\in \langle S \rangle \). By Theorem ELIS \[320\], the set \( S' = S \cup \{w\} \) is again linearly independent. Then \( S' \) is a set of \( m + 1 = t + 1 \) vectors that are linearly independent, while \( B \) is a set of \( t \) vectors that span \( V \). This contradicts Theorem SSLD \[307\].

4. Suppose to the contrary that \( S \) is linearly dependent. Then by Theorem DLDS \[139\] (which can be upgraded, with no changes in the proof, to the setting of a general vector space), there is a vector in \( S \), say \( v_k \) that is equal to a linear combination of the other vectors in \( S \). Let \( S' = S \setminus \{v_k\} \), the set of “other” vectors in \( S \). Then it is easy to show that \( V = \langle S \rangle = \langle S' \rangle \). So \( S' \) is a set of \( m - 1 = t - 1 \) vectors that spans \( V \), while \( B \) is a set of \( t \) linearly independent vectors in \( V \). This contradicts Theorem SSLD \[307\].

There is a tension in the construction of basis. Make a set too big and you will end up with relations of linear dependence among the vectors. Make a set too small and you will not have enough raw material to span the entire vector space. Make a set just the right size (the dimension) and you only need to have linear independence or spanning, and you get the other property for free. These roughly-stated ideas are made precise by Theorem G \[320\].

The structure and proof of this theorem also deserve comment. The hypotheses seem innocuous. We presume we know the dimension of the vector space in hand, then we mostly just look at the size of the set \( S \). From this we get big conclusions about spanning and linear independence. Each of the four proofs relies on ultimately contradicting Theorem SSLD \[307\], so in a way we could think of this entire theorem as a corollary of Theorem SSLD \[307\]. (See Technique LC \[627\].) The proofs of the third and fourth parts parallel each other in style (add \( w \), toss \( v_k \)) and then turn on Theorem ELIS \[320\] before contradicting Theorem SSLD \[307\].

Theorem G \[320\] is useful in both concrete examples and as a tool in other proofs. We will use it often to bypass verifying linear independence or spanning.

**Example BPR**

**Bases for \( P_n \), reprised**

In Example BP \[295\] we claimed that

\[
B = \{1, x, x^2, x^3, \ldots, x^n\}
\]

\[
C = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3, \ldots, 1 + x + x^2 + x^3 + \cdots + x^n\}
\]

were both bases for \( P_n \) (Example VSP \[253\]). Suppose we had first verified that \( B \) was a basis, so we would then know that \( \dim (P_n) = n + 1 \). The size of \( C \) is \( n + 1 \), the right size to be a basis. We could then verify that \( C \) is linearly independent. We would not have to make any special efforts.
to prove that \( C \) spans \( P_n \), since Theorem G \[320\] would allow us to conclude this property of \( C \) directly. Then we would be able to say that \( C \) is a basis of \( P_n \) also.

Example BDM22

Basis by dimension in \( M_{22} \)

In Example DSM22 \[311\] we showed that

\[
B = \left\{ \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]

is a basis for the subspace \( Z \) of \( M_{22} \) (Example VSM \[253\]) given by

\[
Z = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \big| 2a + b + 3c + 4d = 0, -a + 3b - 5c - d = 0 \right\}
\]

This tells us that \( \dim(Z) = 2 \). In this example we will find another basis. We can construct two new matrices in \( Z \) by forming linear combinations of the matrices in \( B \).

\[
2 \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} + (-3) \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -3 \end{bmatrix}
\]

\[
3 \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 3 \\ 3 & 1 \end{bmatrix}
\]

Then the set

\[
C = \left\{ \begin{bmatrix} 2 & 2 \\ 2 & -3 \end{bmatrix}, \begin{bmatrix} -8 & 3 \\ 3 & 1 \end{bmatrix} \right\}
\]

has the right size to be a basis of \( Z \). Let’s see if it is a linearly independent set. The relation of linear dependence

\[
a_1 \begin{bmatrix} 2 & 2 \\ 2 & -3 \end{bmatrix} + a_2 \begin{bmatrix} -8 & 3 \\ 3 & 1 \end{bmatrix} = \mathbf{0}
\]

\[
\begin{bmatrix} 2a_1 - 8a_2 & 2a_1 + 3a_2 \\ 2a_1 + 3a_2 & -3a_1 + a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

leads to the homogeneous system of equations whose coefficient matrix

\[
\begin{bmatrix} 2 & -8 \\ 2 & 3 \\ 2 & 3 \\ -3 & 1 \end{bmatrix}
\]

row-reduces to

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

So with \( a_1 = a_2 = 0 \) as the only solution, the set is linearly independent. Now we can apply Theorem G \[320\] to see that \( C \) also spans \( Z \) and therefore is a second basis for \( Z \).

Example SVP4

Sets of vectors in \( P_4 \)

In Example BSP4 \[295\] we showed that

\[
B = \{ x - 2, x^2 - 4x + 4, x^3 - 6x^2 + 12x - 8, x^4 - 8x^3 + 24x^2 - 32x + 16 \}
\]

is a basis for \( W = \{ p(x) \mid p \in P_4, p(2) = 0 \} \). So \( \dim(W) = 4 \).
The set 
\[ \{ 3x^2 - 5x - 2, 2x^2 - 7x + 6, x^3 - 2x^2 + x - 2 \} \]
is a subset of \( W \) (check this) and it happens to be linearly independent (check this, too). However, by Theorem G \[320\] it cannot span \( W \).

The set 
\[ \{ 3x^2 - 5x - 2, 2x^2 - 7x + 6, x^3 - 2x^2 + x - 2, -x^4 + 2x^3 + 5x^2 - 10x, x^4 - 16 \} \]
is another subset of \( W \) (check this) and Theorem G \[320\] tells us that it must be linearly dependent.

The set 
\[ \{ x - 2, x^2 - 2x, x^3 - 2x^2, x^4 - 2x^3 \} \]
is a third subset of \( W \) (check this) and is linearly independent (check this). Since it has the right size to be a basis, and is linearly independent, Theorem G \[320\] tells us that it also spans \( W \), and therefore is a basis of \( W \).

A simple consequence of Theorem G \[320\] is the observation that proper subspaces have strictly smaller dimensions. Hopefully this may seem intuitively obvious, but it still requires proof, and we will cite this result later.

**Theorem PSSD**

**Proper Subspaces have Smaller Dimension**

Suppose that \( U \) and \( V \) are subspaces of the vector space \( W \), such that \( U \subsetneq V \). Then \( \dim(U) < \dim(V) \).

**Proof** Suppose that \( \dim(U) = m \) and \( \dim(V) = t \). Then \( U \) has a basis \( B \) of size \( m \). If \( m > t \), then by Theorem G \[320\], \( B \) is linearly dependent, which is a contradiction. If \( m = t \), then by Theorem G \[320\], \( B \) spans \( V \). Then \( U = \langle B \rangle = V \), also a contradiction. All that remains is that \( m < t \), which is the desired conclusion.

The final theorem of this subsection is an extremely powerful tool for establishing the equality of two sets that are subspaces. Notice that the hypotheses include the equality of two integers (dimensions) while the conclusion is the equality of two sets (subspaces). It is the extra “structure” of a vector space and its dimension that makes possible this huge leap from an integer equality to a set equality.

**Theorem EDYES**

**Equal Dimensions Yields Equal Subspaces**

Suppose that \( U \) and \( V \) are subspaces of the vector space \( W \), such that \( U \subseteq V \) and \( \dim(U) = \dim(V) \). Then \( U = V \).

**Proof** We give a proof by contradiction (Technique CD \[623\]). Suppose to the contrary that \( U \neq V \). Since \( U \subseteq V \), there must be a vector \( v \) such that \( v \in V \) and \( v \notin U \). Let \( B = \{ u_1, u_2, u_3, \ldots, u_t \} \) be a basis for \( U \). Then, by Theorem ELIS \[320\], the set \( C = B \cup \{ v \} = \{ u_1, u_2, u_3, \ldots, u_t, v \} \) is a linearly independent set of \( t+1 \) vectors in \( V \). However, by hypothesis, \( V \) has the same dimension as \( U \) (namely \( t \)) and therefore Theorem G \[320\] says that \( C \) is too big to be linearly independent. This contradiction shows that \( U = V \).

**Subsection RT**

**Ranks and Transposes**

We now prove one of the most surprising theorems about matrices. Notice the paucity of hypotheses compared to the precision of the conclusion.
Theorem RMRT
Rank of a Matrix is the Rank of the Transpose
Suppose $A$ is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

Proof Suppose we row-reduce $A$ to the matrix $B$ in reduced row-echelon form, and $B$ has $r$ non-zero rows. The quantity $r$ tells us three things about $B$: the number of leading 1’s, the number of non-zero rows and the number of pivot columns. For this proof we will be interested in the latter two.

Theorem BRS and Theorem BCS each has a conclusion that provides a basis, for the row space and the column space, respectively. In each case, these bases contain $r$ vectors. This observation makes the following go.

$$r(A) = \dim \langle C(A) \rangle = \dim \langle R(A) \rangle = \dim \langle C(A^t) \rangle = r(A^t)$$

Jacob Linenthal helped with this proof.

This says that the row space and the column space of a matrix have the same dimension, which should be very surprising. It does not say that column space and the row space are identical. Indeed, if the matrix is not square, then the sizes (number of slots) of the vectors in each space are different, so the sets are not even comparable.

It is not hard to construct by yourself examples of matrices that illustrate Theorem RMRT, since it applies equally well to any matrix. Grab a matrix, row-reduce it, count the nonzero rows or the leading 1’s. That’s the rank. Transpose the matrix, row-reduce that, count the nonzero rows or the leading 1’s. That’s the rank of the transpose. The theorem says the two will be equal. Here’s an example anyway.

Example RRTI
Rank, rank of transpose, Archetype I
Archetype I has a $4 \times 7$ coefficient matrix which row-reduces to

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 2 & 1 & -3 \\
0 & 0 & 1 & 0 & 1 & -3 & 5 \\
0 & 0 & 0 & 1 & 2 & -6 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

so the rank is 3. Row-reducing the transpose yields

$$\begin{bmatrix}
1 & 0 & 0 & -\frac{31}{7} \\
0 & 1 & 0 & \frac{12}{13} \\
0 & 0 & 1 & \frac{13}{13} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

demonstrating that the rank of the transpose is also 3.

Subsection DFS
Dimension of Four Subspaces

That the rank of a matrix equals the rank of its transpose is a fundamental and surprising result. However, applying Theorem FS we can easily determine the dimension of all four fundamental
subspaces associated with a matrix.

**Theorem DFS**

**Dimensions of Four Subspaces**

Suppose that $A$ is an $m \times n$ matrix, and $B$ is a row-equivalent matrix in reduced row-echelon form with $r$ nonzero rows. Then

1. $\dim(\mathcal{N}(A)) = n - r$
2. $\dim(\mathcal{C}(A)) = r$
3. $\dim(\mathcal{R}(A)) = r$
4. $\dim(\mathcal{L}(A)) = m - r$

**Proof**

If $A$ row-reduces to a matrix in reduced row-echelon form with $r$ nonzero rows, then the matrix $C$ of extended echelon form (Definition EEF) will be an $r \times n$ matrix in reduced row-echelon form with no zero rows and $r$ pivot columns (Theorem PEEF). Similarly, the matrix $L$ of extended echelon form will be an $m - r \times m$ matrix in reduced row-echelon form with no zero rows and $m - r$ pivot columns.

\[
\dim(\mathcal{N}(A)) = \dim(\mathcal{N}(C)) = n - r \\
\dim(\mathcal{C}(A)) = \dim(\mathcal{N}(L)) = m - (m - r) = r \\
\dim(\mathcal{R}(A)) = \dim(\mathcal{R}(C)) = r \\
\dim(\mathcal{L}(A)) = \dim(\mathcal{R}(L)) = m - r
\]

There are many different ways to state and prove this result, and indeed, the equality of the dimensions of the column space and row space is just a slight expansion of Theorem RMRT. However, we have restricted our techniques to applying Theorem FS and then determining dimensions with bases provided by Theorem BNS and Theorem BRS. This provides an appealing symmetry to the results and the proof.

**Subsection DS**

**Direct Sums**

Some of the more advanced ideas in linear algebra are closely related to decomposing vector spaces into direct sums of subspaces. With our previous results about bases and dimension, now is the right time to state and collect a few results about direct sums, though we will only mention these results in passing until we get to Section NLT, where they will get a heavy workout.
A direct sum is a short-hand way to describe the relationship between a vector space and two, or more, of its subspaces. As we will use it, it is not a way to construct new vector spaces from others.

**Definition DS**

**Direct Sum**

Suppose that $V$ is a vector space with two subspaces $U$ and $W$ such that for every $v \in V$,

1. There exists vectors $u \in U$, $w \in W$ such that $v = u + w$

2. If $v = u_1 + w_1$ and $v = u_2 + w_2$ where $u_1, u_2 \in U$, $w_1, w_2 \in W$ then $u_1 = u_2$ and $w_1 = w_2$.

Then $V$ is the **direct sum** of $U$ and $W$ and we write $V = U \oplus W$.

(This definition contains Notation DS.)

Informally, when we say $V$ is the direct sum of the subspaces $U$ and $W$, we are saying that each vector of $V$ can always be expressed as the sum of a vector from $U$ and a vector from $W$, and this expression can only be accomplished in one way (i.e. uniquely). This statement should begin to feel something like our definitions of nonsingular matrices (Definition NM 61) and linear independence (Definition LI 280). It should not be hard to imagine the natural extension of this definition to the case of more than two subspaces. Could you provide a careful definition of $V = U_1 \oplus U_2 \oplus U_3 \oplus \ldots \oplus U_m$ (Exercise PD.M50 331)?

**Example SDS**

**Simple direct sum**

In $\mathbb{C}^3$, define

$$v_1 = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Then $\mathbb{C}^3 = \langle \{v_1, v_2\} \rangle \oplus \langle \{v_3\} \rangle$. This statement derives from the fact that $B = \{v_1, v_2, v_3\}$ is basis for $\mathbb{C}^3$. The spanning property of $B$ yields the decomposition of any vector into a sum of vectors from the two subspaces, and the linear independence of $B$ yields the uniqueness of the decomposition. We will illustrate these claims with a numerical example.

Choose $v = \begin{bmatrix} 10 \\ 1 \\ 6 \end{bmatrix}$. Then

$$v = 2v_1 + (-2)v_2 + 1v_3 = (2v_1 + (-2)v_2) + (1v_3)$$

where we have added parentheses for emphasis. Obviously $1v_3 \in \langle \{v_3\} \rangle$, while $2v_1 + (-2)v_2 \in \langle \{v_1, v_2\} \rangle$. Theorem VRRB 288 provides the uniqueness of the scalars in these linear combinations.

**Example SDS** 326 is easy to generalize into a theorem.

**Theorem DSFB**

**Direct Sum From a Basis**

Suppose that $V$ is a vector space with a basis $B = \{v_1, v_2, v_3, \ldots, v_n\}$. Define

$$U = \langle \{v_1, v_2, v_3, \ldots, v_m\} \rangle, \quad W = \langle \{v_{m+1}, v_{m+2}, v_{m+3}, \ldots, v_n\} \rangle$$

Then $V = U \oplus W$.

**Proof** Choose any vector $v \in V$. Then by Theorem VRRB 288 there are unique scalars, $a_1, a_2, a_3, \ldots, a_n$ such that

$$v = a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_nv_n$$

$$= (a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_nv_n) +$$
\[
(a_{m+1}v_{m+1} + a_{m+2}v_{m+2} + a_{m+3}v_{m+3} + \cdots + a_nv_n)
= u + w
\]

where we have implicitly defined \(u\) and \(w\) in the last line. It should be clear that \(u \in U\), and similarly, \(w \in W\) (and not simply by the choice of their names).

Suppose we had another decomposition of \(v\), say \(v = u^* + w^*\). Then we could write \(u^*\) as a linear combination of \(v_1\) through \(v_m\), say using scalars \(b_1, b_2, b_3, \ldots, b_m\). And we could write \(w^*\) as a linear combination of \(v_{m+1}\) through \(v_n\), say using scalars \(c_1, c_2, c_3, \ldots, c_{n-m}\). These two collections of scalars would then together give a linear combination of \(v_1\) through \(v_n\) that equals \(v\). By the uniqueness of \(a_1, a_2, a_3, \ldots, a_n, a_i = b_i\) for \(1 \leq i \leq m\) and \(a_{m+i} = c_i\) for \(1 \leq i \leq n-m\). From the equality of these scalars we conclude that \(u = u^*\) and \(w = w^*\). So with both conditions of Definition DS \[326\] fulfilled we see that \(V = U \oplus W\).

Given one subspace of a vector space, we can always find another subspace that will pair with the first to form a direct sum. The main idea of this theorem, and its proof, is the idea of extending a linearly independent subset into a basis with repeated applications of Theorem ELIS \[320\].

**Theorem DSFOS**

**Direct Sum From One Subspace**

Suppose that \(U\) is a subspace of the vector space \(V\). Then there exists a subspace \(W\) of \(V\) such that \(V = U \oplus W\).

**Proof** If \(U = V\), then \(W = \{0\}\). Otherwise, choose a basis \(B = \{v_1, v_2, v_3, \ldots, v_m\}\) for \(U\). Then since \(B\) is a linearly independent set, Theorem ELIS \[320\] tells us there is a vector \(v_{m+1}\) in \(V\), but not in \(U\), such that \(B \cup \{v_{m+1}\}\) is linearly independent. Define the subspace \(U_1 = \langle B \cup \{v_{m+1}\} \rangle\).

We can repeat this procedure, in the case were \(U_1 \neq V\), creating a new vector \(v_{m+2}\) in \(V\), but not in \(U_1\), and a new subspace \(U_2 = \langle B \cup \{v_{m+1}, v_{m+2}\} \rangle\). If we continue repeating this procedure, eventually, \(U_k = V\) for some \(k\), and we can no longer apply Theorem ELIS \[320\]. No matter, in this case \(B \cup \{v_{m+1}, v_{m+2}, \ldots, v_{m+k}\}\) is a linearly independent set that spans \(V\), i.e. a basis for \(V\).

Define \(W = \langle \{v_{m+1}, v_{m+2}, \ldots, v_{m+k}\} \rangle\). We now are exactly in position to apply Theorem DSFB \[326\] and see that \(V = U \oplus W\).

There are several different ways to define a direct sum. Our next two theorems give equivalences (Technique E \[322\]) for direct sums, and therefore could have been employed as definitions. The first should further cement the notion that a direct sum has some connection with linear independence.

**Theorem DSZV**

**Direct Sums and Zero Vectors**

Suppose \(U\) and \(W\) are subspaces of the vector space \(V\). Then \(V = U \oplus W\) if and only if

1. For every \(v \in V\), there exists vectors \(u \in U\), \(w \in W\) such that \(v = u + w\).
2. Whenever \(0 = u + w\) with \(u \in U\), \(w \in W\) then \(u = w = 0\).

**Proof** The first condition is identical in the definition and the theorem, so we only need to establish the equivalence of the second conditions.

\((\Rightarrow)\) Assume that \(V = U \oplus W\), according to Definition DS \[326\]. By Property Z \[251\], \(0 \in V\) and \(0 = 0 + 0\). If we also assume that \(0 = u + w\), then the uniqueness of the decomposition gives \(u = 0\) and \(w = 0\).

\((\Leftarrow)\) Suppose that \(v \in V\), \(v = u_1 + w_1\) and \(v = u_2 + w_2\) where \(u_1, u_2 \in U\), \(w_1, w_2 \in W\). Then

\[
0 = v - v = (u_1 + w_1) - (u_2 + w_2) = (u_1 - u_2) + (w_1 - w_2)
\]

By Property AT \[252\] and Property AA \[251\].

Version 1.04
By Property AC\textsuperscript{251}, \(u_1 - u_2 \in U\) and \(w_1 - w_2 \in W\). We can now apply our hypothesis, the second statement of the theorem, to conclude that
\[
\begin{align*}
    u_1 - u_2 &= 0 \\
    w_1 - w_2 &= 0
\end{align*}
\]
which establishes the uniqueness needed for the second condition of the definition.

Our second equivalence lends further credence to calling a direct sum a decomposition. The two subspaces of a direct sum have no (nontrivial) elements in common.

Theorem DSZI
Direct Sums and Zero Intersection
Suppose \(U\) and \(W\) are subspaces of the vector space \(V\). Then \(V = U \oplus W\) if and only if

1. For every \(v \in V\), there exists vectors \(u \in U\), \(w \in W\) such that \(v = u + w\).
2. \(U \cap W = \{0\}\).

\[\square\]

Proof The first condition is identical in the definition and the theorem, so we only need to establish the equivalence of the second conditions.

\((\Rightarrow)\) Assume that \(V = U \oplus W\), according to Definition DS\textsuperscript{326}. By Property Z\textsuperscript{251} and Definition SI\textsuperscript{617}, \(\{0\} \subseteq U \cap W\). To establish the opposite inclusion, suppose that \(x \in U \cap W\).

Then, since \(x\) is an element of both \(U\) and \(W\), we can write two decompositions of \(x\) as a vector from \(U\) plus a vector from \(W\),
\[
x = x + 0 \\
x = 0 + x
\]
By the uniqueness of the decomposition, we see (twice) that \(x = 0\) and \(U \cap W \subseteq \{0\}\). Applying Definition SE\textsuperscript{616}, we have \(U \cap W = \{0\}\).

\((\Leftarrow)\) Assume that \(U \cap W = \{0\}\). And assume further that \(v \in V\) is such that \(v = u_1 + w_1\) and \(v = u_2 + w_2\) where \(u_1, u_2 \in U\), \(w_1, w_2 \in W\). Define \(x = u_1 - u_2\). then by Property AC\textsuperscript{251}, \(x \in U\). Also
\[
x = u_1 - u_2 \\
= (v - w_1) - (v - w_2) \\
= (v - v) - (w_1 - w_2) \\
= w_2 - w_1
\]
So \(x \in W\) by Property AC\textsuperscript{251}. Thus, \(x \in U \cap W = \{0\}\) (Definition SI\textsuperscript{617}). So \(x = 0\) and
\[
\begin{align*}
    u_1 - u_2 &= 0 \\
    w_2 - w_1 &= 0
\end{align*}
\]
yielding the desired uniqueness of the second condition of the definition.

If the statement of Theorem DSZV\textsuperscript{327} did not remind you of linear independence, the next theorem should establish the connection.

Theorem DSLI
Direct Sums and Linear Independence
Suppose \(U\) and \(W\) are subspaces of the vector space \(V\) with \(V = U \oplus W\). Suppose that \(R\) is a
linearly independent subset of \( U \) and \( S \) is a linearly independent subset of \( W \). Then \( R \cup S \) is a linearly independent subset of \( V \).

**Proof**  Let \( R = \{u_1, u_2, u_3, \ldots, u_k\} \) and \( S = \{w_1, w_2, w_3, \ldots, w_\ell\} \). Begin with a relation of linear dependence (Definition RLD 280) on the set \( R \cup S \) using scalars \( a_1, a_2, a_3, \ldots, a_k \) and \( b_1, b_2, b_3, \ldots, b_\ell \). Then,

\[
0 = a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_ku_k + b_1w_1 + b_2w_2 + b_3w_3 + \cdots + b_\ell w_\ell
= (a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_ku_k) + (b_1w_1 + b_2w_2 + b_3w_3 + \cdots + b_\ell w_\ell)
= u + w
\]

where we have made an implicit definition of the vectors \( u \in U, \ w \in W \). Applying Theorem DSZV 327 we conclude that

\[
u = a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_ku_k = 0
w = b_1w_1 + b_2w_2 + b_3w_3 + \cdots + b_\ell w_\ell = 0
\]

Now the linear independence of \( R \) and \( S \) (individually) yields

\[
a_1 = a_2 = a_3 = \cdots = a_k = 0 \quad \quad b_1 = b_2 = b_3 = \cdots = b_\ell = 0
\]

Forced to acknowledge that only a trivial linear combination yields the zero vector, Definition LI 280 says the set \( R \cup S \) is linearly independent in \( V \).

Our last theorem in this collection will go some ways towards explaining the word “sum” in the moniker “direct sum,” while also partially explaining why these results appear in a section devoted to a discussion of dimension.

**Theorem DSD**

**Direct Sums and Dimension**

Suppose \( U \) and \( W \) are subspaces of the vector space \( V \) with \( V = U \oplus W \). Then \( \dim(V) = \dim(U) + \dim(W) \).

**Proof**  We will establish this equality of positive integers with two inequalities. We will need a basis of \( U \) (call it \( B \)) and a basis of \( W \) (call it \( C \)).

First, note that \( B \) and \( C \) have sizes equal to the dimensions of the respective subspaces. The union of these two linearly independent sets, \( B \cup C \) will be linearly independent in \( V \) by Theorem DSLI 328. Further, the two bases have no vectors in common by Theorem DSZI 328, since \( B \cap C \subseteq \{0\} \) and the zero vector is never an element of a linearly independent set (Exercise LI.T10 133). So the size of the union is exactly the sum of the dimensions of \( U \) and \( W \). By Theorem G 320 the size of \( B \cup C \) cannot exceed the dimension of \( V \) without being linearly dependent. These observations give us \( \dim(U) + \dim(W) \leq \dim(V) \).

Grab any vector \( v \in V \). Then by Theorem DSZI 328 we can write \( v = u + w \) with \( u \in U \) and \( w \in W \). Individually, we can write \( u \) as a linear combination of the basis elements in \( B \), and similarly, we can write \( w \) as a linear combination of the basis elements in \( C \), since the bases are spanning sets for their respective subspaces. These two sets of scalars will provide a linear combination of all of the vectors in \( B \cup C \) which will equal \( v \). The upshot of this is that \( B \cup C \) is a spanning set for \( V \). By Theorem G 320, the size of \( B \cup C \) cannot be smaller than the dimension of \( V \) without failing to span \( V \). These observations give us \( \dim(U) + \dim(W) \geq \dim(V) \).

There is a certain appealing symmetry in the previous proof, where both linear independence and spanning properties of the bases are used, both of the first two conclusions of Theorem G 320 are employed, and we have quoted both of the two conditions of Theorem DSZI 328.

One final theorem tells us that we can successively decompose direct sums into sums of smaller and smaller subspaces.
Theorem RDS
Repeated Direct Sums
Suppose $V$ is a vector space with subspaces $U$ and $W$ with $V = U \oplus W$. Suppose that $X$ and $Y$ are subspaces of $W$ with $W = X \oplus Y$. Then $V = U \oplus X \oplus Y$.

Proof. Suppose that $v \in V$. Then due to $V = U \oplus W$, there exist vectors $u \in U$ and $w \in W$ such that $v = u + w$. Due to $W = X \oplus Y$, there exist vectors $x \in X$ and $y \in Y$ such that $w = x + y$. All together,

$$v = u + w = u + x + y$$

which would be the first condition of a definition of a 3-way direct product. Now consider the uniqueness. Suppose that

$$v = u_1 + x_1 + y_1 \quad \quad \quad \quad \quad v = u_2 + x_2 + y_2$$

Because $x_1 + y_1 \in W$, $x_2 + y_2 \in W$, and $V = U \oplus W$, we conclude that

$$u_1 = u_2 \quad \quad \quad \quad x_1 + y_1 = x_2 + y_2$$

From the second equality, an application of $W = X \oplus Y$ yields the conclusions $x_1 = x_2$ and $y_1 = y_2$. This establishes the uniqueness of the decomposition of $v$ into a sum of vectors from $U$, $X$ and $Y$.

Remember that when we write $V = U \oplus W$ there always needs to be a “superspace,” in this case $V$. The statement $U \oplus W$ is meaningless. Writing $V = U \oplus W$ is simply a shorthand for a somewhat complicated relationship between $V$, $U$ and $W$, as described in the two conditions of Definition DS [326], or Theorem DSZV [327], or Theorem DSZI [328], or Theorem DSFB [326] and Theorem DSFOS [327] gives us sure-fire ways to build direct sums, while Theorem DSLI [328], Theorem DSD [329] and Theorem RDS [330] tell us interesting properties of direct sums. This subsection has been long on theorems and short on examples. If we were to use the term “lemma” we might have chosen to label some of these results as such, since they will be important tools in other proofs, but may not have much interest on their own (see Technique LC [627]). We will be referencing these results heavily in later sections, and will remind you then to come back for a second look.

Subsection READ
Reading Questions

1. Why does Theorem G [320] have the title it does?
2. What is so surprising about Theorem RMRT [324]?
3. Row-reduce the matrix $A$ to reduced row-echelon form. Without any further computations, compute the dimensions of the four subspaces, $N(A)$, $C(A)$, $R(A)$ and $L(A)$.

$$A = \begin{bmatrix} 1 & -1 & 2 & 8 & 5 \\ 1 & 1 & 1 & 4 & -1 \\ 0 & 2 & -3 & -8 & -6 \\ 2 & 0 & 1 & 8 & 4 \end{bmatrix}$$
Subsection EXC
Exercises

C10 Example SVP4 leaves several details for the reader to check. Verify these five claims. Contributed by Robert Beezer

M50 Mimic Definition DS and construct a reasonable definition of \( V = U_1 \oplus U_2 \oplus U_3 \oplus \ldots \oplus U_m \). Contributed by Robert Beezer

T05 Trivially, if \( U \) and \( V \) are two subspaces of \( W \), then \( \dim(U) = \dim(V) \). Combine this fact, Theorem PSSD, and Theorem EDYES all into one grand combined theorem. You might look to Theorem PIP for stylistic inspiration. (Notice this problem does not ask you to prove anything. It just asks you to roll up three theorems into one compact, logically equivalent statement.) Contributed by Robert Beezer

T10 Prove the following theorem, which could be viewed as a reformulation of parts (3) and (4) of Theorem G, or more appropriately as a corollary of Theorem G (Technique LC). Suppose \( V \) is a vector space and \( S \) is a subset of \( V \) such that the number of vectors in \( S \) equals the dimension of \( V \). Then \( S \) is linearly independent if and only if \( S \) spans \( V \). Contributed by Robert Beezer

T15 Suppose that \( A \) is an \( m \times n \) matrix and let \( \min(m,n) \) denote the minimum of \( m \) and \( n \). Prove that \( r(A) \leq \min(m,n) \). Contributed by Robert Beezer

T20 Suppose that \( A \) is an \( m \times n \) matrix and \( b \in \mathbb{C}^m \). Prove that the linear system \( \mathcal{L}(A, b) \) is consistent if and only if \( r(A) = r([A|b]) \). Contributed by Robert Beezer

T25 Suppose that \( V \) is a vector space with finite dimension. Let \( W \) be any subspace of \( V \). Prove that \( W \) has finite dimension. Contributed by Robert Beezer

T60 Suppose that \( W \) is a vector space with dimension 5, and \( U \) and \( V \) are subspaces of \( W \), each of dimension 3. Prove that \( U \cap V \) contains a non-zero vector. State a more general result. Contributed by Joe Riegsecker

Solution
Subsection SOL
Solutions

\textbf{T20} Contributed by Robert Beezer Statement 331

\((\Rightarrow)\) Suppose first that \(\mathcal{LS}(A, b)\) is consistent. Then by Theorem CSFS 212, \(b \in \mathcal{C}(A)\). This means that \(\mathcal{C}(A) = \mathcal{C}([A \mid b])\) and so it follows that \(r(A) = r([A \mid b])\).

\((\Leftarrow)\) Adding a column to a matrix will only increase the size of its column space, so in all cases, \(\mathcal{C}(A) \subseteq \mathcal{C}([A \mid b])\). However, if we assume that \(r(A) = r([A \mid b])\), then by Theorem EDYES 323 we can conclude that \(\mathcal{C}(A) = \mathcal{C}([A \mid b])\). Then \(b \in \mathcal{C}([A \mid b]) = \mathcal{C}(A)\) so by Theorem CSFS 212, \(\mathcal{LS}(A, b)\) is consistent.

\textbf{T60} Contributed by Robert Beezer Statement 331

Let \(\{u_1, u_2, u_3\}\) and \(\{v_1, v_2, v_3\}\) be bases for \(U\) and \(V\) (respectively). Then, the set \(\{u_1, u_2, u_3, v_1, v_2, v_3\}\) is linearly dependent, since Theorem G 320 says we cannot have 6 linearly independent vectors in a vector space of dimension 5. So we can assert that there is a non-trivial relation of linear dependence,

\[a_1 u_1 + a_2 u_2 + a_3 u_3 + b_1 v_1 + b_2 v_2 + b_3 v_3 = 0\]

where \(a_1, a_2, a_3\) and \(b_1, b_2, b_3\) are not all zero.

We can rearrange this equation as

\[a_1 u_1 + a_2 u_2 + a_3 u_3 = -b_1 v_1 - b_2 v_2 - b_3 v_3\]

This is an equality of two vectors, so we can give this common vector a name, say \(w\),

\[w = a_1 u_1 + a_2 u_2 + a_3 u_3 = -b_1 v_1 - b_2 v_2 - b_3 v_3\]

This is the desired non-zero vector, as we will now show.

First, since \(w = a_1 u_1 + a_2 u_2 + a_3 u_3\), we can see that \(w \in U\). Similarly, \(w = -b_1 v_1 - b_2 v_2 - b_3 v_3\), so \(w \in V\). This establishes that \(w \in U \cap V\) (Definition SI 617).

Is \(w \neq 0\)? Suppose not, in other words, suppose \(w = 0\). Then

\[0 = w = a_1 u_1 + a_2 u_2 + a_3 u_3\]

Because \(\{u_1, u_2, u_3\}\) is a basis for \(U\), it is a linearly independent set and the relation of linear dependence above means we must conclude that \(a_1 = a_2 = a_3 = 0\). By a similar process, we would conclude that \(b_1 = b_2 = b_3 = 0\). But this is a contradiction since \(a_1, a_2, a_3, b_1, b_2, b_3\) were chosen so that some were nonzero. So \(w \neq 0\).

How does this generalize? All we really needed was the original relation of linear dependence that resulted because we had “too many” vectors in \(W\). A more general statement would be: Suppose that \(W\) is a vector space with dimension \(n\), \(U\) is a subspace of dimension \(p\) and \(V\) is a subspace of dimension \(q\). If \(p + q > n\), then \(U \cap V\) contains a non-zero vector.
Chapter D
Determinants

The determinant is a function that takes a square matrix as an input and produces a scalar as an output. So unlike a vector space, it is not an algebraic structure. However, it has many beneficial properties for studying vector spaces, matrices and systems of equations, so it is hard to ignore (though some have tried). While the properties of a determinant can be very useful, they are also complicated to prove.

Section DM
Determinant of a Matrix

First, a slight detour, as we introduce elementary matrices, which will bring us back to the beginning of the course and our old friend, row operations.

Subsection EM
Elementary Matrices

Elementary matrices are very simple, as you might have suspected from their name. Their purpose is to effect row operations (Definition RO 24) on a matrix through matrix multiplication (Definition MM 176). Their definitions look more complicated than they really are, so be sure to read ahead after you read the definition for some explanations and an example.

Definition ELEM
Elementary Matrices

1. $E_{i,j}$ is the square matrix of size $n$ with

\[
[E_{i,j}]_{k\ell} = \begin{cases} 
0 & k \neq i, k \neq j, \ell \neq k \\
1 & k \neq i, k \neq j, \ell = k \\
0 & k = i, \ell \neq j \\
1 & k = i, \ell = j \\
0 & k = j, \ell \neq i \\
1 & k = j, \ell = i 
\end{cases}
\]

2. $E_i (\alpha)$, for $\alpha \neq 0$, is the square matrix of size $n$ with

\[
[E_i (\alpha)]_{k\ell} = \begin{cases} 
0 & k \neq i, \ell \neq k \\
1 & k \neq i, \ell = k \\
\alpha & k = i, \ell = i 
\end{cases}
\]
3. $E_{i,j}(\alpha)$ is the square matrix of size $n$ with

$$
[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 
0 & k \neq j, \ell \neq k \\
1 & k \neq j, \ell = k \\
0 & k = j, \ell \neq i, \ell \neq j \\
1 & k = j, \ell = j \\
\alpha & k = j, \ell = i 
\end{cases}
$$

(This definition contains Notation ELEM.)

Again, these matrices are not as complicated as they appear, since they are mostly perturbations of the $n \times n$ identity matrix (Definition IM [62]). $E_{i,j}$ is the identity matrix with rows (or columns) $i$ and $j$ trading places, $E_i(\alpha)$ is the identity matrix where the diagonal entry in row $i$ and column $i$ has been replaced by $\alpha$, and $E_{i,j}(\alpha)$ is the identity matrix where the entry in row $j$ and column $i$ has been replaced by $\alpha$. (Yes, those subscripts look backwards in the description of $E_{i,j}(\alpha)$). Notice that our notation makes no reference to the size of the elementary matrix, since this will always be apparent from the context, or unimportant.

The *raison d’être* for elementary matrices is to “do” row operations on matrices with matrix multiplication. So here is an example where we will both see some elementary matrices and see how they can accomplish row operations.

**Example EMRO**

**Elementary matrices and row operations**

We will perform a sequence of row operations (Definition RO [24]) on the $3 \times 4$ matrix $A$, while also multiplying the matrix on the left by the appropriate $3 \times 3$ elementary matrix.

$$
A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 3 & 2 & 4 \\ 5 & 0 & 3 & 1 \end{bmatrix}
$$

$$
R_1 \leftrightarrow R_3 : \begin{bmatrix} 5 & 0 & 3 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 1 & 3 & 1 \end{bmatrix} \quad E_{1,3} : \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 & 1 \\ 5 & 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 3 & 1 \\ 1 & 3 & 2 & 4 \end{bmatrix}
$$

$$
2R_2 : \begin{bmatrix} 5 & 0 & 3 & 1 \\ 2 & 6 & 4 & 8 \\ 2 & 1 & 3 & 1 \end{bmatrix} \quad E_2(2) : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 & 3 & 1 \\ 2 & 6 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 3 & 1 \\ 2 & 6 & 4 & 8 \end{bmatrix}
$$

$$
2R_3 + R_1 : \begin{bmatrix} 9 & 2 & 9 & 3 \\ 2 & 6 & 4 & 8 \\ 2 & 1 & 3 & 1 \end{bmatrix} \quad E_{3,1}(2) : \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 9 & 2 & 9 & 3 \\ 2 & 6 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 9 & 2 & 9 & 3 \\ 2 & 6 & 4 & 8 \end{bmatrix}
$$

The next three theorems establish that each elementary matrix effects a row operation via matrix multiplication.

**Theorem EMDRO**

**Elementary Matrices Do Row Operations**

Suppose that $A$ is a matrix, and $B$ is a matrix of the same size that is obtained from $A$ by a single row operation (Definition RO [24]).

1. If the row operation swaps rows $i$ and $j$, then $B = E_{i,j}A$.

2. If the row operation multiplies row $i$ by $\alpha$, then $B = E_i(\alpha)A$. 
3. If the row operation multiplies row \(i\) by \(\alpha\) and adds the result to row \(j\), then \(B = E_{i,j}(\alpha) A\). □

**Proof** In each of the three conclusions, performing the row operation on \(A\) will create the matrix \(B\) where only one or two rows will have changed. So we will establish the equality of the matrix entries row by row, first for the unchanged rows, then for the changed rows, showing in each case that the result of the matrix product is the same as the result of the row operation. Here we go.

Row \(k\) of the product \(E_{i,j}A\), where \(k \neq i, k \neq j\), is unchanged from \(A\),

\[
[E_{i,j}A]_{k\ell} = \sum_{p=1}^{n} [E_{i,j}]_{kp} [A]_{p\ell} = [E_{i,j}]_{kk} [A]_{k\ell} + \sum_{p=1, p \neq k}^{n} [E_{i,j}]_{kp} [A]_{p\ell}
\]

\[= 1 [A]_{k\ell} + \sum_{p=1, p \neq k}^{n} 0 [A]_{p\ell} = [A]_{k\ell}\]

Row \(i\) of the product \(E_{i,j}A\) is row \(j\) of \(A\),

\[
[E_{i,j}A]_{i\ell} = \sum_{p=1}^{n} [E_{i,j}]_{ip} [A]_{p\ell} = [E_{i,j}]_{ij} [A]_{j\ell} + \sum_{p=1, p \neq j}^{n} [E_{i,j}]_{ip} [A]_{p\ell}
\]

\[= 1 [A]_{j\ell} + \sum_{p=1, p \neq j}^{n} 0 [A]_{p\ell} = [A]_{j\ell}\]

Row \(j\) of the product \(E_{i,j}A\) is row \(i\) of \(A\),

\[
[E_{i,j}A]_{j\ell} = \sum_{p=1}^{n} [E_{i,j}]_{jp} [A]_{p\ell} = [E_{i,j}]_{ji} [A]_{i\ell} + \sum_{p=1, p \neq i}^{n} [E_{i,j}]_{jp} [A]_{p\ell}
\]

\[= 1 [A]_{i\ell} + \sum_{p=1, p \neq i}^{n} 0 [A]_{p\ell} = [A]_{i\ell}\]

So the matrix product \(E_{i,j}A\) is the same as the row operation that swaps rows \(i\) and \(j\).

Row \(k\) of the product \(E_{i}(\alpha) A\), where \(k \neq i\), is unchanged from \(A\),

\[
[E_{i}(\alpha) A]_{k\ell} = \sum_{p=1}^{n} [E_{i}(\alpha)]_{kp} [A]_{p\ell} = [E_{i}(\alpha)]_{kk} [A]_{k\ell} + \sum_{p=1, p \neq k}^{n} [E_{i}(\alpha)]_{kp} [A]_{p\ell}
\]

\[= 1 [A]_{k\ell} + \sum_{p=1, p \neq k}^{n} 0 [A]_{p\ell} = [A]_{k\ell}\]

Version 1.04
Row $i$ of the product $E_i(\alpha) A$ is $\alpha$ times row $i$ of $A$,

$$[E_i(\alpha) A]_{i\ell} = \sum_{p=1}^{n} [E_i(\alpha)]_{ip} [A]_{p\ell}$$

Row $k$ of the product $E_i(\alpha) A$, where $k \neq i$, is unchanged from $A$,

$$[E_i(\alpha) A]_{k\ell} = \sum_{p=1, p \neq i}^{n} [E_i(\alpha)]_{ip} [A]_{p\ell}$$

Row $j$ of the product $E_{i,j}(\alpha) A$, is $\alpha$ times row $i$ of $A$ and then added to row $j$ of $A$,

$$[E_{i,j}(\alpha) A]_{j\ell} = \sum_{p=1}^{n} [E_{i,j}(\alpha)]_{jp} [A]_{p\ell}$$

Later in this section we will need two facts about elementary matrices.

**Theorem EMN**

**Elementary Matrices are Nonsingular**

If $E$ is an elementary matrix, then $E$ is nonsingular. □

**Proof**  We can row-reduce each elementary matrix to the identity matrix. Given an elementary matrix of the form $E_{i,j}$, perform the row operation that swaps row $j$ with row $i$. Given an elementary matrix of the form $E_i(\alpha)$, with $\alpha \neq 0$, perform the row operation that multiplies row $i$ by $1/\alpha$. Given an elementary matrix of the form $E_{i,j}(\alpha)$, with $\alpha \neq 0$, perform the row operation that multiplies row $i$ by $-\alpha$ and adds it to row $j$. In each case, the result of the single row operation is the identity matrix. So each elementary matrix is row-equivalent to the identity matrix, and by Theorem NMRRI [62] is nonsingular.
Notice that we have now made use of the nonzero restriction on \( \alpha \) in the definition of \( E_i(\alpha) \). One more key property of elementary matrices.

**Theorem NMPEM**

**Nonsingular Matrices are Products of Elementary Matrices**

Suppose that \( A \) is a nonsingular matrix. Then there exists elementary matrices \( E_1, E_2, E_3, \ldots, E_t \) so that \( A = E_1E_2E_3 \ldots E_t \).

**Proof** Since \( A \) is nonsingular, it is row-equivalent to the identity matrix by [Theorem NMRRI][62], so there is a sequence of \( t \) row operations that converts \( I \) to \( A \). For each of these row operations, form the associated elementary matrix from [Theorem EMDRO][334] and denote these matrices by \( E_1, E_2, E_3, \ldots, E_t \). Applying the first row operation to \( I \) yields the matrix \( E_1I \). The second row operation yields \( E_2(E_1I) \), and the third row operation creates \( E_3E_2E_1I \). The result of the full sequence of \( t \) row operations will yield \( A \), so

\[
A = E_t \ldots E_3E_2E_1I = E_t \ldots E_3E_2E_1
\]

Other than the cosmetic matter of re-indexing these elementary matrices in the opposite order, this is the desired result. \( \square \)

**Subsection DD**

**Definition of the Determinant**

We’ll now turn to the definition of a determinant and do some sample computations. The definition of the determinant function is **recursive**, that is, the determinant of a large matrix is defined in terms of the determinant of smaller matrices. To this end, we will make a few definitions.

**Definition SM**

**SubMatrix**

Suppose that \( A \) is an \( m \times n \) matrix. Then the submatrix \( A(i|j) \) is the \((m-1) \times (n-1)\) matrix obtained from \( A \) by removing row \( i \) and column \( j \).

(This definition contains Notation SM.) \( \triangle \)

**Example SS**

**Some submatrices**

For the matrix

\[
A = \begin{bmatrix}
1 & -2 & 3 & 9 \\
4 & -2 & 0 & 1 \\
3 & 5 & 2 & 1 \\
\end{bmatrix}
\]

we have the submatrices

\[
A(2|3) = \begin{bmatrix}
1 & -2 & 9 \\
3 & 5 & 1 \\
\end{bmatrix} \quad A(3|1) = \begin{bmatrix}
-2 & 3 & 9 \\
-2 & 0 & 1 \\
\end{bmatrix}
\]

\( \triangleright \)

**Definition DM**

**Determinant of a Matrix**

Suppose \( A \) is a square matrix. Then its **determinant**, \( \text{det} (A) = |A| \), is an element of \( \mathbb{C} \) defined recursively by:

If \( A \) is a \( 1 \times 1 \) matrix, then \( \text{det} (A) = [A]_{11} \).

If \( A \) is a matrix of size \( n \) with \( n \geq 2 \), then

\[
\text{det} (A) = [A]_{11} \text{det} (A (1|1)) - [A]_{12} \text{det} (A (1|2)) + [A]_{13} \text{det} (A (1|3)) - \ldots - (-1)^{n+1} [A]_{1n} \text{det} (A (1|n))
\]
Subsection DM.CD Computing Determinants

So to compute the determinant of a $5 \times 5$ matrix we must build 5 submatrices, each of size 4. To compute the determinants of each the $4 \times 4$ matrices we need to create 4 submatrices each, these now of size 3 and so on. To compute the determinant of a $10 \times 10$ matrix would require computing the determinant of $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 3,628,800$ $1 \times 1$ matrices. Fortunately there are better ways. However this does suggest an excellent computer programming exercise to write a recursive procedure to compute a determinant.

Let’s compute the determinant of a reasonable sized matrix by hand.

**Example D33M**

**Determinant of a $3 \times 3$ matrix**

Suppose that we have the $3 \times 3$ matrix

$$A = \begin{bmatrix}
3 & 2 & -1 \\
4 & 1 & 6 \\
-3 & -1 & 2 \\
\end{bmatrix}$$

Then

$$\det (A) = |A| = \begin{vmatrix}
3 & 2 & -1 \\
4 & 1 & 6 \\
-3 & -1 & 2 \\
\end{vmatrix} = 3 \begin{vmatrix}
1 & 6 \\
-1 & 2 \\
\end{vmatrix} - 2 \begin{vmatrix}
4 & 6 \\
-3 & 2 \\
\end{vmatrix} + (-1) \begin{vmatrix}
4 & 1 \\
-3 & -1 \\
\end{vmatrix}$$

$$= 3 (1 \cdot 2 - 6 \cdot -1) - 2 (4 \cdot 2 - 6 \cdot -3) - (4 \cdot -1 - 1 \cdot -3)$$

$$= 3 (2 + 6) - 2 (8 + 18) - (4 + 3)$$

$$= 24 - 52 + 1$$

$$= -27$$

In practice it is a bit silly to decompose a $2 \times 2$ matrix down into a couple of $1 \times 1$ matrices and then compute the exceedingly easy determinant of these puny matrices. So here is a simple theorem.

**Theorem DMST**

**Determinant of Matrices of Size Two**

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det (A) = ad - bc$.

**Proof** Applying Definition DM [337].

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \begin{vmatrix} d \end{vmatrix} - b \begin{vmatrix} c \end{vmatrix} = ad - bc$$

Do you recall seeing the expression $ad - bc$ before? (Hint: Theorem TTMI [191])

Subsection CD Computing Determinants

There are a variety of ways to compute the determinant. We will establish first that we can choose to mimic our definition of the determinant, but by using matrix entries and submatrices based on a row other than the first one.
Theorem DER

Determinant Expansion about Rows

Suppose that $A$ is a square matrix of size $n$. Then

$$\det(A) = (-1)^{i+1} [A]_{1i} \det(A(i|1)) + (-1)^{i+2} [A]_{i2} \det(A(i|2))$$

$$+ (-1)^{i+3} [A]_{i3} \det(A(i|3)) + \cdots + (-1)^{i+n} [A]_{in} \det(A(i|n)) \quad 1 \leq i \leq n$$

which is known as expansion about row $i$.□

Proof Given the recursive definition of the determinant, it should be no surprise that we will use induction for this proof (Technique I [626]). When $n = 1$, there is nothing to prove since there is but one row. When $n = 2$, we just examine expansion about the second row,

$$(-1)^{2+1} [A]_{21} \det(A(2|1)) + (-1)^{2+2} [A]_{22} \det(A(2|2))$$

$$= -[A]_{21} [A]_{12} + [A]_{22} [A]_{11} \quad \text{Definition DM} \ 337$$

$$= [A]_{11} [A]_{22} - [A]_{12} [A]_{21} \quad \text{Theorem DMST} \ 338$$

So the theorem is true for matrices of size $n = 1$ and $n = 2$. Now assume the result is true for all matrices of size $n - 1$ as we derive an expression for expansion about row $i$ for a matrix of size $n$. We will abuse our notation for a submatrix slightly, so $A(i_1, i_2|j_1, j_2)$ will denote the matrix formed by removing rows $i_1$ and $i_2$, along with removing columns $j_1$ and $j_2$. Also, as we take a determinant of a submatrix, we will need to “jump up” the index of summation partway through as we “skip over” a missing column. To do this smoothly we will set

$$\epsilon_{ij} = \begin{cases} 0 & \ell < j \\ 1 & \ell > j \end{cases}$$

Now,

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} [A]_{1j} \det(A(1|j)) \quad \text{Definition DM} \ 337$$

$$= \sum_{j=1}^{n} (-1)^{1+j} [A]_{1j} \sum_{1 \leq \ell \leq n \atop \ell \neq j} (-1)^{i-1+\ell-\epsilon_{ij}} [A]_{i\ell} \det(A(1,i|j, \ell)) \quad \text{Induction, row } i$$

$$= \sum_{1 \leq j \leq n \atop 1 \leq \ell \leq n \atop \ell \neq j} (-1)^{i+j+\ell-\epsilon_{ij}} [A]_{1j} [A]_{i\ell} \det(A(1,i|j, \ell))$$

$$= \sum_{\ell=1}^{n} (-1)^{1+\ell} [A]_{i\ell} \sum_{1 \leq j \leq n \atop j \neq \ell} (-1)^{i-j-\epsilon_{ij}} [A]_{1j} \det(A(1,i|j, \ell))$$

$$= \sum_{\ell=1}^{n} (-1)^{1+\ell} [A]_{i\ell} \sum_{1 \leq j \leq n \atop j \neq \ell} (-1)^{\epsilon_{ij}+j} [A]_{1j} \det(A(i,1|\ell,j)) \quad 2\epsilon_{ij} \text{ is even}$$

$$= \sum_{\ell=1}^{n} (-1)^{1+\ell} [A]_{i\ell} \det(A(i|\ell)) \quad \text{Definition DM} \ 337$$

We can also obtain a formula that computes a determinant by expansion about a column, but this will be simpler if we first prove a result about the interplay of determinants and transposes. Notice how the following proof makes use of the ability to compute a determinant by expanding about any row.
Theorem DT  
Determinant of the Transpose  
Suppose that \( A \) is a square matrix. Then \( \det (A^t) = \det (A) \).  
□

Proof  
As before, with a recursive definition, a proof by induction will be natural [Technique I](#626). For the base case, a square matrix of size 1 is symmetric, so \( A = A^t \), and the determinants will be equal. Now assume the theorem is true for all square matrices of size \( n - 1 \) and consider the determinant of a matrix of size \( n \).

\[
\det (A^t) = \frac{1}{n} \sum_{i=1}^{n} \det (A^t i) \\
= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} [A^t]_{ij} \det (A^t (i|j)) \quad \text{Theorem DER 339} \\
= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} [A]_{ji} \det (A (j|i)) \quad \text{Definition TM 166} \\
= \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{n} (-1)^{i+j} [A]_{ji} \det (A (j|i)) \quad \text{Switch order of summation} \\
= \frac{1}{n} \sum_{j=1}^{n} \det (A) \quad \text{Theorem DER 339} \\
= \det (A)
\]

Now we can easily get the result that a determinant can be computed by expansion about any column as well.

Theorem DEC  
Determinant Expansion about Columns  
Suppose that \( A \) is a square matrix of size \( n \). Then

\[
\det (A) = (-1)^{1+j} [A]_{1j} \det (A (1|j)) + (-1)^{2+j} [A]_{2j} \det (A (2|j)) + \cdots + (-1)^{n+j} [A]_{nj} \det (A (n|j)) \\
1 \leq j \leq n
\]

which is known as expansion about column \( j \).  
□

Proof

\[
\det (A) = \det (A^t) \quad \text{Theorem DT 340} \\
= \sum_{j=1}^{n} [A^t]_{ji} \det (A^t (j|i)) \quad \text{Theorem DER 339} \\
= \sum_{j=1}^{n} [A]_{ij} \det (A (i|j)) \quad \text{Definition TM 166}
\]

That the determinant of an \( n \times n \) matrix can be computed in \( 2n \) different (albeit similar) ways is nothing short of remarkable. For the doubters among us, we will do an example, computing a \( 4 \times 4 \) matrix in two different ways.

Example TCSD  
Two computations, same determinant
Let

\[ A = \begin{bmatrix} -2 & 3 & 0 & 1 \\ 9 & -2 & 0 & 1 \\ 1 & 3 & -2 & -1 \\ 4 & 1 & 2 & 6 \end{bmatrix} \]

Then expanding about the fourth row (Theorem DER \[339\] with \( i = 4 \)) yields,

\[
|A| = (4)(-1)^{4+1} \begin{vmatrix} 3 & 0 & 1 \\ -2 & 0 & 1 \\ 3 & -2 & -1 \end{vmatrix} + (1)(-1)^{4+2} \begin{vmatrix} -2 & 0 & 1 \\ 9 & 0 & 1 \\ 1 & -2 & -1 \end{vmatrix} \\
+ (2)(-1)^{4+3} \begin{vmatrix} -2 & 3 & 1 \\ 9 & -2 & 1 \\ 1 & 3 & -1 \end{vmatrix} + (6)(-1)^{4+4} \begin{vmatrix} -2 & 0 & 1 \\ 9 & 0 & 1 \\ 1 & 3 & -2 \end{vmatrix}
\]

\[
= (-4)(10) + (1)(-22) + (-2)(61) + 6(46) = 92
\]

while expanding about column 3 (Theorem DEC \[340\] with \( j = 3 \)) gives

\[
|A| = (0)(-1)^{1+3} \begin{vmatrix} 9 & -2 & 1 \\ 1 & 3 & -1 \\ 4 & 1 & 6 \end{vmatrix} + (0)(-1)^{2+3} \begin{vmatrix} -2 & 3 & 1 \\ 1 & 3 & -1 \\ 4 & 1 & 6 \end{vmatrix} \\
+ (-2)(-1)^{3+3} \begin{vmatrix} -2 & 3 & 1 \\ 9 & -2 & 1 \\ 4 & 1 & 6 \end{vmatrix} + (2)(-1)^{4+3} \begin{vmatrix} -2 & 0 & 1 \\ 9 & 0 & 1 \\ 1 & 3 & -1 \end{vmatrix}
\]

\[
= 0 + 0 + (-2)(-107) + (-2)(61) = 92
\]

Notice how much easier the second computation was. By choosing to expand about the third column, we have two entries that are zero, so two 3 \( \times \) 3 determinants need not be computed at all! \( \Box \)

When a matrix has all zeros above (or below) the diagonal, exploiting the zeros by expanding about the proper row or column makes computing a determinant insanely easy.

**Example DUTM**

**Determinant of an upper triangular matrix**

Suppose that

\[ T = \begin{bmatrix} 2 & 3 & -1 & 3 & 3 \\ 0 & -1 & 5 & 2 & -1 \\ 0 & 0 & 3 & 9 & 2 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \]

We will compute the determinant of this 5 \( \times \) 5 matrix by consistently expanding about the first column for each submatrix that arises and does not have a zero entry multiplying it.

\[
\det (T) = \begin{vmatrix} 2 & 3 & -1 & 3 & 3 \\ 0 & -1 & 5 & 2 & -1 \\ 0 & 0 & 3 & 9 & 2 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{vmatrix}
\]

\[
= 2(-1)^{1+1} \begin{vmatrix} 1 & 5 & 2 \\ 0 & 3 & 9 \\ 0 & 0 & 0 \end{vmatrix} + (1)(-1)^{2+1} \begin{vmatrix} 2 & 3 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 5 \end{vmatrix}
\]

\[
= 2(-1)(-1)^{1+1} \begin{vmatrix} 3 & 9 \\ 0 & -1 \end{vmatrix}
\]

\[
= 2(-1)(-1)^{1+1} \begin{vmatrix} 3 & 9 \\ 0 & -1 \end{vmatrix}
\]

\[
= 2(-1)(-1)^{1+1} \end{vmatrix} = 92
\]
\[= 2(-1)(3)(-1)^{1+1} \left| \begin{array}{cc} -1 & 3 \\ 0 & 5 \end{array} \right| \]
\[= 2(-1)(3)(-1)(-1)^{1+1} |5| \]
\[= 2(-1)(3)(-1)(5) = 30 \]

If you consult other texts in your study of determinants, you may run into the terms “minor” and “cofactor,” especially in a discussion centered on expansion about rows and columns. We’ve chosen not to make these definitions formally since we’ve been able to get along without them. However, informally, a minor is a determinant of a submatrix, specifically det \((A(i|j))\) and is usually referenced as the minor of \([A]_{ij}\). A cofactor is a signed minor, specifically the cofactor of \([A]_{ij}\) is \((-1)^{i+j} \text{det}(A(i|j))\).

**Subsection READ**  
**Reading Questions**

1. Construct the elementary matrix that will effect the row operation \(-6R_2 + R_3\) on a \(4 \times 7\) matrix.

2. Compute the determinant of the matrix

\[
\begin{bmatrix}
2 & 3 & -1 \\
3 & 8 & 2 \\
4 & -1 & -3 \\
\end{bmatrix}
\]

3. Compute the determinant of the matrix

\[
\begin{bmatrix}
3 & 9 & -2 & 4 & 2 \\
0 & 1 & 4 & -2 & 7 \\
0 & 0 & -2 & 5 & 2 \\
0 & 0 & 0 & -1 & 6 \\
0 & 0 & 0 & 0 & 4 \\
\end{bmatrix}
\]
Subsection EXC
Exercises

C24  Doing the computations by hand, find the determinant of the matrix below.
\[
\begin{bmatrix}
-2 & 3 & -2 \\
-4 & -2 & 1 \\
2 & 4 & 2
\end{bmatrix}
\]
Contributed by Robert Beezer  Solution 344

C25  Doing the computations by hand, find the determinant of the matrix below.
\[
\begin{bmatrix}
3 & -1 & 4 \\
2 & 5 & 1 \\
2 & 0 & 6
\end{bmatrix}
\]
Contributed by Robert Beezer  Solution 344

C26  Doing the computations by hand, find the determinant of the matrix \(A\).
\[
A = \begin{bmatrix}
2 & 0 & 3 & 2 \\
5 & 1 & 2 & 4 \\
3 & 0 & 1 & 2 \\
5 & 3 & 2 & 1
\end{bmatrix}
\]
Contributed by Robert Beezer  Solution 344
Subsection SOL
Solutions

C24 Contributed by Robert Beezer Statement 343
We’ll expand about the first row since there are no zeros to exploit,
\[
\begin{vmatrix}
-2 & 3 & -2 \\
-4 & -2 & 1 \\
2 & 4 & 2 \\
\end{vmatrix}
\begin{vmatrix}
-2 & 1 \\
4 & 2 \\
2 & 4 \\
\end{vmatrix}
+ (-1)(3)
\begin{vmatrix}
-4 & 1 \\
2 & 2 \\
-4 & -2 \\
\end{vmatrix}
\begin{vmatrix}
-4 & -2 \\
2 & 4 \\
\end{vmatrix}
= (-2)((-2)(2) - 1(4)) + (-3)((-4)(2) - 1(2)) + (-2)((-4)(4) - (-2)(2))
= (-2)(-8) + (-3)(-10) + (-2)(-12) = 70
\]

C25 Contributed by Robert Beezer Statement 343
We can expand about any row or column, so the zero entry in the middle of the last row is attractive.
Let’s expand about column 2. By Theorem DER 339 and Theorem DEC 340, you will get the same result by expanding about a different row or column. We will use Theorem DMST 338 twice.
\[
\begin{vmatrix}
3 & -1 & 4 \\
2 & 5 & 1 \\
2 & 0 & 6 \\
\end{vmatrix}
\begin{vmatrix}
2 & 1 \\
2 & 6 \\
3 & 4 \\
\end{vmatrix}
+ (5)(-1)
\begin{vmatrix}
2 & 1 \\
3 & 4 \\
3 & 2 \\
\end{vmatrix}
\begin{vmatrix}
3 & 4 \\
2 & 6 \\
5 & 2 \\
\end{vmatrix}
= (1)(10) + (5)(10) + 0 = 60
\]

C26 Contributed by Robert Beezer Statement 343
With two zeros in column 2, we choose to expand about that column (Theorem DEC 340),
\[
\det(A) = \begin{vmatrix}
2 & 0 & 3 & 2 \\
5 & 1 & 2 & 4 \\
3 & 0 & 1 & 2 \\
5 & 3 & 2 & 1 \\
\end{vmatrix}
= 0(-1)
\begin{vmatrix}
5 & 2 & 4 \\
3 & 1 & 2 \\
5 & 2 & 1 \\
\end{vmatrix}
+ 1(1)
\begin{vmatrix}
2 & 3 & 2 \\
3 & 1 & 2 \\
5 & 2 & 1 \\
\end{vmatrix}
\begin{vmatrix}
2 & 3 & 2 \\
5 & 2 & 4 \\
5 & 2 & 1 \\
\end{vmatrix}
+ 0(-1)
\begin{vmatrix}
5 & 2 & 4 \\
3 & 1 & 2 \\
5 & 2 & 1 \\
\end{vmatrix}
\begin{vmatrix}
2 & 3 & 2 \\
5 & 2 & 4 \\
3 & 1 & 2 \\
\end{vmatrix}
= (1) (2(1(1) - 2(2)) - 3(3(1) - 5(2)) + 2(3(2) - 5(1))) +
(3) (2(2(2) - 4(1)) - 3(5(2) - 4(3)) + 2(5(1) - 3(2)))
= (-6 + 21 + 2) + (3)(0 + 6 - 2) = 29
\]
Section PDM
Properties of Determinants of Matrices

We have seen how to compute the determinant of a matrix, and the incredible fact that we can perform expansion about any row or column to make this computation. In this largely theoretical section, we will state and prove several more intriguing properties about determinants. Our main goal will be the two results in Theorem SMZD [351] and Theorem DRMM [353], but more specifically, we will see how the value of a determinant will allow us to gain insight into the various properties of a square matrix.

Subsection DRO
Determinants and Row Operations

We start easy with a straightforward theorem whose proof presages the style of subsequent proofs in this subsection.

Theorem DZRC
Determinant with Zero Row or Column
Suppose that $A$ is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then $\det(A) = 0$.

Proof Suppose that $A$ is a square matrix of size $n$ and row $i$ has every entry equal to zero. We compute $\det(A)$ via expansion about row $i$.

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} [A]_{ij} \det(A(i|j)) \quad \text{Theorem DER} \ [339]$$

$$= \sum_{j=1}^{n} (-1)^{i+j} 0 \det(A(i|j)) \quad \text{Row } i \text{ is zeros}$$

$$= \sum_{j=1}^{n} 0 = 0$$

The proof for the case of a zero column is entirely similar, or could be derived from an application of Theorem DT [340] employing the transpose of the matrix.

Theorem DRCS
Determinant for Row or Column Swap
Suppose that $A$ is a square matrix. Let $B$ be the square matrix obtained from $A$ by interchanging the location of two rows, or interchanging the location of two columns. Then $\det(B) = -\det(A)$.

Proof Begin with the special case where $A$ is a square matrix of size $n$ and we form $B$ by swapping adjacent rows $i$ and $i+1$ for some $1 \leq i \leq n-1$. Notice that the assumption about swapping adjacent rows means that $B(i+1|j) = A(i|j)$ for all $1 \leq j \leq n$, and $[B]_{i+1,j} = [A]_{ij}$ for all $1 \leq j \leq n$. We compute $\det(B)$ via expansion about row $i+1$.

$$\det(B) = \sum_{j=1}^{n} (-1)^{(i+1)+j} [B]_{i+1,j} \det(B(i+1|j)) \quad \text{Theorem DER} \ [339]$$

$$= \sum_{j=1}^{n} (-1)^{(i+1)+j} [A]_{ij} \det(A(i|j)) \quad \text{Hypothesis}$$
\[
\begin{align*}
&= \sum_{j=1}^{n} (-1)^{i}(-1)^{i+j} [A]_{ij} \det (A (i|j)) \\
&= (-1) \sum_{j=1}^{n} (-1)^{i+j} [A]_{ij} \det (A (i|j)) \\
&= -\det (A)
\end{align*}
\]

So the result holds for the special case where we swap adjacent rows of the matrix. As any computer scientist knows, we can accomplish any rearrangement of an ordered list by swapping adjacent elements. This principle can be demonstrated by naïve sorting algorithms such as “bubble sort.” In any event, we don’t need to discuss every possible reordering, we just need to consider a swap of two rows, say rows \( s \) and \( t \) with \( 1 \leq s < t \leq n \).

Begin with row \( s \), and repeatedly swap it with each row just below it, including row \( t \) and stopping there. This will total \( t-s \) swaps. Now swap the former row \( t \), which currently lives in row \( t-1 \), with each row above it, stopping when it becomes row \( s \). This will total another \( t-s-1 \) swaps. In this way, we create \( B \) through a sequence of \( 2(t-s)-1 \) swaps of adjacent rows, each of which adjusts \( \det (A) \) by a multiplicative factor of \(-1\). So

\[
\det (B) = (-1)^{2(t-s)-1} \det (A) = ((-1)^{2})^{t-s} (-1)^{-1} \det (A) = -\det (A)
\]
as desired.

The proof for the case of swapping two columns is entirely similar, or could be derived from an application of Theorem DT employing the transpose of the matrix.

So Theorem DRCS tells us the effect of the first row operation (Definition RO) on the determinant of a matrix. Here’s the effect of the second row operation.

**Theorem DRCM**

**Determinant for Row or Column Multiples**

Suppose that \( A \) is a square matrix. Let \( B \) be the square matrix obtained from \( A \) by multiplying a single row by the scalar \( \alpha \), or by multiplying a single column by the scalar \( \alpha \). Then \( \det (B) = \alpha \det (A) \).

**Proof** Suppose that \( A \) is a square matrix of size \( n \) and we form the square matrix \( B \) by multiplying each entry of row \( i \) of \( A \) by \( \alpha \). Notice that the other rows of \( A \) and \( B \) are equal, so \( A (i|j) = B (i|j) \), for all \( 1 \leq j \leq n \). We compute \( \det (B) \) via expansion about row \( i \).

\[
\begin{align*}
\det (B) &= \sum_{j=1}^{n} (-1)^{i+j} [B]_{ij} \det (B (i|j)) \\
&= \sum_{j=1}^{n} (-1)^{i+j} [B]_{ij} \det (A (i|j)) \\
&= \sum_{j=1}^{n} (-1)^{i+j} \alpha [A]_{ij} \det (A (i|j)) \\
&= \alpha \sum_{j=1}^{n} (-1)^{i+j} [A]_{ij} \det (A (i|j)) \\
&= \alpha \det (A)
\end{align*}
\]

The proof for the case of a multiple of a column is entirely similar, or could be derived from an application of Theorem DT employing the transpose of the matrix.

Let’s go for understanding the effect of all three row operations. But first we need an intermediate result, but it is an easy one.
Theorem DERC
Determinant with Equal Rows or Columns
Suppose that $A$ is a square matrix with two equal rows, or two equal columns. Then $\det (A) = 0$.
□

Proof  Suppose that $A$ is a square matrix of size $n$ where the two rows $s$ and $t$ are equal. Form the matrix $B$ by swapping rows $r$ and $s$. Notice that as a consequence of our hypothesis, $A = B$. Then

$$
\det (A) = \frac{1}{2} (\det (A) + \det (A))
= \frac{1}{2} (\det (A) - \det (B)) 
\quad \text{(Theorem DRCS [345])}
= \frac{1}{2} (\det (A) - \det (A)) 
\quad \text{Hypothesis, } A = B
= \frac{1}{2} (0) = 0
$$

The proof for the case of two equal columns is entirely similar, or could be derived from an application of Theorem DT [340] employing the transpose of the matrix.  ■

Now explain the third row operation. Here we go.

Theorem DRCMA
Determinant for Row or Column Multiples and Addition
Suppose that $A$ is a square matrix. Let $B$ be the square matrix obtained from $A$ by multiplying a row by the scalar $\alpha$ and then adding it to another row, or by multiplying a column by the scalar $\alpha$ and then adding it to another column. Then $\det (B) = \det (A)$.
□

Proof  Suppose that $A$ is a square matrix of size $n$. Form the matrix $B$ by multiplying row $s$ by $\alpha$ and adding it to row $t$. Let $C$ be the auxiliary matrix where we replace row $t$ of $A$ by row $s$ of $A$. Notice that $A (t|j) = B (t|j) = C (t|j)$ for all $1 \leq j \leq n$. We compute the determinant of $B$ by expansion about row $t$.

$$
\det (B) = \sum_{j=1}^{n} (-1)^{t+j} [B]_{tj} \det (B (t|j)) 
\quad \text{(Theorem DER [339])}
= \sum_{j=1}^{n} (-1)^{t+j} \left( \alpha [A]_{sj} + [A]_{tj} \right) \det (B (s|j))
\quad \text{Hypothesis}
= \sum_{j=1}^{n} (-1)^{t+j} \alpha [A]_{sj} \det (B (t|j))
+ \sum_{j=1}^{n} (-1)^{t+j} [A]_{tj} \det (B (t|j))
= \alpha \sum_{j=1}^{n} (-1)^{t+j} [A]_{sj} \det (B (t|j))
+ \sum_{j=1}^{n} (-1)^{t+j} [A]_{tj} \det (B (t|j))
= \alpha \sum_{j=1}^{n} (-1)^{t+j} [C]_{tj} \det (C (t|j))
+ \sum_{j=1}^{n} (-1)^{t+j} [A]_{tj} \det (A (t|j))
= \alpha \det (C) + \det (A) 
= \alpha 0 + \det (A) = \det (A) 
\quad \text{(Theorem DER [347])}
$$
The proof for the case of adding a multiple of a column is entirely similar, or could be derived from an application of Theorem DT [340] employing the transpose of the matrix.

Is this what you expected? We could argue that the third row operation is the most popular, and yet it has no effect whatsoever on the determinant of a matrix! We can exploit this, along with our understanding of the other two row operations, to provide another approach to computing a determinant. We’ll explain this in the context of an example.

Example DRO
Determinant by row operations
Suppose we desire the determinant of the $4 \times 4$ matrix

\[
A = \begin{bmatrix}
2 & 0 & 2 & 3 \\
1 & 3 & -1 & 1 \\
-1 & 1 & -1 & 2 \\
3 & 5 & 4 & 0
\end{bmatrix}
\]

We will perform a sequence of row operations on this matrix, shooting for an upper triangular matrix, whose determinant will be simply the product of its diagonal entries. For each row operation, we will track the effect on the determinant via Theorem DRCS [345], Theorem DRCM [346], Theorem DRCMA [347].

\[
R_1 \leftrightarrow R_2, \quad A_1 = \begin{bmatrix}
1 & 3 & -1 & 1 \\
2 & 0 & 2 & 3 \\
-1 & 1 & -1 & 2 \\
3 & 5 & 4 & 0
\end{bmatrix}
\]
\[
\det (A) = - \det (A_1) \quad \text{Theorem DRCS [345]}
\]

\[
-2R_1 + R_2, \quad A_2 = \begin{bmatrix}
1 & 3 & -1 & 1 \\
0 & -6 & 4 & 1 \\
-1 & 1 & -1 & 2 \\
3 & 5 & 4 & 0
\end{bmatrix}
\]
\[
= - \det (A_2) \quad \text{Theorem DRCMA [347]}
\]

\[
R_1 + R_3, \quad A_3 = \begin{bmatrix}
1 & 3 & -1 & 1 \\
0 & -6 & 4 & 1 \\
0 & 4 & -2 & 3 \\
3 & 5 & 4 & 0
\end{bmatrix}
\]
\[
= - \det (A_3) \quad \text{Theorem DRCMA [347]}
\]

\[
-3R_1 + R_4, \quad A_4 = \begin{bmatrix}
1 & 3 & -1 & 1 \\
0 & -6 & 4 & 1 \\
0 & 4 & -2 & 3 \\
0 & -4 & 7 & -3
\end{bmatrix}
\]
\[
= - \det (A_4) \quad \text{Theorem DRCMA [347]}
\]

\[
R_3 + R_2, \quad A_5 = \begin{bmatrix}
1 & 3 & -1 & 1 \\
0 & -2 & 2 & 4 \\
0 & 4 & -2 & 3 \\
0 & -4 & 7 & -3
\end{bmatrix}
\]
\[
= - \det (A_5) \quad \text{Theorem DRCMA [347]}
\]

\[
-\frac{1}{2}R_2, \quad A_6 = \begin{bmatrix}
1 & 3 & -1 & 1 \\
0 & 1 & -1 & -2 \\
0 & 4 & -2 & 3 \\
0 & -4 & 7 & -3
\end{bmatrix}
\]
\[
= 2 \det (A_6) \quad \text{Theorem DRCM [346]}
\]

\[
-4R_2 + R_3, \quad A_7 = \begin{bmatrix}
1 & 3 & -1 & 1 \\
0 & 1 & -1 & -2 \\
0 & 0 & 2 & 11 \\
0 & -4 & 7 & -3
\end{bmatrix}
\]
\[
= 2 \det (A_7) \quad \text{Theorem DRCMA [347]}
\]

\[
4R_2 + R_4, \quad A_8 = \begin{bmatrix}
1 & 3 & -1 & 1 \\
0 & 1 & -1 & -2 \\
0 & 0 & 2 & 11 \\
0 & 0 & 3 & -11
\end{bmatrix}
\]
\[
= 2 \det (A_8) \quad \text{Theorem DRCMA [347]}
\]
The matrix $A_{12}$ is upper triangular, so expansion about the first column (repeatedly) will result in $\det(A_{12}) = (1)(1)(1)(1) = 1$ (see Example DTUM 341) and thus, $\det(A) = -110(1) = -110$.

Notice that our sequence of row operations was somewhat ad hoc, such as the transformation to $A_5$. We could have been even more methodical, and strictly followed the process that converts a matrix to reduced row-echelon form (Theorem REMEF 27), eventually achieving the same numerical result with a final matrix that equaled the $4 \times 4$ identity matrix. Notice too that we could have stopped with $A_8$, since at this point we could compute $\det(A_8)$ by two expansions about first columns, followed by a simple determinant of a $2 \times 2$ matrix (Theorem DMST 338).

The beauty of this approach is that computationally we should already have written a procedure to convert matrices to reduced-row echelon form, so all we need to do is track the multiplicative changes to the determinant as the algorithm procedes. Further, for a square matrix of size $n$ this approach requires on the order of $n^3$ multiplications, while a recursive application of expansion about a row or column (Theorem DER 339, Theorem DEC 340) will require in the vicinity of $(n-1)(n!)$ multiplications. So even for very small matrices, a computational approach utilizing row operations will have superior run-time. Tracking, and controlling, the effects of round-off errors is another story, best saved for a numerical linear algebra course.

Subsection DROEM

Determinants, Row Operations, Elementary Matrices

As a final preparation for our two most important theorems about determinants, we prove a handful of facts about the interplay of row operations and matrix multiplication with elementary matrices with regard to the determinant. But first, a simple, but crucial, fact about the identity matrix.

Theorem DIM

Determinant of the Identity Matrix

For every $n \geq 1$, $\det(I_n) = 1$.

Proof It may be overkill, but this is a good situation to run through a proof by induction on $n$ (Technique I 626). Is the result true when $n = 1$? Yes,

$$\det(I_1) = [I_1]_{11} = 1$$

Definition DM 337

Definition IM 62
Now assume the theorem is true for the identity matrix of size \( n - 1 \) and investigate the determinant of the identity matrix of size \( n \) with expansion about row 1,

\[
\det (I_n) = \sum_{j=1}^{n} (-1)^{1+j} [I_n]_{1j} \det (I_n (1|j)) \quad \text{Definition DM [337]}
\]

\[
= (-1)^{1+1} [I_n]_{11} \det (I_n (1|1)) + \sum_{j=2}^{n} (-1)^{1+j} [I_n]_{1j} \det (I_n (1|j))
\]

\[
= 1 \det (I_{n-1}) + \sum_{j=2}^{n} (-1)^{1+j} 0 \det (I_n (1|j)) \quad \text{Definition IM [62]}
\]

\[
= 1(1) + \sum_{j=2}^{n} 0 = 1 \quad \text{Induction Hypothesis}
\]

\[
\square
\]

**Theorem DEM**

**Determinants of Elementary Matrices**

For the three possible versions of an elementary matrix (Definition ELEM [333]) we have the determinants,

1. \( \det (E_{i,j}) = -1 \)
2. \( \det (E_i (\alpha)) = \alpha \)
3. \( \det (E_{i,j} (\alpha)) = 1 \)

\[
\square
\]

**Proof** Swapping rows \( i \) and \( j \) of the identity matrix will create \( E_{i,j} \) (Definition ELEM [333]), so

\[
\det (E_{i,j}) = - \det (I_n) = -1 \quad \text{Theorem DRCS [345]}
\]

\[
\det (E_i (\alpha)) = \alpha \det (I_n) = \alpha(1) = \alpha \quad \text{Theorem DRCM [346]}
\]

Multiplying row \( i \) of the identity matrix by \( \alpha \) and adding to row \( j \) will create \( E_{i} (\alpha) j \) (Definition ELEM [333]), so

\[
\det (E_{i} (\alpha) j) = \det (I_n) = 1 \quad \text{Theorem DRCMA [347]}
\]

\[
\square
\]

**Theorem DEMMM**

**Determinants, Elementary Matrices, Matrix Multiplication**

Suppose that \( A \) is a square matrix of size \( n \) and \( E \) is any elementary matrix of size \( n \). Then

\[
\det (EA) = \det (E) \det (A)
\]
Subsection PDM.DNMMM Determinants, Nonsingular Matrices, Matrix Multiplication

Proof The proof proceeds in three parts, one for each type of elementary matrix, with each part very similar to the other two. First, let $B$ be the matrix obtained from $A$ by swapping rows $i$ and $j$,

$$\det (E_{i,j} A) = \det (B)$$  \hspace{1cm} \text{Theorem EMDRO 334}
$$= - \det (A)$$  \hspace{1cm} \text{Theorem DRCS 345}
$$= \det (E_{i,j}) \det (A)$$  \hspace{1cm} \text{Theorem DEM 350}

Second, let $B$ be the matrix obtained from $A$ by multiplying row $i$ by $\alpha$,

$$\det (E_{i} (\alpha) A) = \det (B)$$  \hspace{1cm} \text{Theorem EMDRO 334}
$$= \alpha \det (A)$$  \hspace{1cm} \text{Theorem DRCM 346}
$$= \det (E_{i} (\alpha)) \det (A)$$  \hspace{1cm} \text{Theorem DEM 350}

Third, let $B$ be the matrix obtained from $A$ by multiplying row $i$ by $\alpha$ and adding to row $j$,

$$\det (E_{i,j} (\alpha) A) = \det (B)$$  \hspace{1cm} \text{Theorem EMDRO 334}
$$= \det (A)$$  \hspace{1cm} \text{Theorem DRCMA 347}
$$= \det (E_{i,j} (\alpha)) \det (A)$$  \hspace{1cm} \text{Theorem DEM 350}

Since the desired result holds for each variety of elementary matrix individually, we are done. ■

Subsection DNMMM
Determinants, Nonsingular Matrices, Matrix Multiplication

If you asked someone with substantial experience working with matrices about the value of the determinant, they’d be likely to quote the following theorem as the first thing to come to mind.

Theorem SMZD
Singular Matrices have Zero Determinants

Let $A$ be a square matrix. Then $A$ is singular if and only if $\det (A) = 0$. □

Proof ($\Rightarrow$) Suppose that $A$ is a singular matrix of size $n$. Then $A$ is row-equivalent to a square matrix $B$ in reduced row-echelon form (Theorem REMEF 27). Since $A$ is singular, the matrix $B$ is not the identity matrix (Theorem NMRRI 62). Therefore, the number of pivot columns is strictly less than $n$, i.e. $r < n$, and so $B$ has at least one row of all zeros.

There is a sequence of row operations $R_1, R_2, R_3, \ldots, R_s$ that will convert $B$ into $A$. For each of these row operations, there is an elementary matrix $E_i$ which effects the row operation by matrix multiplication (Theorem EMDRO 334). Repeated applications of Theorem EMDRO 334 allow us to write

$$A = E_s E_{s-1} \ldots E_2 E_1 B$$

Then

$$\det (A) = \det (E_s E_{s-1} \ldots E_2 E_1 B)$$
$$= \det (E_s) \det (E_{s-1}) \ldots \det (E_2) \det (E_1) \det (B)$$  \hspace{1cm} \text{Theorem DEMMM 350}
$$= \det (E_s) \det (E_{s-1}) \ldots \det (E_2) \det (E_1) 0$$  \hspace{1cm} \text{Theorem DZRC 345}
$$= 0$$

($\Leftarrow$) We will establish the contrapositive of this implication. So begin by assuming that $A$ is nonsingular. Then $A$ is row-equivalent to the identity matrix by Theorem NMRRI 62. As above, there is a sequence of row operations that will convert $I_n$ to $A$, which can be effected by matrix
multiplication by elementary matrices and Theorem DEMMM \[350\] allows us to “distribute” the determinant through this product. Mimicking the first half of the proof, we would arrive at
\[
\det (A) = \det (E_s) \det (E_{s-1}) \ldots \det (E_2) \det (E_1) \det (I_n)
\]
We know that \(\det (I_n) = 1 \neq 0\). From Theorem DEM \[350\] we can infer that the determinant of an elementary matrix is never zero (note the ban on \(\alpha = 0\) for \(E_i (\alpha)\) in Definition ELEM \[333\]). So the product on the right is composed of nonzero scalars, and so is also nonzero. This is the result we needed.

For the case of \(2 \times 2\) matrices you might compare the application of Theorem SMZD \[351\] with the combination of the results stated in Theorem DMST \[338\] and Theorem TTMI \[191\].

Example ZNDAB
Zero and nonzero determinant, Archetypes A and B
The coefficient matrix in Archetype A \[634\] has a zero determinant (check this!) while the coefficient matrix Archetype B \[638\] has a nonzero determinant (check this, too). These matrices are singular and nonsingular, respectively. This is exactly what Theorem SMZD \[351\] says, and continues our list of contrasts between these two archetypes.

Since Theorem SMZD \[351\] is an equivalence (Technique E \[622\]) we can expand on our growing list of equivalences about nonsingular matrices. The addition of the condition \(\det (A) \neq 0\) is one of the best motivations for learning about determinants.

Theorem NME7
Nonsingular Matrix Equivalences, Round 7
Suppose that \(A\) is a square matrix of size \(n\). The following are equivalent.

1. \(A\) is nonsingular.
2. \(A\) row-reduces to the identity matrix.
3. The null space of \(A\) contains only the zero vector, \(N (A) = \{0\}\).
4. The linear system \(\mathcal{L}S(A, b)\) has a unique solution for every possible choice of \(b\).
5. The columns of \(A\) are a linearly independent set.
6. \(A\) is invertible.
7. The column space of \(A\) is \(\mathbb{C}^n\), \(C(A) = \mathbb{C}^n\).
8. The columns of \(A\) are a basis for \(\mathbb{C}^n\).
9. The rank of \(A\) is \(n\), \(r (A) = n\).
10. The nullity of \(A\) is zero, \(n (A) = 0\).
11. The determinant of \(A\) is nonzero, \(\det (A) \neq 0\).

\[\square\]

Proof Theorem SMZD \[351\] says \(A\) is singular if and only if \(\det (A) = 0\). If we negate each of these statements, we arrive at two contrapositives that we can combine as the equivalence, \(A\) is nonsingular if and only if \(\det (A) \neq 0\). This allows us to add a new statement to the list found in Theorem NME6 \[815\].

Computationally, row-reducing a matrix is the most efficient way to determine if a matrix is nonsingular, though the effect of using division in a computer can lead to round-off errors that confuse small quantities with critical zero quantities. Conceptually, the determinant may seem the most efficient way to determine if a matrix is nonsingular. The definition of a determinant uses just addition, subtraction and multiplication, so division is never a problem. And the final test is...
easy: is the determinant zero or not? However, the number of operations involved in computing a
determinant by the definition very quickly becomes so excessive as to be impractical.

Now for the coup de grâce. We will generalize Theorem DEMMM \[350\] to the case of any two
square matrices. You may recall thinking that matrix multiplication was defined in a needlessly
complicated manner. For sure, the definition of a determinant seems even stranger. (Though
Theorem SMZD \[351\] might be forcing you to reconsider.) Read the statement of the next theorem
and contemplate how nicely matrix multiplication and determinants play with each other.

**Theorem DRMM**

**Determinant Respects Matrix Multiplication**

Suppose that \(A\) and \(B\) are square matrices of the same size. Then \(\det(AB) = \det(A) \det(B)\).

**Proof**  Suppose that \(A\) or \(B\) is singular. Then either \(\det(A) = 0\) or \(\det(B) = 0\) by Theorem
SMZD \[351\]. In either case, \(\det(A) \det(B) = 0\). By the contrapositive of Theorem NPNT \[202\],
we know \(AB\) is singular as well. So by Theorem SMZD \[351\], \(\det(AB) = 0\). So in this case, we
have the desired equality.

Now assume that \(A\) and \(B\) are both nonsingular. By Theorem NMPEM \[337\] there are elementary
matrices \(E_1, E_2, E_3, \ldots, E_s\) and \(E_{s+1}, E_{s+2}, E_{s+3}, \ldots, E_{s+t}\) such that

\[
A = E_1E_2E_3\ldots E_s \quad B = E_{s+1}E_{s+2}E_{s+3}\ldots E_{s+t}
\]

Then

\[
\det(AB) = \det(E_1E_2\ldots E_sE_{s+1}E_{s+2}\ldots E_{s+t})
\]
\[
= \det(E_1) \det(E_2)\ldots \det(E_s) \det(E_{s+1}E_{s+2}\ldots E_{s+t}) \quad \text{Theorem DEMMM} \[350\]
\]
\[
= \det(E_1E_2\ldots E_s) \det(E_{s+1}E_{s+2}\ldots E_{s+t}) \quad \text{Theorem DEMMM} \[350\]
\]
\[
= \det(A) \det(B)
\]

It’s an amazing thing that matrix multiplication and the determinant interact this way. Might
it also be true that \(\det(A + B) = \det(A) + \det(B)\)? (See Exercise PDM.M30 \[354\].)

**Subsection READ**

**Reading Questions**

1. Consider the two matrices below, and suppose you already have computed \(\det(A) = -120\). 
What is \(\det(B)\)? Why?

\[
A = \begin{bmatrix}
0 & 8 & 3 & -4 \\
-1 & 2 & -2 & 5 \\
-2 & 8 & 4 & 3 \\
0 & -4 & 2 & -3
\end{bmatrix} \quad B = \begin{bmatrix}
0 & 8 & 3 & -4 \\
0 & -4 & 2 & -3 \\
-2 & 8 & 4 & 3 \\
-1 & 2 & -2 & 5
\end{bmatrix}
\]

2. State the theorem that allows us to make yet another extension to our NMEx series of theo-
rems.

3. What is amazing about the interaction between matrix multiplication and the determinant?
Subsection EXC
Exercises

C30 Each of the archetypes below is a system of equations with a square coefficient matrix, or is a square matrix itself. Compute the determinant of each matrix, noting how Theorem SMZD indicates when the matrix is singular or nonsingular.

Archetype A 634
Archetype B 638
Archetype F 654
Archetype K 676
Archetype L 680

Contributed by Robert Beezer

M20 Construct a $3 \times 3$ nonsingular matrix and call it $A$. Then, for each entry of the matrix, compute the corresponding cofactor, and create a new $3 \times 3$ matrix full of these cofactors by placing the cofactor of an entry in the same location as the entry it was based on. Once complete, call this matrix $C$. Compute $AC^t$. Any observations? Repeat with a new matrix, or perhaps with a $4 \times 4$ matrix.

Contributed by Robert Beezer Solution

M30 Construct an example to show that the following statement is not true for all square matrices $A$ and $B$ of the same size: $\det (A + B) = \det (A) + \det (B)$.

Contributed by Robert Beezer

T10 Theorem NPNT says that if the product of square matrices $AB$ is nonsingular, then the individual matrices $A$ and $B$ are nonsingular also. Construct a new proof of this result making use of theorems about determinants of matrices.

Contributed by Robert Beezer

T15 Use Theorem DRCM to prove Theorem DZRC as a corollary. (See Technique LC.)

Contributed by Robert Beezer

T20 Suppose that $A$ is a square matrix of size $n$ and $\alpha \in \mathbb{C}$ is a scalar. Prove that $\det (\alpha A) = \alpha^n \det (A)$.

Contributed by Robert Beezer

T25 Employ Theorem DT to construct the second half of the proof of Theorem DRCM (the portion about a multiple of a column).

Contributed by Robert Beezer
The result of these computations should be a matrix with the value of \( \det(A) \) in the diagonal entries and zeros elsewhere. The suggestion of using a nonsingular matrix was partially so that it was obvious that the value of the determinant appears on the diagonal.

This result (which is true in general) provides a method for computing the inverse of a nonsingular matrix. Since \( AC^t = \det(A) I_n \), we can multiply by the reciprocal of the determinant (which is nonzero!) and the inverse of \( A \) (it exists!) to arrive at an expression for the matrix inverse:

\[
A^{-1} = \frac{1}{\det(A)} C^t
\]
Chapter E
Eigenvalues

When we have a square matrix of size $n$, $A$, and we multiply it by a vector $x$ from $\mathbb{C}^n$ to form the matrix-vector product (Definition MVP [173]), the result is another vector in $\mathbb{C}^n$. So we can adopt a functional view of this computation — the act of multiplying by a square matrix is a function that converts one vector ($x$) into another one ($Ax$) of the same size. For some vectors, this seemingly complicated computation is really no more complicated than scalar multiplication. The vectors vary according to the choice of $A$, so the question is to determine, for an individual choice of $A$, if there are any such vectors, and if so, which ones. It happens in a variety of situations that these vectors (and the scalars that go along with them) are of special interest.

We will be solving polynomial equations in this chapter, which raises the specter of roots that are complex numbers. This distinct possibility is our main reason for entertaining the complex numbers throughout the course. You might be moved to revisit Section CNO [611] and Section O [151].

Section EE
Eigenvalues and Eigenvectors

We start with the principal definition for this chapter.

Subsection EEM
Eigenvalues and Eigenvectors of a Matrix

Definition EEM
Eigenvalues and Eigenvectors of a Matrix
Suppose that $A$ is a square matrix of size $n$, $x \neq 0$ is a vector in $\mathbb{C}^n$, and $\lambda$ is a scalar in $\mathbb{C}$. Then we say $x$ is an eigenvector of $A$ with eigenvalue $\lambda$ if

$$Ax = \lambda x$$

Before going any further, perhaps we should convince you that such things ever happen at all. Understand the next example, but do not concern yourself with where the pieces come from. We will have methods soon enough to be able to discover these eigenvectors ourselves.

Example SEE
Some eigenvalues and eigenvectors
Consider the matrix

\[
A = \begin{bmatrix}
204 & 98 & -26 & -10 \\
-280 & -134 & 36 & 14 \\
716 & 348 & -90 & -36 \\
-472 & -232 & 60 & 28
\end{bmatrix}
\]

and the vectors

\[
\begin{align*}
x &= \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix} \\
y &= \begin{bmatrix} -3 \\ 4 \\ -10 \\ 4 \end{bmatrix} \\
z &= \begin{bmatrix} -3 \\ 7 \\ 0 \\ 8 \end{bmatrix} \\
w &= \begin{bmatrix} 1 \\ -1 \\ 4 \\ 0 \end{bmatrix}
\end{align*}
\]

Then

\[
Ax = \begin{bmatrix} 204 & 98 & -26 & -10 \\ -280 & -134 & 36 & 14 \\ 716 & 348 & -90 & -36 \\ -472 & -232 & 60 & 28 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2 \\ 20 \end{bmatrix} = 4x
\]

so \(x\) is an eigenvector of \(A\) with eigenvalue \(\lambda = 4\). Also,

\[
Ay = \begin{bmatrix} 204 & 98 & -26 & -10 \\ -280 & -134 & 36 & 14 \\ 716 & 348 & -90 & -36 \\ -472 & -232 & 60 & 28 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ -10 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0y
\]

so \(y\) is an eigenvector of \(A\) with eigenvalue \(\lambda = 0\). Also,

\[
Az = \begin{bmatrix} 204 & 98 & -26 & -10 \\ -280 & -134 & 36 & 14 \\ 716 & 348 & -90 & -36 \\ -472 & -232 & 60 & 28 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -6 \\ 14 \\ 0 \\ 16 \end{bmatrix} = 2z
\]

so \(z\) is an eigenvector of \(A\) with eigenvalue \(\lambda = 2\). Also,

\[
Aw = \begin{bmatrix} 204 & 98 & -26 & -10 \\ -280 & -134 & 36 & 14 \\ 716 & 348 & -90 & -36 \\ -472 & -232 & 60 & 28 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 4 \\ 0 \end{bmatrix} = 2w
\]

so \(w\) is an eigenvector of \(A\) with eigenvalue \(\lambda = 2\).

So we have demonstrated four eigenvectors of \(A\). Are there more? Yes, any nonzero scalar multiple of an eigenvector is again an eigenvector. In this example, set \(u = 30x\). Then

\[
Au = A(30x) = 30Ax = 30(4x) = 4(30x) = 4u
\]

so that \(u\) is also an eigenvector of \(A\) for the same eigenvalue, \(\lambda = 4\).

The vectors \(z\) and \(w\) are both eigenvectors of \(A\) for the same eigenvalue \(\lambda = 2\), yet this is not as simple as the two vectors just being scalar multiples of each other (they aren’t). Look what happens when we add them together, to form \(v = z + w\), and multiply by \(A\),

\[
Av = A(z + w) = Az + Aw = 2z + 2w
\]
so that \( \mathbf{v} \) is also an eigenvector of \( A \) for the eigenvalue \( \lambda = 2 \). So it would appear that the set of eigenvectors that are associated with a fixed eigenvalue is closed under the vector space operations of \( \mathbb{C}^n \). Hmmm.

The vector \( \mathbf{y} \) is an eigenvector of \( A \) for the eigenvalue \( \lambda = 0 \), so we can use Theorem ZSSM \[258\] to write \( A\mathbf{y} = 0\mathbf{y} = \mathbf{0} \). But this also means that \( \mathbf{y} \in \mathcal{N}(A) \). There would appear to be a connection here also.

Example SEE \[356\] hints at a number of intriguing properties, and there are many more. We will explore the general properties of eigenvalues and eigenvectors in Section PEE \[378\], but in this section we will concern ourselves with the question of actually computing eigenvalues and eigenvectors. First we need a bit of background material on polynomials and matrices.

Subsection PM
Polynomials and Matrices

A polynomial is a combination of powers, multiplication by scalar coefficients, and addition (with subtraction just being the inverse of addition). We never have occasion to divide when computing the value of a polynomial. So it is with matrices. We can add and subtract matrices, we can multiply matrices by scalars, and we can form powers of square matrices by repeated applications of matrix multiplication. We do not normally divide matrices (though sometimes we can multiply by an inverse). If a matrix is square, all the operations constituting a polynomial will preserve the size of the matrix. So it is natural to consider evaluating a polynomial with a matrix, effectively replacing the variable of the polynomial by a matrix. We’ll demonstrate with an example,

Example PM
Polynomial of a matrix

Let

\[
p(x) = 14 + 19x - 3x^2 - 7x^3 + x^4
\]

and we will compute \( p(D) \). First, the necessary powers of \( D \). Notice that \( D^0 \) is defined to be the multiplicative identity, \( I_3 \), as will be the case in general.

\[
D^0 = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
D^1 = D = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix}
\]

\[
D^2 = DD^1 = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 & -6 \\ 5 & 1 & 0 \\ -3 & -8 & -7 \end{bmatrix}
\]

\[
D^3 = DD^2 = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & -6 \\ 5 & 1 & 0 \\ -3 & -8 & -7 \end{bmatrix} = \begin{bmatrix} 19 & -12 & -8 \\ -4 & 15 & 8 \\ 12 & -4 & 11 \end{bmatrix}
\]

\[
D^4 = DD^3 = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 19 & -12 & -8 \\ -4 & 15 & 8 \\ 12 & -4 & 11 \end{bmatrix} = \begin{bmatrix} -7 & 49 & 54 \\ -5 & -4 & -30 \\ -49 & 47 & 43 \end{bmatrix}
\]
Then
\[
p(D) = 14 + 19D - 3D^2 - 7D^3 + D^4
\]
\[
= 14 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 19 \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} - 3 \begin{bmatrix} -2 & -1 & -6 \\ 5 & 1 & 0 \\ 1 & -8 & -7 \end{bmatrix}
\]
\[
- 7 \begin{bmatrix} 19 & -12 & -8 \\ -4 & 15 & 8 \\ 12 & -4 & 11 \end{bmatrix} + 54 \begin{bmatrix} 7 & 49 & 54 \\ -5 & -4 & -30 \\ -49 & 47 & 43 \end{bmatrix}
\]
\[
= \begin{bmatrix} -139 & 193 & 166 \\ 27 & -98 & -124 \\ -193 & 118 & 20 \end{bmatrix}
\]

Notice that \( p(x) \) factors as
\[
p(x) = 14 + 19x - 3x^2 - 7x^3 + x^4 = (x - 2)(x - 7)(x + 1)^2
\]

Because \( D \) commutes with itself \((DD = DD)\), we can use distributivity of matrix multiplication across matrix addition \(\text{Theorem MMDAA[179]}\) without being careful with any of the matrix products, and just as easily evaluate \( p(D) \) using the factored form of \( p(x) \),
\[
p(D) = 14 + 19D - 3D^2 - 7D^3 + D^4 = (D - 2I_3)(D - 7I_3)(D + I_3)^2
\]
\[
= \begin{bmatrix} -3 & 3 & 2 \\ 1 & -2 & -2 \\ -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} -8 & 3 & 2 \\ -1 & -7 & -2 \\ -3 & 1 & -6 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 \\ 1 & 1 & -2 \\ -3 & 1 & 2 \end{bmatrix}^2
\]
\[
= \begin{bmatrix} -139 & 193 & 166 \\ 27 & -98 & -124 \\ -193 & 118 & 20 \end{bmatrix}
\]

This example is not meant to be too profound. It is meant to show you that it is natural to evaluate a polynomial with a matrix, and that the factored form of the polynomial is as good as (or maybe better than) the expanded form. And do not forget that constant terms in polynomials are really multiples of the identity matrix when we are evaluating the polynomial with a matrix. 

\[ \blacksquare \]

Subsection EEE
Existence of Eigenvalues and Eigenvectors

Before we embark on computing eigenvalues and eigenvectors, we will prove that every matrix has at least one eigenvalue (and an eigenvector to go with it). Later, in \[\text{Theorem MNEM[385]}\], we will determine the maximum number of eigenvalues a matrix may have.

The determinant \(\text{Definition D[307]}\) will be a powerful tool in \[\text{Subsection EE.CEE[362]}\] when it comes time to compute eigenvalues. However, it is possible, with some more advanced machinery, to compute eigenvalues without ever making use of the determinant. Sheldon Axler does just that in his book, \textit{Linear Algebra Done Right}. Here and now, we give Axler’s “determinant-free” proof that every matrix has an eigenvalue. The result is not too startling, but the proof is most enjoyable.

\textbf{Theorem EMHE}
Every Matrix Has an Eigenvalue

Suppose \( A \) is a square matrix. Then \( A \) has at least one eigenvalue. 

\textbf{Proof} Suppose that \( A \) has size \( n \), and choose \( x \) as \textit{any} nonzero vector from \( \mathbb{C}^n \). (Notice how much latitude we have in our choice of \( x \). Only the zero vector is off-limits.) Consider the set
\[
S = \{ x, Ax, A^2x, A^3 x, \ldots, A^nx \}
\]
This is a set of \( n + 1 \) vectors from \( \mathbb{C}^n \), so by Theorem MVSLD [126], \( S \) is linearly dependent. Let \( a_0, a_1, a_2, \ldots, a_n \) be a collection of \( n + 1 \) scalars from \( \mathbb{C} \), not all zero, that provide a relation of linear dependence on \( S \). In other words,

\[
a_0x + a_1Ax + a_2A^2x + a_3A^3x + \cdots + a_nA^n x = 0
\]

Some of the \( a_i \) are nonzero. Suppose that just \( a_0 \neq 0 \), and \( a_1 = a_2 = a_3 = \cdots = a_n = 0 \). Then \( a_0x = 0 \) and by Theorem SMEZV [259], either \( a_0 = 0 \) or \( x = 0 \), which are both contradictions. So \( a_i \neq 0 \) for some \( i \geq 1 \). Let \( m \) be the largest integer such that \( a_m \neq 0 \). From this discussion we know that \( m \geq 1 \). We can also assume that \( a_m = 1 \) for if not, replace each \( a_i \) by \( a_i/a_m \) to obtain scalars that serve equally well in providing a relation of linear dependence on \( S \).

Define the polynomial

\[
p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_m x^m
\]

Because we have consistently used \( \mathbb{C} \) as our set of scalars (rather than \( \mathbb{R} \)), we know that we can factor \( p(x) \) into linear factors of the form \((x - b_i)\), where \( b_i \in \mathbb{C} \). So there are scalars, \( b_1, b_2, b_3, \ldots, b_m \), from \( \mathbb{C} \) so that,

\[
p(x) = (x - b_m)(x - b_{m-1}) \cdots (x - b_3)(x - b_2)(x - b_1)
\]

Put it all together and

\[
0 = a_0x + a_1Ax + a_2A^2x + a_3A^3x + \cdots + a_nA^n x
\]

\[
= a_0x + a_1Ax + a_2A^2x + a_3A^3x + \cdots + a_mA^m x
\]

\[
= (a_0I_n + a_1A + a_2A^2 + a_3A^3 + \cdots + a_mA^m) x
\]

\[
= p(A)x
\]

\[
= (A - b_mI_n)(A - b_{m-1}I_n) \cdots (A - b_3I_n)(A - b_2I_n)(A - b_1I_n)x
\]

Let \( k \) be the smallest integer such that

\[
(A - b_kI_n)(A - b_{k-1}I_n) \cdots (A - b_3I_n)(A - b_2I_n)(A - b_1I_n)x = 0.
\]

From the preceding equation, we know that \( k \leq m \). Define the vector \( z \) by

\[
z = (A - b_{k-1}I_n) \cdots (A - b_3I_n)(A - b_2I_n)(A - b_1I_n)x
\]

Notice that by the definition of \( k \), the vector \( z \) must be nonzero. In the case where \( k = 1 \), we understand that \( z \) is defined by \( z = x \), and \( z \) is still nonzero. Now

\[
(A - b_kI_n)z = (A - b_kI_n)(A - b_{k-1}I_n) \cdots (A - b_3I_n)(A - b_2I_n)(A - b_1I_n)x = 0
\]

which allows us to write

\[
Az = (A + 0)z = (A - b_kI_n + b_kI_n)z = (A - b_kI_n)z + b_kI_n z = 0 + b_kI_n z = b_kI_n z = b_k z
\]

Since \( z \neq 0 \), this equation says that \( z \) is an eigenvector of \( A \) for the eigenvalue \( \lambda = b_k \) (Definition EEM [356]), so we have shown that any square matrix \( A \) does have at least one eigenvalue.

The proof of Theorem EMHE [359] is constructive (it contains an unambiguous procedure that leads to an eigenvalue), but it is not meant to be practical. We will illustrate the theorem with an example, the purpose being to provide a companion for studying the proof and not to suggest this is the best procedure for computing an eigenvalue.
Example CAEHW
Computing an eigenvalue the hard way
This example illustrates the proof of Theorem EMHE \[359\], so will employ the same notation as the proof — look there for full explanations. It is not meant to be an example of a reasonable computational approach to finding eigenvalues and eigenvectors. OK, warnings in place, here we go.

Let
\[
A = \begin{bmatrix}
-7 & -1 & 11 & 0 & -4 \\
4 & 1 & 0 & 2 & 0 \\
-10 & -1 & 14 & 0 & -4 \\
8 & 2 & -15 & -1 & 5 \\
-10 & -1 & 16 & 0 & -6
\end{bmatrix}
\]

and choose
\[
x = \begin{bmatrix}
3 \\
0 \\
3 \\
-5 \\
4
\end{bmatrix}
\]

It is important to notice that the choice of \(x\) could be anything, so long as it is not the zero vector. We have not chosen \(x\) totally at random, but so as to make our illustration of the theorem as general as possible. You could replicate this example with your own choice and the computations are guaranteed to be reasonable, provided you have a computational tool that will factor a fifth degree polynomial for you.

The set
\[
S = \{x, Ax, A^2x, A^3x, A^4x, A^5x\}
\]

is guaranteed to be linearly dependent, as it has six vectors from \(\mathbb{C}^5\) (Theorem MVSLD \[126\]). We will search for a non-trivial relation of linear dependence by solving a homogeneous system of equations whose coefficient matrix has the vectors of \(S\) as columns through row operations,

\[
\begin{bmatrix}
3 & -4 & 6 & -10 & 18 & -34 \\
0 & 2 & -6 & -30 & 62 \\
3 & -4 & 6 & -10 & 18 & -34 \\
-5 & 4 & -2 & -10 & 10 & -26 \\
4 & -6 & 10 & -18 & 34 & -66
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & -2 & 6 & -14 & 30 \\
0 & 1 & -3 & 7 & -15 & 31 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

There are four free variables for describing solutions to this homogeneous system, so we have our pick of solutions. The most expedient choice would be to set \(x_3 = 1\) and \(x_4 = x_5 = x_6 = 0\). However, we will again opt to maximize the generality of our illustration of Theorem EMHE \[359\] and choose \(x_3 = -8\), \(x_4 = -3\), \(x_5 = 1\) and \(x_6 = 0\). The leads to a solution with \(x_1 = 16\) and \(x_2 = 12\).

This relation of linear dependence then says that
\[
0 = 16x + 12Ax - 8A^2x - 3A^3x + A^4x + 0A^5x
\]

So we define \(p(x) = 16 + 12x - 8x^2 - 3x^3 + x^4\), and as advertised in the proof of Theorem EMHE \[359\], we have a polynomial of degree \(m = 4 > 1\) such that \(p(A)x = 0\). Now we need to factor \(p(x)\) over \(\mathbb{C}\). If you made your own choice of \(x\) at the start, this is where you might have a fifth degree polynomial.

Version 1.04
polynomial, and where you might need to use a computational tool to find roots and factors. We have
\[ p(x) = 16 + 12x - 8x^2 - 3x^3 + x^4 = (x - 4)(x + 2)(x - 2)(x + 1) \]
So we know that
\[ 0 = p(A)x = (A - 4I_5)(A + 2I_5)(A - 2I_5)(A + 1I_5)x \]
We apply one factor at a time, until we get the zero vector, so as to determine the value of \( k \) described in the proof of Theorem EMHE \[359\].

\[
\begin{pmatrix}
-6 & -1 & 11 & 0 & -4 \\
4 & 2 & 0 & 2 & 0 \\
10 & -1 & 15 & 0 & -4 \\
8 & 2 & -15 & 0 & 5 \\
10 & -1 & 16 & 0 & -5
\end{pmatrix}
\begin{pmatrix}
3 \\
0 \\
3 \\
5 \\
4
\end{pmatrix} =
\begin{pmatrix}
-1 \\
2 \\
3 \\
-5 \\
-2
\end{pmatrix}
\]
\[
\begin{pmatrix}
-9 & -1 & 11 & 0 & -4 \\
4 & -1 & 0 & 2 & 0 \\
10 & -1 & 12 & 0 & -4 \\
8 & 2 & -15 & 3 & 5 \\
10 & -1 & 16 & 0 & -8
\end{pmatrix}
\begin{pmatrix}
-1 \\
2 \\
3 \\
5 \\
4
\end{pmatrix} =
\begin{pmatrix}
4 \\
8 \\
-1 \\
4 \\
8
\end{pmatrix}
\]
\[
\begin{pmatrix}
-5 & -1 & 11 & 0 & -4 \\
4 & 3 & 0 & 2 & 0 \\
10 & -1 & 16 & 0 & -4 \\
8 & 2 & -15 & 1 & 5 \\
10 & -1 & 16 & 0 & -4
\end{pmatrix}
\begin{pmatrix}
4 \\
0 \\
4 \\
4 \\
8
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

So \( k = 3 \) and
\[
\begin{pmatrix}
4 \\
-8 \\
4 \\
4 \\
8
\end{pmatrix}
\]
is an eigenvector of \( A \) for the eigenvalue \( \lambda = -2 \), as you can check by doing the computation \( Az \). If you work through this example with your own choice of the vector \( x \) (strongly recommended) then the eigenvalue you will find may be different, but will be in the set \( \{3, 0, 1, -1, -2\} \). See Exercise EE.M60 \[373\] for a suggested starting vector.

**Subsection CEE**

**Computing Eigenvalues and Eigenvectors**

Fortunately, we need not rely on the procedure of Theorem EMHE \[359\] each time we need an eigenvalue. It is the determinant, and specifically Theorem SMZD \[351\], that provides the main tool for computing eigenvalues. Here is an informal sequence of equivalences that is the key to determining the eigenvalues and eigenvectors of a matrix,

\[ Ax = \lambda x \iff Ax - \lambda I_n x = 0 \iff (A - \lambda I_n) x = 0 \]

So, for an eigenvalue \( \lambda \) and associated eigenvector \( x \neq 0 \), the vector \( x \) will be a nonzero element of the null space of \( A - \lambda I_n \), while the matrix \( A - \lambda I_n \) will be singular and therefore have zero determinant. These ideas are made precise in Theorem EMRCP \[363\] and Theorem EMNS \[364\], but for now this brief discussion should suffice as motivation for the following definition and example.
Definition CP
Characteristic Polynomial
Suppose that $A$ is a square matrix of size $n$. Then the characteristic polynomial of $A$ is the polynomial $p_A(x)$ defined by

$$p_A(x) = \det(A - xI_n)$$

\[\triangle\]

Example CPMS3
Characteristic polynomial of a matrix, size 3
Consider

$$F = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix}$$

Then

$$p_F(x) = \det(F - xI_3)$$

$$= \begin{vmatrix} -13 - x & -8 & -4 \\ 12 & 7 - x & 4 \\ 24 & 16 & 7 - x \end{vmatrix}$$

Definition CP 363

$$= (-13 - x)\begin{vmatrix} 7 - x & 4 \\ 16 & 7 - x \end{vmatrix} + (-8)(-1)\begin{vmatrix} 12 & 4 \\ 24 & 7 - x \end{vmatrix}$$

Definition DM 337

$$+ (-4)\begin{vmatrix} 12 & 7 - x \\ 24 & 16 \end{vmatrix}$$

Theorem DMST 338

$$= (-13 - x)((7 - x)(7 - x) - 4(16))$$

$$+ (-8)(-1)(12(7 - x) - 4(24))$$

$$+ (-4)(12(16) - (7 - x)(24))$$

$$= 3 + 5x + x^2 - x^3$$

$$= -(x - 3)(x + 1)^2$$

\[\boxplus\]

The characteristic polynomial is our main computational tool for finding eigenvalues, and will sometimes be used to aid us in determining the properties of eigenvalues.

Theorem EMRCP
Eigenvalues of a Matrix are Roots of Characteristic Polynomials
Suppose $A$ is a square matrix. Then $\lambda$ is an eigenvalue of $A$ if and only if $p_A(\lambda) = 0$.

Proof Suppose $A$ has size $n$.

\begin{align*}
\lambda &= \text{an eigenvalue of } A \\
\iff & \text{there exists } x \neq 0 \text{ so that } Ax = \lambda x \quad \text{Definition EEM } 356 \\
\iff & \text{there exists } x \neq 0 \text{ so that } Ax - \lambda x = 0 \quad \text{Theorem MMIM } 179 \\
\iff & \text{there exists } x \neq 0 \text{ so that } Ax - \lambda I_n x = 0 \quad \text{Theorem MMDAA } 179 \\
\iff & \text{there exists } x \neq 0 \text{ so that } (A - \lambda I_n)x = 0 \quad \text{Theorem SMZD } 351 \\
\iff & A - \lambda I_n \text{ is singular} \\
\iff & \det(A - \lambda I_n) = 0 \\
\iff & p_A(\lambda) = 0 \\
\Box
\end{align*}

Example EMS3
Eigenvalues of a matrix, size 3
We found the characteristic polynomial of
\[ F = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \]
to be \( p_F(x) = -(x - 3)(x + 1)^2 \). Factored, we can find all of its roots easily, they are \( x = 3 \) and \( x = -1 \). By Theorem EMRCP, \( \lambda = 3 \) and \( \lambda = -1 \) are both eigenvalues of \( F \), and these are the only eigenvalues of \( F \). We’ve found them all.

Let us now turn our attention to the computation of eigenvectors.

**Definition EM**
**Eigenspace of a Matrix**
Suppose that \( A \) is a square matrix and \( \lambda \) is an eigenvalue of \( A \). Then the **eigenspace** of \( A \) for \( \lambda \), \( \mathcal{E}_A(\lambda) \), is the set of all the eigenvectors of \( A \) for \( \lambda \), together with the inclusion of the zero vector.

Example SEE hinted that the set of eigenvectors for a single eigenvalue might have some closure properties, and with the addition of the non-eigenvector, \( 0 \), we indeed get a whole subspace.

**Theorem EMS**
**Eigenspace for a Matrix is a Subspace**
Suppose \( A \) is a square matrix of size \( n \) and \( \lambda \) is an eigenvalue of \( A \). Then the eigenspace \( \mathcal{E}_A(\lambda) \) is a subspace of the vector space \( \mathbb{C}^n \).

**Proof**
We will check the three conditions of Theorem TSS. First, Definition EM explicitly includes the zero vector in \( \mathcal{E}_A(\lambda) \), so the set is non-empty.

Suppose that \( x, y \in \mathcal{E}_A(\lambda) \), that is, \( x \) and \( y \) are two eigenvectors of \( A \) for \( \lambda \). Then
\[
A(x + y) = Ax + Ay = \lambda x + \lambda y = \lambda(x + y)
\]
So either \( x + y = 0 \), or \( x + y \) is an eigenvector of \( A \) for \( \lambda \). So, in either event, \( x + y \in \mathcal{E}_A(\lambda) \), and we have additive closure.

Suppose that \( \alpha \in \mathbb{C} \), and that \( x \in \mathcal{E}_A(\lambda) \), that is, \( x \) is an eigenvector of \( A \) for \( \lambda \). Then
\[
A(\alpha x) = \alpha(Ax) = \alpha \lambda x = \lambda (\alpha x)
\]
So either \( \alpha x = 0 \), or \( \alpha x \) is an eigenvector of \( A \) for \( \lambda \). So, in either event, \( \alpha x \in \mathcal{E}_A(\lambda) \), and we have scalar closure.

With the three conditions of Theorem TSS met, we know \( \mathcal{E}_A(\lambda) \) is a subspace.

**Theorem EMNS**
**Eigenspace of a Matrix is a Null Space**
Suppose \( A \) is a square matrix of size \( n \) and \( \lambda \) is an eigenvalue of \( A \). Then
\[
\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)
\]
**Proof**
The conclusion of this theorem is an equality of sets, so normally we would follow the advice of Definition SE. However, in this case we can construct a sequence of equivalences which will together provide the two subset inclusions we need. First, notice that \( 0 \in \mathcal{E}_A(\lambda) \) by
Subsection EE.ECEE Examples of Computing Eigenvalues and Eigenvectors 365

Definition EM \[364\] and \(0 \in \mathcal{N}(A - \lambda I_n)\) by Theorem HSC \[52\]. Now consider any nonzero vector \(x \in \mathbb{C}^n\),

\[
\begin{align*}
x \in \mathcal{E}_A(\lambda) & \iff Ax = \lambda x \\
& \iff Ax - \lambda x = 0 \\
& \iff (A - \lambda I_n)x = 0 \\
& \iff x \in \mathcal{N}(A - \lambda I_n) \\
& \iff (A - \lambda I_n - \lambda I_n)x = 0
\end{align*}
\]

You might notice the close parallels (and differences) between the proofs of Theorem EMRCP \[363\] and Theorem EMNS \[364\]. Since Theorem EMNS \[364\] describes the set of all the eigenvectors of \(A\) as a null space we can use techniques such as Theorem BNS \[128\] to provide concise descriptions of eigenspaces.

Example ESMS3

Eigenspaces of a matrix, size 3

Example CPMS3 \[363\] and Example EMS3 \[363\] describe the characteristic polynomial and eigenvalues of the \(3 \times 3\) matrix

\[
F = \begin{bmatrix}
-13 & -8 & -4 \\
12 & 7 & 4 \\
24 & 16 & 7
\end{bmatrix}
\]

We will now take the each eigenvalue in turn and compute its eigenspace. To do this, we row-reduce the matrix \(F - \lambda I_3\) in order to determine solutions to the homogeneous system \(LS(F - \lambda I_3, 0)\) and then express the eigenspace as the null space of \(F - \lambda I_3\) \(\text{Theorem EMNS 364}\). Theorem BNS \[128\] then tells us how to write the null space as the span of a basis.

\[
\begin{align*}
\lambda = 3 & \quad F - 3I_3 = \begin{bmatrix}
-16 & -8 & -4 \\
12 & 4 & 4 \\
24 & 16 & 4
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & \frac{1}{2} \\
0 & 0 & 0
\end{bmatrix} \\
\mathcal{E}_F(3) & = \mathcal{N}(F - 3I_3) = \left\langle \begin{bmatrix}
-\frac{1}{2} \\
1 \\
1
\end{bmatrix} \right\rangle = \left\langle \begin{bmatrix}
-1 \\
1 \\
2
\end{bmatrix} \right\rangle
\end{align*}
\]

\[
\begin{align*}
\lambda = -1 & \quad F + I_3 = \begin{bmatrix}
-12 & -8 & -4 \\
12 & 8 & 4 \\
24 & 16 & 8
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 2 & \frac{1}{3} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \\
\mathcal{E}_F(-1) & = \mathcal{N}(F + I_3) = \left\langle \begin{bmatrix}
-\frac{2}{3} \\
1 \\
0
\end{bmatrix} , \begin{bmatrix}
-\frac{1}{3} \\
0 \\
1
\end{bmatrix} \right\rangle = \left\langle \begin{bmatrix}
-2 \\
3 \\
0
\end{bmatrix} , \begin{bmatrix}
-1 \\
0 \\
3
\end{bmatrix} \right\rangle
\end{align*}
\]

Eigenspaces in hand, we can easily compute eigenvectors by forming nontrivial linear combinations of the basis vectors describing each eigenspace. In particular, notice that we can “pretty up” our basis vectors by using scalar multiples to clear out fractions.

Subsection ECEE

Examples of Computing Eigenvalues and Eigenvectors

No theorems in this section, just a selection of examples meant to illustrate the range of possibilities for the eigenvalues and eigenvectors of a matrix. These examples can all be done by hand, though the computation of the characteristic polynomial would be very time-consuming and error-prone. It can also be difficult to factor an arbitrary polynomial, though if we were to suggest that most
of our eigenvalues are going to be integers, then it can be easier to hunt for roots. These examples are meant to look similar to a concatenation of Example CPMS3, Example EMS3 and Example ESMS3. First, we will sneak in a pair of definitions so we can illustrate them throughout this sequence of examples.

**Definition AME**

**Algebraic Multiplicity of an Eigenvalue**

Suppose that $A$ is a square matrix and $\lambda$ is an eigenvalue of $A$. Then the algebraic multiplicity of $\lambda$, $\alpha_A(\lambda)$, is the highest power of $(x - \lambda)$ that divides the characteristic polynomial, $p_A(x)$.

Since an eigenvalue $\lambda$ is a root of the characteristic polynomial, there is always a factor of $(x - \lambda)$, and the algebraic multiplicity is just the power of this factor in a factorization of $p_A(x)$. So in particular, $\alpha_A(\lambda) \geq 1$. Compare the definition of algebraic multiplicity with the next definition.

**Definition GME**

**Geometric Multiplicity of an Eigenvalue**

Suppose that $A$ is a square matrix and $\lambda$ is an eigenvalue of $A$. Then the geometric multiplicity of $\lambda$, $\gamma_A(\lambda)$, is the dimension of the eigenspace $E_A(\lambda)$.

Since every eigenvalue must have at least one eigenvector, the associated eigenspace cannot be trivial, and so $\gamma_A(\lambda) \geq 1$.

**Example EMMS4**

**Eigenvalue multiplicities, matrix of size 4**

Consider the matrix

$$B = \begin{bmatrix} -2 & 1 & -2 & -4 \\ 12 & 1 & 4 & 9 \\ 6 & 5 & -2 & -4 \\ 3 & -4 & 5 & 10 \end{bmatrix}$$

then

$$p_B(x) = 8 - 20x + 18x^2 - 7x^3 + x^4 = (x - 1)(x - 2)^3$$

So the eigenvalues are $\lambda = 1, 2$ with algebraic multiplicities $\alpha_B(1) = 1$ and $\alpha_B(2) = 3$.

Computing eigenvectors,

$$\lambda = 1 \quad B - 1I_4 = \begin{bmatrix} -3 & 1 & -2 & -4 \\ 12 & 0 & 4 & 9 \\ 6 & 5 & -3 & -4 \\ 3 & -4 & 5 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_B(1) = N(B - 1I_4) = \begin{BMatrix} \begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bMatrix} \end{BMatrix} = \begin{BMatrix} \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bMatrix} \end{BMatrix}$$

$$\lambda = 2 \quad B - 2I_4 = \begin{bmatrix} -4 & 1 & -2 & -4 \\ 12 & -1 & 4 & 9 \\ 6 & 5 & -4 & -4 \\ 3 & -4 & 5 & 8 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_B(2) = N(B - 2I_4) = \begin{BMatrix} \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bMatrix} \end{BMatrix} = \begin{BMatrix} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bMatrix} \end{BMatrix}$$

So each eigenspace has dimension 1 and so $\gamma_B(1) = 1$ and $\gamma_B(2) = 1$. This example is of interest because of the discrepancy between the two multiplicities for $\lambda = 2$. In many of our examples the algebraic and geometric multiplicities will be equal for all of the eigenvalues (as it was for $\lambda = 1$ in...
this example), so keep this example in mind. We will have some explanations for this phenomenon later (see Example NDMS4 [397]).

Example ESMS4
Eigenvalues, symmetric matrix of size 4
Consider the matrix
\[
C = \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{bmatrix}
\]
then
\[
p_C(x) = -3 + 4x + 2x^2 - 4x^3 + x^4 = (x - 3)(x - 1)^2(x + 1)
\]
So the eigenvalues are \(\lambda = 3, 1, -1\) with algebraic multiplicities \(\alpha_C(3) = 1, \alpha_C(1) = 2\) and \(\alpha_C(-1) = 1\).

Computing eigenvectors,
\[
\lambda = 3 \quad C - 3I_4 = \begin{bmatrix}
-2 & 0 & 1 & 1 \\
0 & -2 & 1 & 1 \\
1 & 1 & -2 & 0 \\
1 & 1 & 0 & -2
\end{bmatrix} \text{ RREF } \begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
E_C(3) = N(C - 3I_4) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}
\]
\[
\lambda = 1 \quad C - I_4 = \begin{bmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix} \text{ RREF } \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
E_C(1) = N(C - I_4) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}
\]
\[
\lambda = -1 \quad C + I_4 = \begin{bmatrix}
2 & 0 & 1 & 1 \\
0 & 2 & 1 & 1 \\
1 & 1 & 2 & 0 \\
1 & 1 & 0 & 2
\end{bmatrix} \text{ RREF } \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
E_C(-1) = N(C + I_4) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}
\]
So the eigenspace dimensions yield geometric multiplicities \(\gamma_C(3) = 1, \gamma_C(1) = 2\) and \(\gamma_C(-1) = 1\), the same as for the algebraic multiplicities. This example is of interest because \(A\) is a symmetric matrix, and will be the subject of Theorem HMRE [386].

Example HMEM5
High multiplicity eigenvalues, matrix of size 5
Consider the matrix
\[
E = \begin{bmatrix}
29 & 14 & 2 & 6 & -9 \\
-47 & -22 & -1 & -11 & 13 \\
19 & 10 & 5 & 4 & -8 \\
-19 & -10 & -3 & -2 & 8 \\
7 & 4 & 3 & 1 & -3
\end{bmatrix}
\]
then
\[ p_E(x) = -16 + 16x + 8x^2 - 16x^3 + 7x^4 - x^5 = -(x - 2)^4(x + 1) \]

So the eigenvalues are \( \lambda = 2, -1 \) with algebraic multiplicities \( \alpha_E(2) = 4 \) and \( \alpha_E(-1) = 1 \).

Computing eigenvectors,
\[ \lambda = 2 \quad E - 2I_5 = \begin{bmatrix} 27 & 14 & 2 & 6 & -9 \\ -47 & -24 & -1 & -11 & 13 \\ 19 & 10 & 3 & 4 & -8 \\ -19 & -10 & -3 & -4 & 8 \\ 7 & 4 & 3 & 1 & -5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\( E_E(2) = \mathcal{N}(E - 2I_5) = \langle \begin{bmatrix} -1 \\ \frac{3}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \rangle \)

\[ \lambda = -1 \quad E + 1I_5 = \begin{bmatrix} 30 & 14 & 2 & 6 & -9 \\ -47 & -21 & -1 & -11 & 13 \\ 19 & 10 & 6 & 4 & -8 \\ -19 & -10 & -3 & -1 & 8 \\ 7 & 4 & 3 & 1 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\( E_E(-1) = \mathcal{N}(E + 1I_5) = \langle \begin{bmatrix} -2 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rangle \)

So the eigenspace dimensions yield geometric multiplicities \( \gamma_E(2) = 2 \) and \( \gamma_E(-1) = 1 \). This example is of interest because \( \lambda = 2 \) has such a large algebraic multiplicity, which is also not equal to its geometric multiplicity.

Example CEMS6
Complex eigenvalues, matrix of size 6
Consider the matrix

then
\[ p_F(x) = -50 + 55x + 13x^2 - 50x^3 + 32x^4 - 9x^5 + x^6 = (x - 2)(x + 1)(x^2 - 4x + 5)^2 (x - (2 + i))(x - (2 - i))^2 \]

So the eigenvalues are \( \lambda = 2, -1, 2 + i, 2 - i \) with algebraic multiplicities \( \alpha_F(2) = 1, \alpha_F(-1) = 1, \alpha_F(2 + i) = 2 \) and \( \alpha_F(2 - i) = 2 \).

Computing eigenvectors,
\[ \lambda = 2 \]

$\mathcal{E}_F(2) = \mathcal{N}(F - 2I_6) = \left\{ \begin{bmatrix} -\frac{1}{5} \\ 0 \\ -\frac{3}{5} \\ -\frac{4}{5} \\ 1 \\ -1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ -4 \\ 5 \end{bmatrix} \right\}$

$\lambda = -1$


$\mathcal{E}_F(-1) = \mathcal{N}(F + I_6) = \left\{ \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

$\lambda = 2 + i$


$\mathcal{E}_F(2 + i) = \mathcal{N}(F - (2 + i)I_6) = \left\{ \begin{bmatrix} \frac{1}{5}(7 + i) \\ \frac{1}{5}(9 + 2i) \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -7 - i \\ 9 + 2i \\ -5 \\ 5 \\ -5 \\ 5 \end{bmatrix} \right\}$
Subsection EE.ECEE

Examples of Computing Eigenvalues and Eigenvectors

\[ \lambda = 2 - i \]

\[
F - (2 - i)I_6 = \begin{bmatrix}
-61 + i & -34 & 41 & 12 & 25 & 30 \\
1 & 5 + i & -46 & -36 & -11 & -29 \\
-233 & -119 & 56 + i & -35 & 75 & 54 \\
157 & 81 & -43 & 19 + i & -51 & -39 \\
-91 & -48 & 32 & -5 & 30 + i & 26 \\
209 & 107 & -55 & 28 & -69 & -52 + i
\end{bmatrix}
\]

\[ RREF \rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \frac{1}{5}(7 - i) \\
0 & 1 & 0 & 0 & 0 & \frac{1}{5}(-9 + 2i) \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[ E_F(2 - i) = N(F - (2 - i)I_6) = \begin{bmatrix}
\frac{1}{5}(7 + i) \\
\frac{1}{5}(9 - 2i) \\
-1 \\
1 \\
-1
\end{bmatrix} = \begin{bmatrix}
-7 + i \\
9 - 2i \\
-5 \\
-5
\end{bmatrix}
\]

So the eigenspace dimensions yield geometric multiplicities \( \gamma_F(2) = 1, \gamma_F(-1) = 1, \gamma_F(2 + i) = 1 \) and \( \gamma_F(2 - i) = 1 \). This example demonstrates some of the possibilities for the appearance of complex eigenvalues, even when all the entries of the matrix are real. Notice how all the numbers in the analysis of \( \lambda = 2 - i \) are conjugates of the corresponding number in the analysis of \( \lambda = 2 + i \). This is the content of the upcoming Theorem ERMCP 382.

Example DEMS5

Distinct eigenvalues, matrix of size 5

Consider the matrix

\[
H = \begin{bmatrix}
15 & 18 & -8 & 6 & -5 \\
5 & 3 & 1 & -1 & -3 \\
0 & -4 & 5 & -4 & -2 \\
-43 & -46 & 17 & -14 & 15 \\
26 & 30 & -12 & 8 & -10
\end{bmatrix}
\]

then

\[ p_H(x) = -6x + x^2 + 7x^3 - x^4 - x^5 = x(x - 2)(x - 1)(x + 1)(x + 3) \]

So the eigenvalues are \( \lambda = 2, 1, 0, -1, -3 \) with algebraic multiplicities \( \alpha_H(2) = 1, \alpha_H(1) = 1, \alpha_H(0) = 1, \alpha_H(-1) = 1 \) and \( \alpha_H(-3) = 1 \).

Computing eigenvectors,

\[
\lambda = 2 \quad H - 2I_5 = \begin{bmatrix}
13 & 18 & -8 & 6 & -5 \\
5 & 1 & 1 & -1 & -3 \\
0 & -4 & 3 & -4 & -2 \\
-43 & -46 & 17 & -16 & 15 \\
26 & 30 & -12 & 8 & -12
\end{bmatrix} \rightarrow RREF \rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[ E_H(2) = N(H - 2I_5) = \begin{bmatrix}
1 \\
-1 \\
-2 \\
-1 \\
1
\end{bmatrix}
\]
Subsection EE.ECEE  Examples of Computing Eigenvalues and Eigenvectors  371

\[
\lambda = 1 \quad H - 1I_5 = \begin{bmatrix}
14 & 18 & -8 & 6 & -5 \\
5 & 2 & 1 & -1 & -3 \\
0 & -4 & 4 & -4 & -2 \\
-43 & -46 & 17 & -15 & 15 \\
26 & 30 & -12 & 8 & -11
\end{bmatrix} \quad \text{RREF} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 & -1/2 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\mathcal{E}_H (1) = \mathcal{N}(H - 1I_5) = \left\{ \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ -1 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \\ -2 \end{bmatrix} \right\}
\]

\[
\lambda = 0 \quad H - 0I_5 = \begin{bmatrix}
15 & 18 & -8 & 6 & -5 \\
5 & 3 & 1 & -1 & -3 \\
0 & -4 & 5 & -4 & -2 \\
-43 & -46 & 17 & -14 & 15 \\
26 & 30 & -12 & 8 & -10
\end{bmatrix} \quad \text{RREF} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & -2 \\
0 & 0 & 1 & 0 & -2 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\mathcal{E}_H (0) = \mathcal{N}(H - 0I_5) = \left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

\[
\lambda = -1 \quad H + 1I_5 = \begin{bmatrix}
16 & 18 & -8 & 6 & -5 \\
5 & 4 & 1 & -1 & -3 \\
0 & -4 & 6 & -4 & -2 \\
-43 & -46 & 17 & -13 & 15 \\
26 & 30 & -12 & 8 & -9
\end{bmatrix} \quad \text{RREF} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 & -1/2 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1/2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\mathcal{E}_H (-1) = \mathcal{N}(H + 1I_5) = \left\{ \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ -1 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 2 \end{bmatrix} \right\}
\]

\[
\lambda = -3 \quad H + 3I_5 = \begin{bmatrix}
18 & 18 & -8 & 6 & -5 \\
5 & 6 & 1 & -1 & -3 \\
0 & -4 & 8 & -4 & -2 \\
-43 & -46 & 17 & -11 & 15 \\
26 & 30 & -12 & 8 & -7
\end{bmatrix} \quad \text{RREF} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 1/2 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\mathcal{E}_H (-3) = \mathcal{N}(H + 3I_5) = \left\{ \begin{bmatrix} 1 \\ -1/2 \\ -1 \\ -2 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 2 \\ 4 \\ -2 \end{bmatrix} \right\}
\]

So the eigenspace dimensions yield geometric multiplicities \(\gamma_H (2) = 1\), \(\gamma_H (1) = 1\), \(\gamma_H (0) = 1\), \(\gamma_H (-1) = 1\) and \(\gamma_H (-3) = 1\), identical to the algebraic multiplicities. This example is of interest for two reasons. First, \(\lambda = 0\) is an eigenvalue, illustrating the upcoming Theorem SMZE [379]. Second, all the eigenvalues are distinct, yielding algebraic and geometric multiplicities of 1 for each eigenvalue, illustrating Theorem DED [398].
Subsection READ
Reading Questions

Suppose $A$ is the $2 \times 2$ matrix

\[ A = \begin{bmatrix} -5 & 8 \\ -4 & 7 \end{bmatrix} \]

1. Find the eigenvalues of $A$.
2. Find the eigenspaces of $A$.
3. For the polynomial $p(x) = 3x^2 - x + 2$, compute $p(A)$. 
Subsection EXC
Exercises

C19  Find the eigenvalues, eigenspaces, algebraic multiplicities and geometric multiplicities for the matrix below. It is possible to do all these computations by hand, and it would be instructive to do so.

\[ C = \begin{bmatrix} -1 & 2 \\ -6 & 6 \end{bmatrix} \]

Contributed by Robert Beezer  Solution 374

C20  Find the eigenvalues, eigenspaces, algebraic multiplicities and geometric multiplicities for the matrix below. It is possible to do all these computations by hand, and it would be instructive to do so.

\[ B = \begin{bmatrix} -12 & 30 \\ -5 & 13 \end{bmatrix} \]

Contributed by Robert Beezer  Solution 374

C21  The matrix \( A \) below has \( \lambda = 2 \) as an eigenvalue. Find the geometric multiplicity of \( \lambda = 2 \) using your calculator only for row-reducing matrices.

\[ A = \begin{bmatrix} 18 & -15 & 33 & -15 \\ -4 & 8 & -6 & 6 \\ -9 & 9 & -16 & 9 \\ 5 & -6 & 9 & -4 \end{bmatrix} \]

Contributed by Robert Beezer  Solution 375

C22  Without using a calculator, find the eigenvalues of the matrix \( B \).

\[ B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \]

Contributed by Robert Beezer  Solution 375

M60  Repeat Example CAEHW 361 by choosing \( \mathbf{x} = \begin{bmatrix} 0 \\ 8 \\ 2 \\ 1 \\ 2 \end{bmatrix} \) and then arrive at an eigenvalue and eigenvector of the matrix \( A \). The hard way.

Contributed by Robert Beezer  Solution 375

T10  A matrix \( A \) is idempotent if \( A^2 = A \). Show that the only possible eigenvalues of an idempotent matrix are \( \lambda = 0 \) and \( \lambda = 1 \). Then give an example of a matrix that is idempotent and has both of these two values as eigenvalues.

Contributed by Robert Beezer  Solution 376

T20  Suppose that \( \lambda \) and \( \rho \) are two different eigenvalues of the square matrix \( A \). Prove that the intersection of the eigenspaces for these two eigenvalues is trivial. That is, \( \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho) = \{0\} \).

Contributed by Robert Beezer  Solution 376
Subsection SOL Solutions

C19 Contributed by Robert Beezer Statement

First compute the characteristic polynomial,

\[ p_C(x) = \det(C - xI_2) = \begin{vmatrix} -1 - x & 2 \\ -6 & 6 - x \end{vmatrix} \]

\[ = (-1 - x)(6 - x) - (2)(-6) = x^2 - 5x + 6 = (x - 3)(x - 2) \]

So the eigenvalues of \( C \) are the solutions to \( p_C(x) = 0 \), namely, \( \lambda = 2 \) and \( \lambda = 3 \).

To obtain the eigenspaces, construct the appropriate singular matrices and find expressions for the null spaces of these matrices.

\[ \lambda = 2 \]

\[ C - (2)I_2 = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix} \overset{\text{RREF}}{\rightarrow} \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & 0 \end{bmatrix} \]

\[ E_C(2) = \mathcal{N}(C - (2)I_2) = \left\{ \begin{bmatrix} \frac{7}{3} \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} \]

\[ \lambda = 3 \]

\[ C - (3)I_2 = \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix} \overset{\text{RREF}}{\rightarrow} \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix} \]

\[ E_C(3) = \mathcal{N}(C - (3)I_2) = \left\{ \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \]

C20 Contributed by Robert Beezer Statement

The characteristic polynomial of \( B \) is

\[ p_B(x) = \det(B - xI_2) = \begin{vmatrix} -12 - x & 30 \\ -5 & 13 - x \end{vmatrix} \]

\[ = (-12 - x)(13 - x) - (30)(-5) = x^2 - x - 6 = (x - 3)(x + 2) \]

From this we find eigenvalues \( \lambda = 3, -2 \) with algebraic multiplicities \( \alpha_B(3) = 1 \) and \( \alpha_B(-2) = 1 \).

For eigenvectors and geometric multiplicities, we study the null spaces of \( B - \lambda I_2 \) (Theorem EMNS).

\[ \lambda = 3 \]

\[ B - 3I_2 = \begin{bmatrix} -15 & 30 \\ -5 & 10 \end{bmatrix} \overset{\text{RREF}}{\rightarrow} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \]

\[ E_B(3) = \mathcal{N}(B - 3I_2) = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \]

\[ \lambda = -2 \]

\[ B + 2I_2 = \begin{bmatrix} -10 & 30 \\ -5 & 15 \end{bmatrix} \overset{\text{RREF}}{\rightarrow} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \]
Each eigenspace has dimension one, so we have geometric multiplicities \( \gamma_B(3) = 1 \) and \( \gamma_B(-2) = 1 \).

If \( \lambda = 2 \) is an eigenvalue of \( A \), the matrix \( A - 2I_4 \) will be singular, and its null space will be the eigenspace of \( A \). So we form this matrix and row-reduce,

\[
A - 2I_4 = \begin{bmatrix}
16 & -15 & 33 & -15 \\
-4 & 6 & -6 & 6 \\
-9 & 9 & -18 & 9 \\
5 & -6 & 9 & -6
\end{bmatrix}
\]

With two free variables, we know a basis of the null space (Theorem BNS [128]) will contain two vectors. Thus the null space of \( A - 2I_4 \) has dimension two, and so the eigenspace of \( \lambda = 2 \) has dimension two also (Theorem EMNS [364]), \( \gamma_A(2) = 2 \).

The characteristic polynomial (Definition CP [363]) is

\[
p_B(x) = \det(B - xI_2) = (2 - x)(1 - x) - (1)(-1) = x^2 - 3x + 3
\]

where the factorization can be obtained by finding the roots of \( p_B(x) = 0 \) with the quadratic equation. By Theorem EMRCP [363] the eigenvalues of \( B \) are the complex numbers \( \lambda_1 = \frac{3 + 3i}{2} \) and \( \lambda_2 = \frac{3 - 3i}{2} \).

Form the matrix \( C \) whose columns are \( x, Ax, A^2x, A^3x, A^4x, A^5x \) and row-reduce the matrix,

\[
\begin{bmatrix}
0 & 6 & 32 & 102 & 320 & 966 \\
8 & 10 & 24 & 58 & 168 & 490 \\
2 & 12 & 50 & 156 & 482 & 1452 \\
1 & -5 & -47 & -149 & -479 & -1445 \\
2 & 12 & 50 & 156 & 482 & 1452
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & -3 & -9 & -30 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 3 & 10 & 30 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The simplest possible relation of linear dependence on the columns of \( C \) comes from using scalars \( \alpha_4 = 1 \) and \( \alpha_5 = \alpha_6 = 0 \) for the free variables in a solution to \( LS(C, 0) \). The remainder of this solution is \( \alpha_1 = 3, \alpha_2 = -1, \alpha_3 = -3 \). This solution gives rise to the polynomial

\[
p(x) = 3 - x - 3x^2 + x^3 = (x - 3)(x - 1)(x + 1)
\]

which then has the property that \( p(A)x = 0 \).

No matter how you choose to order the factors of \( p(x) \), the value of \( k \) (in the language of Theorem EMHE [359] and Example CAEHW [361]) is \( k = 2 \). For each of the three possibilities, we list the resulting eigenvector and the associated eigenvalue:

\[
(C - 3I_5)(C - I_5)z = \begin{bmatrix}
8 \\
8 \\
8 \\
-24 \\
8
\end{bmatrix}
\]

\( \lambda = -1 \)
\[(C - 3I_5)(C + I_5)\mathbf{z} = \begin{bmatrix} 20 \\ -20 \\ 20 \\ -40 \\ 20 \end{bmatrix} \quad \lambda = 1\]

\[(C + I_5)(C - I_5)\mathbf{z} = \begin{bmatrix} 32 \\ 16 \\ 48 \\ -48 \\ 48 \end{bmatrix} \quad \lambda = 3\]

Note that each of these eigenvectors can be simplified by an appropriate scalar multiple, but we have shown here the actual vector obtained by the product specified in the theorem.

**T10** Contributed by Robert Beezer Statement [373]

Suppose that \(\lambda\) is an eigenvalue of \(A\). Then there is an eigenvector \(\mathbf{x}\), such that \(A\mathbf{x} = \lambda\mathbf{x}\). We have,

\[
\lambda\mathbf{x} = A\mathbf{x} \\
= A^2\mathbf{x} \\
= A(A\mathbf{x}) \\
= A(\lambda\mathbf{x}) \\
= \lambda(A\mathbf{x}) \\
= \lambda(\lambda\mathbf{x}) \\
= \lambda^2\mathbf{x}
\]

From this we get

\[
0 = \lambda^2\mathbf{x} - \lambda\mathbf{x} \\
= (\lambda^2 - \lambda)\mathbf{x}
\]

Since \(\mathbf{x}\) is an eigenvector, it is nonzero, and Theorem SMEZY [259] leaves us with the conclusion that \(\lambda^2 - \lambda = 0\), and the solutions to this quadratic polynomial equation in \(\lambda\) are \(\lambda = 0\) and \(\lambda = 1\).

The matrix

\[
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]

is idempotent (check this!) and since it is a diagonal matrix, its eigenvalues are the diagonal entries, \(\lambda = 0\) and \(\lambda = 1\), so each of these possible values for an eigenvalue of an idempotent matrix actually occurs as an eigenvalue of some idempotent matrix.

**T20** Contributed by Robert Beezer Statement [373]

This problem asks you to prove that two sets are equal, so use Definition SE [616].

First show that \(\{0\} \subseteq \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho)\). Choose \(\mathbf{x} \in \{0\}\). Then \(\mathbf{x} = 0\). Eigenspaces are subspaces (Theorem EMS [364]), so both \(\mathcal{E}_A(\lambda)\) and \(\mathcal{E}_A(\rho)\) contain the zero vector, and therefore \(\mathbf{x} \in \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho)\) (Definition SI [617]).

To show that \(\mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho) \subseteq \{0\}\), suppose that \(\mathbf{x} \in \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho)\). Then \(\mathbf{x}\) is an eigenvector of \(A\) for both \(\lambda\) and \(\rho\) (Definition SI [617]) and so

\[
\mathbf{x} = 1\mathbf{x} \\
= \frac{1}{\lambda - \rho} (\lambda - \rho)\mathbf{x} \quad \lambda \neq \rho, \; \lambda - \rho \neq 0 \\
= \frac{1}{\lambda - \rho} (\lambda\mathbf{x} - \rho\mathbf{x}) \quad \text{Property DSAC [75]}
\]
\[
\frac{1}{\lambda - \rho} (Ax - Ax) = \frac{1}{\lambda - \rho} (0) = 0
\]

\(x\) eigenvector of \(A\) for \(\lambda, \rho\)

So \(x = 0\), and trivially, \(x \in \{0\}\).
Section PEE
Properties of Eigenvalues and Eigenvectors

The previous section introduced eigenvalues and eigenvectors, and concentrated on their existence and determination. This section will be more about theorems, and the various properties eigenvalues and eigenvectors enjoy. Like a good 4 × 100 meter relay, we will lead-off with one of our better theorems and save the very best for the anchor leg.

**Theorem EDELI**

Eigenvalues with Distinct Eigenvalues are Linearly Independent

Suppose that $A$ is an $n \times n$ square matrix and $S = \{x_1, x_2, x_3, \ldots, x_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then $S$ is a linearly independent set.

**Proof**

If $p = 1$, then the set $S = \{x_1\}$ is linearly independent since eigenvectors are nonzero (Definition EEM [356]), so assume for the remainder that $p \geq 2$.

We will prove this result by contradiction (Technique CD [623]). Suppose to the contrary that $S$ is a linearly dependent set. Define $k$ to be an integer such that $\{x_1, x_2, x_3, \ldots, x_k\}$ is linearly independent and $\{x_1, x_2, x_3, \ldots, x_k\}$ is linearly dependent. We have to ask if there is even such an integer? Since eigenvectors are nonzero, the set $\{x_i\}$ is linearly independent. Think of adding in vectors to this set, one at a time, $x_2, x_3, x_4, \ldots$ Since we are assuming that $S$ is linearly dependent, eventually this set will convert from being linearly independent to being linearly dependent. In other words, it is the addition of the vector $x_k$ that converts the set from linear independence to linear dependence. So there is such a $k$, and furthermore $2 \leq k \leq p$.

Since $\{x_1, x_2, x_3, \ldots, x_k\}$ is linearly dependent there are scalars, $a_1, a_2, a_3, \ldots, a_k$, some nonzero, so that

$$0 = a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_kx_k$$

Then,

$$0 = (A - \lambda_k I_n) 0 = (A - \lambda_k I_n) (a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_kx_k)$$

$$= (A - \lambda_k I_n) a_1x_1 + (A - \lambda_k I_n) a_2x_2 + \cdots + (A - \lambda_k I_n) a_kx_k$$

$$= a_1(A - \lambda_k I_n)x_1 + a_2(A - \lambda_k I_n)x_2 + \cdots + a_k(A - \lambda_k I_n)x_k$$

$$= a_1(Ax_1 - \lambda_k I_n x_1) + a_2(Ax_2 - \lambda_k I_n x_2) + \cdots + a_k(Ax_k - \lambda_k I_n x_k)$$

$$= a_1(Ax_1 - \lambda_k x_1) + a_2(Ax_2 - \lambda_k x_2) + \cdots + a_k(Ax_k - \lambda_k x_k)$$

$$= a_1(\lambda_1 x_1 - \lambda_k x_1) + a_2(\lambda_2 x_2 - \lambda_k x_2) + \cdots + a_k(\lambda_k x_k - \lambda_k x_k)$$

$$= a_1(\lambda_1 - \lambda_k) x_1 + a_2(\lambda_2 - \lambda_k) x_2 + \cdots + a_k(\lambda_k - \lambda_k) x_k$$

$$= a_1(\lambda_1 - \lambda_k) + a_2(\lambda_2 - \lambda_k) x_2 + \cdots + a_k(\lambda_k - \lambda_k) x_k$$

$$= a_1(\lambda_1 - \lambda_k) x_1 + a_2(\lambda_2 - \lambda_k) x_2 + \cdots + a_k(\lambda_k - \lambda_k) x_k$$

$$= a_1(\lambda_1 - \lambda_k) x_1 + a_2(\lambda_2 - \lambda_k) x_2 + \cdots + a_k(\lambda_k - \lambda_k) x_k$$

$$= a_1(\lambda_1 - \lambda_k) x_1 + a_2(\lambda_2 - \lambda_k) x_2 + \cdots + a_k(\lambda_k - \lambda_k) x_k$$

This is a relation of linear dependence on the linearly independent set $\{x_1, x_2, x_3, \ldots, x_{k-1}\}$, so the scalars must all be zero. That is, $a_i(\lambda_i - \lambda_k) = 0$ for $1 \leq i \leq k - 1$. However, we have the hypothesis that the eigenvalues are distinct, so $\lambda_i \neq \lambda_k$ for $1 \leq i \leq k - 1$. Thus $a_i = 0$ for $1 \leq i \leq k - 1$.

This reduces the original relation of linear dependence on $\{x_1, x_2, x_3, \ldots, x_k\}$ to the simpler equation $a_kx_k = 0$. By Theorem SMEZV [259], we conclude that $a_k = 0$ or $x_k = 0$. Eigenvectors are never the zero vector (Definition EEM [356]), so $a_k = 0$. So all of the scalars $a_i$, $1 \leq i \leq k$ are zero, contradicting their introduction as the scalars creating a nontrivial relation of linear dependence on
the set \( \{x_1, x_2, x_3, \ldots, x_k\} \). With a contradiction in hand, we conclude that \( S \) must be linearly independent. 

There is a simple connection between the eigenvalues of a matrix and whether or not the matrix is nonsingular.

**Theorem SMZE**  
**Singular Matrices have Zero Eigenvalues**  
Suppose \( A \) is a square matrix. Then \( A \) is singular if and only if \( \lambda = 0 \) is an eigenvalue of \( A \). 

**Proof** We have the following equivalences:

\[
\begin{align*}
A \text{ is singular} & \iff \text{there exists } x \neq 0, A x = 0 & \text{Definition NSM} \ 54 \\
& \iff \text{there exists } x \neq 0, A x = 0 & \text{Theorem ZSSM} \ 258 \\
& \iff \lambda = 0 \text{ is an eigenvalue of } A & \text{Definition EEM} \ 356 
\end{align*}
\]

With an equivalence about singular matrices we can update our list of equivalences about nonsingular matrices.

**Theorem NME8**  
**Nonsingular Matrix Equivalences, Round 8**  
Suppose that \( A \) is a square matrix of size \( n \). The following are equivalent.

1. \( A \) is nonsingular.
2. \( A \) row-reduces to the identity matrix.
3. The null space of \( A \) contains only the zero vector, \( \mathcal{N}(A) = \{0\} \).
4. The linear system \( \mathcal{L}(A, b) \) has a unique solution for every possible choice of \( b \).
5. The columns of \( A \) are a linearly independent set.
6. \( A \) is invertible.
7. The column space of \( A \) is \( \mathbb{C}^n \), \( \mathcal{C}(A) = \mathbb{C}^n \).
8. The columns of \( A \) are a basis for \( \mathbb{C}^n \).
9. The rank of \( A \) is \( n \), \( r(A) = n \).
10. The nullity of \( A \) is zero, \( n(A) = 0 \).
11. The determinant of \( A \) is nonzero, \( \det(A) \neq 0 \).
12. \( \lambda = 0 \) is not an eigenvalue of \( A \).

**Proof** The equivalence of the first and last statements is the contrapositive of Theorem SMZE \ 379, so we are able to improve on Theorem NME7 \ 352. 

Certain changes to a matrix change its eigenvalues in a predictable way.

**Theorem ESMM**  
**Eigenvalues of a Scalar Multiple of a Matrix**  
Suppose \( A \) is a square matrix and \( \lambda \) is an eigenvalue of \( A \). Then \( \alpha \lambda \) is an eigenvalue of \( \alpha A \). 

**Proof** Let \( x \neq 0 \) be one eigenvector of \( A \) for \( \lambda \). Then

\[
(\alpha A) x = \alpha (A x) = \alpha (\lambda x) = \alpha (\lambda x) \quad \text{x eigenvector of } A
\]

\[
\text{Theorem MMSMM} \ 180
\]

\[
\text{Version 1.04}
\]
\[ (\alpha \lambda) x \quad \text{Property SMAC} \]

So \( x \neq 0 \) is an eigenvector of \( \alpha A \) for the eigenvalue \( \alpha \lambda \).

Unfortunately, there are not parallel theorems about the sum or product of arbitrary matrices. But we can prove a similar result for powers of a matrix.

**Theorem EOMP**

**Eigenvalues Of Matrix Powers**

Suppose \( A \) is a square matrix, \( \lambda \) is an eigenvalue of \( A \), and \( s \geq 0 \) is an integer. Then \( \lambda^s \) is an eigenvalue of \( A^s \).

**Proof** Let \( x \neq 0 \) be one eigenvector of \( A \) for \( \lambda \). Suppose \( A \) has size \( n \). Then we proceed by induction on \( s \) (Technique I [626]). First, for \( s = 0 \),

\[
A^0 x = A^0 x = I_n x = x = \lambda^0 x = \lambda x
\]

so \( \lambda^s \) is an eigenvalue of \( A^s \) in this special case. If we assume the theorem is true for \( s \), then we find

\[
A^{s+1} x = A^s Ax = A^s (\lambda x) = \lambda (A^s x) = \lambda (\lambda^s x) = (\lambda \lambda^s) x = \lambda^{s+1} x
\]

so \( x \neq 0 \) is an eigenvector of \( A^{s+1} \) for \( \lambda^{s+1} \), and induction tells us the theorem is true for all \( s \geq 0 \).

While we cannot prove that the sum of two arbitrary matrices behaves in any reasonable way with regard to eigenvalues, we can work with the sum of dissimilar powers of the same matrix. We have already seen two connections between eigenvalues and polynomials, in the proof of **Theorem EMHE** [359] and the characteristic polynomial (Definition CP [363]). Our next theorem strengthens this connection.

**Theorem EPM**

**Eigenvalues of the Polynomial of a Matrix**

Suppose \( A \) is a square matrix and \( \lambda \) is an eigenvalue of \( A \). Let \( q(x) \) be a polynomial in the variable \( x \). Then \( q(\lambda) \) is an eigenvalue of the matrix \( q(A) \).

**Proof** Let \( x \neq 0 \) be one eigenvector of \( A \) for \( \lambda \), and write \( q(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m \). Then

\[
q(A)x = (a_0 A^0 + a_1 A^1 + a_2 A^2 + \cdots + a_m A^m) x
\]

\[
= (a_0 A^0 x + (a_1 A^1) x + (a_2 A^2) x + \cdots + (a_m A^m) x) \quad \text{Theorem MMDAA} [179]
\]

\[
= a_0 (A^0 x) + a_1 (A^1 x) + a_2 (A^2 x) + \cdots + a_m (A^m x) \quad \text{Theorem MMSMM} [180]
\]

\[
= a_0 (\lambda^0 x) + a_1 (\lambda^1 x) + a_2 (\lambda^2 x) + \cdots + a_m (\lambda^m x) \quad \text{Theorem EOMP} [380]
\]

\[
= (a_0 \lambda^0) x + (a_1 \lambda^1) x + (a_2 \lambda^2) x + \cdots + (a_m \lambda^m) x \quad \text{Property SMAC} [75]
\]
\[(a_0\lambda^0 + a_1\lambda^1 + a_2\lambda^2 + \cdots + a_m\lambda^m)x \quad \text{Property DSAC 75}\]

So \(x \neq 0\) is an eigenvector of \(q(A)\) for the eigenvalue \(q(\lambda)\).

Example BDE

Building desired eigenvalues

In Example ESMS4 367 the \(4 \times 4\) symmetric matrix

\[
C = \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
\end{bmatrix}
\]

is shown to have the three eigenvalues \(\lambda = 3, 1, -1\). Suppose we wanted a \(4 \times 4\) matrix that has the three eigenvalues \(\lambda = 4, 0, -2\). We can employ Theorem EPM 380 by finding a polynomial that converts 3 to 4, 1 to 0, and \(-1\) to \(-2\). Such a polynomial is called an interpolating polynomial, and in this example we can use

\[
r(x) = \frac{1}{4}x^2 + x - \frac{5}{4}
\]

We will not discuss how to concoct this polynomial, but a text on numerical analysis should provide the details. In our case, simply verify that \(r(3) = 4\), \(r(1) = 0\) and \(r(-1) = -2\).

Now compute

\[
r(C) = \frac{1}{4}C^2 + C - \frac{5}{4}I_4
\]

\[
= \begin{bmatrix}
3 & 2 & 2 & 2 \\
2 & 3 & 2 & 2 \\
2 & 2 & 3 & 2 \\
2 & 2 & 2 & 3 \\
\end{bmatrix}
+ \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
\end{bmatrix}
- \frac{5}{4} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & 3 & 3 \\
1 & 1 & 3 & 3 \\
3 & 3 & 1 & 1 \\
3 & 3 & 1 & 1 \\
\end{bmatrix}
\]

Theorem EPM 380 tells us that if \(r(x)\) transforms the eigenvalues in the desired manner, then \(r(C)\) will have the desired eigenvalues. You can check this by computing the eigenvalues of \(r(C)\) directly. Furthermore, notice that the multiplicities are the same, and the eigenspaces of \(C\) and \(r(C)\) are identical.

Inverses and transposes also behave predictably with regard to their eigenvalues.

Theorem EIM

Eigenvalues of the Inverse of a Matrix

Suppose \(A\) is a square nonsingular matrix and \(\lambda\) is an eigenvalue of \(A\). Then \(\frac{1}{\lambda}\) is an eigenvalue of the matrix \(A^{-1}\).

Proof Notice that since \(A\) is assumed nonsingular, \(A^{-1}\) exists by Theorem NI 204, but more importantly, \(\frac{1}{\lambda}\) does not involve division by zero since Theorem SMZE 379 prohibits this possibility.

Let \(x \neq 0\) be one eigenvector of \(A\) for \(\lambda\). Suppose \(A\) has size \(n\). Then

\[
A^{-1}x = A^{-1}(1x) \quad \text{Property OC 75}
\]

\[
= A^{-1}\left(\frac{1}{\lambda}\lambda x\right) \quad \text{Property MICN 613}
\]

\[
= \frac{1}{\lambda}A^{-1}(\lambda x) \quad \text{Theorem MMSMM 180}
\]

\[
= \frac{1}{\lambda}A^{-1}(Ax) \quad \text{Definition EEM 356}
\]
So \( \mathbf{x} \neq 0 \) is an eigenvector of \( A^{-1} \) for the eigenvalue \( \frac{1}{\lambda} \).

The theorems above have a similar style to them, a style you should consider using when confronted with a need to prove a theorem about eigenvalues and eigenvectors. So far we have been able to reserve the characteristic polynomial for strictly computational purposes. However, the next theorem, whose statement resembles the preceding theorems, has an easier proof if we employ the characteristic polynomial and results about determinants.

**Theorem ETM**

**Eigenvalues of the Transpose of a Matrix**

Suppose \( A \) is a square matrix and \( \lambda \) is an eigenvalue of \( A \). Then \( \lambda \) is an eigenvalue of the matrix \( A^t \).

**Proof** Let \( \mathbf{x} \neq 0 \) be one eigenvector of \( A \) for \( \lambda \). Suppose \( A \) has size \( n \). Then

\[
p_A(x) = \det(A - xI_n) = \det((A - xI_n)^t) = \det(A^t - (xI_n)^t) = \det(A^t - xI_n^n) = \det(A^t - xI_n).
\]

So \( A \) and \( A^t \) have the same characteristic polynomial, and by **Theorem EMRCP** [363], their eigenvalues are identical and have equal algebraic multiplicities. Notice that what we have proved here is a bit stronger than the stated conclusion in the theorem.

If a matrix has only real entries, then the computation of the characteristic polynomial (Definition CP [363]) will result in a polynomial with coefficients that are real numbers. Complex numbers could result as roots of this polynomial, but they are roots of quadratic factors with real coefficients, and as such, come in conjugate pairs. The next theorem proves this, and a bit more, without mentioning the characteristic polynomial.

**Theorem ERMCP**

**Eigenvalues of Real Matrices come in Conjugate Pairs**

Suppose \( A \) is a square matrix with real entries and \( \mathbf{x} \) is an eigenvector of \( A \) for the eigenvalue \( \lambda \). Then \( \bar{x} \) is an eigenvector of \( A \) for the eigenvalue \( \bar{\lambda} \).

**Proof**

\[
A\mathbf{x} = \bar{\lambda}\mathbf{x} \quad A \text{ has real entries}
\]

\[
= \bar{\lambda}\mathbf{x} \quad \text{Theorem MMCC [181]}
\]

\[
= \bar{\lambda}\mathbf{x} \quad \mathbf{x} \text{ eigenvector of } A
\]

\[
= \bar{\lambda}\mathbf{x} \quad \text{Theorem CRSM [151]}
\]

So \( \mathbf{x} \) is an eigenvector of \( A \) for the eigenvalue \( \bar{\lambda} \).

This phenomenon is amply illustrated in **Example CEMS6 [368]**, where the four complex eigenvalues come in two pairs, and the two basis vectors of the eigenspaces are complex conjugates of each other. **Theorem ERMCP [382]** can be a time-saver for computing eigenvalues and eigenvectors of real matrices with complex eigenvalues, since the conjugate eigenvalue and eigenspace can be inferred from the theorem rather than computed.
A polynomial of degree $n$ will have exactly $n$ roots. From this fact about polynomial equations we can say more about the algebraic multiplicities of eigenvalues.

**Theorem DCP**

**Degree of the Characteristic Polynomial**

Suppose that $A$ is a square matrix of size $n$. Then the characteristic polynomial of $A$, $p_A(x)$, has degree $n$.

**Proof** We will prove a more general result by induction. Then the theorem will be true as a special case. We will carefully state this result as a proposition indexed by $m$, $m \geq 1$.

$P(m)$: Suppose that $A$ is an $m \times m$ matrix whose entries are complex numbers or linear polynomials in the variable $x$ of the form $c - x$, where $c$ is a complex number. Suppose further that there are exactly $k$ entries that contain $x$ and that no row or column contains more than one such entry. Then, when $k = m$, $\det(A)$ is a polynomial in $x$ of degree $m$, with leading coefficient ±1, and when $k < m$, $\det(A)$ is a polynomial in $x$ of degree $k$ or less.

Base Case: Suppose $A$ is a $1 \times 1$ matrix. Then its determinant is equal to the lone entry $\det(A)$ (Definition DM [337]). When $k = m = 1$, the entry is of the form $c - x$, a polynomial in $x$ of degree $m = 1$ with leading coefficient $-1$. When $k < m$, then $k = 0$ and the entry is simply a complex number, a polynomial of degree $0 < k$. So $P(1)$ is true.

Induction Step: Assume $P(m)$ is true, and that $A$ is an $(m+1) \times (m+1)$ matrix with $k$ entries of the form $c - x$. There are two cases to consider.

Suppose $k = m + 1$. Then every row and every column will contain an entry of the form $c - x$. We consider the determinant of $A$ by an expansion about this first row. The term associated with entry $t$ of this row will be of the form

$$(c - x)(-1)^{1+t} \det(A(1|t))$$

The submatrix $A(1|t)$ is an $m \times m$ matrix with $k = m$ terms of the form $c - x$, no more than one per row or column. By the induction hypothesis, $\det(A(1|t))$ will be a polynomial in $x$ of degree $m$ with leading coefficient $\pm 1$. So this entire term is then a polynomial of degree $m + 1$ with leading coefficient $\pm 1$.

The remaining terms (which constitute the sum that is the determinant of $A$) are products of complex numbers from the first row with cofactors built from submatrices that lack the first row of $A$ and lack some column of $A$, other than column $t$. As such, these submatrices are $m \times m$ matrices with $k = m - 1 < m$ entries of the form $c - x$, no more than one per row or column. Applying the induction hypothesis, we see that these terms are polynomials in $x$ of degree $m - 1$ or less. Adding the single term from the entry in column $t$ with all these others, we see that $\det(A)$ is a polynomial in $x$ of degree $m + 1$ and leading coefficient $\pm 1$.

The second case occurs when $k < m + 1$. Now there is a row of $A$ that does not contain an entry of the form $c - x$. We consider the determinant of $A$ by expanding about this row. The cofactors employed are built from submatrices that are $m \times m$ matrices with either $k$ or $k - 1$ entries of the form $c - x$, no more than one per row or column. In either case, $k \leq m$, and we can apply the induction hypothesis to see that the determinants computed for the cofactors are all polynomials of degree $k$ or less. Summing these contributions to the determinant of $A$ yields a polynomial in $x$ of degree $k$ or less, as desired.

**Theorem NEM**

**Number of Eigenvalues of a Matrix**
Suppose that \( A \) is a square matrix of size \( n \) with distinct eigenvalues \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k \). Then
\[
\sum_{i=1}^{k} \alpha_A(\lambda_i) = n
\]

**Proof** By the definition of the algebraic multiplicity (Definition AME [366]), we can factor the characteristic polynomial as
\[
p_A(x) = c(x - \lambda_1)^{\alpha_A(\lambda_1)}(x - \lambda_2)^{\alpha_A(\lambda_2)}(x - \lambda_3)^{\alpha_A(\lambda_3)}\cdots(x - \lambda_k)^{\alpha_A(\lambda_k)}
\]
where \( c \) is a nonzero constant. (We could prove that \( c = (-1)^n \), but we do not need that specificity right now. See Exercise PEE.T30 [388].) The left-hand side is a polynomial of degree \( n \) by Theorem DCP [383] and the right-hand side is a polynomial of degree \( \sum_{i=1}^{k} \alpha_A(\lambda_i) \). So the equality of the polynomials’ degrees gives the equality \( \sum_{i=1}^{k} \alpha_A(\lambda_i) = n \).

---

**Theorem ME**

**Multiplicities of an Eigenvalue**

Suppose that \( A \) is a square matrix of size \( n \) and \( \lambda \) is an eigenvalue. Then
\[
1 \leq \gamma_A(\lambda) \leq \alpha_A(\lambda) \leq n
\]

**Proof** Since \( \lambda \) is an eigenvalue of \( A \), there is an eigenvector of \( A \) for \( \lambda \), \( \mathbf{x} \). Then \( \mathbf{x} \in \mathcal{E}_A(\lambda) \), so \( \gamma_A(\lambda) \geq 1 \), since we can extend \( \{\mathbf{x}\} \) into a basis of \( \mathcal{E}_A(\lambda) \) (Theorem ELIS [320]).

To show that \( \gamma_A(\lambda) \leq \alpha_A(\lambda) \) is the most involved portion of this proof. To this end, let \( g = \gamma_A(\lambda) \) and let \( \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_g \) be a basis for the eigenspace of \( \lambda \), \( \mathcal{E}_A(\lambda) \). Construct another \( n - g \) vectors, \( \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots, \mathbf{y}_{n-g} \), so that
\[
\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_g, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots, \mathbf{y}_{n-g}\}
\]
is a basis of \( \mathbb{C}^n \). This can be done by repeated applications of Theorem ELIS [320]. Finally, define a matrix \( S \) by
\[
S = [\mathbf{x}_1|\mathbf{x}_2|\mathbf{x}_3|\ldots|\mathbf{x}_g]|\mathbf{y}_1|\mathbf{y}_2|\mathbf{y}_3|\ldots|\mathbf{y}_{n-g}] = [\mathbf{x}_1|\mathbf{x}_2|\mathbf{x}_3|\ldots|\mathbf{x}_g]|R
\]
where \( R \) is an \( n \times (n-g) \) matrix whose columns are \( \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots, \mathbf{y}_{n-g} \). The columns of \( S \) are linearly independent by design, so \( S \) is nonsingular (Theorem NMLIC [126]) and therefore invertible (Theorem MI [204]). Then,
\[
[e_1|e_2|e_3|\ldots|e_n] = I_n
\]
\[
= S^{-1}S
\]
\[
= S^{-1}[\mathbf{x}_1|\mathbf{x}_2|\mathbf{x}_3|\ldots|\mathbf{x}_g]|R
\]
\[
= [S^{-1}\mathbf{x}_1|S^{-1}\mathbf{x}_2|S^{-1}\mathbf{x}_3|\ldots|S^{-1}\mathbf{x}_g]|S^{-1}R
\]
So
\[
S^{-1}\mathbf{x}_i = e_i \quad 1 \leq i \leq g
\]
Preparations in place, we compute the characteristic polynomial of \( A \),
\[
p_A(x) = \det (A - xI_n)
\]
\[
= 1 \det (A - xI_n)
\]
\[
= \det (I_n) \det (A - xI_n)
\]
\[
= \det (S^{-1}S) \det (A - xI_n)
\]
\[
= \det (S^{-1}) \det (S) \det (A - xI_n)
\]
\[
\text{Definition CF } 363
\]
\[
\text{Definition MI } 189
\]
\[
\text{Definition DM } 337
\]
\[
\text{Definition OCN } 612
\]
What can we learn then about the matrix $S$?

$S^{-1}AS = S^{-1}A[x_1|x_2|x_3|\ldots|x_g]|R$

$= S^{-1}[Ax_1|Ax_2|Ax_3|\ldots|Ax_g]|AR$

$= S^{-1}[\lambda x_1|\lambda x_2|\lambda x_3|\ldots|\lambda x_g]|AR$

$= [S^{-1}\lambda x_1|S^{-1}\lambda x_2|S^{-1}\lambda x_3|\ldots|S^{-1}\lambda x_g]|S^{-1}AR$

$= [\lambda S^{-1}x_1|\lambda S^{-1}x_2|\lambda S^{-1}x_3|\ldots|\lambda S^{-1}x_g]|S^{-1}AR$

$= [\lambda e_1|\lambda e_2|\lambda e_3|\ldots|\lambda e_g]|S^{-1}AR$

Now imagine computing the characteristic polynomial of $A$ by computing the characteristic polynomial of $S^{-1}AS$ using the form just obtained. The first $g$ columns of $S^{-1}AS$ are all zero, save for a $\lambda$ on the diagonal. So if we compute the determinant by expanding about the first column, successively, we will get successive factors of $(\lambda - x)$. More precisely, let $T$ be the square matrix of size $n - g$ that is formed from the last $n - g$ rows and last $n - g$ columns of $S^{-1}AR$. Then

$p_A(x) = p_{S^{-1}AS}(x) = (\lambda - x)^gp_T(x)\cdot$

This says that $(x - \lambda)$ is a factor of the characteristic polynomial at least $g$ times, so the algebraic multiplicity of $\lambda$ as an eigenvalue of $A$ is greater than or equal to $g$ (Definition AME 366). In other words,

$\gamma_A(\lambda) = g \leq \alpha_A(\lambda)$

as desired.

Theorem NEM 383 says that the sum of the algebraic multiplicities for all the eigenvalues of $A$ is equal to $n$. Since the algebraic multiplicity is a positive quantity, no single algebraic multiplicity can exceed $n$ without the sum of all of the algebraic multiplicities doing the same.

Theorem MNEM

Maximum Number of Eigenvalues of a Matrix

Suppose that $A$ is a square matrix of size $n$. Then $A$ cannot have more than $n$ distinct eigenvalues.

Proof Suppose that $A$ has $k$ distinct eigenvalues, $\lambda_1$, $\lambda_2$, $\lambda_3$, \ldots, $\lambda_k$. Then

$$k = \sum_{i=1}^{k} 1 \leq \sum_{i=1}^{k} \alpha_A(\lambda_i) \leq n$$

Theorem ME 384

Theorem NEM 383
Recall that a matrix is Hermitian (or self-adjoint) if \( A = (\overline{A})^t \) (Definition HM 208). In the case where \( A \) is a matrix whose entries are all real numbers, being Hermitian is identical to being symmetric (Definition SYM 166). Keep this in mind as you read the next two theorems. Their hypotheses could be changed to “suppose \( A \) is a real symmetric matrix.”

**Theorem HMRE**

**Hermitian Matrices have Real Eigenvalues**

Suppose that \( A \) is a Hermitian matrix and \( \lambda \) is an eigenvalue of \( A \). Then \( \lambda \in \mathbb{R} \).

\[ \lambda \langle x, x \rangle = \langle \lambda x, x \rangle \]

**Proof** Let \( x \neq 0 \) be one eigenvector of \( A \) for \( \lambda \). Then

\[
\begin{align*}
\lambda \langle x, x \rangle &= \langle \lambda x, x \rangle \\
&= \langle Ax, x \rangle \\
&= (Ax)^t x \\
&= x^t A^t x \\
&= x^t (\overline{A})^t x \\
&= x^t \overline{A} x \\
&= \langle x, Ax \rangle \\
&= \langle x, \lambda x \rangle \\
&= \overline{\lambda} \langle x, x \rangle
\end{align*}
\]

Since \( x \neq 0 \), Theorem PIP 156 says that \( \langle x, x \rangle \neq 0 \), so we can “cancel” \( \langle x, x \rangle \) from both sides of this equality. This leaves \( \lambda = \overline{\lambda} \), so \( \lambda \) has a complex part equal to zero, and therefore is a real number.

**Look back and compare Example ESMS4 367 and Example CEMS6 368.** In Example CEMS6 368, the matrix has only real entries, yet the characteristic polynomial has roots that are complex numbers, and so the matrix has complex eigenvalues. However, in Example ESMS4 367, the matrix has only real entries, but is also symmetric. So by Theorem HMRE 386, we were guaranteed eigenvalues that are real numbers.

In many physical problems, a matrix of interest will be real and symmetric, or Hermitian. Then if the eigenvalues are to represent physical quantities of interest, Theorem HMRE 386 guarantees that these values will not be complex numbers.

The eigenvectors of a Hermitian matrix also enjoy a pleasing property that we will exploit later.

**Theorem HMOE**

**Hermitian Matrices have Orthogonal Eigenvectors**

Suppose that \( A \) is a Hermitian matrix and \( x \) and \( y \) are two eigenvectors of \( A \) for different eigenvalues. Then \( x \) and \( y \) are orthogonal vectors.

**Proof** Let \( x \neq 0 \) be an eigenvector of \( A \) for \( \lambda \) and let \( y \neq 0 \) be an eigenvector of \( A \) for \( \rho \). By Theorem HMRE 386, we know that \( \rho \) must be a real number. Then

\[
\begin{align*}
\lambda \langle x, y \rangle &= \langle \lambda x, y \rangle \\
&= \langle Ax, y \rangle \\
&= (Ax)^t y \\
&= x^t A^t y \\
&= x^t (\overline{A})^t y \\
&= \overline{\lambda} \langle x, y \rangle
\end{align*}
\]

Version 1.04
= x^t A y  \quad \text{Theorem TT [167]}
= x^t A y  \quad \text{Theorem MMCC [181]}
= \langle x, A y \rangle  \quad \text{Theorem MMIP [181]}
= \langle x, \rho y \rangle  \quad y \text{ eigenvector of } A
= \rho \langle x, y \rangle  \quad \text{Theorem IPSM [153]}
= \rho \langle x, y \rangle  \quad \text{Theorem HMRE [386]}

Since \( \lambda \neq \rho \), we conclude that \( \langle x, y \rangle = 0 \) and so \( x \) and \( y \) are orthogonal vectors (Definition OV [156]).

Subsection READ
Reading Questions

1. How can you identify a nonsingular matrix just by looking at its eigenvalues?
2. How many different eigenvalues may a square matrix of size \( n \) have?
3. What is amazing about the eigenvalues of a Hermitian matrix and why is it amazing?
Suppose that $A$ is a square matrix. Prove that the constant term of the characteristic polynomial of $A$ is equal to the determinant of $A$.

Contributed by Robert Beezer

Suppose that $A$ is a square matrix. Prove that a single vector may not be an eigenvector of $A$ for two different eigenvalues.

Contributed by Robert Beezer

Theorem DCP tells us that the characteristic polynomial of a square matrix of size $n$ has degree $n$. By suitably augmenting the proof of Theorem DCP prove that the coefficient of $x^n$ in the characteristic polynomial is $(-1)^n$.

Contributed by Robert Beezer
**Subsection SOL Solutions**

**T10** Contributed by Robert Beezer Statement 388
Suppose that the characteristic polynomial of $A$ is

$$p_A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

Then

$$a_0 = a_0 + a_1(0) + a_2(0)^2 + \cdots + a_n(0)^n$$

$$= p_A(0)$$

$$= \det (A - 0I_n)$$

$$= \det (A)$$

**T20** Contributed by Robert Beezer Statement 388
Suppose that the vector $x \neq 0$ is an eigenvector of $A$ for the two eigenvalues $\lambda$ and $\rho$, where $\lambda \neq \rho$.

Then $\lambda - \rho \neq 0$, so

$$0 \neq (\lambda - \rho)x$$

$$= \lambda x - \rho x$$

$$= A\lambda x - A\rho x$$

$$= 0$$

which is a contradiction.
Section SD
Similarity and Diagonalization

This section’s topic will perhaps seem out of place at first, but we will make the connection soon with eigenvalues and eigenvectors. This is also our first look at one of the central ideas of Chapter R [473].

Subsection SM
Similar Matrices

The notion of matrices being “similar” is a lot like saying two matrices are row-equivalent. Two similar matrices are not equal, but they share many important properties. This section, and later sections in Chapter R [473] will be devoted in part to discovering just what these common properties are.

First, the main definition for this section.

Definition SIM
Similar Matrices
Suppose $A$ and $B$ are two square matrices of size $n$. Then $A$ and $B$ are similar if there exists a nonsingular matrix of size $n$, $S$, such that $A = S^{-1}BS$. △

We will say “$A$ is similar to $B$ via $S$” when we want to emphasize the role of $S$ in the relationship between $A$ and $B$. Also, it doesn’t matter if we say $A$ is similar to $B$, or $B$ is similar to $A$. If one statement is true then so is the other, as can be seen by using $S^{-1}$ in place of $S$ (see Theorem SER [391] for the careful proof). Finally, we will refer to $S^{-1}BS$ as a similarity transformation when we want to emphasize the way $S$ changes $B$. OK, enough about language, let’s build a few examples.

Example SMS5
Similar matrices of size 5
If you wondered if there are examples of similar matrices, then it won’t be hard to convince you they exist. Define

$$B = \begin{bmatrix} -4 & 1 & -3 & -2 & 2 \\ 1 & 2 & -1 & 3 & -2 \\ -4 & 1 & 3 & 2 & 2 \\ -3 & 4 & -2 & -1 & -3 \\ 3 & 1 & -1 & 1 & -4 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 1 & 3 & -1 & 1 & 1 \\ -2 & -3 & 3 & 1 & -2 \\ 1 & 3 & -1 & 2 & 1 \end{bmatrix}$$

Check that $S$ is nonsingular and then compute $A = S^{-1}BS$

$$\begin{bmatrix} 10 & 1 & 0 & 2 & -5 \\ -1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 1 & -3 \\ 0 & 0 & -1 & 0 & 1 \\ -4 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 1 & -3 & -2 & 2 \\ 1 & 2 & -1 & 3 & -2 \\ -4 & 1 & 3 & 2 & 2 \\ -3 & 4 & -2 & -1 & -3 \\ 3 & 1 & -1 & 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 1 & 3 & -1 & 1 & 1 \\ -2 & -3 & 3 & 1 & -2 \\ 1 & 3 & -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 27 & -29 & -80 & -25 \\ -2 & 6 & 6 & 10 & -2 \\ -3 & 11 & -9 & -14 & -9 \\ -1 & -13 & 0 & -10 & -1 \\ 11 & 35 & 6 & 49 & 19 \end{bmatrix}$$
So by this construction, we know that $A$ and $B$ are similar.

Let’s do that again.

**Example SMS3**

**Similar matrices of size 3**

Define

$$B = \begin{bmatrix}
-13 & -8 & -4 \\
12 & 7 & 4 \\
24 & 16 & 7
\end{bmatrix} \quad S = \begin{bmatrix}
1 & 1 & 2 \\
-2 & -1 & -3 \\
1 & -2 & 0
\end{bmatrix}$$

Check that $S$ is nonsingular and then compute

$$A = S^{-1}BS$$

$$= \begin{bmatrix}
-6 & -4 & -1 \\
-3 & -2 & -1 \\
5 & 3 & 1
\end{bmatrix} \begin{bmatrix}
-13 & -8 & -4 \\
12 & 7 & 4 \\
24 & 16 & 7
\end{bmatrix} \begin{bmatrix}
1 & 1 & 2 \\
-2 & -1 & -3 \\
1 & -2 & 0
\end{bmatrix}$$

$$= \begin{bmatrix}
-1 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & -1
\end{bmatrix}$$

So by this construction, we know that $A$ and $B$ are similar. But before we move on, look at how pleasing the form of $A$ is. Not convinced? Then consider that several computations related to $A$ are especially easy. For example, in the spirit of Example DUTM [341], $\det(A) = (-1)(3)(-1) = 3$. Similarly, the characteristic polynomial is straightforward to compute by hand, $p_A(x) = (-1 - x)(3 - x)(-1 - x) = -(x - 3)(x + 1)^2$ and since the result is already factored, the eigenvalues are transparently $\lambda = 3, -1$. Finally, the eigenvectors of $A$ are just the standard unit vectors (Definition SUV [190]).

**Subsection PSM**

**Properties of Similar Matrices**

Similar matrices share many properties and it is these theorems that justify the choice of the word “similar.” First we will show that similarity is an equivalence relation. Equivalence relations are important in the study of various algebras and can always be regarded as a kind of weak version of equality. Sort of alike, but not quite equal. The notion of two matrices being row-equivalent is an example of an equivalence relation we have been working with since the beginning of the course (see Exercise RREF.T11 [36]). Row-equivalent matrices are not equal, but they are a lot alike. For example, row-equivalent matrices have the same rank. Formally, an equivalence relation requires three conditions hold: reflexive, symmetric and transitive. We will illustrate these as we prove that similarity is an equivalence relation.

**Theorem SER**

**Similarity is an Equivalence Relation**

Suppose $A$, $B$ and $C$ are square matrices of size $n$. Then

1. $A$ is similar to $A$. (Reflexive)
2. If $A$ is similar to $B$, then $B$ is similar to $A$. (Symmetric)
3. If $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$. (Transitive)

**Proof** To see that $A$ is similar to $A$, we need only demonstrate a nonsingular matrix that effects a similarity transformation of $A$ to $A$. $I_n$ is nonsingular (since it row-reduces to the identity matrix, Theorem NMRRI [62]), and

$$I_n^{-1}AI_n = I_nAI_n = A$$
If we assume that $A$ is similar to $B$, then we know there is a nonsingular matrix $S$ so that $A = S^{-1}BS$ by Definition SIM [390]. By Theorem MIMI [196], $S^{-1}$ is invertible, and by Theorem NI [204] is therefore nonsingular. So

$$(S^{-1})^{-1}A(S^{-1}) = SAS^{-1}$$

Theorem MIMI [196]

$$= SS^{-1}BSS^{-1}$$

Definition SIM [390]

$$= (SS^{-1}) B (SS^{-1})$$

Theorem MMA [180]

$$= I_nB I_n$$

Definition MI [189]

$$= B$$

Theorem MMIM [179]

and we see that $B$ is similar to $A$.

Assume that $A$ is similar to $B$, and $B$ is similar to $C$. This gives us the existence of two nonsingular matrices, $S$ and $R$, such that $A = S^{-1}BS$ and $B = R^{-1}CR$, by Definition SIM [390]. (Notice how we have to assume $S \neq R$, as will usually be the case.) Since $S$ and $R$ are invertible, so too $RS$ is invertible by Theorem SS [195] and then nonsingular by Theorem NI [204]. Now

$$(RS)^{-1}C(RS) = S^{-1}R^{-1}CRS$$

Theorem SS [195]

$$= S^{-1}(R^{-1}CR) S$$

Theorem MMA [180]

$$= S^{-1}BS$$

Definition SIM [390]

$$= A$$

so $A$ is similar to $C$ via the nonsingular matrix $RS$.

Here’s another theorem that tells us exactly what sorts of properties similar matrices share.

**Theorem SMEE**

**Similar Matrices have Equal Eigenvalues**

Suppose $A$ and $B$ are similar matrices. Then the characteristic polynomials of $A$ and $B$ are equal, that is $p_A(x) = p_B(x)$.

**Proof** Suppose $A$ and $B$ have size $n$ and are similar via the nonsingular matrix $S$, so $A = S^{-1}BS$ by Definition SIM [390].

$$p_A(x) = \det(A - xI_n)$$

Definition CP [363]

$$= \det(S^{-1}BS - xI_n)$$

Definition SIM [390]

$$= \det(S^{-1}BS - xS^{-1}I_nS)$$

Theorem MMIM [179]

$$= \det(S^{-1}BS - S^{-1}xI_nS)$$

Theorem MMSMM [180]

$$= \det(S^{-1})(B - xI_n)S$$

Theorem MMDAA [179]

$$= \det(S^{-1}) \det(B - xI_n) \det(S)$$

Theorem DRMM [353]

$$= \det(S^{-1}) \det(S) \det(B - xI_n)$$

Property MCCN [612]

$$= \det(S^{-1}S) \det(B - xI_n)$$

Theorem DRMM [353]

$$= \det(I_n) \det(B - xI_n)$$

Definition MI [189]

$$= 1 \det(B - xI_n)$$

Definition DM [337]

$$= p_B(x)$$

Definition CP [363]

So similar matrices not only have the same set of eigenvalues, the algebraic multiplicities of these eigenvalues will also be the same. However, be careful with this theorem. It is tempting to think the converse is true, and argue that if two matrices have the same eigenvalues, then they are similar. Not so, as the following example illustrates.

**Example EENS**

**Equal eigenvalues, not similar**
Define
\[ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]
and check that
\[ p_A(x) = p_B(x) = 1 - 2x + x^2 = (x - 1)^2 \]
and so \( A \) and \( B \) have equal characteristic polynomials. If the converse of Theorem SMEE \[392\] were true, then \( A \) and \( B \) would be similar. Suppose this is the case. In other words, there is a nonsingular matrix \( S \) so that \( A = S^{-1}BS \). Then
\[ A = S^{-1}BS = S^{-1}I_2S = S^{-1}S = I_2 \neq A \]
and this contradiction tells us that the converse of Theorem SMEE \[392\] is false.

Subsection D
Diagonalization

Good things happen when a matrix is similar to a diagonal matrix. For example, the eigenvalues of the matrix are the entries on the diagonal of the diagonal matrix. And it can be a much simpler matter to compute high powers of the matrix. Diagonalizable matrices are also of interest in more abstract settings. Here are the relevant definitions, then our main theorem for this section.

Definition DIM
Diagonal Matrix
Suppose that \( A \) is a square matrix. Then \( A \) is a \textbf{diagonal matrix} if \( A_{ij} = 0 \) whenever \( i \neq j \).

Definition DZM
Diagonalizable Matrix
Suppose \( A \) is a square matrix. Then \( A \) is \textbf{diagonalizable} if \( A \) is similar to a diagonal matrix.

Example DAB
Diagonalization of Archetype B
Archetype B \[638\] has a \( 3 \times 3 \) coefficient matrix
\[ B = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \]
and is similar to a diagonal matrix, as can be seen by the following computation with the nonsingular matrix \( S \),
\[ S^{-1}BS = \begin{bmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \]
Example SMS3 \[391\] provides yet another example of a matrix that is subjected to a similarity transformation and the result is a diagonal matrix. Alright, just how would we find the magic
matrix $S$ that can be used in a similarity transformation to produce a diagonal matrix? Before you read the statement of the next theorem, you might study the eigenvalues and eigenvectors of Archetype B [638] and compute the eigenvalues and eigenvectors of the matrix in Example SMS3 [391].

Theorem DC
Diagonalization Characterization
Suppose $A$ is a square matrix of size $n$. Then $A$ is diagonalizable if and only if there exists a linearly independent set $S$ that contains $n$ eigenvectors of $A$.

Proof
(\Rightarrow) Let $S = \{x_1, x_2, x_3, \ldots, x_n\}$ be a linearly independent set of eigenvectors of $A$ for the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$. Recall Definition SUV [190] and define

$$R = [x_1 | x_2 | x_3 | \ldots | x_n]$$

$$D = \begin{bmatrix}
\lambda_1 & 0 & 0 & \cdots & 0 \\
0 & \lambda_2 & 0 & \cdots & 0 \\
0 & 0 & \lambda_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_n
\end{bmatrix} = [\lambda_1 e_1 | \lambda_2 e_2 | \lambda_3 e_3 | \ldots | \lambda_n e_n]$$

The columns of $R$ are the vectors of the linearly independent set $S$ and so by Theorem NMLIC [126] the matrix $R$ is nonsingular. By Theorem NI [204] we know $R^{-1}$ exists.

$$R^{-1}AR = R^{-1}A [x_1 | x_2 | x_3 | \ldots | x_n]$$

$$= R^{-1}[Ax_1 | Ax_2 | Ax_3 | \ldots | Ax_n]$$

$$= R^{-1}[\lambda_1 x_1 | \lambda_2 x_2 | \lambda_3 x_3 | \ldots | \lambda_n x_n]$$

$$= R^{-1}[\lambda_1 xe_1 | \lambda_2 xe_2 | \lambda_3 xe_3 | \ldots | \lambda_n xe_n]$$

$$= R^{-1}[R(\lambda_1 x_1) | R(\lambda_2 x_2) | R(\lambda_3 x_3) | \ldots | R(\lambda_n x_n)]$$

$$= R^{-1}R[\lambda_1 e_1 | \lambda_2 e_2 | \lambda_3 e_3 | \ldots | \lambda_n e_n]$$

$$= I_n D$$

$$= D$$

This says that $A$ is similar to the diagonal matrix $D$ via the nonsingular matrix $R$. Thus $A$ is diagonalizable (Definition DZM [393]).

(\Rightarrow) Suppose that $A$ is diagonalizable, so there is a nonsingular matrix of size $n$

$$T = [y_1 | y_2 | y_3 | \ldots | y_n]$$

and a diagonal matrix (recall Definition SUV [190])

$$E = \begin{bmatrix}
d_1 & 0 & 0 & \cdots & 0 \\
0 & d_2 & 0 & \cdots & 0 \\
0 & 0 & d_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & d_n
\end{bmatrix} = [d_1 e_1 | d_2 e_2 | d_3 e_3 | \ldots | d_n e_n]$$

such that $T^{-1}AT = E$. Then consider,

$$[Ay_1 | Ay_2 | Ay_3 | \ldots | Ay_n] = A [y_1 | y_2 | y_3 | \ldots | y_n]$$

$$= AT$$

$$= I_n AT$$

$$= TT^{-1}AT$$

Definition MM [176]
Definition MI [189]
This equality of matrices (Definition ME [163]) allows us to conclude that the individual columns are equal vectors (Definition CVE [73]). That is, $Ay_i = d_i y_i$ for $1 \leq i \leq n$. In other words, $y_i$ is an eigenvector of $A$ for the eigenvalue $d_i$, $1 \leq i \leq n$. (Why can’t $y_i = 0$?). Because $T$ is nonsingular, the set containing $T$’s columns, $S = \{y_1, y_2, y_3, \ldots, y_n\}$, is a linearly independent set (Theorem NMLIC [126]). So the set $S$ has all the required properties.

Notice that the proof of Theorem DC [394] is constructive. To diagonalize a matrix, we need only locate $n$ linearly independent eigenvectors. Then we can construct a nonsingular matrix using the eigenvectors as columns ($R$) so that $R^{-1}AR$ is a diagonal matrix ($D$). The entries on the diagonal of $D$ will be the eigenvalues of the eigenvectors used to create $R$, in the same order as the eigenvectors appear in $R$. We illustrate this by diagonalizing some matrices.

**Example DMS3**

**Diagonalizing a matrix of size 3**

Consider the matrix

$$F = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix}$$

of Example CPMS3 [363], Example EMS3 [363] and Example ESMS3 [365]. $F$’s eigenvalues and eigenspaces are

$$\lambda = 3 \quad \mathcal{E}_F(3) = \left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{2}{3} \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{2}{3} \\ 0 \end{bmatrix} \right\}$$

$$\lambda = -1 \quad \mathcal{E}_F(-1) = \left\{ \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$$

Define the matrix $S$ to be the $3 \times 3$ matrix whose columns are the three basis vectors in the eigenspaces for $F$,

$$S = \begin{bmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Check that $S$ is nonsingular (row-reduces to the identity matrix, Theorem NMRRI [62] or has a nonzero determinant, Theorem SMZD [351]). Then the three columns of $S$ are a linearly independent set (Theorem NMLIC [126]). By Theorem DC [394], we now know that $F$ is diagonalizable. Furthermore, the construction in the proof of Theorem DC [394] tells us that if we apply the matrix $S$ to $F$ in a similarity transformation, the result will be a diagonal matrix with the eigenvalues of $F$ on the diagonal. The eigenvalues appear on the diagonal of the matrix in the same order as the eigenvectors appear in $S$. So,

$$S^{-1}FS = \begin{bmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 4 & 2 \\ -3 & -1 & -1 \\ -6 & -4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$
Recall that sets of eigenvectors whose eigenvalues are distinct form a linearly independent set. Since the eigenspace is a set of eigenvectors for \( A \). A vector cannot be an eigenvector for two different eigenvalues (see Example SMS5 390). Let \( A \) be diagonalizable if and only if \( \gamma_A (\lambda) = \alpha_A (\lambda) \) for every eigenvalue \( \lambda \) of \( A \).

\[
\begin{bmatrix}
3 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

Then \( S = S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_k \) is a set of eigenvectors for \( A \). A vector cannot be an eigenvector for two different eigenvalues (see Exercise EE.T20 373) so the sets \( S_i \) have no vectors in common. Thus the size of \( S \) is

\[
\sum_{i=1}^{k} \gamma_A (\lambda_i) = \sum_{i=1}^{k} \alpha_A (\lambda_i)
\]

Hypothesis

\[
= n \quad \text{Theorem NEM 383}
\]

We now want to show that \( S \) is a linearly independent set. So we will begin with a relation of linear dependence on \( S \), using doubly-subscripted scalars and eigenvectors,

\[
0 = (a_{i1} x_{i1} + a_{i2} x_{i2} + \cdots + a_{i\gamma_A (\lambda_i) x_{1\gamma_A (\lambda_i)})} + (a_{21} x_{21} + a_{22} x_{22} + \cdots + a_{2\gamma_A (\lambda_2) x_{2\gamma_A (\lambda_2))}) + \cdots + (a_{k1} x_{k1} + a_{k2} x_{k2} + \cdots + a_{k\gamma_A (\lambda_k) x_{k\gamma_A (\lambda_k))})
\]

Define the vectors \( y_i, 1 \leq i \leq k \) by

\[
\begin{align*}
y_1 &= (a_{i1} x_{i1} + a_{i2} x_{i2} + a_{i3} x_{i3} + \cdots + a_{i\gamma_A (\lambda_i) x_{1\gamma_A (\lambda_i)})} \\
y_2 &= (a_{21} x_{21} + a_{22} x_{22} + a_{23} x_{23} + \cdots + a_{2\gamma_A (\lambda_2) x_{2\gamma_A (\lambda_2))}) \\
y_3 &= (a_{31} x_{31} + a_{32} x_{32} + a_{33} x_{33} + \cdots + a_{3\gamma_A (\lambda_3) x_{3\gamma_A (\lambda_3))}) \\
&\vdots \\
y_k &= (a_{k1} x_{k1} + a_{k2} x_{k2} + a_{k3} x_{k3} + \cdots + a_{k\gamma_A (\lambda_k) x_{k\gamma_A (\lambda_k))})
\end{align*}
\]

Then the relation of linear dependence becomes

\[
0 = y_1 + y_2 + y_3 + \cdots + y_k
\]

Since the eigenspace \( \mathcal{E}_A (\lambda_i) \) is closed under vector addition and scalar multiplication, \( y_i \in \mathcal{E}_A (\lambda_i), 1 \leq i \leq k \). Thus, for each \( i \), the vector \( y_i \) is an eigenvector of \( A \) for \( \lambda_i \), or is the zero vector. Recall that sets of eigenvectors whose eigenvalues are distinct form a linearly independent set by Theorem EDELI 378. Should any (or some) \( y_i \) be nonzero, the previous equation would
provide a nontrivial relation of linear dependence on a set of eigenvectors with distinct eigenvalues, contradicting Theorem EDELI [378]. Thus $y_i = 0$, $1 \leq i \leq k$.

Each of the $k$ equations, $y_i = 0$ is a relation of linear dependence on the corresponding set $S_i$, a set of basis vectors for the eigenspace $E_A(\lambda_i)$, which is therefore linearly independent. From these relations of linear dependence on linearly independent sets we conclude that the scalars are all zero, more precisely, $a_{ij} = 0$, $1 \leq j \leq \gamma_A(\lambda_i)$ for $1 \leq i \leq k$. This establishes that our original relation of linear dependence on $S$ has only the trivial relation of linear dependence, and hence $S$ is a linearly independent set.

We have determined that $S$ is a set of $n$ linearly independent eigenvectors for $A$, and so by Theorem DC [394] is diagonalizable.

($\Rightarrow$) Now we assume that $A$ is diagonalizable. Aiming for a contradiction (Technique CD [623]), suppose that there is at least one eigenvalue, say $\lambda_i$, such that $\gamma_A(\lambda_i) \neq \alpha_A(\lambda_i)$. By Theorem ME [384] we must have $\gamma_A(\lambda_i) < \alpha_A(\lambda_i)$, and $\gamma_A(\lambda_i) \leq \alpha_A(\lambda_i)$ for $1 \leq i \leq k$, $i \neq t$.

Since $A$ is diagonalizable, Theorem DC [394] guarantees a set of $n$ linearly independent vectors, all of which are eigenvectors of $A$. Let $n_i$ denote the number of eigenvectors in $S$ that are eigenvectors for $\lambda_i$, and recall that a vector cannot be an eigenvector for two different eigenvalues (Exercise EE.T20 [373]). $S$ is a linearly independent set, so the the subset $S_i$ containing the $n_i$ eigenvectors for $\lambda_i$ must also be linearly independent. Because the eigenspace $E_A(\lambda_i)$ has dimension $\gamma_A(\lambda_i)$ and $S_i$ is a linearly independent subset in $E_A(\lambda_i)$, $n_i \leq \gamma_A(\lambda_i)$, $1 \leq i \leq k$. Now,

$$n = n_1 + n_2 + n_3 + \cdots + n_t + \cdots + n_k$$

Size of $S$

$$\leq \gamma_A(\lambda_1) + \gamma_A(\lambda_2) + \gamma_A(\lambda_3) + \cdots + \gamma_A(\lambda_t) + \cdots + \gamma_A(\lambda_k)$$

$S_i$ linearly independent

$$< \alpha_A(\lambda_1) + \alpha_A(\lambda_2) + \alpha_A(\lambda_3) + \cdots + \alpha_A(\lambda_t) + \cdots + \alpha_A(\lambda_k)$$

Assumption about $\lambda_i$

$$= n$$

Theorem NEM [383]

This is a contradiction (we can’t have $n < n$!) and so our assumption that some eigenspace had less than full dimension was false.

Example SEE [356], Example CAEHW [361], Example ESMS3 [365], Example ESMS4 [367]. Example DEMS5 [370], Archetype B [638], Archetype F [654], Archetype K [676] and Archetype L [680] are all examples of matrices that are diagonalizable and that illustrate Theorem DMFE [396]. While we have provided many examples of matrices that are diagonalizable, especially among the archetypes, there are many matrices that are not diagonalizable. Here’s one now.

Example NDMS4

A non-diagonalizable matrix of size 4

In Example EMMS4 [366] the matrix

$$B = \begin{bmatrix}
-2 & 1 & -2 & -4 \\
12 & 1 & 4 & 9 \\
6 & 5 & -2 & -4 \\
3 & -4 & 5 & 10
\end{bmatrix}$$

was determined to have characteristic polynomial

$$p_B(x) = (x - 1)(x - 2)^3$$

and an eigenspace for $\lambda = 2$ of

$$E_B(2) = \left\langle \begin{bmatrix} -1/2 \\ 1 \\ 1/2 \end{bmatrix} \right\rangle$$

So the geometric multiplicity of $\lambda = 2$ is $\gamma_B(2) = 1$, while the algebraic multiplicity is $\alpha_B(2) = 3$. By Theorem DMFE [396], the matrix $B$ is not diagonalizable.

Archetype A [634] is the lone archetype with a square matrix that is not diagonalizable, as the algebraic and geometric multiplicities of the eigenvalue $\lambda = 0$ differ. Example HMEM5 [367].
is another example of a matrix that cannot be diagonalized due to the difference between the geometric and algebraic multiplicities of $\lambda = 2$, as is Example CEMS6 \[368\] which has two complex eigenvalues, each with differing multiplicities. Likewise, Example EMMS4 \[366\] has an eigenvalue with different algebraic and geometric multiplicities and so cannot be diagonalized.

**Theorem DED**

**Distinct Eigenvalues implies Diagonalizable**

Suppose $A$ is a square matrix of size $n$ with $n$ distinct eigenvalues. Then $A$ is diagonalizable. \(\square\)

**Proof** Let $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ denote the $n$ distinct eigenvalues of $A$. Then by Theorem NEM \[383\] we have $n = \sum_{i=1}^{n} \alpha_A(\lambda_i)$, which implies that $\alpha_A(\lambda_i) = 1$, $1 \leq i \leq n$. From Theorem ME \[384\] it follows that $\gamma_A(\lambda_i) = 1$, $1 \leq i \leq n$. So $\gamma_A(\lambda_i) = \alpha_A(\lambda_i)$, $1 \leq i \leq n$ and Theorem DMFE \[396\] says $A$ is diagonalizable. \(\blacksquare\)

**Example DEHD**

**Distinct eigenvalues, hence diagonalizable**

In Example DEMS5 \[370\] the matrix

$$H = \begin{bmatrix}
15 & 18 & -8 & 6 & -5 \\
5 & 3 & 1 & -1 & -3 \\
0 & -4 & 5 & -4 & -2 \\
-43 & -46 & 17 & -14 & 15 \\
26 & 30 & -12 & 8 & -10
\end{bmatrix}$$

has characteristic polynomial

$$p_H(x) = x(x - 2)(x - 1)(x + 1)(x + 3)$$

and so is a $5 \times 5$ matrix with 5 distinct eigenvalues. By Theorem DED \[398\] we know $H$ must be diagonalizable. But just for practice, we exhibit the diagonalization itself. The matrix $S$ contains eigenvectors of $H$ as columns, one from each eigenspace, guaranteeing linear independent columns and thus the nonsingularity of $S$. The diagonal matrix has the eigenvalues of $H$ in the same order that their respective eigenvectors appear as the columns of $S$. Notice that we are using the versions of the eigenvectors from Example DEMS5 \[370\] that have integer entries.

$$S^{-1}HS$$

$$= \begin{bmatrix}
2 & 1 & -1 & 1 & 1 \\
-1 & 0 & 2 & 0 & -1 \\
-2 & 0 & 2 & -1 & -2 \\
-4 & -1 & 0 & -2 & -1 \\
2 & 2 & 1 & 2 & 1
\end{bmatrix}^{-1} \begin{bmatrix}
15 & 18 & -8 & 6 & -5 \\
5 & 3 & 1 & -1 & -3 \\
0 & -4 & 5 & -4 & -2 \\
-43 & -46 & 17 & -14 & 15 \\
26 & 30 & -12 & 8 & -10
\end{bmatrix} \begin{bmatrix}
2 & 1 & -1 & 1 & 1 \\
-1 & 0 & 2 & 0 & -1 \\
-2 & 0 & 2 & -1 & -2 \\
-4 & -1 & 0 & -2 & -1 \\
2 & 2 & 1 & 2 & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
-3 & -3 & 1 & -1 & 1 \\
-1 & -2 & 1 & 0 & 1 \\
-5 & -4 & 1 & -1 & 2 \\
10 & 10 & -3 & 2 & -4 \\
-7 & -6 & 1 & -1 & 3
\end{bmatrix}^{-1} \begin{bmatrix}
15 & 18 & -8 & 6 & -5 \\
5 & 3 & 1 & -1 & -3 \\
0 & -4 & 5 & -4 & -2 \\
-43 & -46 & 17 & -14 & 15 \\
26 & 30 & -12 & 8 & -10
\end{bmatrix} \begin{bmatrix}
2 & 1 & -1 & 1 & 1 \\
-1 & 0 & 2 & 0 & -1 \\
-2 & 0 & 2 & -1 & -2 \\
-4 & -1 & 0 & -2 & -1 \\
2 & 2 & 1 & 2 & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
-3 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 2
\end{bmatrix}$$

Archetype B \[638\] is another example of a matrix that has as many distinct eigenvalues as its size, and is hence diagonalizable by Theorem DED \[398\].
Powers of a diagonal matrix are easy to compute, and when a matrix is diagonalizable, it is almost as easy. We could state a theorem here perhaps, but we will settle instead for an example that makes the point just as well.

**Example HPDM**

**High power of a diagonalizable matrix**

Suppose that

\[
A = \begin{bmatrix}
19 & 0 & 6 & 13 \\
-33 & -1 & -9 & -21 \\
21 & -4 & 12 & 21 \\
-36 & 2 & -14 & -28 \\
\end{bmatrix}
\]

and we wish to compute \(A^{20}\). Normally this would require 19 matrix multiplications, but since \(A\) is diagonalizable, we can simplify the computations substantially. First, we diagonalize \(A\). With

\[
S = \begin{bmatrix}
1 & -1 & 2 & -1 \\
-2 & 3 & -3 & 3 \\
1 & 1 & 3 & 3 \\
-2 & 1 & -4 & 0 \\
\end{bmatrix}
\]

we find

\[
D = S^{-1}AS = \begin{bmatrix}
-6 & 1 & -3 & -6 \\
0 & 2 & -2 & -3 \\
3 & 0 & 1 & 2 \\
-1 & -1 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
19 & 0 & 6 & 13 \\
-33 & -1 & -9 & -21 \\
21 & -4 & 12 & 21 \\
-36 & 2 & -14 & -28 \\
\end{bmatrix}
\begin{bmatrix}
1 & -1 & 2 & -1 \\
-2 & 3 & -3 & 3 \\
1 & 1 & 3 & 3 \\
-2 & 1 & -4 & 0 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Now we find an alternate expression for \(A^{20}\),

\[
A^{20} = AAA\ldots A \\
= I_nAI_nAI_nAI_n\ldots I_nAI_n \\
= (SS^{-1})A(SS^{-1})A(SS^{-1})A(SS^{-1})\ldots (SS^{-1})A(SS^{-1}) \\
= S(S^{-1}AS)(S^{-1}AS)(S^{-1}AS)\ldots (S^{-1}AS)S^{-1} \\
= SDDD\ldots DS^{-1} \\
= SD^{20}S^{-1}
\]

and since \(D\) is a diagonal matrix, powers are much easier to compute,

\[
= S \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}^{20} S^{-1}
\]

\[
= S \begin{bmatrix}
(-1)^{20} & 0 & 0 & 0 \\
0 & (0)^{20} & 0 & 0 \\
0 & 0 & (2)^{20} & 0 \\
0 & 0 & 0 & (1)^{20} \\
\end{bmatrix} S^{-1}
\]

\[
= \begin{bmatrix}
1 & -1 & 2 & -1 \\
-2 & 3 & -3 & 3 \\
1 & 1 & 3 & 3 \\
-2 & 1 & -4 & 0 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
-6 & 1 & -3 & -6 \\
0 & 2 & -2 & -3 \\
3 & 0 & 1 & 2 \\
-1 & -1 & 1 & 1 \\
\end{bmatrix}
\]
Notice how we effectively replaced the twentieth power of $A$ by the twentieth power of $D$, and how a high power of a diagonal matrix is just a collection of powers of scalars on the diagonal. The price we pay for this simplification is the need to diagonalize the matrix (by computing eigenvalues and eigenvectors) and finding the inverse of the matrix of eigenvectors. And we still need to do two matrix products. But the higher the power, the greater the savings.

We close this section with a comment about an important theorem that we prove in one of the Topics sections. Every Hermitian matrix (Definition HM [208]) is diagonalizable (Definition DZM [393]), and the similarity transformation that accomplishes the diagonalization uses a unitary matrix (Definition UM [205]). This means that for every Hermitian matrix of size $n$ there is a basis of $\mathbb{C}^n$ that is composed entirely of eigenvectors for the matrix and also forms an orthonormal set (Definition ONS [160]). Notice that for matrices with only real entries, we only need the hypothesis that the matrix is symmetric (Definition SYM [166]) to reach this conclusion (Example ESMS4 [367]). Can you imagine a prettier basis for use with a matrix? I can’t. A precise statement of this result applies to a slightly broader class of matrices, known as “normal” matrices, see (link to definition here). With this expanded category of matrices, the result becomes an equivalence (Technique E [622]). See (link to theorem here).

Subsection READ
Reading Questions

1. What is an equivalence relation?

2. State a condition that is equivalent to a matrix being diagonalizable, but is not the definition.

3. Find a diagonal matrix similar to

\[
A = \begin{bmatrix}
-5 & 8 \\
-4 & 7
\end{bmatrix}
\]
Subsection EXC
Exercises

C20  Consider the matrix $A$ below. First, show that $A$ is diagonalizable by computing the geometric multiplicities of the eigenvalues and quoting the relevant theorem. Second, find a diagonal matrix $D$ and a nonsingular matrix $S$ so that $S^{-1}AS = D$. (See Exercise EE.C20 [373] for some of the necessary computations.)

$$A = \begin{bmatrix} 18 & -15 & 33 & -15 \\ -4 & 8 & -6 & 6 \\ -9 & 9 & -16 & 9 \\ 5 & -6 & 9 & -4 \end{bmatrix}$$

Contributed by Robert Beezer  Solution [402]

C21  Determine if the matrix $A$ below is diagonalizable. If the matrix is diagonalizable, then find a diagonal matrix $D$ that is similar to $A$, and provide the invertible matrix $S$ that performs the similarity transformation. You should use your calculator to find the eigenvalues of the matrix, but try only using the row-reducing function of your calculator to assist with finding eigenvectors.

$$A = \begin{bmatrix} 1 & 9 & 9 & 24 \\ -3 & -27 & -29 & -68 \\ 1 & 11 & 13 & 26 \\ 1 & 7 & 7 & 18 \end{bmatrix}$$

Contributed by Robert Beezer  Solution [402]

C22  Consider the matrix $A$ below. Find the eigenvalues of $A$ using a calculator and use these to construct the characteristic polynomial of $A$, $p_A(x)$. State the algebraic multiplicity of each eigenvalue. Find all of the eigenspaces for $A$ by computing expressions for null spaces, only using your calculator to row-reduce matrices. State the geometric multiplicity of each eigenvalue. Is $A$ diagonalizable? If not, explain why. If so, find a diagonal matrix $D$ that is similar to $A$.

$$A = \begin{bmatrix} 19 & 25 & 30 & 5 \\ -23 & -30 & -35 & -5 \\ 7 & 9 & 10 & 1 \\ -3 & -4 & -5 & -1 \end{bmatrix}$$

Contributed by Robert Beezer  Solution [403]

T15  Suppose that $A$ and $B$ are similar matrices. Prove that $A^3$ and $B^3$ are similar matrices. Generalize.

Contributed by Robert Beezer  Solution [404]

T16  Suppose that $A$ and $B$ are similar matrices, with $A$ nonsingular. Prove that $B$ is nonsingular, and that $A^{-1}$ is similar to $B^{-1}$.

Contributed by Robert Beezer

T17  Suppose that $B$ is a nonsingular matrix. Prove that $AB$ is similar to $BA$.

Contributed by Robert Beezer  Solution [404]
Subsection SOL Solutions

C20 Contributed by Robert Beezer Statement 401
Using a calculator, we find that $A$ has three distinct eigenvalues, $\lambda = 3$, 2, $-1$, with $\lambda = 2$ having algebraic multiplicity two, $\alpha_A(2) = 2$. The eigenvalues $\lambda = 3, -1$ have algebraic multiplicity one, and so by Theorem ME 384 we can conclude that their geometric multiplicities are one as well. Together with the computation of the geometric multiplicity of $\lambda = 2$ from Exercise EE.C20 373, we know

$\gamma_A(3) = \alpha_A(3) = 1 \quad \gamma_A(2) = \alpha_A(2) = 2 \quad \gamma_A(-1) = \alpha_A(-1) = 1$

This satisfies the hypotheses of Theorem DMFE 396, and so we can conclude that $A$ is diagonalizable.

A calculator will give us four eigenvectors of $A$, the two for $\lambda = 2$ being linearly independent presumably. Or, by hand, we could find basis vectors for the three eigenspaces. For $\lambda = 3, -1$ the eigenspaces have dimension one, and so any eigenvector for these eigenvalues will be multiples of the ones we use below. For $\lambda = 2$ there are many different bases for the eigenspace, so your answer could vary. Our eigenvectors are the basis vectors we would have obtained if we had actually constructed a basis in Exercise EE.C20 373 rather than just computing the dimension.

By the construction in the proof of Theorem DC 394, the required matrix $S$ has columns that are four linearly independent eigenvectors of $A$ and the diagonal matrix has the eigenvalues on the diagonal (in the same order as the eigenvectors in $S$). Here are the pieces, “doing” the diagonalization,

$$
\begin{bmatrix}
-1 & 0 & -3 & 6 \\
-2 & -1 & -1 & 0 \\
0 & 0 & 1 & -3 \\
1 & 1 & 0 & 1 \\
\end{bmatrix}^{-1}
\begin{bmatrix}
18 & -15 & 33 & -15 \\
-4 & 8 & -6 & 6 \\
-9 & 9 & -16 & 9 \\
5 & -6 & 9 & -4 \\
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & -3 & 6 \\
-2 & -1 & -1 & 0 \\
0 & 0 & 1 & -3 \\
1 & 1 & 0 & 1 \\
\end{bmatrix} =
\begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & -1 \\
\end{bmatrix}
$$

C21 Contributed by Robert Beezer Statement 401
A calculator will provide the eigenvalues $\lambda = 2, 1, 0$, so we can reconstruct the characteristic polynomial as

$$
p_A(x) = (x - 2)^2(x - 1)x
$$

so the algebraic multiplicities of the eigenvalues are

$\alpha_A(2) = 2 \quad \alpha_A(1) = 1 \quad \alpha_A(0) = 1$

Now compute eigenspaces by hand, obtaining null spaces for each of the three eigenvalues by constructing the correct singular matrix (Theorem EMNS 364).

$$
A - 2I_A =
\begin{bmatrix}
-1 & 9 & 9 & 24 \\
-3 & -29 & -29 & -68 \\
1 & 11 & 11 & 26 \\
1 & 7 & 7 & 16 \\
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 0 & \frac{-3}{2} \\
0 & 1 & 1 & \frac{3}{2} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

$$
E_A(2) = N(A - 2I_A) =
\left\{\begin{bmatrix}
\frac{3}{2} \\
0 \\
1 \\
0 \\
\end{bmatrix},
\begin{bmatrix}
0 \\
-1 \\
1 \\
0 \\
\end{bmatrix}\right\} =
\left\{\begin{bmatrix}
3 \\
0 \\
0 \\
0 \\
\end{bmatrix},
\begin{bmatrix}
0 \\
-5 \\
1 \\
1 \\
\end{bmatrix}\right\}
$$

$$
A - 1I_A =
\begin{bmatrix}
0 & 9 & 9 & 24 \\
-3 & -28 & -29 & -68 \\
1 & 11 & 12 & 26 \\
1 & 7 & 7 & 17 \\
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 0 & \frac{-5}{3} \\
0 & 1 & 0 & \frac{13}{3} \\
0 & 0 & 1 & \frac{-5}{3} \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$
\[ \mathcal{E}_A(1) = \mathcal{N}(A - I_4) = \left\langle \left\{ \begin{bmatrix} 5 \\ -13 \\ 5 \\ 3 \end{bmatrix} \right\} \right\rangle = \left\langle \left\{ \begin{bmatrix} 5 \\ -13 \\ 5 \\ 3 \end{bmatrix} \right\} \right\rangle \]

\[
A - 0I_4 = \begin{bmatrix} 1 & 9 & 9 & 24 \\ -3 & -27 & -29 & -68 \\ 1 & 11 & 13 & 26 \\ 1 & 7 & 7 & 18 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[ \mathcal{E}_A(0) = \mathcal{N}(A - I_4) = \left\langle \left\{ \begin{bmatrix} 3 \\ -5 \\ 2 \\ 1 \end{bmatrix} \right\} \right\rangle \]

From this we can compute the dimensions of the eigenspaces to obtain the geometric multiplicities,

\[ \gamma_A(2) = 2 \quad \gamma_A(1) = 1 \quad \gamma_A(0) = 1 \]

For each eigenvalue, the algebraic and geometric multiplicities are equal and so by Theorem DMFE, we now know that \( A \) is diagonalizable. The construction in Theorem DC suggests we form a matrix whose columns are eigenvectors of \( A \)

\[
S = \begin{bmatrix} 3 & 0 & 5 & 3 \\ -5 & -1 & -13 & -5 \\ 0 & 1 & 5 & 2 \\ 2 & 0 & 3 & 1 \end{bmatrix}
\]

Since \( \det(S) = -1 \neq 0 \), we know that \( S \) is nonsingular (Theorem SMZD), so the columns of \( S \) are a set of 4 linearly independent eigenvectors of \( A \). By the proof of Theorem SMZD we know

\[
S^{-1}AS = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

a diagonal matrix with the eigenvalues of \( A \) along the diagonal, in the same order as the associated eigenvectors appear as columns of \( S \).

**C22 Contributed by Robert Beezer Statement**

A calculator will report \( \lambda = 0 \) as an eigenvalue of algebraic multiplicity of 2, and \( \lambda = -1 \) as an eigenvalue of algebraic multiplicity 2 as well. Since eigenvalues are roots of the characteristic polynomial (Theorem EMRCP) we have the factored version

\[ p_A(x) = (x - 0)^2(x - (-1))^2 = x^2(x^2 + 2x + 1) = x^4 + 2x^3 + x^2 \]

The eigenspaces are then

\[
A - (0)I_4 = \begin{bmatrix} 19 & 25 & 30 & 5 \\ -23 & -30 & -35 & -5 \\ 7 & 9 & 10 & 1 \\ -3 & -4 & -5 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[ \mathcal{E}_A(0) = \mathcal{N}(C - (0)I_4) = \left\langle \left\{ \begin{bmatrix} 5 \\ -5 \\ 1 \\ 0 \end{bmatrix} \right\} \right\rangle \]

\[ \lambda = -1 \]
\[ A - (-1)I_4 = \begin{bmatrix} 20 & 25 & 30 & 5 \\ -23 & -29 & -35 & -5 \\ 7 & 9 & 11 & 1 \\ -3 & -4 & -5 & 0 \end{bmatrix} \overset{\text{RREF}}{\rightarrow} \begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ E_A(-1) = N(C - (-1)I_4) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\} \]

Each eigenspace above is described by a spanning set obtained through an application of Theorem BNS \[128\] and so is a basis for the eigenspace. In each case the dimension, and therefore the geometric multiplicity, is 2.

For each of the two eigenvalues, the algebraic and geometric multiplicities are equal. Theorem DMFE \[396\] says that in this situation the matrix is diagonalizable. We know from Theorem DC \[394\] that when we diagonalize \( A \) the diagonal matrix will have the eigenvalues of \( A \) on the diagonal (in some order). So we can claim that

\[ D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \]

\textbf{T15} \hspace{1em} Contributed by Robert Beezer \hspace{1em} Statement \[401\]

By Definition SIM \[390\] we know that there is a nonsingular matrix \( S \) so that \( A = S^{-1}BS \). Then

\[ A^3 = (S^{-1}BS)^3 \]
\[ = (S^{-1}BS)(S^{-1}BS)(S^{-1}BS) \]
\[ = S^{-1}B(SS^{-1})B(SS^{-1})BS \]
\[ = S^{-1}B(I_3)B(I_3)BS \]
\[ = S^{-1}BBBS \]
\[ = S^{-1}B^3S \]

This equation says that \( A^3 \) is similar to \( B^3 \) (via the matrix \( S \)).

More generally, if \( A \) is similar to \( B \), and \( m \) is a non-negative integer, then \( A^m \) is similar to \( B^m \). This can be proved using induction (Technique I \[626\]).

\textbf{T17} \hspace{1em} Contributed by Robert Beezer \hspace{1em} Statement \[401\]

The nonsingular (invertible) matrix \( B \) will provide the desired similarity transformation,

\[ B^{-1} (BA) B = (B^{-1}B)(AB) \]
\[ = I_n AB \]
\[ = AB \]
Chapter LT
Linear Transformations

In the next linear algebra course you take, the first lecture might be a reminder about what a vector space is (Definition VS [251]), their ten properties, basic theorems and then some examples. The second lecture would likely be all about linear transformations. While it may seem we have waited a long time to present what must be a central topic, in truth we have already been working with linear transformations for some time.

Functions are important objects in the study of calculus, but have been absent from this course until now (well, not really, it just seems that way). In your study of more advanced mathematics it is nearly impossible to escape the use of functions — they are as fundamental as sets are.

Section LT
Linear Transformations

Here comes a key definition.

Subsection LT
Linear Transformations

Definition LT
Linear Transformation

A linear transformation, \( T: U \rightarrow V \), is a function that carries elements of the vector space \( U \) (called the domain) to the vector space \( V \) (called the codomain), and which has two additional properties

1. \( T(u_1 + u_2) = T(u_1) + T(u_2) \) for all \( u_1, u_2 \in U \)

2. \( T(\alpha u) = \alpha T(u) \) for all \( u \in U \) and all \( \alpha \in \mathbb{C} \)

(This definition contains Notation LT.)

The two defining conditions in the definition of a linear transformation should “feel linear,” whatever that means. Conversely, these two conditions could be taken as a exactly what it means to be linear. As every vector space property derives from vector addition and scalar multiplication, so too, every property of a linear transformation derives from these two defining properties. While these conditions may be reminiscent of how we test subspaces, they really are quite different, so do not confuse the two.

Here are two diagrams that convey the essence of the two defining properties of a linear transformation. In each case, begin in the upper left-hand corner, and follow the arrows around the
rectangle to the lower-right hand corner, taking two different routes and doing the indicated operations labeled on the arrows. There are two results there. For a linear transformation these two expressions are always equal.

\[
\begin{align*}
\mathbf{u}_1, \mathbf{u}_2 & \xrightarrow{T} T(\mathbf{u}_1), T(\mathbf{u}_2) \\
\downarrow & \quad \downarrow + \\
\mathbf{u}_1 + \mathbf{u}_2 & \xrightarrow{T} T(\mathbf{u}_1) + T(\mathbf{u}_2), \\
& \quad T(\mathbf{u}_1 + \mathbf{u}_2)
\end{align*}
\]

A couple of words about notation. \( T \) is the name of the linear transformation, and should be used when we want to discuss the function as a whole. \( T(\mathbf{u}) \) is how we talk about the output of the function, it is a vector in the vector space \( V \). When we write \( T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) \), the plus sign on the left is the operation of vector addition in the vector space \( U \), since \( \mathbf{x} \) and \( \mathbf{y} \) are elements of \( U \). The plus sign on the right is the operation of vector addition in the vector space \( V \), since \( T(\mathbf{x}) \) and \( T(\mathbf{y}) \) are elements of the vector space \( V \). These two instances of vector addition might be wildly different.

Let’s examine several examples and begin to form a catalog of known linear transformations to work with.

Example ALT

A linear transformation

Define \( T: \mathbb{C}^3 \to \mathbb{C}^2 \) by describing the output of the function for a generic input with the formula

\[
T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_3 \\ -4x_2 \end{bmatrix}
\]

and check the two defining properties.

\[
T(\mathbf{x} + \mathbf{y}) = T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + T \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}
= T \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}
= \begin{bmatrix} 2(x_1 + y_1) + (x_3 + y_3) \\ -4(x_2 + y_2) \end{bmatrix}
= \begin{bmatrix} 2x_1 + x_3 + 2y_1 + y_3 \\ -4x_2 + (-4)y_2 \end{bmatrix}
= \begin{bmatrix} 2x_1 + x_3 \\ -4x_2 \end{bmatrix} + \begin{bmatrix} 2y_1 + y_3 \\ -4y_2 \end{bmatrix}
= T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + T \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}
= T(\mathbf{x}) + T(\mathbf{y})
\]
and

\[ T(\alpha \mathbf{x}) = T \left( \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = T \left( \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix} \right) = \begin{bmatrix} 2(\alpha x_1) + (\alpha x_3) \\ -4(\alpha x_2) \end{bmatrix} = \begin{bmatrix} \alpha(2x_1 + x_3) \\ \alpha(-4x_2) \end{bmatrix} = \alpha \begin{bmatrix} 2x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \alpha T(\mathbf{x}) \]

So by Definition LT [405], \( T \) is a linear transformation.

It can be just as instructive to look at functions that are \textit{not} linear transformations. Since the defining conditions must be true for \textit{all} vectors and scalars, it is enough to find just one situation where the properties fail.

**Example NLT**

**Not a linear transformation**

Define \( S : \mathbb{C}^3 \rightarrow \mathbb{C}^3 \) by

\[ S \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 4x_1 + 2x_2 \\ 0 \\ x_1 + 3x_3 - 2 \end{bmatrix} \]

This function “looks” linear, but consider

\[ 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 24 \\ 0 \\ 24 \end{bmatrix} \]

while

\[ 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = S \left( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 24 \\ 0 \\ 28 \end{bmatrix} \]

So the second required property fails for the choice of \( \alpha = 3 \) and \( \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) and by Definition LT [405], \( S \) is not a linear transformation. It is just about as easy to find an example where the first defining property fails (try it!). Notice that it is the “-2” in the third component of the definition of \( S \) that prevents the function from being a linear transformation.

**Example LTPM**

**Linear transformation, polynomials to matrices**

Define a linear transformation \( T : P_3 \rightarrow M_{22} \) by

\[ T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a + b & a - 2c \\ d & b - d \end{bmatrix} \]
We verify the two defining conditions of a linear transformation.

\[ T(x + y) = T((a_1 + b_1)x + c_1x^2 + d_1x^3) + (a_2 + b_2x + c_2x^2 + d_2x^3) \]
\[ = T((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 + (d_1 + d_2)x^3) \]
\[ = \begin{bmatrix} (a_1 + a_2) & (b_1 + b_2) \\ (d_1 + d_2) & (b_1 + b_2) \end{bmatrix} \begin{bmatrix} a_1 + 2c_1 \\ b_1 - d_1 \end{bmatrix} + \begin{bmatrix} a_2 + b_2 & a_2 - 2c_2 \\ d_2 & b_2 - d_2 \end{bmatrix} \]
\[ = T((a_1 + 2c_1) + (b_1 - d_1)x^2 + d_1x^3) + T((a_2 + b_2)x + c_2x^2 + d_2x^3) \]
\[ = T(x) + T(y) \]

and

\[ T(ax) = T(ax) \]
\[ = T((aa) + (ab)x + (ac)x^2 + (ad)x^3) \]
\[ = \begin{bmatrix} (aa) + (ab) \\ (ad) \end{bmatrix} \begin{bmatrix} a + 2c \\ a - d \end{bmatrix} \]
\[ = \begin{bmatrix} a + b & a - 2c \\ d & b - d \end{bmatrix} \]
\[ = aT(x) \]

So by Definition LT 405, \( T \) is a linear transformation.

**Example LTTP**
Linear transformation, polynomials to polynomials

Define a function \( S: P_1 \rightarrow P_5 \) by

\[ S(p(x)) = (x - 2)p(x) \]

Then

\[ S(p(x) + q(x)) = (x - 2)(p(x) + q(x)) = (x - 2)p(x) + (x - 2)q(x) = S(p(x)) + S(q(x)) \]
\[ S(\alpha p(x)) = (x - 2)(\alpha p(x)) = (x - 2)\alpha p(x) = \alpha(x - 2)p(x) = \alpha S(p(x)) \]

So by Definition LT 405, \( S \) is a linear transformation.

Linear transformations have many amazing properties, which we will investigate through the next few sections. However, as a taste of things to come, here is a theorem we can prove now and put to use immediately.

**Theorem LTTZZ**
**Linear Transformations Take Zero to Zero**

Suppose \( T: U \rightarrow V \) is a linear transformation. Then \( T(0) = 0 \).

**Proof**  The two zero vectors in the conclusion of the theorem are different. The first is from \( U \) while the second is from \( V \). We will subscript the zero vectors in this proof to highlight the distinction. Think about your objects. (This proof is contributed by Mark Shoemaker).

\[ T(0_U) = T(0U) \]
\[ = 0T(0_U) \]
\[ = 0 \]
\[ \text{Theorem ZSSM 258 in } U \]
\[ \text{Definition LT 405} \]
Return to Example NLT 407 and compute $S \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ to quickly see again that $S$ is not a linear transformation, while in Example LTPM 407 and compute $S(0 + 0x + 0x^2 + 0x^3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ as an example of Theorem LTTZZ 408 at work.

Subsection MLT
Matrices and Linear Transformations

If you give me a matrix, then I can quickly build you a linear transformation. Always. First a motivating example and then the theorem.

Example LTM
Linear transformation from a matrix
Let $A = \begin{bmatrix} 3 & -1 & 8 & 1 \\ 2 & 0 & 5 & -2 \\ 1 & 1 & 3 & -7 \end{bmatrix}$ and define a function $P: \mathbb{C}^4 \rightarrow \mathbb{C}^3$ by $P(x) = Ax$

So we are using an old friend, the matrix-vector product (Definition MVP 173) as a way to convert a vector with 4 components into a vector with 3 components. Applying Definition MVP 173 allows us to write the defining formula for $P$ in a slightly different form,

$$P(x) = Ax = \begin{bmatrix} 3 & -1 & 8 & 1 \\ 2 & 0 & 5 & -2 \\ 1 & 1 & 3 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix}$$

So we recognize the action of the function $P$ as using the components of the vector $(x_1, x_2, x_3, x_4)$ as scalars to form the output of $P$ as a linear combination of the four columns of the matrix $A$, which are all members of $\mathbb{C}^3$, so the result is a vector in $\mathbb{C}^3$. We can rearrange this expression further, using our definitions of operations in $\mathbb{C}^3$ (Section VO 72).

$$P(x) = Ax = \begin{bmatrix} 3 & -1 & 8 & 1 \\ 2 & 0 & 5 & -2 \\ 1 & 1 & 3 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix}$$

You might recognize this final expression as being similar in style to some previous examples (Example ALT 406) and some linear transformations defined in the archetypes (Archetype M 683 through Archetype R 695). But the expression that says the output of this linear transformation
is a linear combination of the columns of $A$ is probably the most powerful way of thinking about examples of this type.

Almost forgot — we should verify that $P$ is indeed a linear transformation. This is easy with two matrix properties from Section MM [173].

$$P(x + y) = A(x + y)$$  \hspace{1cm} \text{Definition of } P
$$= Ax + Ay$$  \hspace{1cm} \text{Theorem MMDAA [179]}$$= P(x) + P(y)$$  \hspace{1cm} \text{Definition of } P$$

and

$$P(\alpha x) = A(\alpha x)$$  \hspace{1cm} \text{Definition of } P
$$= \alpha (Ax)$$  \hspace{1cm} \text{Theorem MMSMM [180]}$$= \alpha P(x)$$  \hspace{1cm} \text{Definition of } P$$

So by Definition LT [405], $P$ is a linear transformation.

So the multiplication of a vector by a matrix “transforms” the input vector into an output vector, possibly of a different size, by performing a linear combination. And this transformation happens in a “linear” fashion. This “functional” view of the matrix-vector product is the most important shift you can make right now in how you think about linear algebra. Here’s the theorem, whose proof is very nearly an exact copy of the verification in the last example.

**Theorem MBLT**
Matrices Build Linear Transformations
Suppose that $A$ is an $m \times n$ matrix. Define a function $T: \mathbb{C}^n \mapsto \mathbb{C}^m$ by $T(x) = Ax$. Then $T$ is a linear transformation. □

**Proof**

$$T(x + y) = A(x + y)$$  \hspace{1cm} \text{Definition of } T
$$= Ax + Ay$$  \hspace{1cm} \text{Theorem MMDAA [179]}$$= T(x) + T(y)$$  \hspace{1cm} \text{Definition of } T$$

and

$$T(\alpha x) = A(\alpha x)$$  \hspace{1cm} \text{Definition of } T
$$= \alpha (Ax)$$  \hspace{1cm} \text{Theorem MMSMM [180]}$$= \alpha T(x)$$  \hspace{1cm} \text{Definition of } T$$

So by Definition LT [405], $T$ is a linear transformation. □

So Theorem MBLT [410] gives us a rapid way to construct linear transformations. Grab an $m \times n$ matrix $A$, define $T(x) = Ax$ and Theorem MBLT [410] tells us that $T$ is a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^m$, without any further checking.

We can turn Theorem MBLT [410] around. You give me a linear transformation and I will give you a matrix.

**Example MFLT**
Matrix from a linear transformation
Define the function $R: \mathbb{C}^3 \mapsto \mathbb{C}^4$ by

$$R \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 - 3x_2 + 4x_3 \\ x_1 + x_2 + x_3 \\ -x_1 + 5x_2 - 3x_3 \\ x_2 - 4x_3 \end{bmatrix}$$
You could verify that $R$ is a linear transformation by applying the definition, but we will instead massage the expression defining a typical output until we recognize the form of a known class of linear transformations.

$$R\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 2x_1 - 3x_2 + 4x_3 \\ x_1 + x_2 + x_3 \\ -x_1 + 5x_2 - 3x_3 \\ x_2 - 4x_3 \end{bmatrix} = \begin{pmatrix} 2x_1 \\ x_1 \\ -x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} -3x_2 \\ x_2 \\ 5x_2 \\ x_2 \end{pmatrix} + \begin{pmatrix} 4x_3 \\ x_3 \\ -3x_3 \\ -4x_3 \end{pmatrix}$$

Definition CVA [73]

$$= x_1 \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 5 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ 1 \\ -3 \\ -4 \end{pmatrix}$$

Definition CVSM [74]

$$= \begin{pmatrix} 2 & -3 & 4 \\ 1 & 1 & 1 \\ -1 & 5 & -3 \\ 0 & 1 & -4 \end{pmatrix}$$

Definition MVP [173]

So if we define the matrix

$$B = \begin{pmatrix} 2 & -3 & 4 \\ 1 & 1 & 1 \\ -1 & 5 & -3 \\ 0 & 1 & -4 \end{pmatrix}$$

then $R(x) = Bx$. By Theorem MBLT [410], we can easily recognize $R$ as a linear transformation since it has the form described in the hypothesis of the theorem.

Example MFLT [410] was not accident. Consider any one of the archetypes where both the domain and codomain are sets of column vectors (Archetype M [683] through Archetype R [695]) and you should be able to mimic the previous example. Here’s the theorem, which is notable since it is our first occasion to use the full power of the defining properties of a linear transformation when our hypothesis includes a linear transformation.

**Theorem MLTCV**
Matrix of a Linear Transformation, Column Vectors
Suppose that $T : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix $A$ such that $T(x) = Ax$.

**Proof** The conclusion says a certain matrix exists. What better way to prove something exists than to actually build it? So our proof will be constructive, and the procedure that we will use abstractly in the proof can be used concretely in specific examples.

Let $e_1, e_2, e_3, \ldots, e_n$ be the columns of the identity matrix of size $n$, $I_n$ (Definition SUV [190]). Evaluate the linear transformation $T$ with each of these standard unit vectors as an input, and record the result. In other words, define $n$ vectors in $\mathbb{C}^m$, $A_i$, $1 \leq i \leq n$ by

$$A_i = T(e_i)$$

Then package up these vectors as the columns of a matrix

$$A = [A_1 | A_2 | A_3 | \ldots | A_n]$$

Does $A$ have the desired properties? First, $A$ is clearly an $m \times n$ matrix. Then

$$T(x) = T(I_n x) = T([e_1 | e_2 | e_3 | \ldots | e_n] x)$$

Theorem MMIM [179]

Definition SUV [190]
\[ T ([x_1] e_1 + [x_2] e_2 + [x_3] e_3 + \cdots + [x_n] e_n) = T ([x_1] e_1) + T ([x_2] e_2) + T ([x_3] e_3) + \cdots + T ([x_n] e_n) \]
\[ = [x_1] T (e_1) + [x_2] T (e_2) + [x_3] T (e_3) + \cdots + [x_n] T (e_n) \]
\[ = [x_1] A_1 + [x_2] A_2 + [x_3] A_3 + \cdots + [x_n] A_n \]

Definition MVP \[ 173 \]

as desired.

So if we were to restrict our study of linear transformations to those where the domain and codomain are both vector spaces of column vectors (Definition VSCV \[ 72 \]), every matrix leads to a linear transformation of this type (Theorem MBLT \[ 410 \]), while every such linear transformation leads to a matrix (Theorem MLTCV \[ 411 \]). So matrices and linear transformations are fundamentally the same. We call the matrix \( A \) of Theorem MLTCV \[ 411 \] the matrix representation of \( T \).

We have defined linear transformations for more general vector spaces than just \( \mathbb{C}^m \), can we extend this correspondence between linear transformations and matrices to more general linear transformations (more general domains and codomains)? Yes, and this is the main theme of Chapter R \[ 473 \]. Stay tuned. For now, let’s illustrate Theorem MLTCV \[ 411 \] with an example.

Example MOLT
Matrix of a linear transformation
Suppose \( S : \mathbb{C}^3 \rightarrow \mathbb{C}^4 \) is defined by
\[
S \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 3x_1 - 2x_2 + 5x_3 \\ x_1 + x_2 + x_3 \\ 9x_1 - 2x_2 + 5x_3 \\ 4x_2 \end{bmatrix}
\]

Then
\[
C_1 = S (e_1) = S \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 1 \\ 9 \\ 0 \end{bmatrix}
\]
\[
C_2 = S (e_2) = S \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 1 \\ -2 \\ 4 \end{bmatrix}
\]
\[
C_3 = S (e_3) = S \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 1 \\ 5 \\ 0 \end{bmatrix}
\]

so define
\[
C = [C_1|C_2|C_3] = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 1 & 1 \\ 9 & -2 & 5 \\ 0 & 4 & 0 \end{bmatrix}
\]

and Theorem MLTCV \[ 411 \] guarantees that \( S (x) = C x \).

As an illuminating exercise, let \( z = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} \) and compute \( S (z) \) two different ways. First, return to the definition of \( S \) and evaluate \( S (z) \) directly. Then do the matrix-vector product \( C z \). In both cases you should obtain the vector \( S (z) = \begin{bmatrix} 27 \\ 2 \\ 39 \\ -12 \end{bmatrix} \).
Subsection LT.LTLC  
Linear Transformations and Linear Combinations

It is the interaction between linear transformations and linear combinations that lies at the heart of many of the important theorems of linear algebra. The next theorem distills the essence of this. The proof is not deep, the result is hardly startling, but it will be referenced frequently. We have already passed by one occasion to employ it, in the proof of Theorem MLTCV. Paraphrasing, this theorem says that we can “push” linear transformations “down into” linear combinations, or “pull” linear transformations “up out” of linear combinations. We’ll have opportunities to both push and pull.

**Theorem LTLC**  
Linear Transformations and Linear Combinations

Suppose that $T: U \rightarrow V$ is a linear transformation, $u_1, u_2, u_3, \ldots, u_t$ are vectors from $U$ and $a_1, a_2, a_3, \ldots, a_t$ are scalars from $C$. Then

$$T(a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_tu_t) = a_1T(u_1) + a_2T(u_2) + a_3T(u_3) + \cdots + a_tT(u_t)$$

□

**Proof**

$$T(a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_tu_t)$$

$$= T(a_1u_1) + T(a_2u_2) + T(a_3u_3) + \cdots + T(a_tu_t) \quad \text{Definition LT}$$

$$= a_1T(u_1) + a_2T(u_2) + a_3T(u_3) + \cdots + a_tT(u_t) \quad \text{Definition LT}$$

□

Our next theorem says, informally, that it is enough to know how a linear transformation behaves for inputs from a basis of the domain, and all other outputs are described by a linear combination of these values. Again, the theorem and its proof are not remarkable, but the insight that goes along with it is fundamental.

**Theorem LTDB**  
Linear Transformation Defined on a Basis

Suppose that $T: U \rightarrow V$ is a linear transformation, $B = \{u_1, u_2, u_3, \ldots, u_n\}$ is a basis for $U$ and $w$ is a vector from $U$. Let $a_1, a_2, a_3, \ldots, a_n$ be the scalars from $C$ such that

$$w = a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_nu_n$$

Then

$$T(w) = a_1T(u_1) + a_2T(u_2) + a_3T(u_3) + \cdots + a_nT(u_n)$$

□

**Proof** For any $w \in U$, Theorem VRRB says there are (unique) scalars such that $w$ is a linear combination of the basis vectors in $B$. The result then follows from a straightforward application of Theorem LTLC to the linear combination.

**Example LTDB1**  
Linear transformation defined on a basis

Suppose you are told that $T: C^3 \rightarrow C^2$ is a linear transformation and given the three values,

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \quad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$
Because

\[ B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} \]

is a basis for \( \mathbb{C}^3 \) (Theorem SUVB [294]), Theorem LTDB [413] says we can compute any output of \( T \) with just this information. For example, consider,

\[ w = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = (2) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

so

\[ T(w) = (2) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 13 \\ -10 \end{bmatrix} \]

Doing it again,

\[ w = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = (5) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

so

\[ T(w) = (5) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 \\ 13 \end{bmatrix} \]

Any other value of \( T \) could be computed in a similar manner. So rather than being given a formula for the outputs of \( T \), the requirement that \( T \) behave as a linear transformation, along with its values on a handful of vectors (the basis), are just as sufficient as a formula for computing any value of the function. You might notice some parallels between this example and Example MOLT [412] or Theorem MLTCV [411].

Example LTDB2
Linear transformation defined on a basis
Suppose you are told that \( R: \mathbb{C}^3 \rightarrow \mathbb{C}^2 \) is a linear transformation and given the three values,

\[ R \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}, \quad R \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \quad R \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

You can check that

\[ D = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \]

is a basis for \( \mathbb{C}^3 \) (make the vectors the columns of a square matrix and check that the matrix is nonsingular, Theorem CNMB [299]). By Theorem LTDB [413] we can compute any output of \( R \) with just this information. However, we have to work just a bit harder to take an input vector and express it as a linear combination of the vectors in \( D \). For example, consider,

\[ y = \begin{bmatrix} 8 \\ -3 \\ 5 \end{bmatrix} \]

Then we must first write \( y \) as a linear combination of the vectors in \( D \) and solve for the unknown scalars, to arrive at

\[ y = \begin{bmatrix} 8 \\ -3 \\ 5 \end{bmatrix} = (3) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \]

Then Theorem LTDB [413] gives us

\[ R(y) = (3) \begin{bmatrix} 5 \\ -1 \end{bmatrix} + (-2) \begin{bmatrix} 0 \\ 4 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 17 \\ -8 \end{bmatrix} \]
Any other value of $R$ could be computed in a similar manner.

Here is a third example of a linear transformation defined by its action on a basis, only with more abstract vector spaces involved.

**Example LTDB3**

**Linear transformation defined on a basis**

The set $W = \{ p(x) \in P_3 \mid p(1) = 0, p(3) = 0 \} \subseteq P_3$. This subspace has $C = \{ 3 - 4x + x^2, 12 - 13x + x^3 \}$ as a basis (check this!). Suppose we *define* a linear transformation $S : P_3 \mapsto M_{22}$ by the values

$$S(3 - 4x + x^2) = \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix}, \quad S(12 - 13x + x^3) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

To illustrate a sample computation of $S$, consider $q(x) = 9 - 6x - 5x^2 + 2x^3$. Verify that $q(x)$ is an element of $W$ (does it have roots at $x = 1$ and $x = 3$?), then find the scalars needed to write it as a linear combination of the basis vectors in $C$. Because

$q(x) = 9 - 6x - 5x^2 + 2x^3 = (-5)(3 - 4x + x^2) + (2)(12 - 13x + x^3)$

**Theorem LTDB** [413] gives us

$$S(q) = (-5) \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix} + (2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -5 & 17 \\ -8 & 0 \end{pmatrix}$$

And all the other outputs of $S$ could be computed in the same manner. Every output of $S$ will have a zero in the second row, second column. Can you see why this is so? ☑

**Subsection PI**

**Pre-Images**

The definition of a function requires that for each input in the domain there is *exactly* one output in the codomain. However, the correspondence does not have to behave the other way around. A member of the codomain might have many inputs from the domain that create it, or it may have none at all. To formalize our discussion of this aspect of linear transformations, we define the pre-image.

**Definition PI**

**Pre-Image**

Suppose that $T : U \mapsto V$ is a linear transformation. For each $v$, define the pre-image of $v$ to be the subset of $U$ given by

$$T^{-1}(v) = \{ u \in U \mid T(u) = v \}$$

△

In other words, $T^{-1}(v)$ is the set of all those vectors in the domain $U$ that get “sent” to the vector $v$.

TODO: All preimages form a partition of $U$, an equivalence relation is about. Maybe to exercises.

**Example SPIAS**

**Sample pre-images, Archetype S**

Archetype S [698] is the linear transformation defined by

$$T : \mathbb{C}^3 \mapsto M_{22}, \quad T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a - b & 2a + 2b + c \\ 3a + b + c & -2a - 6b - 2c \end{pmatrix}$$
We could compute a pre-image for every element of the codomain $M_{22}$. However, even in a free textbook, we do not have the room to do that, so we will compute just two.

Choose

$$\mathbf{v} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \in M_{22}$$

for no particular reason. What is $T^{-1}(\mathbf{v})$? Suppose $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in T^{-1}(\mathbf{v})$. That $\mathbf{T}(\mathbf{u}) = \mathbf{v}$ becomes

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \mathbf{v} = \mathbf{T}(\mathbf{u}) = \mathbf{T} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 - u_2 & 2u_1 + 2u_2 + u_3 \\ 3u_1 + u_2 + u_3 & -2u_1 - 6u_2 - 2u_3 \end{bmatrix}$$

Using matrix equality (Definition ME [163]), we arrive at a system of four equations in the three unknowns $u_1, u_2, u_3$ with an augmented matrix that we can row-reduce in the hunt for solutions,

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & 2 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ -2 & -6 & -2 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & \frac{1}{4} & \frac{5}{4} \\ 0 & 1 & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We recognize this system as having infinitely many solutions described by the single free variable $u_3$. Eventually obtaining the vector form of the solutions (Theorem VFSLS [88]), we can describe the preimage precisely as,

$$T^{-1}(\mathbf{v}) = \{ \mathbf{u} \in \mathbb{C}^3 \mid \mathbf{T}(\mathbf{u}) = \mathbf{v} \} = \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \mid u_1 = \frac{5}{4} - \frac{1}{4}u_3, u_2 = \frac{3}{4} - \frac{1}{4}u_3 \right\} = \left\{ \begin{bmatrix} \frac{5}{4} - \frac{1}{4}u_3 \\ \frac{3}{4} - \frac{1}{4}u_3 \\ u_3 \end{bmatrix} \mid u_3 \in \mathbb{C}^3 \right\} = \left\{ \begin{bmatrix} \frac{5}{4} \\ \frac{3}{4} \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix} \mid u_3 \in \mathbb{C}^3 \right\} = \left\{ \begin{bmatrix} \frac{5}{4} \\ \frac{3}{4} \\ 0 \end{bmatrix} + \left\langle \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix} \right\rangle \right\}$$

This last line is merely a suggestive way of describing the set on the previous line. You might create three or four vectors in the preimage, and evaluate $\mathbf{T}$ with each. Was the result what you expected? For a hint of things to come, you might try evaluating $\mathbf{T}$ with just the lone vector in the spanning set above. What was the result? Now take a look back at Theorem PSSPHS [93]. Hmmmm.

OK, let’s compute another preimage, but with a different outcome this time. Choose

$$\mathbf{v} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \in M_{22}$$

What is $T^{-1}(\mathbf{v})$? Suppose $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in T^{-1}(\mathbf{v})$. That $\mathbf{T}(\mathbf{u}) = \mathbf{v}$ becomes

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \mathbf{v} = \mathbf{T}(\mathbf{u}) = \mathbf{T} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1 - u_2 & 2u_1 + 2u_2 + u_3 \\ 3u_1 + u_2 + u_3 & -2u_1 - 6u_2 - 2u_3 \end{bmatrix}$$
Using matrix equality (Definition ME [163]), we arrive at a system of four equations in the three unknowns \( u_1, u_2, u_3 \) with an augmented matrix that we can row-reduce in the hunt for solutions,

\[
\begin{bmatrix}
1 & -1 & 0 & 1 \\
2 & 2 & 1 & 1 \\
3 & 1 & 1 & 2 \\
-2 & -6 & -2 & 4 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

By Theorem RCLS [45] we recognize this system as inconsistent. So no vector \( u \) is a member of \( T^{-1}(v) \) and so \( T^{-1}(v) = \emptyset \)

The preimage is just a set, it is almost never a subspace of \( U \) (you might think about just when \( T^{-1}(v) \) is a subspace, see Exercise ILT.T10 [436]). We will describe its properties going forward, and it will be central to the main ideas of this chapter.

### Subsection NLTFO

#### New Linear Transformations From Old

We can combine linear transformations in natural ways to create new linear transformations. So we will define these combinations and then prove that the results really are still linear transformations. First the sum of two linear transformations.

**Definition LTA**

**Linear Transformation Addition**

Suppose that \( T: U \rightarrow V \) and \( S: U \rightarrow V \) are two linear transformations with the same domain and codomain. Then their sum is the function \( T + S: U \rightarrow V \) whose outputs are defined by

\[
(T + S)(u) = T(u) + S(u)
\]

Notice that the first plus sign in the definition is the operation being defined, while the second one is the vector addition in \( V \). (Vector addition in \( U \) will appear just now in the proof that \( T + S \) is a linear transformation.) Definition LTA [417] only provides a function. It would be nice to know that when the constituents \( (T, S) \) are linear transformations, then so too is \( T + S \).

**Theorem SLTLT**

**Sum of Linear Transformations is a Linear Transformation**

Suppose that \( T: U \rightarrow V \) and \( S: U \rightarrow V \) are two linear transformations with the same domain and codomain. Then \( T + S: U \rightarrow V \) is a linear transformation. □

**Proof** We simply check the defining properties of a linear transformation (Definition LT [405]). This is a good place to consistently ask yourself which objects are being combined with which operations.

\[
(T + S)(x + y) = T(x + y) + S(x + y) = T(x) + T(y) + S(x) + S(y) = T(x) + S(x) + T(y) + S(y) = (T + S)(x) + (T + S)(y)
\]

and

\[
(T + S)(\alpha x) = T(\alpha x) + S(\alpha x) = \alpha T(x) + \alpha S(x)
\]
Example STLT
Sum of two linear transformations
Suppose that \( T: \mathbb{C}^2 \mapsto \mathbb{C}^3 \) and \( S: \mathbb{C}^2 \mapsto \mathbb{C}^3 \) are defined by

\[
T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 - 4x_2 \\ 5x_1 + 2x_2 \end{bmatrix} \\
S \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 4x_1 - x_2 \\ x_1 + 3x_2 \\ -7x_1 + 5x_2 \end{bmatrix}
\]

Then by Definition LTA 417, we have

\[
(T + S) \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) + S \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 5x_1 + x_2 \\ 4x_1 - x_2 \\ -2x_1 + 7x_2 \end{bmatrix}
\]

and by Theorem SLTLT 417 we know \( T + S \) is also a linear transformation from \( \mathbb{C}^2 \) to \( \mathbb{C}^3 \). □

Definition LTSM
Linear Transformation Scalar Multiplication
Suppose that \( T: U \mapsto V \) is a linear transformation and \( \alpha \in \mathbb{C} \). Then the scalar multiple is the function \( \alpha T: U \mapsto V \) whose outputs are defined by

\[
(\alpha T)(u) = \alpha T(u)
\]

Given that \( T \) is a linear transformation, it would be nice to know that \( \alpha T \) is also a linear transformation.

Theorem MLTLT
Multiple of a Linear Transformation is a Linear Transformation
Suppose that \( T: U \mapsto V \) is a linear transformation and \( \alpha \in \mathbb{C} \). Then \( (\alpha T): U \mapsto V \) is a linear transformation.

Proof We simply check the defining properties of a linear transformation (Definition LT 405). This is another good place to consistently ask yourself which objects are being combined with which operations.

\[
(\alpha T)(x + y) = \alpha (T(x + y)) \\
= \alpha (T(x) + T(y)) \\
= \alpha T(x) + \alpha T(y) \\
= (\alpha T)(x) + (\alpha T)(y)
\]

and

\[
(\alpha T)(\beta x) = \alpha T(\beta x) \\
= \alpha (\beta T(x)) \\
= (\alpha \beta) T(x) \\
= (\beta \alpha) T(x) \\
= \beta (\alpha T(x)) \\
= \beta ((\alpha T)(x))
\]
Example SMLT
Scalar multiple of a linear transformation
Suppose that \( T: \mathbb{C}^4 \mapsto \mathbb{C}^3 \) is defined by

\[
T \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right) = \begin{bmatrix} x_1 + 2x_2 - x_3 + 2x_4 \\ x_1 + 5x_2 - 3x_3 + x_4 \\ -2x_1 + 3x_2 - 4x_3 + 2x_4 \end{bmatrix}
\]

For the sake of an example, choose \( \alpha = 2 \), so by Definition LTSM 418, we have

\[
\alpha T \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right) = 2T \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right) = 2 \begin{bmatrix} x_1 + 2x_2 - x_3 + 2x_4 \\ x_1 + 5x_2 - 3x_3 + x_4 \\ -2x_1 + 3x_2 - 4x_3 + 2x_4 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 - 2x_3 + 4x_4 \\ 2x_1 + 10x_2 - 6x_3 + 2x_4 \\ -4x_1 + 6x_2 - 8x_3 + 4x_4 \end{bmatrix}
\]

and by Theorem MLTLT 418 we know \( 2T \) is also a linear transformation from \( \mathbb{C}^4 \) to \( \mathbb{C}^3 \). □

Now, let’s imagine we have two vector spaces, \( U \) and \( V \), and we collect every possible linear transformation from \( U \) to \( V \) into one big set, and call it \( LT(U, V) \). Definition LTA 417 and Definition LTSM 418 tell us how we can “add” and “scalar multiply” two elements of \( LT(U, V) \). Theorem SLTLT 417 and Theorem MLTLT 418 tell us that if we do these operations, then the resulting functions are linear transformations that are also in \( LT(U, V) \). Hmmmm, sounds like a vector space to me! A set of objects, an addition and a scalar multiplication. Why not?

Theorem VSLT
Vector Space of Linear Transformations
Suppose that \( U \) and \( V \) are vector spaces. Then the set of all linear transformations from \( U \) to \( V \), \( LT(U, V) \) is a vector space when the operations are those given in Definition LTA 417 and Definition LTSM 418. □

Proof Theorem SLTLT 417 and Theorem MLTLT 418 provide two of the ten properties in Definition VS 251. However, we still need to verify the remaining eight properties. By and large, the proofs are straightforward and rely on concocting the obvious object, or by reducing the question to the same vector space property in the vector space \( V \).

The zero vector is of some interest, though. What linear transformation would we add to any other linear transformation, so as to keep the second one unchanged? The answer is \( Z: U \mapsto V \) defined by \( Z(u) = 0_V \) for every \( u \in U \). Notice how we do not need to know any specifics about \( U \) and \( V \) to make this definition. □

Definition LTC
Linear Transformation Composition
Suppose that \( T: U \mapsto V \) and \( S: V \mapsto W \) are linear transformations. Then the composition of \( S \) and \( T \) is the function \( (S \circ T): U \mapsto W \) whose outputs are defined by

\[
(S \circ T)(u) = S(T(u))
\]

Given that \( T \) and \( S \) are linear transformations, it would be nice to know that \( S \circ T \) is also a linear transformation.

Theorem CLTLT
Composition of Linear Transformations is a Linear Transformation
Suppose that \( T: U \mapsto V \) and \( S: V \mapsto W \) are linear transformations. Then \( (S \circ T): U \mapsto W \) is a linear transformation. □

Proof We simply check the defining properties of a linear transformation (Definition LT 405).

\[
(S \circ T)(x + y) = S(T(x + y))
\]

Definition LTC 419
Then by Definition LTC \[419\]
\[S(\alpha x) = \alpha S(x)\]
by Theorem CLTLT \[419\]
\[(S \circ T)(\alpha x) = S(T(\alpha x))\]
\[= S(\alpha T(x))\]
\[=\alpha S(T(x))\]
\[=\alpha (S \circ T)(x)\]

\textbf{Example CTLT}

**Composition of two linear transformations**

Suppose that \(T: \mathbb{C}^2 \rightarrow \mathbb{C}^4\) and \(S: \mathbb{C}^4 \rightarrow \mathbb{C}^3\) are defined by

\[
T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 - 4x_2 \\ 5x_1 + 2x_2 \\ 6x_1 - 3x_2 \end{pmatrix}
\]

\[
S\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 + x_3 - x_4 \\ 5x_1 - 3x_2 + 8x_3 - 2x_4 \\ -4x_1 + 3x_2 - 4x_3 + 5x_4 \end{pmatrix}
\]

Then by Definition LTC \[419\]
\[(S \circ T)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = S(T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix})\]

\[
= S\begin{pmatrix} x_1 + 2x_2 \\ 3x_1 - 4x_2 \\ 5x_1 + 2x_2 \\ 6x_1 - 3x_2 \end{pmatrix}
\]

\[
= \begin{pmatrix} 2x_1 + 2x_2 - (3x_1 - 4x_2) + (5x_1 + 2x_2) - (6x_1 - 3x_2) \\ 5(x_1 + 2x_2) - 3(3x_1 - 4x_2) + 8(5x_1 + 2x_2) - 2(6x_1 - 3x_2) \\ -4(x_1 + 2x_2) + 3(3x_1 - 4x_2) - 4(5x_1 + 2x_2) + 5(6x_1 - 3x_2) \end{pmatrix}
\]

\[
= \begin{pmatrix} -2x_1 + 12x_2 \\ 24x_1 + 44x_2 \\ 15x_1 - 43x_2 \end{pmatrix}
\]

and by Theorem CLTLT \[419\] \(S \circ T\) is a linear transformation from \(\mathbb{C}^2\) to \(\mathbb{C}^3\). 

Here is an interesting exercise that will presage an important result later. In Example STLT \[418\] compute (via Theorem MLTCV \[411\]) the matrix of \(T\), \(S\) and \(T + S\). Do you see a relationship between these three matrices?

In Example SMLT \[419\] compute (via Theorem MLTCV \[411\]) the matrix of \(T\) and \(2T\). Do you see a relationship between these two matrices?

Here’s the tough one. In Example CTLT \[420\] compute (via Theorem MLTCV \[411\]) the matrix of \(T\), \(S\) and \(S \circ T\). Do you see a relationship between these three matrices???
2. Determine the matrix representation of the linear transformation $S$ below.

$$S: \mathbb{C}^2 \rightarrow \mathbb{C}^3, \quad S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 5x_2 \\ 8x_1 - 3x_2 \\ -4x_1 \end{bmatrix}$$

3. Theorem LTLC\[413\] has a fairly simple proof. Yet the result itself is very powerful. Comment on why we might say this.
Subsection EXC
Exercises

C15  The archetypes below are all linear transformations whose domains and codomains are vector spaces of column vectors (Definition VSCV [72]). For each one, compute the matrix representation described in the proof of Theorem MLTCV [411].

Archetype M [683]
Archetype N [685]
Archetype O [687]
Archetype P [690]
Archetype Q [692]
Archetype R [695]
Contributed by Robert Beezer

C20  Let \( \mathbf{w} = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} \). Referring to Example MOLT [412], compute \( S(\mathbf{w}) \) two different ways. First use the definition of \( S \), then compute the matrix-vector product \( C\mathbf{w} \) (Definition MVP [173]).
Contributed by Robert Beezer  Solution [424]

C25  Define the linear transformation \( T : \mathbb{C}^3 \to \mathbb{C}^2 \), \( T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{bmatrix} \)
Verify that \( T \) is a linear transformation.
Contributed by Robert Beezer  Solution [424]

C26  Verify that the function below is a linear transformation.
\[ T : P_2 \to \mathbb{C}^2, \quad T \left( a + bx + cx^2 \right) = \begin{bmatrix} 2a - b \\ b + c \end{bmatrix} \]
Contributed by Robert Beezer  Solution [424]

C30  Define the linear transformation
\[ T : \mathbb{C}^3 \to \mathbb{C}^2, \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{bmatrix} \]
Compute the preimages, \( T^{-1} \left( \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) \) and \( T^{-1} \left( \begin{bmatrix} 4 \\ -8 \end{bmatrix} \right) \).
Contributed by Robert Beezer  Solution [424]

C31  For the linear transformation \( S \) compute the pre-images.
\[ S : \mathbb{C}^3 \to \mathbb{C}^3, \quad S \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} a - 2b - c \\ 3a - b + 2c \\ a + b + 2c \end{bmatrix} \]
\[ S^{-1} \left( \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} \right) \quad S^{-1} \left( \begin{bmatrix} -5 \\ 5 \\ 7 \end{bmatrix} \right) \]
Contributed by Robert Beezer  Solution [425]
M10 Define two linear transformations, $T: \mathbb{C}^4 \to \mathbb{C}^3$ and $S: \mathbb{C}^3 \to \mathbb{C}^2$ by

$$S \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 - 2x_2 + 3x_3 \\ 5x_1 + 4x_2 + 2x_3 \end{bmatrix}$$

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} -x_1 + 3x_2 + x_3 + 9x_4 \\ 2x_1 + x_3 + 7x_4 \\ 4x_1 + 2x_2 + x_3 + 2x_4 \end{bmatrix}$$

Using the proof of Theorem MLTCV compute the matrix representations of the three linear transformations $T$, $S$ and $S \circ T$. Discover and comment on the relationship between these three matrices.

Contributed by Robert Beezer Solution
In both cases the result will be \( S(w) = \begin{bmatrix} 2 \\ -9 \\ 4 \end{bmatrix} \).

We can rewrite \( T \) as follows:

\[
T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ -10 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -2 \\ -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

and Theorem MBLT [410] tell us that any function of this form is a linear transformation.

Check the two conditions of Definition LT [405].

\[
T(u + v) = T((a + bx + cx^2) + (d + ex + fx^2)) = T((a + d) + (b + e)x + (c + f)x^2)
\]

\[
= \begin{bmatrix} 2(a + d) - (b + e) \\ (b + e) + (c + f) \end{bmatrix}
\]

\[
= \begin{bmatrix} (2a - b) + (2d - e) \\ (b + c) + (e + f) \end{bmatrix}
\]

\[
= 2a - b + 2d - e + b + c + e + f
\]

\[
= T(u) + T(v)
\]

and

\[
T(\alpha u) = T(\alpha(a + bx + cx^2)) = T((\alpha a) + (\alpha b)x + (\alpha c)x^2)
\]

\[
= \begin{bmatrix} 2(\alpha a) - (\alpha b) \\ (\alpha b) + (\alpha c) \end{bmatrix}
\]

\[
= \alpha \begin{bmatrix} 2a - b \\ b + c \end{bmatrix}
\]

\[
= \alpha T(u)
\]

So \( T \) is indeed a linear transformation.

For the first pre-image, we want \( x \in \mathbb{C}^3 \) such that \( T(x) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \). This becomes,

\[
\begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}
\]

Vector equality gives a system of two linear equations in three variables, represented by the augmented matrix

\[
\begin{bmatrix} 2 & -1 & 5 & 2 \\ -4 & 2 & -10 & 3 \end{bmatrix} \rightarrow RREF \begin{bmatrix} 1 & -\frac{1}{2} & \frac{5}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]
so the system is inconsistent and the pre-image is the empty set. For the second pre-image the same procedure leads to an augmented matrix with a different vector of constants

\[
\begin{bmatrix}
2 & -1 & 5 & 4 \\
-4 & 2 & -10 & -8
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & -\frac{1}{2} & \frac{5}{2} & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

This system is consistent and has infinitely many solutions, as we can see from the presence of the two free variables \((x_2 \text{ and } x_3)\) both to zero. We apply Theorem VFSLS \([88]\) to obtain

\[
T^{-1}\left(\begin{bmatrix} 4 \\ -8 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -\frac{5}{7} \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \mid x_2, x_3 \in \mathbb{C} \right\}
\]

C31 Contributed by Robert Beezer Statement \([422]\)

We work from the definition of the pre-image, Definition PI \([415]\). Setting

\[
S \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}
\]

we arrive at a system of three equations in three variables, with an augmented matrix that we row-reduce in a search for solutions,

\[
\begin{bmatrix}
1 & -2 & -1 \\
3 & -1 & 2 \\
1 & 1 & 2
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

With a leading 1 in the last column, this system is inconsistent (Theorem RCLS \([45]\)), and there are no values of \(a, b\) and \(c\) that will create an element of the pre-image. So the preimage is the empty set.

We work from the definition of the pre-image, Definition PI \([415]\). Setting

\[
S \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \\ 7 \end{bmatrix}
\]

we arrive at a system of three equations in three variables, with an augmented matrix that we row-reduce in a search for solutions,

\[
\begin{bmatrix}
1 & -2 & -1 & -5 \\
3 & -1 & 2 & 5 \\
1 & 1 & 2 & 7
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 0 & 3 \\
0 & 1 & 1 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The solution set to this system, which is also the desired pre-image, can be expressed using the vector form of the solutions (Theorem VFSLS \([88]\))

\[
S^{-1}\left(\begin{bmatrix} -5 \\ 5 \\ 7 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \mid c \in \mathbb{C} \right\} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \langle\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\} \rangle
\]

Does the final expression for this set remind you of Theorem KPI \([431]\)?

M10 Contributed by Robert Beezer Statement \([423]\)

\[
\begin{bmatrix}
1 & -2 & 3 \\
5 & 4 & 2
\end{bmatrix}
\begin{bmatrix}
-1 & 3 & 1 & 9 \\
2 & 0 & 1 & 7 \\
4 & 2 & 1 & 2
\end{bmatrix}
= \begin{bmatrix}
7 & 9 & 2 & 1 \\
11 & 19 & 11 & 77
\end{bmatrix}
\]
Section ILT
Injective Linear Transformations

Some linear transformations possess one, or both, of two key properties, which go by the names injective and surjective. We will see that they are closely related to ideas like linear independence and spanning, and subspaces like the null space and the column space. In this section we will define an injective linear transformation and analyze the resulting consequences. The next section will do the same for the surjective property. In the final section of this chapter we will see what happens when we have the two properties simultaneously.

As usual, we lead with a definition.

Definition ILT
Injective Linear Transformation
Suppose \( T: U \rightarrow V \) is a linear transformation. Then \( T \) is injective if whenever \( T(x) = T(y) \), then \( x = y \).

Given an arbitrary function, it is possible for two different inputs to yield the same output (think about the function \( f(x) = x^2 \) and the inputs \( x = 3 \) and \( x = -3 \)). For an injective function, this never happens. If we have equal outputs \( (T(x) = T(y)) \) then we must have achieved those equal outputs by employing equal inputs \( (x = y) \). Some authors prefer the term one-to-one where we use injective, and we will sometimes refer to an injective linear transformation as an injection.

Subsection EILT
Examples ofInjective Linear Transformations

It is perhaps most instructive to examine a linear transformation that is not injective first.

Example NIAQ
Not injective, Archetype Q
Archetype Q [692] is the linear transformation

\[
T: \mathbb{C}^5 \rightarrow \mathbb{C}^5, \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_1 + 3x_2 + 3x_3 - 6x_4 + 3x_5 \\ -16x_1 + 9x_2 + 12x_3 - 28x_4 + 28x_5 \\ -19x_1 + 7x_2 + 14x_3 - 32x_4 + 37x_5 \\ -21x_1 + 9x_2 + 15x_3 - 35x_4 + 39x_5 \\ -9x_1 + 5x_2 + 7x_3 - 16x_4 + 16x_5 \end{pmatrix}
\]

Notice that for

\[
x = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 4 \end{pmatrix}, \quad y = \begin{pmatrix} 4 \\ 7 \\ 0 \\ 5 \\ 7 \end{pmatrix}
\]

we have

\[
T\begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 47 \\ 55 \\ 72 \\ 77 \\ 31 \end{pmatrix}, \quad T\begin{pmatrix} 4 \\ 7 \\ 0 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 47 \\ 72 \\ 77 \\ 31 \end{pmatrix}
\]

So we have two vectors from the domain, \( x \neq y \), yet \( T(x) = T(y) \), in violation of Definition ILT [426]. This is another example where you should not concern yourself with how \( x \) and \( y \) were
selected, as this will be explained shortly. However, do understand why these two vectors provide enough evidence to conclude that $T$ is not injective.

To show that a linear transformation is not injective, it is enough to find a single pair of inputs that get sent to the identical output, as in Example NIAQ 426. However, to show that a linear transformation is injective we must establish that this coincidence of outputs never occurs. Here is an example that shows how to establish this.

**Example IAR**

**Injective, Archetype R**

Archetype R 695 is the linear transformation

$$T: \mathbb{C}^5 \rightarrow \mathbb{C}^5, \quad T \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \right) = \left( \begin{array}{c} -65x_1 + 128x_2 + 10x_3 - 262x_4 + 40x_5 \\ 36x_1 - 73x_2 - x_3 + 151x_4 - 16x_5 \\ -44x_1 + 88x_2 + 5x_3 - 180x_4 + 24x_5 \\ 34x_1 - 68x_2 - 3x_3 + 140x_4 - 18x_5 \\ 12x_1 - 24x_2 - x_3 + 49x_4 - 5x_5 \end{array} \right)$$

To establish that $R$ is injective we must begin with the assumption that $T(x) = T(y)$ and somehow arrive from this at the conclusion that $x = y$. Here we go,

$$T(x) = T(y)$$

$$T \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \right) = T \left( \begin{array}{c} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{array} \right)$$

$$\left( \begin{array}{c} -65x_1 + 128x_2 + 10x_3 - 262x_4 + 40x_5 \\ 36x_1 - 73x_2 - x_3 + 151x_4 - 16x_5 \\ -44x_1 + 88x_2 + 5x_3 - 180x_4 + 24x_5 \\ 34x_1 - 68x_2 - 3x_3 + 140x_4 - 18x_5 \\ 12x_1 - 24x_2 - x_3 + 49x_4 - 5x_5 \end{array} \right) = \left( \begin{array}{c} -65y_1 + 128y_2 + 10y_3 - 262y_4 + 40y_5 \\ 36y_1 - 73y_2 - y_3 + 151y_4 - 16y_5 \\ -44y_1 + 88y_2 + 5y_3 - 180y_4 + 24y_5 \\ 34y_1 - 68y_2 - 3y_3 + 140y_4 - 18y_5 \\ 12y_1 - 24y_2 - y_3 + 49y_4 - 5y_5 \end{array} \right)$$

$$\left( \begin{array}{c} -65(x_1 - y_1) + 128(x_2 - y_2) + 10(x_3 - y_3) - 262(x_4 - y_4) + 40(x_5 - y_5) \\ 36(x_1 - y_1) - 73(x_2 - y_2) - (x_3 - y_3) + 151(x_4 - y_4) - 16(x_5 - y_5) \\ -44(x_1 - y_1) + 88(x_2 - y_2) + 5(x_3 - y_3) - 180(x_4 - y_4) + 24(x_5 - y_5) \\ 34(x_1 - y_1) - 68(x_2 - y_2) - 3(x_3 - y_3) + 140(x_4 - y_4) - 18(x_5 - y_5) \\ 12(x_1 - y_1) - 24(x_2 - y_2) - (x_3 - y_3) + 49(x_4 - y_4) - 5(x_5 - y_5) \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right)$$

Now we recognize that we have a homogeneous system of 5 equations in 5 variables (the terms $x_i - y_i$ are the variables), so we row-reduce the coefficient matrix to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
So the only solution is the trivial solution
\[ x_1 - y_1 = 0 \quad x_2 - y_2 = 0 \quad x_3 - y_3 = 0 \quad x_4 - y_4 = 0 \quad x_5 - y_5 = 0 \]
and we conclude that indeed \( x = y \). By Definition ILT 426, \( T \) is injective.

Let’s now examine an injective linear transformation between abstract vector spaces.

**Example IAV**

**Injective, Archetype V**

[Archetype V 704] is defined by

\[ T : P_3 \rightarrow M_{22}, \quad T (a + bx + cx^2 + dx^3) = \begin{bmatrix} a + b & a - 2c \\ d & b - d \end{bmatrix} \]

To establish that the linear transformation is injective, begin by supposing that two polynomial inputs yield the same output matrix,

\[ T (a_1 + b_1x + c_1x^2 + d_1x^3) = T (a_2 + b_2x + c_2x^2 + d_2x^3) \]

Then

\[
\mathcal{O} = \begin{bmatrix} 0 & 0 \\
0 & 0 \end{bmatrix} = T (a_1 + b_1x + c_1x^2 + d_1x^3) - T (a_2 + b_2x + c_2x^2 + d_2x^3) \quad \text{Hypothesis}
\]

\[ = T ((a_1 + b_1x + c_1x^2 + d_1x^3) - (a_2 + b_2x + c_2x^2 + d_2x^3)) \quad \text{Definition LT 405}
\]

\[ = T ((a_1 - a_2) + (b_1 - b_2)x + (c_1 - c_2)x^2 + (d_1 - d_2)x^3) \quad \text{Operations in } P_3
\]

\[ = \begin{bmatrix} (a_1 - a_2) + (b_1 - b_2) & (a_1 - a_2) - 2(c_1 - c_2) \\
(d_1 - d_2) & (b_1 - b_2) - (d_1 - d_2) \end{bmatrix} \quad \text{Definition of } T
\]

This single matrix equality translates to the homogeneous system of equations in the variables \( a_i - b_i \),

\[
\begin{align*}
(a_1 - a_2) + (b_1 - b_2) &= 0 \\
(a_1 - a_2) - 2(c_1 - c_2) &= 0 \\
(d_1 - d_2) &= 0 \\
(b_1 - b_2) - (d_1 - d_2) &= 0
\end{align*}
\]

This system of equations can be rewritten as the matrix equation

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & -2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
(a_1 - a_2) \\
b_1 - b_2 \\
(c_1 - c_2) \\
d_1 - d_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Since the coefficient matrix is nonsingular (check this) the only solution is trivial, i.e.

\[
a_1 - a_2 = 0 \quad b_1 - b_2 = 0 \quad c_1 - c_2 = 0 \quad d_1 - d_2 = 0
\]

so that

\[
a_1 = a_2 \quad b_1 = b_2 \quad c_1 = c_2 \quad d_1 = d_2
\]

so the two inputs must be equal polynomials. By Definition ILT 426, \( T \) is injective. \( \Box \)
Subsection KLT
Kernel of a Linear Transformation

For a linear transformation $T: U \rightarrow V$, the kernel is a subset of the domain $U$. Informally, it is the set of all inputs that the transformation sends to the zero vector of the codomain. It will have some natural connections with the null space of a matrix, so we will keep the same notation, and if you think about your objects, then there should be little confusion. Here’s the careful definition.

Definition KLT
Kernel of a Linear Transformation
Suppose $T: U \rightarrow V$ is a linear transformation. Then the kernel of $T$ is the set $K(T) = \{ u \in U \mid T(u) = 0 \}$. (This definition contains Notation KLT.)

Notice that the kernel of $T$ is just the preimage of $0$, $T^{-1}(0)$ (Definition PI [415]). Here’s an example.

Example NKAO
Nontrivial kernel, Archetype O
Archetype O [687] is the linear transformation

$$T: \mathbb{C}^3 \rightarrow \mathbb{C}^5, \quad T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} -x_1 + x_2 - 3x_3 \\ -x_1 + 2x_2 - 4x_3 \\ x_1 + x_2 + x_3 \\ 2x_1 + 3x_2 + x_3 \\ x_1 + 2x_3 \end{bmatrix}$$

To determine the elements of $\mathbb{C}^3$ in $K(T)$, find those vectors $u$ such that $T(u) = 0$, that is,

$$T(u) = 0$$

Vector equality (Definition CVE [73]) leads us to a homogeneous system of 5 equations in the variables $u_i$,

$$-u_1 + u_2 - 3u_3 = 0$$
$$-u_1 + 2u_2 - 4u_3 = 0$$
$$u_1 + u_2 + u_3 = 0$$
$$2u_1 + 3u_2 + u_3 = 0$$
$$u_1 + 2u_3 = 0$$

Row-reducing the coefficient matrix gives
The kernel of $T$ is the set of solutions to this homogeneous system of equations, which by Theorem BNS can be expressed as

$$\mathcal{K}(T) = \langle \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \rangle$$

We know that the span of a set of vectors is always a subspace (Theorem SSS), so the kernel computed in Example NKAO is also a subspace. This is no accident, the kernel of a linear transformation is always a subspace.

**Theorem KLTS**

**Kernel of a Linear Transformation is a Subspace**

Suppose that $T: U \mapsto V$ is a linear transformation. Then the kernel of $T$, $\mathcal{K}(T)$, is a subspace of $U$.

**Proof** We can apply the three-part test of Theorem TSS. First $T(0_U) = 0_V$ by Theorem LTTZZ, so $0_U \in \mathcal{K}(T)$ and we know that the kernel is non-empty.

Suppose we assume that $x, y \in \mathcal{K}(T)$. Is $x + y \in \mathcal{K}(T)$?

$$T(x + y) = T(x) + T(y) \quad \text{Definition LT}$$

$$= 0 + 0 \quad x, y \in \mathcal{K}(T)$$

$$= 0 \quad \text{Property Z}$$

This qualifies $x + y$ for membership in $\mathcal{K}(T)$. So we have additive closure.

Suppose we assume that $\alpha \in \mathbb{C}$ and $x \in \mathcal{K}(T)$. Is $\alpha x \in \mathcal{K}(T)$?

$$T(\alpha x) = \alpha T(x) \quad \text{Definition LT}$$

$$= \alpha 0 \quad x \in \mathcal{K}(T)$$

$$= 0 \quad \text{Theorem ZVSM}$$

This qualifies $\alpha x$ for membership in $\mathcal{K}(T)$. So we have scalar closure and Theorem TSS tells us that $\mathcal{K}(T)$ is a subspace of $U$.

Let’s compute another kernel, now that we know in advance that it will be a subspace.

**Example TKAP**

**Trivial kernel, Archetype P**

Archetype P is the linear transformation

$$T: \mathbb{C}^3 \mapsto \mathbb{C}^5, \quad T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 + x_3 \\ -x_1 + 2x_2 + 2x_3 \\ x_1 + x_2 + 3x_3 \\ 2x_1 + 3x_2 + x_3 \\ -2x_1 + x_2 + 3x_3 \end{bmatrix}$$

To determine the elements of $\mathbb{C}^3$ in $\mathcal{K}(T)$, find those vectors $u$ such that $T(u) = 0$, that is,

$$T(u) = 0$$

$$\begin{bmatrix} -u_1 + u_2 + u_3 \\ -u_1 + 2u_2 + 2u_3 \\ u_1 + u_2 + 3u_3 \\ 2u_1 + 3u_2 + u_3 \\ -2u_1 + u_2 + 3u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
Vector equality (Definition CVE [73]) leads us to a homogeneous system of 5 equations in the variables $u_i$,

\[-u_1 + u_2 + u_3 = 0\]
\[-u_1 + 2u_2 + 2u_3 = 0\]
\[u_1 + u_2 + 3u_3 = 0\]
\[2u_1 + 3u_2 + u_3 = 0\]
\[-2u_1 + u_2 + 3u_3 = 0\]

Row-reducing the coefficient matrix gives

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

The kernel of $T$ is the set of solutions to this homogeneous system of equations, which is simply the trivial solution $u = 0$, so

\[\mathcal{K}(T) = \{0\} = \{\}\]

Our next theorem says that if a preimage is a non-empty set then we can construct it by picking any one element and adding on elements of the kernel.

**Theorem KPI**

**Kernel and Pre-Image**

Suppose $T: U \mapsto V$ is a linear transformation and $v \in V$. If the preimage $T^{-1}(v)$ is non-empty, and $u \in T^{-1}(v)$ then

\[T^{-1}(v) = \{u + z | z \in \mathcal{K}(T)\} = u + \mathcal{K}(T)\]

**Proof** Let $M = \{u + z | z \in \mathcal{K}(T)\}$. First, we show that $M \subseteq T^{-1}(v)$. Suppose that $w \in M$, so $w$ has the form $w = u + z$, where $z \in \mathcal{K}(T)$. Then

\[T(w) = T(u + z) = T(u) + T(z) = v + 0 = v \quad \text{Definition LT [405]} \]

which qualifies $w$ for membership in the preimage of $v$, $w \in T^{-1}(v)$.

For the opposite inclusion, suppose $x \in T^{-1}(v)$. Then

\[T(x - u) = T(x) - T(u) = v - v = 0 \quad \text{Definition LT [405]} \]

This qualifies $x - u$ for membership in the kernel of $T$, $\mathcal{K}(T)$. So there is a vector $z \in \mathcal{K}(T)$ such that $x - u = z$. Rearranging this equation gives $x = u + z$ and so $x \in M$. So $T^{-1}(v) \subseteq M$ and we see that $M = T^{-1}(v)$, as desired.

This theorem, and its proof, should remind you very much of Theorem PSPHS [93]. Additionally, you might go back and review Example SPIAS [415]. Can you tell now which is the only preimage to be a subspace?
The next theorem is one we will cite frequently, as it characterizes injections by the size of the kernel.

**Theorem KILT**  
**Kernel of an Injective Linear Transformation**

Suppose that $T: U \rightarrow V$ is a linear transformation. Then $T$ is injective if and only if the kernel of $T$ is trivial, $\mathcal{K}(T) = \{0\}$.

**Proof** ($\Rightarrow$) Suppose $x \in \mathcal{K}(T)$. Then by Definition KLT, $T(x) = 0$. By Theorem LTTZZ, $T(0) = 0$. Now, since $T(x) = T(0)$, we can apply Definition ILT to conclude that $x = 0$. Therefore $\mathcal{K}(T) = \{0\}$.

($\Leftarrow$) To establish that $T$ is injective, appeal to Definition ILT and begin with the assumption that $T(x) = T(y)$. Then

\[
0 = T(x) - T(y) \quad \text{Hypothesis}
\]

\[
= T(x - y) \quad \text{Definition LT}
\]

so by Definition KLT and the hypothesis that the kernel is trivial,

\[
x - y \in \mathcal{K}(T) = \{0\}
\]

which means that

\[
0 = x - y
\]

\[
x = y
\]

thus establishing that $T$ is injective.

**Example NIAQR**  
**Not injective, Archetype Q, revisited**

We are now in a position to revisit our first example in this section, Example NIAQ. In that example, we showed that Archetype Q is not injective by constructing two vectors, which when used to evaluate the linear transformation provided the same output, thus violating Definition ILT. Just where did those two vectors come from?

The key is the vector

\[
z = \begin{bmatrix}
3 \\
4 \\
1 \\
3 \\
3
\end{bmatrix}
\]

which you can check is an element of $\mathcal{K}(T)$ for Archetype Q. Choose a vector $x$ at random, and then compute $y = x + z$ (verify this computation back in Example NIAQ). Then

\[
T(y) = T(x + z) = T(x) + T(z) = T(x) + 0 = T(x)
\]

Whenever the kernel of a linear transformation is non-trivial, we can employ this device and conclude that the linear transformation is not injective. This is another way of viewing Theorem KILT. For an injective linear transformation, the kernel is trivial and our only choice for $z$ is the zero vector, which will not help us create two different inputs for $T$ that yield identical outputs. For every one of the archetypes that is not injective, there is an example presented of exactly this form.

**Example NIAO**  
**Not injective, Archetype O**
In Example NKAO the kernel of Archetype O was determined to be
\[ \left\langle \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\rangle \]
a subspace of \( \mathbb{C}^3 \) with dimension 1. Since the kernel is not trivial, Theorem KILT tells us that \( T \) is not injective.

Example IAP
Injective, Archetype P
In Example TKAP it was shown that the linear transformation in Archetype P has a trivial kernel. So by Theorem KILT, \( T \) is injective.

Subsection ILTLI
Injective Linear Transformations and Linear Independence

There is a connection between injective linear transformations and linear independent sets that we will make precise in the next two theorems. However, more informally, we can get a feel for this connection when we think about how each property is defined. A set of vectors is linearly independent if the only relation of linear dependence is the trivial one. A linear transformation is injective if the only way two input vectors can produce the same output is if the trivial way, when both input vectors are equal.

Theorem ILTLI
Injective Linear Transformations and Linear Independence
Suppose that \( T: U \rightarrow V \) is an injective linear transformation and \( S = \{ u_1, u_2, u_3, \ldots, u_t \} \) is a linearly independent subset of \( U \). Then \( R = \{ T(u_1), T(u_2), T(u_3), \ldots, T(u_t) \} \) is a linearly independent subset of \( V \).

Proof Begin with a relation of linear dependence on \( R \) (Definition RLD, Definition LI),
\[ a_1 T(u_1) + a_2 T(u_2) + a_3 T(u_3) + \cdots + a_t T(u_t) = 0 \]
\[ T(a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_t u_t) = 0 \] (Theorem LTLC)
\[ a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_t u_t \in K(T) \]
\[ a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_t u_t \in \{0\} \] (Definition KLT)
\[ a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_t u_t = 0 \] (Theorem KILT)

Since this is a relation of linear dependence on the linearly independent set \( S \), we can conclude that
\[ a_1 = 0 \quad a_2 = 0 \quad a_3 = 0 \quad \ldots \quad a_t = 0 \]
and this establishes that \( R \) is a linearly independent set.

Theorem ILTB
Injective Linear Transformations and Bases
Suppose that \( T: U \rightarrow V \) is a linear transformation and \( B = \{ u_1, u_2, u_3, \ldots, u_m \} \) is a basis of \( U \). Then \( T \) is injective if and only if \( C = \{ T(u_1), T(u_2), T(u_3), \ldots, T(u_m) \} \) is a linearly independent subset of \( V \).

Proof (\( \Rightarrow \)) Assume \( T \) is injective. Since \( B \) is a basis, we know \( B \) is linearly independent (Definition B). Then Theorem ILTLI says that \( C \) is a linearly independent subset of \( V \).
(\( \Leftarrow \)) Assume that \( C \) is linearly independent. To establish that \( T \) is injective, we will show that the kernel of \( T \) is trivial. Suppose that \( u \in K(T) \). As an element of \( U \), we...
can write \( u \) as a linear combination of the basis vectors in \( B \) (uniquely). So there are are scalars, \( a_1, a_2, a_3, \ldots, a_m \), such that

\[
u = a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_m u_m\]

Then,

\[
0 = T(u) = T(a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_m u_m) = a_1 T(u_1) + a_2 T(u_2) + a_3 T(u_3) + \cdots + a_m T(u_m)
\]

This is a relation of linear dependence (Definition RLD [280]) on the linearly independent set \( C \), so the scalars are all zero: \( a_1 = a_2 = a_3 = \cdots = a_m = 0 \). Then

\[
u = a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_m u_m = 0 + 0 + 0 + \cdots + 0 = 0
\]

Since \( u \) was chosen as an arbitrary vector from \( \mathcal{K}(T) \), we have \( \mathcal{K}(T) = \{0\} \) and Theorem KILT [432] tells us that \( T \) is injective.

Subsection ILTD
Injective Linear Transformations and Dimension

**Theorem ILTD**

**Injective Linear Transformations and Dimension**

Suppose that \( T: U \rightarrow V \) is an injective linear transformation. Then \( \dim(U) \leq \dim(V) \).

**Proof** Suppose to the contrary that \( m = \dim(U) > \dim(V) = t \). Let \( B \) be a basis of \( U \), which will then contain \( m \) vectors. Apply \( T \) to each element of \( B \) to form a set \( C \) that is a subset of \( V \). By Theorem ILTB [433], \( C \) is linearly independent and therefore must contain \( m \) distinct vectors. So we have found a set of \( m \) linearly independent vectors in \( V \), a vector space of dimension \( t \), with \( m > t \). However, this contradicts Theorem G [320], so our assumption is false and \( \dim(U) \leq \dim(V) \).  

**Example NIDAU**

**Not injective by dimension, Archetype U**

The linear transformation in Archetype U [702] is

\[
T: M_{23} \rightarrow \mathbb{C}^4, \quad T\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a + 2b + 12c - 3d + e + 6f \\ 2a - b - c + d - 11f \\ a + b + 7c + 2d + e - 3f \\ a + 2b + 12c + 5e - 5f \end{bmatrix}
\]

Since \( \dim(M_{23}) = 6 > 4 = \dim(\mathbb{C}^4) \), \( T \) cannot be injective for then \( T \) would violate Theorem ILTD [434].

Notice that the previous example made no use of the actual formula defining the function. Merely a comparison of the dimensions of the domain and codomain are enough to conclude that the linear transformation is not injective. Archetype M [683] and Archetype N [685] are two more examples of linear transformations that have “big” domains and “small” codomains, resulting in “collisions” of outputs and thus are non-injective linear transformations.
Subsection CILT
Composition of Injective Linear Transformations

In Subsection LT.NLTF0 we saw how to combine linear transformations to build new linear transformations, specifically, how to build the composition of two linear transformations (Definition LTC). It will be useful later to know that the composition of injective linear transformations is again injective, so we prove that here.

**Theorem CILTI**
Composition of Injective Linear Transformations is Injective
Suppose that \( T: U \rightarrow V \) and \( S: V \rightarrow W \) are injective linear transformations. Then \( (S \circ T): U \rightarrow W \) is an injective linear transformation.

**Proof** That the composition is a linear transformation was established in Theorem CLLT, so we need only establish that the composition is injective. Applying Definition ILT, choose \( x, y \) from \( U \). Then

\[
\begin{align*}
(S \circ T)(x) &= (S \circ T)(y) \\
S(T(x)) &= S(T(y)) & \text{Definition LTC} \\
\Rightarrow T(x) &= T(y) & \text{Definition ILT for } S \\
\Rightarrow x &= y & \text{Definition ILT for } T
\end{align*}
\]

---

Subsection READ
Reading Questions

1. Suppose \( T: \mathbb{C}^8 \rightarrow \mathbb{C}^5 \) is a linear transformation. Why can’t \( T \) be injective?

2. Describe the kernel of an injective linear transformation.

3. Theorem KPI should remind you of Theorem PSPHS. Why do we say this?
Subsection EXC
Exercises

C10  Each archetype below is a linear transformation. Compute the kernel for each.

Archetype M 683
Archetype N 685
Archetype O 687
Archetype P 690
Archetype Q 692
Archetype R 695
Archetype S 698
Archetype T 700
Archetype U 702
Archetype V 704
Archetype W 706
Archetype X 708

Contributed by Robert Beezer

C20  The linear transformation \( T: \mathbb{C}^4 \to \mathbb{C}^3 \) is not injective. Find two inputs \( x, y \in \mathbb{C}^4 \) that yield the same output (that is \( T(x) = T(y) \)).

\[
T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 + x_3 \\ -x_1 + 3x_2 + x_3 - x_4 \\ 3x_1 + 2x_2 + 2x_3 - 2x_4 \end{bmatrix}
\]

Contributed by Robert Beezer  Solution 438

C25  Define the linear transformation

\[
T: \mathbb{C}^3 \to \mathbb{C}^2, \quad T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{bmatrix}
\]

Find a basis for the kernel of \( T \), \( \mathcal{K}(T) \). Is \( T \) injective?

Contributed by Robert Beezer  Solution 438

C40  Show that the linear transformation \( R \) is not injective by finding two different elements of the domain, \( x \) and \( y \), such that \( R(x) = R(y) \). (\( S_{22} \) is the vector space of symmetric \( 2 \times 2 \) matrices.)

\[
R: S_{22} \to P_1 \quad R \begin{bmatrix} a & b \\ b & c \end{bmatrix} = (2a - b + c) + (a + b + 2c)x
\]

Contributed by Robert Beezer  Solution 439

T10  Suppose \( T: U \to V \) is a linear transformation. For which vectors \( v \in V \) is \( T^{-1}(v) \) a subspace of of \( U \)?

Contributed by Robert Beezer

T15  Suppose that that \( T: U \to V \) and \( S: V \to W \) are linear transformations. Prove the following relationship between null spaces.

\( \mathcal{K}(T) \subseteq \mathcal{K}(S \circ T) \)

Contributed by Robert Beezer  Solution 439
T20  Suppose that $A$ is an $m \times n$ matrix. Define the linear transformation $T$ by

$$T : \mathbb{C}^n \rightarrow \mathbb{C}^m, \quad T(x) = Ax$$

Prove that the kernel of $T$ equals the null space of $A$, $K(T) = \mathcal{N}(A)$.

Contributed by Andy Zimmer  Solution 439
C20 Contributed by Robert Beezer Statement 436
A linear transformation that is not injective will have a non-trivial kernel (Theorem KILT 432), and this is the key to finding the desired inputs. We need one non-trivial element of the kernel, so suppose that \( z \in \mathbb{C}^4 \) is an element of the kernel,

\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = 0 = T(z) = 
\begin{bmatrix}
2z_1 + z_2 + z_3 \\
-z_1 + 3z_2 + z_3 - z_4 \\
3z_1 + z_2 + 2z_3 - 2z_4
\end{bmatrix}
\]

Vector equality Definition CVE 73 leads to the homogeneous system of three equations in four variables,

\[
\begin{align*}
2z_1 + z_2 + z_3 &= 0 \\
-z_1 + 3z_2 + z_3 - z_4 &= 0 \\
3z_1 + z_2 + 2z_3 - 2z_4 &= 0
\end{align*}
\]

The coefficient matrix of this system row-reduces as

\[
\begin{bmatrix}
2 & 1 & 1 & 0 \\
-1 & 3 & 1 & -1 \\
3 & 1 & 2 & -2
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & -3
\end{bmatrix}
\]

From this we can find a solution (we only need one), that is an element of \( K(T) \),

\[
z = 
\begin{bmatrix}
-1 \\
-1 \\
3 \\
1
\end{bmatrix}
\]

Now, we choose a vector \( x \) at random and set \( y = x + z \),

\[
x = 
\begin{bmatrix}
2 \\
3 \\
4 \\
-2
\end{bmatrix} \quad y = x + z = 
\begin{bmatrix}
2 \\
3 \\
4 \\
-2
\end{bmatrix} + 
\begin{bmatrix}
-1 \\
-1 \\
3 \\
1
\end{bmatrix} = 
\begin{bmatrix}
1 \\
2 \\
7 \\
-1
\end{bmatrix}
\]

and you can check that

\[
T(x) = \begin{bmatrix}
11 \\
13 \\
21
\end{bmatrix} = T(y)
\]

A quicker solution is to take two elements of the kernel (in this case, scalar multiples of \( z \)) which both get sent to \( 0 \) by \( T \). Quicker yet, take \( 0 \) and \( z \) as \( x \) and \( y \), which also both get sent to \( 0 \) by \( T \).

C25 Contributed by Robert Beezer Statement 436
To find the kernel, we require all \( x \in \mathbb{C}^3 \) such that \( T(x) = 0 \). This condition is

\[
\begin{bmatrix}
2x_1 - x_2 + 5x_3 \\
-4x_1 + 2x_2 - 10x_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

This leads to a homogeneous system of two linear equations in three variables, whose coefficient matrix row-reduces to

\[
\begin{bmatrix}
1 & -1 & 5 \\
0 & -\frac{1}{2} & \frac{5}{2}
\end{bmatrix}
\]
With two free variables, Theorem BNS yields the basis for the null space
\[
\left\{ \begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \\ \frac{7}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}
\]

With \( n(T) \neq 0 \), \( K(T) \neq \{0\} \), so Theorem KILT says \( T \) is not injective.

C40 Contributed by Robert Beezer  Statement

We choose \( x \) to be any vector we like. A particularly cocky choice would be to choose \( x = 0 \), but we will instead choose

\[
x = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 4 \end{bmatrix}
\]

Then \( R(x) = 9 + 9x \). Now compute the kernel of \( R \), which by Theorem KILT we expect to be nontrivial. Setting \( R \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) \) equal to the zero vector, \( 0 = 0 + 0x \), and equating coefficients leads to a homogenous system of equations. Row-reducing the coefficient matrix of this system will allow us to determine the values of \( a \), \( b \) and \( c \) that create elements of the null space of \( R \),

\[
\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}
\]

We only need a single element of the null space of this coefficient matrix, so we will not compute a precise description of the whole null space. Instead, choose the free variable \( c = 2 \). Then

\[
z = \begin{bmatrix} -2 \\ -2 \\ -2 \\ 2 \end{bmatrix}
\]

is the corresponding element of the kernel. We compute the desired \( y \) as

\[
y = x + z = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -3 \\ 6 \end{bmatrix}
\]

Then check that \( R(y) = 9 + 9x \).

T15 Contributed by Robert Beezer  Statement

We are asked to prove that \( K(T) \) is a subset of \( K(S \circ T) \). Employing Definition SSET, choose \( x \in K(T) \). Then we know that \( T(x) = 0 \). So

\[
(S \circ T)(x) = S(T(x)) = S(0) = 0 \quad \text{Definition LTC, Theorem LTTZZ}
\]

This qualifies \( x \) for membership in \( K(S \circ T) \).

T20 Contributed by Andy Zimmer  Statement

This is an equality of sets, so we want to establish two subset conditions (Definition SE).

First, show \( N(A) \subseteq K(T) \). Choose \( x \in N(A) \). Check to see if \( x \in K(T) \),

\[
T(x) = Ax = 0 \quad \text{Definition of } T \quad x \in N(A)
\]

So by Definition KLT, \( x \in K(T) \) and thus \( N(A) \subseteq N(T) \).

Now, show \( K(T) \subseteq N(A) \). Choose \( x \in K(T) \). Check to see if \( x \in N(A) \),

\[
Ax = T(x) = 0 \quad \text{Definition of } T \quad x \in K(T)
\]

So by Definition NSM, \( x \in N(A) \) and thus \( N(T) \subseteq N(A) \).
The companion to an injection is a surjection. Surjective linear transformations are closely related to spanning sets and ranges. So as you read this section reflect back on Section ILT [426] and note the parallels and the contrasts. In the next section, Section IVLT [456], we will combine the two properties.

As usual, we lead with a definition.

**Definition SLT**

**Surjective Linear Transformation**

Suppose $T: U \rightarrow V$ is a linear transformation. Then $T$ is **surjective** if for every $v \in V$ there exists $u \in U$ so that $T(u) = v$.

Given an arbitrary function, it is possible for there to be an element of the codomain that is not an output of the function (think about the function $y = f(x) = x^2$ and the codomain element $y = -3$). For a surjective function, this never happens. If we choose any element of the codomain ($v \in V$) then there must be an input from the domain ($u \in U$) which will create the output when used to evaluate the linear transformation ($T(u) = v$). Some authors prefer the term **onto** where we use surjective, and we will sometimes refer to a surjective linear transformation as a surjection.

**Subsection ESLT**

**Examples of Surjective Linear Transformations**

It is perhaps most instructive to examine a linear transformation that is not surjective first.

**Example NSAQ**

**Not surjective, Archetype Q**

Archetype Q [692] is the linear transformation

$$ T: \mathbb{C}^5 \rightarrow \mathbb{C}^5, \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_1 + 3x_2 + 3x_3 - 6x_4 + 3x_5 \\ -16x_1 + 9x_2 + 12x_3 - 28x_4 + 28x_5 \\ -19x_1 + 7x_2 + 14x_3 - 32x_4 + 37x_5 \\ -21x_1 + 9x_2 + 15x_3 - 35x_4 + 39x_5 \\ -9x_1 + 5x_2 + 7x_3 - 16x_4 + 16x_5 \end{pmatrix} $$

We will demonstrate that

$$ v = \begin{pmatrix} -1 \\ 2 \\ 3 \\ -1 \\ 4 \end{pmatrix} $$

is an unobtainable element of the codomain. Suppose to the contrary that $u$ is an element of the domain such that $T(u) = v$. Then

$$ \begin{pmatrix} -1 \\ 2 \\ 3 \\ -1 \\ 4 \end{pmatrix} = v = T(u) = T \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} $$
Form the augmented matrix of the system, and row-reduce to

Now we recognize the appropriate input vector $u$ as a solution to a linear system of equations. Form the augmented matrix of the system, and row-reduce to

With a leading 1 in the last column, Theorem RCLS tells us the system is inconsistent. From the absence of any solutions we conclude that no such vector $u$ exists, and by Definition SLT, $T$ is not surjective.

Again, do not concern yourself with how $v$ was selected, as this will be explained shortly. However, do understand why this vector provides enough evidence to conclude that $T$ is not surjective.

To show that a linear transformation is not surjective, it is enough to find a single element of the codomain that is never created by any input, as in Example NSAQ. However, to show that a linear transformation is surjective we must establish that every element of the codomain occurs as an output of the linear transformation for some appropriate input.

Example SAR

Surjective, Archetype R

Archetype R is the linear transformation

To establish that $R$ is surjective we must begin with a totally arbitrary element of the codomain, $v$ and somehow find an input vector $u$ such that $T(u) = v$. We desire,

$$T(u) = v$$
We recognize this equation as a system of equations in the variables $u_i$, but our vector of constants contains symbols. In general, we would have to row-reduce the augmented matrix by hand, due to the symbolic final column. However, in this particular example, the $5 \times 5$ coefficient matrix is nonsingular and so has an inverse (Theorem NI [204], Definition MI [189]).

$$
\begin{bmatrix}
-65 & 128 & 10 & -262 & 40 \\
36 & -73 & -1 & 151 & -16 \\
-44 & 88 & 5 & -180 & 24 \\
34 & -68 & -3 & 140 & -18 \\
12 & -24 & -1 & 49 & -5
\end{bmatrix}
^{-1}
= 
\begin{bmatrix} 
-47 & 92 & 1 & -181 & -14 \\
27 & -55 & \frac{7}{2} & \frac{221}{2} & 11 \\
-32 & 64 & -1 & -126 & -12 \\
25 & -50 & \frac{3}{2} & \frac{199}{2} & 9 \\
9 & -18 & \frac{1}{2} & \frac{71}{2} & 4
\end{bmatrix}
$$

so we find that

$$
\begin{bmatrix} 
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 
\end{bmatrix}
= 
\begin{bmatrix} 
-47v_1 + 92v_2 + v_3 - 181v_4 - 14v_5 \\
27v_1 - 55v_2 + \frac{7}{2}v_3 + \frac{221}{2}v_4 + 11v_5 \\
-32v_1 + 64v_2 - v_3 - 126v_4 - 12v_5 \\
25v_1 - 50v_2 + \frac{3}{2}v_3 + \frac{199}{2}v_4 + 9v_5 \\
9v_1 - 18v_2 + \frac{1}{2}v_3 + \frac{71}{2}v_4 + 4v_5 
\end{bmatrix}
$$

This establishes that if we are given any output vector $v$, we can use its components in this final expression to formulate a vector $u$ such that $T(u) = v$. So by Definition SLT [440] we now know that $T$ is surjective. You might try to verify this condition in its full generality (i.e. evaluate $T$ with this final expression and see if you get $v$ as the result), or test it more specifically for some numerical vector $v$ (see Exercise SLT.C20 [452]).

Let’s now examine a surjective linear transformation between abstract vector spaces.

**Example SAV**

**Surjective, Archetype V**

Archetype V [704] is defined by

$$
T : P_3 \mapsto M_{22}, \quad T (a + bx + cx^2 + dx^3) = \begin{bmatrix} a + b & a - 2c \\
d & b - d \end{bmatrix}
$$

To establish that the linear transformation is surjective, begin by choosing an arbitrary output. In this example, we need to choose an arbitrary $2 \times 2$ matrix, say

$$
v = \begin{bmatrix} x & y \\
z & w \end{bmatrix}
$$

and we would like to find an input polynomial

$$
u = a + bx + cx^2 + dx^3
$$

so that $T(u) = v$. So we have,

$$
\begin{bmatrix} x & y \\
z & w \end{bmatrix}
= v
= T(u)
= T (a + bx + cx^2 + dx^3)
= \begin{bmatrix} a + b & a - 2c \\
d & b - d \end{bmatrix}
$$
Matrix equality leads us to the system of four equations in the four unknowns, \(x, y, z, w\),

\[
\begin{align*}
  a + b &= x \\
  a - 2c &= y \\
  d &= z \\
  b - d &= w
\end{align*}
\]

which can be rewritten as a matrix equation,

\[
\begin{bmatrix}
  1 & 1 & 0 & 0 \\
  1 & 0 & -2 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
  a \\
  b \\
  c \\
  d
\end{bmatrix} =
\begin{bmatrix}
  x \\
  y \\
  z \\
  w
\end{bmatrix}
\]

The coefficient matrix is nonsingular, hence it has an inverse,

\[
\begin{bmatrix}
  1 & 1 & 0 & 0 \\
  1 & 0 & -2 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 1 & 0 & -1
\end{bmatrix}^{-1} =
\begin{bmatrix}
  1 & 0 & -1 & -1 \\
  0 & 0 & 1 & 1 \\
  \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
  0 & 0 & 1 & 0
\end{bmatrix}
\]

so we have

\[
\begin{bmatrix}
  a \\
  b \\
  c \\
  d
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & -1 & -1 \\
  0 & 0 & 1 & 1 \\
  \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
  0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  z \\
  w
\end{bmatrix}
\]

So the input polynomial \(u = (x - z - w) + (z + w)x + \frac{1}{2}(x - y - z - w)x^2 + zx^3\) will yield the output matrix \(v\), no matter what form \(v\) takes. This means by Definition SLT 440 that \(T\) is surjective. All the same, let’s do a concrete demonstration and evaluate \(T\) with \(u\),

\[
T(u) = T \left( (x - z - w) + (z + w)x + \frac{1}{2}(x - y - z - w)x^2 + zx^3 \right)
\]

\[
= \begin{bmatrix}
  (x - z - w) + (z + w)(x - z - w) - 2(\frac{1}{2}(x - y - z - w)) \\
  z \\
  (z + w) - z
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  x & y \\
  z & w
\end{bmatrix}
\]

\(= v\)

Subsection RLT
Range of a Linear Transformation

For a linear transformation \(T : U \rightarrow V\), the range is a subset of the codomain \(V\). Informally, it is the set of all outputs that the transformation creates when fed every possible input from the domain. It will have some natural connections with the column space of a matrix, so we will keep the same notation, and if you think about your objects, then there should be little confusion. Here’s the careful definition.
Definition RLT
Range of a Linear Transformation
Suppose $T : U \mapsto V$ is a linear transformation. Then the range of $T$ is the set

$$\mathcal{R}(T) = \{ T(u) \mid u \in U \}$$

(This definition contains Notation RLT.)

Example RAO
Range, Archetype O
Archetype O \[687\] is the linear transformation

$$T : \mathbb{C}^3 \mapsto \mathbb{C}^5, \quad T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} -x_1 + x_2 - 3x_3 \\ -x_1 + 2x_2 - 4x_3 \\ x_1 + x_2 + 3x_3 \\ 2x_1 + 3x_2 + x_3 \\ x_1 + 2x_3 \end{bmatrix}$$

To determine the elements of $\mathbb{C}^5$ in $\mathcal{R}(T)$, find those vectors $v$ such that $T(u) = v$ for some $u \in \mathbb{C}^3$,

$$v = T(u) = \begin{bmatrix} -u_1 + u_2 - 3u_3 \\ -u_1 + 2u_2 - 4u_3 \\ u_1 + u_2 + u_3 \\ 2u_1 + 3u_2 + u_3 \\ u_1 \end{bmatrix} = \begin{bmatrix} -u_1 \\ -u_1 \\ 2u_1 \\ u_1 \end{bmatrix} + \begin{bmatrix} u_2 \\ u_2 \\ 3u_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3u_3 \\ -4u_3 \\ u_3 \\ 2u_3 \end{bmatrix} = u_1 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 3 \\ -4 \\ 1 \\ 1 \end{bmatrix}$$

This says that every output of $T(v)$ can be written as a linear combination of the three vectors

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 1 \\ 2 \end{bmatrix}$$

using the scalars $u_1, u_2, u_3$. Furthermore, since $u$ can be any element of $\mathbb{C}^3$, every such linear combination is an output. This means that

$$\mathcal{R}(T) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 1 \\ 2 \end{bmatrix} \right\}$$

The three vectors in this spanning set for $\mathcal{R}(T)$ form a linearly dependent set (check this!). So we can find a more economical presentation by any of the various methods from Section CRS \[211\].
Subsection SLT.RLT  Range of a Linear Transformation  

We will place the vectors into a matrix as rows, row-reduce, toss out zero rows and appeal to Theorem BRS, so we can describe the range of $T$ with a basis,

$$\mathcal{R}(T) = \langle \begin{pmatrix} 1 \\ 0 \\ -3 \\ -7 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 5 \\ 1 \end{pmatrix} \rangle$$

We know that the span of a set of vectors is always a subspace (Theorem SSS), so the range computed in Example RAO is also a subspace. This is no accident, the range of a linear transformation is always a subspace.

**Theorem RLTS**

**Range of a Linear Transformation is a Subspace**

Suppose that $T: U \rightarrow V$ is a linear transformation. Then the range of $T$, $\mathcal{R}(T)$, is a subspace of $V$. □

**Proof** We can apply the three-part test of Theorem TSS. First, $0_U \in U$ and $T(0_U) = 0_V$ by Theorem LTTZZ, so $0_V \in \mathcal{R}(T)$ and we know that the range is non-empty.

Suppose we assume that $x, y \in \mathcal{R}(T)$. Is $x + y \in \mathcal{R}(T)$? If $x, y \in \mathcal{R}(T)$ then we know there are vectors $w, z \in U$ such that $T(w) = x$ and $T(z) = y$. Because $U$ is a vector space, additive closure (Property AC) implies that $w + z \in U$. Then

$$T(w + z) = T(w) + T(z)$$

Definition LT

$$= x + y$$

Definition of $w$ and $z$

So we have found an input, $w + z$, which when fed into $T$ creates $x + y$ as an output. This qualifies $x + y$ for membership in $\mathcal{R}(T)$. So we have additive closure.

Suppose we assume that $\alpha \in \mathbb{C}$ and $x \in \mathcal{R}(T)$. Is $\alpha x \in \mathcal{R}(T)$? If $x \in \mathcal{R}(T)$, then there is a vector $w \in U$ such that $T(w) = x$. Because $U$ is a vector space, scalar closure implies that $\alpha w \in U$. Then

$$T(\alpha w) = \alpha T(w)$$

Definition LT

$$= \alpha x$$

Definition of $w$

So we have found an input $(\alpha w)$ which when fed into $T$ creates $\alpha x$ as an output. This qualifies $\alpha x$ for membership in $\mathcal{R}(T)$. So we have scalar closure and Theorem TSS tells us that $\mathcal{R}(T)$ is a subspace of $V$. ■

Let’s compute another range, now that we know in advance that it will be a subspace.

**Example FRAN**

**Full range, Archetype N**

Archetype N is the linear transformation

$$T: \mathbb{C}^5 \rightarrow \mathbb{C}^3, \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{bmatrix} 2x_1 + x_2 + 3x_3 - 4x_4 + 5x_5 \\ x_1 - 2x_2 + 3x_3 - 9x_4 + 3x_5 \\ 3x_1 + 4x_3 - 6x_4 + 5x_5 \end{bmatrix}$$

To determine the elements of $\mathbb{C}^3$ in $\mathcal{R}(T)$, find those vectors $v$ such that $T(u) = v$ for some $u \in \mathbb{C}^5$,

$$v = T(u)$$
This says that every output of $T(v)$ can be written as a linear combination of the five vectors

\[
\begin{bmatrix}
2 \\ 1 \\ -2 \\ 3 \\
1 \\ -2 \\ 0 \\ 4 \\
3 \\ 1 \\ 3 \\ 3 \\
2 \\ 1 \\ 0 \\ 4 \\
1 \\ 3 \\ 3 \\ 4 \\
\end{bmatrix}
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\
-4 \\ 3 \\ 5 \\ 0 \\
3 \\ 4 \\ 5 \\ 5 \\
\end{bmatrix}
\]

using the scalars $u_1$, $u_2$, $u_3$, $u_4$, $u_5$. Furthermore, since $u$ can be any element of $\mathbb{C}^5$, every such linear combination is an output. This means that

\[
\mathcal{R}(T) = \left\langle \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ -6 \\ -4 \\ -9 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 5 \\ 5 \\ 5 \end{bmatrix} \right\rangle
\]

The five vectors in this spanning set for $\mathcal{R}(T)$ form a linearly dependent set (Theorem MVSLD [126]). So we can find a more economical presentation by any of the various methods from Section CRS [211] and Section FS [231]. We will place the vectors into a matrix as rows, row-reduce, toss out zero rows and appeal to Theorem BRS [220], so we can describe the range of $T$ with a (nice) basis,

\[
\mathcal{R}(T) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \right\rangle = \mathbb{C}^3
\]

In contrast to injective linear transformations having small (trivial) kernels (Theorem KILT [132]), surjective linear transformations have large ranges, as indicated in the next theorem.

**Theorem RSLT**  
**Range of a Surjective Linear Transformation**  
Suppose that $T: U \rightarrow V$ is a linear transformation. Then $T$ is surjective if and only if the range of $T$ equals the codomain, $\mathcal{R}(T) = V$. \hfill \Box

**Proof**  
$(\Rightarrow)$ By Definition RLT [444], we know that $\mathcal{R}(T) \subseteq V$. To establish the reverse inclusion, assume $v \in V$. Then since $T$ is surjective (Definition SLT [440]), there exists a vector $u \in U$ so that $T(u) = v$. However, the existence of $u$ gains $v$ membership in $\mathcal{R}(T)$, so $V \subseteq \mathcal{R}(T)$. Thus, $\mathcal{R}(T) = V$.

$(\Leftarrow)$ To establish that $T$ is surjective, choose $v \in V$. Since we are assuming that $\mathcal{R}(T) = V$, $v \in \mathcal{R}(T)$. This says there is a vector $u \in U$ so that $T(u) = v$, i.e. $T$ is surjective. \hfill \blacksquare

**Example NSAQR**  
**Not surjective, Archetype Q, revisited**  
We are now in a position to revisit our first example in this section, Example NSAQ [440]. In that example, we showed that Archetype Q [692] is not surjective by constructing a vector in the codomain where no element of the domain could be used to evaluate the linear transformation to create the output, thus violating Definition SLT [440]. Just where did this vector come from?
The short answer is that the vector
\[ \mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \\ 4 \end{bmatrix} \]
was constructed to lie outside of the range of \( T \). How was this accomplished? First, the range of \( T \) is given by
\[ \mathcal{R}(T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\} \]
Suppose an element of the range \( \mathbf{v}^* \) has its first 4 components equal to \(-1, 2, 3, -1\), in that order. Then to be an element of \( \mathcal{R}(T) \), we would have
\[ \mathbf{v}^* = (-1) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + (2) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + (3) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \\ -8 \end{bmatrix} \]
So the only vector in the range with these first four components specified, must have \(-8\) in the fifth component. To set the fifth component to any other value (say, 4) will result in a vector (\( \mathbf{v} \) in Example NSAQ [440]) outside of the range. Any attempt to find an input for \( T \) that will produce \( \mathbf{v} \) as an output will be doomed to failure.

Whenever the range of a linear transformation is not the whole codomain, we can employ this device and conclude that the linear transformation is not surjective. This is another way of viewing Theorem RSLT [446]. For a surjective linear transformation, the range is all of the codomain and there is no choice for a vector \( \mathbf{v} \) that lies in \( V \), yet not in the range. For every one of the archetypes that is not surjective, there is an example presented of exactly this form.

Example NSAO
Not surjective, Archetype O
In Example RAO [444] the range of Archetype O [687] was determined to be
\[ \mathcal{R}(T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ -7 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \\ 1 \end{bmatrix} \right\} \]
a subspace of dimension 2 in \( \mathbb{C}^5 \). Since \( \mathcal{R}(T) \neq \mathbb{C}^5 \), Theorem RSLT [446] says \( T \) is not surjective.

Example SAN
Surjective, Archetype N
The range of Archetype N [685] was computed in Example FRAN [445] to be
\[ \mathcal{R}(T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \]
Since the basis for this subspace is the set of standard unit vectors for \( \mathbb{C}^3 \) (Theorem SUVB [294]), we have \( \mathcal{R}(T) = \mathbb{C}^3 \) and by Theorem RSLT [446], \( T \) is surjective.
Subsection SSSLT
Spanning Sets and Surjective Linear Transformations

Just as injective linear transformations are allied with linear independence (Theorem ILTLI 433, Theorem ILTB 433), surjective linear transformations are allied with spanning sets.

Theorem SSRLT
Spanning Set for Range of a Linear Transformation
Suppose that \( T: U \rightarrow V \) is a linear transformation and \( S = \{ u_1, u_2, u_3, \ldots, u_t \} \) spans \( U \). Then
\[
R = \{ T(u_1), T(u_2), T(u_3), \ldots, T(u_t) \}
\]
spans \( \mathbb{R}(T) \).

Proof We need to establish that \( \mathbb{R}(T) = \langle R \rangle \), a set equality. First we establish that \( \mathbb{R}(T) \subseteq \langle R \rangle \). To this end, choose \( v \in \mathbb{R}(T) \). Then there exists a vector \( u \in U \), such that \( T(u) = v \) (Definition RLT 444). Because \( S \) spans \( U \) there are scalars, \( a_1, a_2, a_3, \ldots, a_t \), such that
\[
u = a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_t u_t
\]
Then
\[
v = T(u) = T(a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_t u_t) = a_1 T(u_1) + a_2 T(u_2) + a_3 T(u_3) + \cdots + a_t T(u_t)
\]
which establishes that \( v \in \langle R \rangle \) (Definition SS 270). So \( \mathbb{R}(T) \subseteq \langle R \rangle \).

To establish the opposite inclusion, choose an element of the span of \( R \), say \( v \in \langle R \rangle \). Then there are scalars \( b_1, b_2, b_3, \ldots, b_t \) so that
\[
v = b_1 T(u_1) + b_2 T(u_2) + b_3 T(u_3) + \cdots + b_t T(u_t) = T(b_1 u_1 + b_2 u_2 + b_3 u_3 + \cdots + b_t u_t)
\]
This demonstrates that \( v \) is an output of the linear transformation \( T \), so \( v \in \mathbb{R}(T) \). Therefore \( \langle R \rangle \subseteq \mathbb{R}(T) \), so we have the set equality \( \mathbb{R}(T) = \langle R \rangle \) (Definition SE 616). In other words, \( R \) spans \( \mathbb{R}(T) \) (Definition TSVS 284).

Theorem SSRLT 448 provides an easy way to begin the construction of a basis for the range of a linear transformation, since the construction of a spanning set requires simply evaluating the linear transformation on a spanning set of the domain. In practice the best choice for a spanning set of the domain would be as small as possible, in other words, a basis. The resulting spanning set for the codomain may not be linearly independent, so to find a basis for the range might require tossing out redundant vectors from the spanning set. Here’s an example.

Example BRLT
A basis for the range of a linear transformation
Define the linear transformation \( T: M_{22} \rightarrow P_2 \) by
\[
T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a + 2b + 8c + d) + (-3a + 2b + 5d)x + (a + b + 5c)x^2
\]
A convenient spanning set for \( M_{22} \) is the basis
\[
S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]
So by Theorem SSRLT [448], a spanning set for $\mathcal{R}(T)$ is

$$
R = \left\{ T\left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right), T\left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right), T\left( \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right), T\left( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \right\}
$$

$$
= \{ 1 - 3x + x^2, 2 + 2x + x^2, 8 + 5x^2, 1 + 5x \}
$$

The set $R$ is not linearly independent, so if we desire a basis for $\mathcal{R}(T)$, we need to eliminate some redundant vectors. Two particular relations of linear dependence on $R$ are

$$
(-2)(1 - 3x + x^2) + (-3)(2 + 2x + x^2) + (8 + 5x^2) = 0 + 0 + 0x^2 = 0
$$

$$
(1 - 3x + x^2) + (-1)(2 + 2x + x^2) + (1 + 5x) = 0 + 0 + 0x^2 = 0
$$

These, individually, allow us to remove $8 + 5x^2$ and $1 + 5x$ from $R$ with out destroying the property that $R$ spans $\mathcal{R}(T)$. The two remaining vectors are linearly independent (check this!), so we can write

$$
\mathcal{R}(T) = \langle \{ 1 - 3x + x^2, 2 + 2x + x^2 \} \rangle
$$

and see that $\dim(\mathcal{R}(T)) = 2$.

Elements of the range are precisely those elements of the codomain with non-empty preimages.

**Theorem RPI**

**Range and Pre-Image**

Suppose that $T: U \rightarrow V$ is a linear transformation. Then

$$
v \in \mathcal{R}(T) \text{ if and only if } T^{-1}(v) \neq \emptyset
$$

**Proof**  
$(\Rightarrow)$ If $v \in \mathcal{R}(T)$, then there is a vector $u \in U$ such that $T(u) = v$. This qualifies $u$ for membership in $T^{-1}(v)$, and thus the preimage of $v$ is not empty.

$(\Leftarrow)$ Suppose the preimage of $v$ is not empty, so we can choose a vector $u \in U$ such that $T(u) = v$. Then $v \in \mathcal{R}(T)$.

**Theorem SLTB**

**Surjective Linear Transformations and Bases**

Suppose that $T: U \rightarrow V$ is a linear transformation and $B = \{ u_1, u_2, u_3, \ldots, u_m \}$ is a basis of $U$. Then $T$ is surjective if and only if $C = \{ T(u_1), T(u_2), T(u_3), \ldots, T(u_m) \}$ is a spanning set for $V$.

**Proof**  
$(\Rightarrow)$ Assume $T$ is surjective. Since $B$ is a basis, we know $B$ is a spanning set of $U$ (Definition B [294]). Then Theorem SSRLT [448] says that $C$ spans $\mathcal{R}(T)$. But the hypothesis that $T$ is surjective means $V = \mathcal{R}(T)$ (Theorem RSLT [446]), so $C$ spans $V$.

$(\Leftarrow)$ Assume that $C$ spans $V$. To establish that $T$ is surjective, we will show that every element of $V$ is an output of $T$ for some input (Definition SLT [440]). Suppose that $v \in V$. As an element of $V$, we can write $v$ as a linear combination of the spanning set $C$. So there are are scalars, $b_1, b_2, b_3, \ldots, b_m$, such that

$$
v = b_1T(u_1) + b_2T(u_2) + b_3T(u_3) + \cdots + b_mT(u_m)
$$

Now define the vector $u \in U$ by

$$
u = b_1u_1 + b_2u_2 + b_3u_3 + \cdots + b_mu_m
$$

Then

$$
T(u) = T(b_1u_1 + b_2u_2 + b_3u_3 + \cdots + b_mu_m)
$$

$$
= b_1T(u_1) + b_2T(u_2) + b_3T(u_3) + \cdots + b_mT(u_m)
$$

(Theorem LTLC [413])

$$
= v
$$

So, given any choice of a vector $v \in V$, we can design an input $u \in U$ to produce $v$ as an output of $T$. Thus, by Definition SLT [440], $T$ is surjective.
Theorem SLTD
Surjective Linear Transformations and Dimension
Suppose that \( T: U \rightarrow V \) is a surjective linear transformation. Then \( \dim(U) \geq \dim(V) \).

**Proof** Suppose to the contrary that \( m = \dim(U) < \dim(V) = t \). Let \( B \) be a basis of \( U \), which will then contain \( m \) vectors. Apply \( T \) to each element of \( B \) to form a set \( C \) that is a subset of \( V \). By Theorem SLTB \([449]\), \( C \) is spanning set of \( V \) with \( m \) or fewer vectors. So we have a set of \( m \) or fewer vectors that span \( V \), a vector space of dimension \( t \), with \( m < t \). However, this contradicts Theorem G \([320]\), so our assumption is false and \( \dim(U) \geq \dim(V) \). 

Example NSDAT
Not surjective by dimension, Archetype T
The linear transformation in Archetype T \([700]\) is

\[
T: P_4 \rightarrow P_5, \quad T(p(x)) = (x - 2)p(x)
\]

Since \( \dim(P_4) = 5 < 6 = \dim(P_5) \), \( T \) cannot be surjective for then it would violate Theorem SLTD \([450]\).

Notice that the previous example made no use of the actual formula defining the function. Merely a comparison of the dimensions of the domain and codomain are enough to conclude that the linear transformation is not surjective. Archetype O \([687]\) and Archetype P \([690]\) are two more examples of linear transformations that have “small” domains and “big” codomains, resulting in an inability to create all possible outputs and thus they are non-surjective linear transformations.

Subsection CSLT
Composition of Surjective Linear Transformations

In Subsection LT.NLTFO \([417]\) we saw how to combine linear transformations to build new linear transformations, specifically, how to build the composition of two linear transformations (Definition LTC \([419]\)). It will be useful later to know that the composition of surjective linear transformations is again surjective, so we prove that here.

Theorem CSLTS
Composition of Surjective Linear Transformations is Surjective
Suppose that \( T: U \rightarrow V \) and \( S: V \rightarrow W \) are surjective linear transformations. Then \((S \circ T): U \rightarrow W\) is a surjective linear transformation.

**Proof** That the composition is a linear transformation was established in Theorem CLTLT \([419]\), so we need only establish that the composition is surjective. Applying Definition SLT \([440]\), choose \( w \in W \).

Because \( S \) is surjective, there must be a vector \( v \in V \), such that \( S(v) = w \). With the existence of \( v \) established, that \( T \) is surjective guarantees a vector \( u \in U \) such that \( T(u) = v \). Now,

\[
(S \circ T)(u) = S(T(u)) = S(v) = w
\]

This establishes that any element of the codomain \( w \) can be created by evaluating \( S \circ T \) with the right input \( u \). Thus, by Definition SLT \([440]\), \( S \circ T \) is surjective.
1. Suppose $T: \mathbb{C}^5 \rightarrow \mathbb{C}^8$ is a linear transformation. Why can’t $T$ be surjective?

2. What is the relationship between a surjective linear transformation and its range?

3. Compare and contrast injective and surjective linear transformations.
Subsection EXC
Exercises

C10 Each archetype below is a linear transformation. Compute the range for each.

Archetype M [683]
Archetype N [685]
Archetype Q [687]
Archetype P [690]
Archetype Q [692]
Archetype R [695]
Archetype S [698]
Archetype T [700]
Archetype U [702]
Archetype V [704]
Archetype W [706]
Archetype X [708]

Contributed by Robert Beezer

C20 Example SAR [441] concludes with an expression for a vector $u \in \mathbb{C}^5$ that we believe will create the vector $v \in \mathbb{C}^5$ when used to evaluate $T$. That is, $T(u) = v$. Verify this assertion by actually evaluating $T$ with $u$. If you don’t have the patience to push around all these symbols, try choosing a numerical instance of $v$, compute $u$, and then compute $T(u)$, which should result in $v$.

Contributed by Robert Beezer

C22 The linear transformation $S: \mathbb{C}^4 \mapsto \mathbb{C}^3$ is not surjective. Find an output $w \in \mathbb{C}^3$ that has an empty pre-image (that is $S^{-1}(w) = \emptyset$).

$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 + 3x_3 - 4x_4 \\ x_1 + 3x_2 + 4x_3 + 3x_4 \\ -x_1 + 2x_2 + x_3 + 7x_4 \end{bmatrix}$

Contributed by Robert Beezer Solution [454]

C25 Define the linear transformation $T: \mathbb{C}^3 \mapsto \mathbb{C}^2$, $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{bmatrix}$

Find a basis for the range of $T$, $\mathcal{R}(T)$. Is $T$ surjective?

Contributed by Robert Beezer Solution [454]

C40 Show that the linear transformation $T$ is not surjective by finding an element of the codomain, $v$, such that there is no vector $u$ with $T(u) = v$. (15 points)

$T: \mathbb{C}^3 \mapsto \mathbb{C}^3$, $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 2a + 3b - c \\ 2b - 2c \\ a - b + 2c \end{bmatrix}$

Contributed by Robert Beezer Solution [455]

T15 Suppose that that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations. Prove the following relationship between ranges. (15 points)

$\mathcal{R}(S \circ T) \subseteq \mathcal{R}(S)$

Version 1.04
Suppose that $A$ is an $m \times n$ matrix. Define the linear transformation $T$ by

$$T : \mathbb{C}^n \rightarrow \mathbb{C}^m, \quad T(x) = Ax$$

Prove that the range of $T$ equals the column space of $A$, $\mathcal{R}(T) = \mathcal{C}(A)$. 

Contributed by Andy Zimmer  Solution 455
To find an element of $\mathbb{C}^3$ with an empty pre-image, we will compute the range of the linear transformation $R(S)$ and then find an element outside of this set.

By Theorem SSRLT [448] we can evaluate $S$ with the elements of a spanning set of the domain and create a spanning set for the range.

So

$$R(S) = \langle \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 7 \end{bmatrix} \rangle$$

This spanning set is obviously linearly dependent, so we can reduce it to a basis for $R(S)$ using Theorem BRS [220], where the elements of the spanning set are placed as the rows of a matrix. The result is that

$$R(S) = \langle \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \rangle$$

Therefore, the unique vector in $R(S)$ with a first slot equal to 6 and a second slot equal to 15 will be the linear combination

$$6 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 15 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \\ 9 \end{bmatrix}$$

So, any vector with first two components equal to 6 and 15, but with a third component different from 9, such as

$$w = \begin{bmatrix} 6 \\ 15 \\ -63 \end{bmatrix}$$

will not be an element of the range of $S$ and will therefore have an empty pre-image. Another strategy on this problem is to guess. Almost any vector will lie outside the range of $T$, you have to be unlucky to randomly choose an element of the range. This is because the codomain has dimension 3, while the range is “much smaller” at a dimension of 2. You still need to check that your guess lies outside of the range, which generally will involve solving a system of equations that turns out to be inconsistent.
With \( r(T) \neq 2 \), \( \mathcal{R}(T) \neq \mathbb{C}^2 \), so Theorem RSLT \[446\] says \( T \) is not surjective.

**C40** Contributed by Robert Beezer Statement [452]

We wish to find an output vector \( v \) that has no associated input. This is the same as requiring that there is no solution to the equality

\[
v = T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a + 3b - c \\ 2b - 2c \\ a - b + 2c \end{pmatrix} = a \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + c \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}
\]

In other words, we would like to find an element of \( \mathbb{C}^3 \) not in the set

\[
Y = \left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \\ \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -1 \\ \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} \right\}
\]

If we make these vectors the rows of a matrix, and row-reduce, Theorem BRS \[220\] provides an alternate description of \( Y \),

\[
Y = \left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \\ \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -5 \end{pmatrix} \right\}
\]

If we add these vectors together, and then change the third component of the result, we will create a vector that lies outside of \( Y \), say \( v = \begin{pmatrix} 2 \\ 4 \\ 9 \end{pmatrix} \).

**T15** Contributed by Robert Beezer Statement [452]

This question asks us to establish that one set \( \mathcal{R}(S \circ T) \) is a subset of another \( \mathcal{R}(S) \). Choose an element in the “smaller” set, say \( w \in \mathcal{R}(S \circ T) \). Then we know that there is a vector \( u \in U \) such that

\[
w = (S \circ T)(u) = S(T(u))
\]

Now define \( v = T(u) \), so that then

\[
S(v) = S(T(u)) = w
\]

This statement is sufficient to show that \( w \in \mathcal{R}(S) \), so \( w \) is an element of the “larger” set, and \( \mathcal{R}(S \circ T) \subseteq \mathcal{R}(S) \).

**T20** Contributed by Andy Zimmer Statement [453]

This is an equality of sets, so we want to establish two subset conditions (Definition SE \[616\]).

First, show \( \mathcal{C}(A) \subseteq \mathcal{R}(T) \). Choose \( y \in \mathcal{C}(A) \). Then by Definition CSM \[211\] and Definition MVP \[173\] there is a vector \( x \in \mathbb{C}^n \) such that \( Ax = y \). Then

\[
T(x) = Ax = y
\]

This statement qualifies \( y \) as a member of \( \mathcal{R}(T) \) (Definition RLT \[444\]), so \( \mathcal{C}(A) \subseteq \mathcal{R}(T) \).

Now, show \( \mathcal{R}(T) \subseteq \mathcal{C}(A) \). Choose \( y \in \mathcal{R}(T) \). Then by Definition RLT \[444\], there is a vector \( x \) in \( \mathbb{C}^n \) such that \( T(x) = y \). Then

\[
Ax = T(x) = y
\]

So by Definition CSM \[211\] and Definition MVP \[173\], \( y \) qualifies for membership in \( \mathcal{C}(A) \) and so \( \mathcal{R}(T) \subseteq \mathcal{C}(A) \).
In this section we will conclude our introduction to linear transformations by bringing together the twin properties of injectivity and surjectivity and consider linear transformations with both of these properties.

One preliminary definition, and then we will have our main definition for this section.

Definition IDLT
Identity Linear Transformation
The identity linear transformation on the vector space $W$ is defined as

$$I_W : W \mapsto W, \quad I_W(w) = w$$

Informally, $I_W$ is the “do-nothing” function. You should check that $I_W$ is really a linear transformation, as claimed, and then compute its kernel and range to see that it is both injective and surjective. All of these facts should be straightforward to verify (Exercise IVLT.T05 [468]). With this in hand we can make our main definition.

Definition IVLT
Invertible Linear Transformations
Suppose that $T : U \mapsto V$ is a linear transformation. If there is a function $S : V \mapsto U$ such that

$$S \circ T = I_U \quad \quad T \circ S = I_V$$

then $T$ is invertible. In this case, we call $S$ the inverse of $T$ and write $S = T^{-1}$.

Informally, a linear transformation $T$ is invertible if there is a companion linear transformation, $S$, which “undoes” the action of $T$. When the two linear transformations are applied consecutively (composition), in either order, the result is to have no real effect. It is entirely analogous to squaring a positive number and then taking its (positive) square root.

Here is an example of a linear transformation that is invertible. As usual at the beginning of a section, do not be concerned with where $S$ came from, just understand how it illustrates Definition IVLT.

Example AIVLT
An invertible linear transformation
Archetype V [704] is the linear transformation

$$T : P_3 \mapsto M_{22}, \quad T \left( a + bx + cx^2 + dx^3 \right) = \begin{bmatrix} a + b & a - 2c \\ d & b - d \end{bmatrix}$$

Define the function $S : M_{22} \mapsto P_3$ defined by

$$S \left( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) = (a - c - d) + (c + d)x + \frac{1}{2}(a - b - c - d)x^2 + cx^3$$

Then

$$(T \circ S) \left( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) = T \left( S \left( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) \right)$$
\[
= T \left( (a - c - d) + (c + d)x + \frac{1}{2}(a - b - c - d)x^2 + cx^3 \right)
\]
\[
= \left[ (a - c - d) + (c + d) \begin{array}{c}
(c - a - d) - 2\left(\frac{1}{2}(a - b - c - d)\right)
\end{array}
\right]
\]
\[
= \left[ \begin{array}{c}
a - b \\
2a + 2b + c \\
3a + b + c \\
-2a - 6b - 2c
\end{array} \right]
\]
\[
= I_{M_{22}} \left( \begin{array}{c}
a \\
b \\
c \\
d
\end{array} \right)
\]

And
\[
(S \circ T) (a + bx + cx^2 + dx^3) = S \left( T (a + bx + cx^2 + dx^3) \right)
\]
\[
= S \left( \begin{array}{c}
a + b \\
a - 2c \\
d \\
b - d
\end{array} \right)
\]
\[
= ((a + b) - d - (b - d)) + (d + (b - d))x
\]
\[
+ \left( \frac{1}{2}((a + b) - (a - 2c) - d - (b - d)) \right) x^2 + (d)x^3
\]
\[
= a + bx + cx^2 + dx^3
\]
\[
= I_{P_3} \left( a + bx + cx^2 + dx^3 \right)
\]

For now, understand why these computations show that \( T \) is invertible, and that \( S = T^{-1} \). Maybe even be amazed by how \( S \) works so perfectly in concert with \( T \)! We will see later just how to arrive at the correct form of \( S \) (when it is possible).

It can be as instructive to study a linear transformation that is not invertible.

**Example ANILT**

**A non-invertible linear transformation**

Consider the linear transformation \( T: \mathbb{C}^3 \mapsto M_{22} \) defined by

\[
T \left( \begin{array}{c}
a \\
b \\
c
\end{array} \right) = \left[ \begin{array}{c}
a - b \\
2a + 2b + c \\
3a + b + c \\
-2a - 6b - 2c
\end{array} \right]
\]

Suppose we were to search for an inverse function \( S: M_{22} \mapsto \mathbb{C}^3 \).

First verify that the \( 2 \times 2 \) matrix \( A = \begin{bmatrix} 5 & 3 \\ 8 & 2 \end{bmatrix} \) is not in the range of \( T \). This will amount to finding an input to \( T \), \( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \), such that

\[
a - b = 5 \\
2a + 2b + c = 3 \\
3a + b + c = 8 \\
-2a - 6b - 2c = 2
\]

As this system of equations is inconsistent, there is no input column vector, and \( A \notin R(T) \). How should we define \( S(A) \)? Note that

\[
T (S(A)) = (T \circ S) (A) = I_{M_{22}} (A) = A
\]

So any definition we would provide for \( S(A) \) must then be a column vector that \( T \) sends to \( A \) and we would have \( A \in R(T) \), contrary to the definition of \( T \). This is enough to see that there is no function \( S \) that will allow us to conclude that \( T \) is invertible, since we cannot provide a consistent definition for \( S(A) \) if we assume \( T \) is invertible.
Even though we now know that $T$ is not invertible, let’s not leave this example just yet. Check that
\[
T \left( \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 3 & 2 \\ 5 & 2 \end{bmatrix} B \quad \text{and} \quad T \left( \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix} \right) = \begin{bmatrix} 3 & 2 \\ 5 & 2 \end{bmatrix} B
\]

How would we define $S(B)$?

\[
S(B) = S \left( T \left( \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \right) \right) = (S \circ T) \left( \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \right) = I^3 \left( \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}
\]

or

\[
S(B) = S \left( T \left( \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix} \right) \right) = (S \circ T) \left( \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix} \right) = I^3 \left( \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}
\]

Which definition should we provide for $S(B)$? Both are necessary. But then $S$ is not a function. So we have a second reason to know that there is no function $S$ that will allow us to conclude that $T$ is invertible. It happens that there are infinitely many column vectors that $S$ would have to take to $B$. Construct the kernel of $T$,

\[
\mathcal{K}(T) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} \right\}
\]

Now choose either of the two inputs used above for $T$ and add to it a scalar multiple of the basis vector for the kernel of $T$. For example,

\[
x = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + (-2) \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}
\]

then verify that $T(x) = B$. Practice creating a few more inputs for $T$ that would be sent to $B$, and see why it is hopeless to think that we could ever provide a reasonable definition for $S(B)$! There is a “whole subspace’s worth” of values that $S(B)$ would have to take on. 

In Example ANILT\textsuperscript{457} you may have noticed that $T$ is not surjective, since the matrix $A$ was not in the range of $T$. And $T$ is not injective since there are two different input column vectors that $T$ sends to the matrix $B$. Linear transformations $T$ that are not surjective lead to putative inverse functions $S$ that are undefined on inputs outside of the range of $T$. Linear transformations $T$ that are not injective lead to putative inverse functions $S$ that are multiply-defined on each of their inputs. We will formalize these ideas in Theorem ILTIS\textsuperscript{459}.

But first notice in Definition IVLT\textsuperscript{456} that we only require the inverse (when it exists) to be a function. When it does exist, it too is a linear transformation.

**Theorem II.LTLT**

**Inverse of a Linear Transformation is a Linear Transformation**

Suppose that $T: U \rightarrow V$ is an invertible linear transformation. Then the function $T^{-1}: V \rightarrow U$ is a linear transformation. \[\square\]

**Proof** We work through verifying Definition LT\textsuperscript{405} for $T^{-1}$, using the fact that $T$ is a linear transformation to obtain the second equality in each half of the proof. To this end, suppose $x, y \in V$ and $\alpha \in \mathbb{C}$.

\[
T^{-1}(x + y) = T^{-1}(T(T^{-1}(x)) + T(T^{-1}(y))) \\
= T^{-1}(T(T^{-1}(x) + T^{-1}(y))) \\
= T^{-1}(x) + T^{-1}(y) \\
\]

Definition IVLT\textsuperscript{456}

\[
T^{-1}(\alpha x) = T^{-1}(\alpha(T(T^{-1}(x)))) \\
= \alpha T^{-1}(T(T^{-1}(x))) \\
= \alpha x \\
\]

Definition IVLT\textsuperscript{456}
Now check the second defining property of a linear transformation for $T^{-1}$,
\[
T^{-1}(\alpha x) = T^{-1}(\alpha T(T^{-1}(x))) = T^{-1}(T(\alpha T^{-1}(x))) = \alpha T^{-1}(x) \quad \text{(Definition IVLT 456, Definition LT 405, Definition IVLT 456)}
\]

So $T^{-1}$ fulfills the requirements of Definition LT 405 and is therefore a linear transformation. So when $T$ has an inverse, $T^{-1}$ is also a linear transformation. Additionally, $T^{-1}$ is invertible and its inverse is what you might expect.

**Theorem IILT**

**Inverse of an Invertible Linear Transformation**

Suppose that $T: U \mapsto V$ is an invertible linear transformation. Then $T^{-1}$ is an invertible linear transformation and $(T^{-1})^{-1} = T$.

**Proof** Because $T$ is invertible, Definition IVLT 456 tells us there is a function $T^{-1}: V \mapsto U$ such that
\[
T^{-1} \circ T = I_U \quad \text{and} \quad T \circ T^{-1} = I_V \quad \text{(Definition IVLT 456, Definition LTC 419)}
\]

Additionally, Theorem ILTILT 458 tells us that $T^{-1}$ is more than just a function, it is a linear transformation. Now view these two statements as properties of the linear transformation $T^{-1}$. In light of Definition IVLT 456, they together say that $T^{-1}$ is invertible (let $T$ play the role of $S$ in the statement of the definition). Furthermore, the inverse of $T^{-1}$ is then $T$, i.e. $(T^{-1})^{-1} = T$.

Subsection IV

**Invertibility**


We now know what an inverse linear transformation is, but just which linear transformations have inverses? Here is a theorem we have been preparing for all chapter long.

**Theorem ILTIS**

**Invertible Linear Transformations are Injective and Surjective**

Suppose $T: U \mapsto V$ is a linear transformation. Then $T$ is invertible if and only if $T$ is injective and surjective.

**Proof** ($\Rightarrow$) Since $T$ is presumed invertible, we can employ its inverse, $T^{-1}$ (Definition IVLT 456). To see that $T$ is injective, suppose $x, y \in U$ and assume that $T(x) = T(y)$,
\[
x = I_U(x) = (T^{-1} \circ T)(x) = T^{-1}(T(x)) = T^{-1}(T(y)) = (T^{-1} \circ T)(y) = I_U(y) = y \quad \text{(Definition ILT 426, Definition LTC 419, Definition IDLT 456)}
\]

So by Definition ILT 426 $T$ is injective. To check that $T$ is surjective, suppose $v \in V$. Then $T^{-1}(v)$ is a vector in $U$. Compute
\[
T(T^{-1}(v)) = (T \circ T^{-1})(v) = I_V(v) = v \quad \text{(Definition LTC 419, Definition IVLT 456)}
\]
injectivity limits the preimage to a singleton. Since our choice of \( v \) was empty set, or an infinite set. But surjectivity requires that the preimage not be empty, and then is the key to this half of this proof. Normally the preimage of a vector from the codomain might be every pre-image for \( T \) is defined, verifying that it is the inverse of \( T \) was arbitrary we have function equality, \( S \circ T = I_U \).

Now choose \( v \in V \). Define \( u \) to be the single vector in the set \( T^{-1}(v) \), in other words, \( u = S(v) \). Then \( T(u) = v \), so \( (T \circ S)(v) = T(S(v)) = T(u) = v = I_V(v) \) and since our choice of \( v \) was arbitrary we have function equality, \( T \circ S = I_V \).

We will make frequent use of this characterization of invertible linear transformations. The next theorem is a good example of this, and we will use it often, too.

**Theorem CIVLT**

**Composition of Invertible Linear Transformations**

Suppose that \( T : U \mapsto V \) and \( S : V \mapsto W \) are invertible linear transformations. Then the composition, \((S \circ T) : U \mapsto W\) is an invertible linear transformation.

**Proof** Since \( S \) and \( T \) are both linear transformations, \( S \circ T \) is also a linear transformation by **Theorem CLTLT**. Since \( S \) and \( T \) are both invertible, **Theorem ILTIS** says that \( S \) and \( T \) are both injective and surjective. Then **Theorem CLTLT** says \( S \circ T \) is injective, and **Theorem CSLTS** says \( S \circ T \) is surjective. Now apply the “other half” of **Theorem ILTIS** and conclude that \( S \circ T \) is invertible.

When a composition is invertible, the inverse is easy to construct.

**Theorem ICLT**

**Inverse of a Composition of Linear Transformations**

Suppose that \( T : U \mapsto V \) and \( S : V \mapsto W \) are invertible linear transformations. Then \( S \circ T \) is invertible and \((S \circ T)^{-1} = T^{-1} \circ S^{-1}\).

**Proof** Compute, for all \( w \in W \)

\[
\left((S \circ T) \circ (T^{-1} \circ S^{-1})\right)(w) = S\left(T\left(T^{-1}\left(S^{-1}(w)\right)\right)\right)
= S\left(I_V\left(S^{-1}(w)\right)\right)
= S\left(S^{-1}(w)\right)
= w
\]

So there is an element from \( U \), when used as an input to \( T \) (namely \( T^{-1}(v) \)) that produces the desired output, \( v \), and hence \( T \) is surjective by **Definition SLT**.

\((\Leftarrow)\) Now assume that \( T \) is both injective and surjective. We will build a function \( S : V \mapsto U \) that will establish that \( T \) is invertible. To this end, choose any \( v \in V \). Since \( T \) is surjective, **Theorem RSLT** says \( R(T) = V \), so we have \( v \in R(T) \). **Theorem RPI** says that the pre-image of \( v \), \( T^{-1}(v) \), is nonempty. So we can choose a vector from the pre-image of \( v \), say \( u \).

In other words, there exists \( u \in T^{-1}(v) \).

Since \( T^{-1}(v) \) is non-empty, **Theorem KPI** then says that \[
T^{-1}(v) = \{ u + z \mid z \in \mathcal{K}(T) \}
\]

However, because \( T \) is injective, by **Theorem KILT** the kernel is trivial, \( \mathcal{K}(T) = \{0\} \). So the pre-image is a set with just one element, \( T^{-1}(v) = \{u\} \). Now we can define \( S \) by \( S(v) = u \). This is the key to this half of this proof. Normally the preimage of a vector from the codomain might be an empty set, or an infinite set. But surjectivity requires that the preimage not be empty, and then injectivity limits the preimage to a singleton. Since our choice of \( v \) was arbitrary, we know that every pre-image for \( T \) is a set with a single element. This allows us to construct \( S \) as a function. Now that it is defined, verifying that it is the inverse of \( T \) will be easy. Here we go.

Choose \( u \in U \). Define \( v = T(u) \). Then \( T^{-1}(v) = \{u\} \), so that \( S(v) = u \) and,

\[
(S \circ T)(u) = S(T(u)) = S(v) = u = I_U(u)
\]

and since our choice of \( u \) was arbitrary we have function equality, \( S \circ T = I_U \).
\[(S \circ T) \circ (T^{-1} \circ S^{-1}) = I_W\] and also

\[
\left( (T^{-1} \circ S^{-1}) \circ (S \circ T) \right)(u) = T^{-1}(S^{-1}(S(T(u))))
\]

= \[T^{-1}(I_V(T(u)))\] (Definition IVLT 456)

= \[T^{-1}(T(u))\] (Definition IDLT 456)

= \[u\] (Definition IVLT 456)

= \[I_U(u)\] (Definition IDLT 456)

so \[(T^{-1} \circ S^{-1}) \circ (S \circ T) = I_U.\] By Definition IVLT 456, \(S \circ T\) is invertible and \((S \circ T)^{-1} = T^{-1} \circ S^{-1} \). □

Notice that this theorem not only establishes what the inverse of \(S \circ T\) is, it also duplicates the conclusion of Theorem CIVLT 460 and also establishes the invertibility of \(S \circ T\). But somehow, the proof of Theorem CIVLT 460 is nicer way to get this property.

Does Theorem ICLT 460 remind you of the flavor of any theorem we have seen about matrices? (Hint: Think about getting dressed.) Hmmm.

Subsection SI
Structure and Isomorphism

A vector space is defined (Definition VS 251) as a set of objects (“vectors”) endowed with a definition of vector addition (+) and a definition of scalar multiplication (written with juxtaposition). Many of our definitions about vector spaces involve linear combinations (Definition LC 269), such as the span of a set (Definition SS 270) and linear independence (Definition LI 280). Other definitions are built up from these ideas, such as bases (Definition B 294) and dimension (Definition D 307). The defining properties of a linear transformation require that a function “respect” the operations of the two vector spaces that are the domain and the codomain (Definition LT 405). Finally, an invertible linear transformation is one that can be “undone” — it has a companion that reverses its effect. In this subsection we are going to begin to roll all these ideas into one.

A vector space has “structure” derived from definitions of the two operations and the requirement that these operations interact in ways that satisfy the ten properties of Definition VS 251. When two different vector spaces have an invertible linear transformation defined between them, then we can translate questions about linear combinations (spans, linear independence, bases, dimension) from the first vector space to the second. The answers obtained in the second vector space can then be translated back, via the inverse linear transformation, and interpreted in the setting of the first vector space. We say that these invertible linear transformations “preserve structure.” And we say that the two vector spaces are “structurally the same.” The precise term is “isomorphic,” from Greek meaning “of the same form.” Let’s begin to try to understand this important concept.

Definition IVS
Isomorphic Vector Spaces

Two vector spaces \(U\) and \(V\) are isomorphic if there exists an invertible linear transformation \(T\) with domain \(U\) and codomain \(V\), \(T: U \rightarrow V\). In this case, we write \(U \cong V\), and the linear transformation \(T\) is known as an isomorphism between \(U\) and \(V\). △

A few comments on this definition. First, be careful with your language (Technique L 620). Two vector spaces are isomorphic, or not. It is a yes/no situation and the term only applies to a pair of vector spaces. Any invertible linear transformation can be called an isomorphism, it is a term that applies to functions. Second, a given pair of vector spaces there might be several different isomorphisms between the two vector spaces. But it only takes the existence of one to call the pair isomorphic. Third, \(U\) isomorphic to \(V\), or \(V\) isomorphic to \(U\)? Doesn’t matter, since
the inverse linear transformation will provide the needed isomorphism in the “opposite” direction. 
Being “isomorphic to” is an equivalence relation on the set of all vector spaces (see Theorem SER
391 for a reminder about equivalence relations).

**Example IVSAV**

**Isomorphic vector spaces, Archetype V**

Archetype V [704] is a linear transformation from \( P_3 \) to \( M_{2,2} \),

\[
T: P_3 \mapsto M_{2,2}, \quad T \left( a + bx + cx^2 + dx^3 \right) = \begin{bmatrix} a + b & a - 2c \\ d & b - d \end{bmatrix}
\]

Since it is injective and surjective, Theorem ILTIS [459] tells us that it is an invertible linear 
transformation. By Definition IVS [461] we say \( P_3 \) and \( M_{2,2} \) are isomorphic.

At a basic level, the term “isomorphic” is nothing more than a codeword for the presence of an 
invertible linear transformation. However, it is also a description of a powerful idea, and this power 
only becomes apparent in the course of studying examples and related theorems. In this example, 
we are led to believe that there is nothing “structurally” different about \( P_3 \) and \( M_{2,2} \). In a certain 
sense they are the same. Not equal, but the same. One is as good as the other. One is just as 
interesting as the other.

Here is an extremely basic application of this idea. Suppose we want to compute the following 
linear combination of polynomials in \( P_3 \),

\[
5(2 + 3x - 4x^2 + 5x^3) + (-3)(3 - 5x + 3x^2 + x^3)
\]

Rather than doing it straight-away (which is very easy), we will apply the transformation \( T \) to 
convert into a linear combination of matrices, and then compute in \( M_{2,2} \) according to the definitions 
of the vector space operations there (Example VSM [253]),

\[
T \left( 5(2 + 3x - 4x^2 + 5x^3) + (-3)(3 - 5x + 3x^2 + x^3) \right)
= 5T \left( 2 + 3x - 4x^2 + 5x^3 \right) + (-3)T \left( 3 - 5x + 3x^2 + x^3 \right)
= \begin{bmatrix} 5 & 10 \\ 5 & -2 \end{bmatrix} + \begin{bmatrix} -2 & -3 \\ 1 & -6 \end{bmatrix}
= \begin{bmatrix} 31 & 59 \\ 22 & 8 \end{bmatrix}
\]

Operations in \( M_{2,2} \)

Now we will translate our answer back to \( P_3 \) by applying \( T^{-1} \), which we found in Example AIVLT
456.

\[
T^{-1}: M_{2,2} \mapsto P_3, \quad T^{-1} \left( \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix} \right) = (a - c - d) + (c + d)x + \frac{1}{2}(a - b - c - d)x^2 + cx^3
\]

We compute,

\[
T^{-1} \left( \begin{bmatrix} 31 \\ 22 \\ 59 \\ 8 \end{bmatrix} \right) = 1 + 30x - 29x^2 + 22x^3
\]

which is, as expected, exactly what we would have computed for the original linear combination 
had we just used the definitions of the operations in \( P_3 \) (Example VSP [253]). Notice this is meant 
only as an illustration and not a suggested route for doing this particular computation. ☒

Checking the dimensions of two vector spaces can be a quick way to establish that they are not 
isomorphic. Here’s the theorem.

**Theorem IVSED**

**Isomorphic Vector Spaces have Equal Dimension**

Suppose \( U \) and \( V \) are isomorphic vector spaces. Then \( \dim(U) = \dim(V) \).

**Proof** If \( U \) and \( V \) are isomorphic, there is an invertible linear transformation \( T: U \rightarrow V \) 
(Definition IVS [461]). \( T \) is injective by Theorem ILTIS [459] and so by Theorem ILTD [434].
dim (U) \leq dim (V). Similarly, T is surjective by Theorem ILTIS \[459\] and so by Theorem SLTD \[450\], dim (U) \geq dim (V). The net effect of these two inequalities is that dim (U) = dim (V). \[\blacksquare\]

The contrapositive of Theorem IVSED \[462\] says that if U and V have different dimensions, then they are not isomorphic. Dimension is the simplest “structural” characteristic that will allow you to distinguish non-isomorphic vector spaces. For example \( P_6 \) is not isomorphic to \( M_{34} \) since their dimensions (7 and 12, respectively) are not equal. With tools developed in Section VR \[473\] we will be able to establish that the converse of Theorem IVSED \[462\] is true. Think about that one for a moment.

Subsection RNLT
Rank and Nullity of a Linear Transformation

Just as a matrix has a rank and a nullity, so too do linear transformations. And just like the rank and nullity of a matrix are related (they sum to the number of columns, Theorem RPNC \[314\]) the rank and nullity of a linear transformation are related. Here are the definitions and theorems, see the Archetypes (Appendix A \[630\]) for loads of examples.

**Definition ROLT**
Rank Of a Linear Transformation

Suppose that \( T: U \to V \) is a linear transformation. Then the **rank** of \( T \), \( r (T) \), is the dimension of the range of \( T \),

\[
r (T) = \dim (R(T))
\]

(This definition contains Notation ROLT.)

**Definition NOLT**
Nullity Of a Linear Transformation

Suppose that \( T: U \to V \) is a linear transformation. Then the **nullity** of \( T \), \( n (T) \), is the dimension of the kernel of \( T \),

\[
n (T) = \dim (K(T))
\]

(This definition contains Notation NOLT.)

Here are two quick theorems.

**Theorem ROSLT**
Rank Of a Surjective Linear Transformation

Suppose that \( T: U \to V \) is a linear transformation. Then the rank of \( T \), \( r (T) \), is the dimension of \( V \),

\[
r (T) = \dim (V), \text{ if and only if } T \text{ is surjective.}
\]

**Proof** By Theorem RSLT \[446\], \( T \) is surjective if and only if \( R(T) = V \). Applying Definition ROLT \[463\], \( R(T) = V \) if and only if \( r (T) = \dim (R(T)) = \dim (V) \).

**Theorem NOILT**
Nullity Of an Injective Linear Transformation

Suppose that \( T: U \to V \) is an injective linear transformation. Then the nullity of \( T \), \( n (T) \), is zero, \( n (T) = 0 \), if and only if \( T \) is injective.

**Proof** By Theorem KILT \[432\], \( T \) is injective if and only if \( K(T) = \{0\} \). Applying Definition NOLT \[463\], \( K(T) = \{0\} \) if and only if \( n (T) = 0 \).

Just as injectivity and surjectivity come together in invertible linear transformations, there is a clear relationship between rank and nullity of a linear transformation. If one is big, the other is small.
Theorem RPNDD
Rank Plus Nullity is Domain Dimension
Suppose that \( T: U \rightarrow V \) is a linear transformation. Then

\[
 r(T) + n(T) = \dim(U)
\]

\(\square\)

**Proof** Let \( r = r(T) \) and \( s = n(T) \). Suppose that \( R = \{ v_1, v_2, v_3, \ldots, v_r \} \subseteq V \) is a basis of the range of \( T \), \( \mathcal{R}(T) \), and \( S = \{ u_1, u_2, u_3, \ldots, u_s \} \subseteq U \) is a basis of the kernel of \( T \), \( \ker(T) \). Note that \( R \) and \( S \) are possibly empty, which means that some of the sums in this proof are “empty” and are equal to the zero vector.

Because the elements of \( R \) are all in the range of \( T \), each must have a non-empty pre-image by Theorem RPI. Choose vectors \( w_i \in U, 1 \leq i \leq r \) such that \( w_i \in T^{-1}(v_i) \). So \( T(w_i) = v_i \), \( 1 \leq i \leq r \). Consider the set

\[
 B = \{ u_1, u_2, u_3, \ldots, u_s, w_1, w_2, w_3, \ldots, w_r \}
\]

We claim that \( B \) is a basis for \( U \).

To establish linear independence for \( B \), begin with a relation of linear dependence on \( B \). So suppose there are scalars \( a_1, a_2, a_3, \ldots, a_s \) and \( b_1, b_2, b_3, \ldots, b_r \)

\[
 0 = a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_s u_s + b_1 w_1 + b_2 w_2 + b_3 w_3 + \cdots + b_r w_r
\]

Then

\[
 0 = T(0)
 0 = T(a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_s u_s + b_1 w_1 + b_2 w_2 + b_3 w_3 + \cdots + b_r w_r)
 0 = a_1 T(u_1) + a_2 T(u_2) + a_3 T(u_3) + \cdots + a_s T(u_s) + b_1 T(w_1) + b_2 T(w_2) + b_3 T(w_3) + \cdots + b_r T(w_r)
 0 = a_1 0 + a_2 0 + \cdots + a_s 0 + b_1 T(w_1) + b_2 T(w_2) + b_3 T(w_3) + \cdots + b_r T(w_r)
 0 = 0 + 0 + \cdots + 0 + b_1 T(w_1) + b_2 T(w_2) + b_3 T(w_3) + \cdots + b_r T(w_r)
 0 = 0 + 0 + \cdots + 0 + b_1 T(w_1) + b_2 T(w_2) + b_3 T(w_3) + \cdots + b_r T(w_r)
 0 = b_1 v_1 + b_2 v_2 + b_3 v_3 + \cdots + b_r v_r
\]

This is a relation of linear dependence on \( R \) (Definition RLD), and since \( R \) is a linearly independent set (Definition LI), we see that \( b_1 = b_2 = b_3 = \cdots = b_r = 0 \). Then the original relation of linear dependence on \( B \) becomes

\[
 0 = a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_s u_s + 0 w_1 + 0 w_2 + \cdots + 0 w_r
 0 = a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_s u_s + 0 + 0 + \cdots + 0
 0 = a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_s u_s + 0 + 0 + \cdots + 0
\]

But this is again a relation of linear independence (Definition RLD), now on the set \( S \). Since \( S \) is linearly independent (Definition LI), we have \( a_1 = a_2 = a_3 = \cdots = a_r = 0 \). Since we now know that all the scalars in the relation of linear dependence on \( B \) must be zero, we have established the linear independence of \( S \) through Definition LI.

To now establish that \( B \) spans \( U \), choose an arbitrary vector \( u \in U \). Then \( T(u) \in \mathcal{R}(T) \), so there are scalars \( c_1, c_2, c_3, \ldots, c_r \) such that

\[
 T(u) = c_1 v_1 + c_2 v_2 + c_3 v_3 + \cdots + c_r v_r
\]

Version 1.04
Use the scalars $c_1, c_2, c_3, \ldots, c_r$ to define a vector $y \in U$,

$$y = c_1w_1 + c_2w_2 + c_3w_3 + \cdots + c_rw_r.$$  

Then

$$T(u - y) = T(u) - T(y)$$

$$= T(u) - T(c_1w_1 + c_2w_2 + c_3w_3 + \cdots + c_rw_r)$$

$$= T(u) - (c_1T(w_1) + c_2T(w_2) + \cdots + c_rT(w_r))$$

$$= T(u) - (c_1v_1 + c_2v_2 + c_3v_3 + \cdots + c_rv_r)$$

$$= T(u) - T(u)$$

$$= 0$$

Substitution

Theorem LTLC 413

Hence $u - y$ is sent to the zero vector by $T$ and hence is an element of the kernel of $T$. As such it can be written as a linear combination of the basis vectors for $\mathcal{K}(T)$, the elements of the set $S$. So there are scalars $d_1, d_2, d_3, \ldots, d_s$ such that

$$u - y = d_1u_1 + d_2u_2 + d_3u_3 + \cdots + d_su_s$$

Then

$$u = (u - y) + y$$

$$= d_1u_1 + d_2u_2 + d_3u_3 + \cdots + d_su_s + c_1w_1 + c_2w_2 + c_3w_3 + \cdots + c_rw_r$$

This says that for any vector, $u$, from $U$, there exist scalars $(d_1, d_2, d_3, \ldots, d_s, c_1, c_2, c_3, \ldots, c_r)$ that form $u$ as a linear combination of the vectors in the set $B$. In other words, $B$ spans $U$ (Definition SS 270).

So $B$ is a basis (Definition B 294) of $U$ with $s + r$ vectors, and thus

$$\dim(U) = s + r = n(T) + r(T)$$

as desired. ■

Theorem RPNC 314 said that the rank and nullity of a matrix sum to the number of columns of the matrix. This result is now an easy consequence of Theorem RPNDT 464 when we consider the linear transformation $T : \mathbb{C}^m \rightarrow \mathbb{C}^n$ defined with the $m \times n$ matrix $A$ by $T(x) = Ax$. The range and kernel of $T$ are identical to the column space and null space of the matrix $A$ (Exercise ILT.T20 437, Exercise SLT.T20 453), so the rank and nullity of the matrix $A$ are identical to the rank and nullity of the linear transformation $T$. The dimension of the domain of $T$ is the dimension of $\mathbb{C}^n$, exactly the number of columns for the matrix $A$.

This theorem can be especially useful in determining basic properties of linear transformations. For example, suppose that $T : \mathbb{C}^6 \rightarrow \mathbb{C}^6$ is a linear transformation and you are able to quickly establish that the kernel is trivial. Then $n(T) = 0$. First this means that $T$ is injective by Theorem NOILT 463. Also, Theorem RPNDT 464 becomes

$$6 = \dim(\mathbb{C}^6) = r(T) + n(T) = r(T) + 0 = r(T)$$

So the rank of $T$ is equal to the rank of the codomain, and by Theorem ROSLT 463, we know $T$ is surjective. Finally, we know $T$ is invertible by Theorem ILTIS 459. So from the determination that the kernel is trivial, and consideration of various dimensions, the theorems of this section allow us to conclude the existence of an inverse linear transformation for $T$.

Similarly, Theorem RPNDT 464 can be used to provide alternative proofs for Theorem ILTD 434, Theorem SLTD 450 and Theorem IVSED 462. It would be an interesting exercise to construct these proofs.

It would be instructive to study the archetypes that are linear transformations and see how many of their properties can be deduced just from considering only the dimensions of the domain and codomain. Then add in just knowledge of either the nullity or rank, and so how much more you can learn about the linear transformation. The table preceding all of the archetypes (Appendix A 630) could be a good place to start this analysis.
Subsection SLELT
Systems of Linear Equations and Linear Transformations

This subsection does not really belong in this section, or any other section, for that matter. It is just the right time to have a discussion about the connections between the central topic of linear algebra, linear transformations, and our motivating topic from Chapter SLE, systems of linear equations. We will discuss several theorems we have seen already, but we will also make some forward-looking statements that will be justified in Chapter R.

Archetype D and Archetype E are ideal examples to illustrate connections with linear transformations. Both have the same coefficient matrix,

\[
D = \begin{bmatrix}
2 & 1 & 7 & -7 \\
-3 & 4 & -5 & -6 \\
1 & 1 & 4 & -5 \\
\end{bmatrix}
\]

To apply the theory of linear transformations to these two archetypes, employ matrix multiplication and define the linear transformation,

\[
T: \mathbb{C}^4 \rightarrow \mathbb{C}^3, \quad T(x) = Dx = x_1 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -6 \\ -5 \end{bmatrix}
\]

Theorem MBLT tells us that \(T\) is indeed a linear transformation. Archetype D asks for solutions to \(LS(D, b)\), where \(b = \begin{bmatrix} 8 \\ -12 \\ -4 \end{bmatrix}\). In the language of linear transformations this is equivalent to asking for \(T^{-1}(b)\). In the language of vectors and matrices it asks for a linear combination of the four columns of \(D\) that will equal \(b\). One solution listed is \(w = \begin{bmatrix} 7 \\ 8 \\ 1 \\ 3 \end{bmatrix}\). With a non-empty preimage, Theorem KPI tells us that the complete solution set of the linear system is the preimage of \(b\),

\[w + \mathcal{K}(T) = \{ w + z \mid z \in \mathcal{K}(T) \}\]

The kernel of the linear transformation \(T\) is exactly the null space of the matrix \(D\) (see Exercise ILT.T20), so this approach to the solution set should be reminiscent of Theorem PSPHS. The kernel of the linear transformation is the preimage of the zero vector, exactly equal to the solution set of the homogeneous system \(LS(D, 0)\). Since \(D\) has a null space of dimension two, every preimage (and in particular the preimage of \(b\)) is as “big” as a subspace of dimension two (but is not a subspace).

Archetype E is identical to Archetype D but with a different vector of constants, \(d = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}\). We can use the same linear transformation \(T\) to discuss this system of equations since the coefficient matrix is identical. Now the set of solutions to \(LS(D, d)\) is the pre-image of \(d\), \(T^{-1}(d)\). However, the vector \(d\) is not in the range of the linear transformation (nor is it in the column space of the matrix, since these two sets are equal by Exercise SLT.T20). So the empty pre-image is equivalent to the inconsistency of the linear system.

These two archetypes each have three equations in four variables, so either the resulting linear systems are inconsistent, or they are consistent and application of Theorem CMVEI tells us that the system has infinitely many solutions. Considering these same parameters for the linear transformation, the dimension of the domain, \(\mathbb{C}^4\), is four, while the codomain, \(\mathbb{C}^3\), has dimension three. Then

\[n(T) = \dim(\mathbb{C}^4) - r(T)\]
So the kernel of $T$ is nontrivial simply by considering the dimensions of the domain (number of variables) and the codomain (number of equations). Pre-images of elements of the codomain that are not in the range of $T$ are empty (inconsistent systems). For elements of the codomain that are in the range of $T$ (consistent systems), Theorem KPI \[431\] tells us that the pre-images are built from the kernel, and with a non-trivial kernel, these pre-images are infinite (infinitely many solutions).

When do systems of equations have unique solutions? Consider the system of linear equations $LS(C, f)$ and the linear transformation $S(x) = Cx$. If $S$ has a trivial kernel, then pre-images will either be empty or be finite sets with single elements. Correspondingly, the coefficient matrix $C$ will have a trivial null space and solution sets will either be empty (inconsistent) or contain a single solution (unique solution). Should the matrix be square and have a trivial null space then we recognize the matrix as being nonsingular. A square matrix means that the corresponding linear transformation, $T$, has equal-sized domain and codomain. With a nullity of zero, $T$ is injective, and also Theorem RPNDD \[464\] tells us that rank of $T$ is equal to the dimension of the domain, which in turn is equal to the dimension of the codomain. In other words, $T$ is surjective. Injective and surjective, and Theorem ILTIS \[459\] tells us that $T$ is invertible. Just as we can use the inverse of the coefficient matrix to find the unique solution of any linear system with a nonsingular coefficient matrix (Theorem SNCM \[204\]), we can use the inverse of the linear transformation to construct the unique element of any pre-image (proof of Theorem ILTIS \[459\]).

The executive summary of this discussion is that to every coefficient matrix of a system of linear equations we can associate a natural linear transformation. Solution sets for systems with this coefficient matrix are preimages of elements of the codomain of the linear transformation. For every theorem about systems of linear equations there is an analogue about linear transformations. The theory of linear transformations provides all the tools to recreate the theory of solutions to linear systems of equations.

We will continue this adventure in Chapter R \[473\].

Subsection READ
Reading Questions

1. What conditions allow us to easily determine if a linear transformation is invertible?
2. What does it mean to say two vector spaces are isomorphic? Both technically, and informally?
3. How do linear transformations relate to systems of linear equations?
Subsection EXC
Exercises

C10  The archetypes below are linear transformations of the form $T: U \mapsto V$ that are invertible. For each, the inverse linear transformation is given explicitly as part of the archetype’s description. Verify for each linear transformation that

\[ T^{-1} \circ T = I_U \quad \quad T \circ T^{-1} = I_V \]

Archetype R [695],
Archetype V [704],
Archetype W [706]
Contributed by Robert Beezer

C20  Determine if the linear transformation $T: P_2 \mapsto M_{22}$ is (a) injective, (b) surjective, (c) invertible.

\[ T(a + bx + cx^2) = \begin{bmatrix} a + 2b - 2c & 2a + 2b \\ -a + b - 4c & 3a + 2b + 2c \end{bmatrix} \]

Contributed by Robert Beezer  Solution [470]

C21  Determine if the linear transformation $S: P_3 \mapsto M_{22}$ is (a) injective, (b) surjective, (c) invertible.

\[ S(a + bx + cx^2 + dx^3) = \begin{bmatrix} -a + 4b + c + 2d & 4a - b + 6c - d \\ a + 5b - 2c + 2d & a + 2c + 5d \end{bmatrix} \]

Contributed by Robert Beezer  Solution [470]

C50  Consider the linear transformation $S: M_{12} \mapsto P_1$ from the set of $1 \times 2$ matrices to the set of polynomials of degree at most 1, defined by

\[ S \begin{bmatrix} a \\ b \end{bmatrix} = (3a + b) + (5a + 2b)x \]

Prove that $S$ is invertible. Then show that the linear transformation

\[ R: P_1 \mapsto M_{12}, \quad R(r + sx) = \begin{bmatrix} (2r - s) & (-5r + 3s) \end{bmatrix} \]

is the inverse of $S$, that is $S^{-1} = R$.

Contributed by Robert Beezer  Solution [471]

M30  The linear transformation $S$ below is invertible. Find a formula for the inverse linear transformation, $S^{-1}$.

\[ S: P_1 \mapsto M_{1,2}, \quad S(a + bx) = \begin{bmatrix} 3a + b & 2a + b \end{bmatrix} \]

Contributed by Robert Beezer  Solution [471]

M31  The linear transformation $R: M_{12} \mapsto M_{21}$ is invertible. Determine a formula for the inverse linear transformation $R^{-1}: M_{21} \mapsto M_{12}$. (15 points)

\[ R \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a + 3b \\ 4a + 11b \end{bmatrix} \]

Contributed by Robert Beezer  Solution [472]

T05  Prove that the identity linear transformation (Definition IDLT [456]) is both injective and surjective, and hence invertible.

Contributed by Robert Beezer
\textbf{T15} Suppose that }T: U \rightarrow V\text{ is a surjective linear transformation and }\dim(U) = \dim(V).\text{ Prove that }T\text{ is injective.}

Contributed by Robert Beezer Solution \[472\]

\textbf{T16} Suppose that }T: U \rightarrow V\text{ is an injective linear transformation and }\dim(U) = \dim(V).\text{ Prove that }T\text{ is surjective.}

Contributed by Robert Beezer
C20 Contributed by Robert Beezer  Statement 468
(a) We will compute the kernel of $T$. Suppose that $a + bx + cx^2 \in K(T)$. Then

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = T (a + bx + cx^2) = \begin{bmatrix} a + 2b - 2c & 2a + 2b \\ -a + b - 4c & 3a + 2b + 2c \end{bmatrix}$$

and matrix equality (Theorem ME 384) yields the homogeneous system of four equations in three variables,

$$\begin{align*}
    a + 2b - 2c &= 0 \\
    2a + 2b &= 0 \\
    -a + b - 4c &= 0 \\
    3a + 2b + 2c &= 0
\end{align*}$$

The coefficient matrix of this system row-reduces as

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 2 & 0 \\ -1 & 1 & -4 \\ 3 & 2 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From the existence of non-trivial solutions to this system, we can infer non-zero polynomials in $K(T)$. By Theorem KILT 432, we then know that $T$ is not injective.

(b) Since $3 = \dim (P_2) < \dim (M_{22}) = 4$, by Theorem SLTD 450 $T$ is not surjective.

(c) Since $T$ is not surjective, it is not invertible by Theorem ILTIS 459.

C21 Contributed by Robert Beezer  Statement 468
(a) To check injectivity, we compute the kernel of $S$. To this end, suppose that $a + bx + cx^2 + dx^3 \in K(S)$, so

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = S (a + bx + cx^2 + dx^3) = \begin{bmatrix} -a + 4b + c + 2d & 4a - b + 6c - d \\ a + 5b - 2c + 2d & a + 2c + 5d \end{bmatrix}$$

this creates the homogeneous system of four equations in four variables,

$$\begin{align*}
    -a + 4b + c + 2d &= 0 \\
    4a - b + 6c - d &= 0 \\
    a + 5b - 2c + 2d &= 0 \\
    a + 2c + 5d &= 0
\end{align*}$$

The coefficient matrix of this system row-reduces as,

$$\begin{bmatrix} -1 & 4 & 1 & 2 \\ 4 & -1 & 6 & -1 \\ 1 & 5 & -2 & 2 \\ 1 & 0 & 2 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We recognize the coefficient matrix as being nonsingular, so the only solution to the system is $a = b = c = d = 0$, and the kernel of $S$ is trivial, $K(S) = \{ 0 + 0x + 0x^2 + 0x^3 \}$. By Theorem KILT 432, we see that $S$ is injective.
(b) We can establish that $S$ is surjective by considering the rank and nullity of $S$.

$$
r(S) = \dim(P_3) - n(S) \hspace{1cm} \text{Theorem RPNDD} \ [464]$

$$
= 4 - 0
= \dim(M_{22})
$$

So, $R(S)$ is a subspace of $M_{22}$ (Theorem RLTS \ [445]) whose dimension equals that of $M_{22}$. By Theorem EDYES \ [323], we gain the set equality $R(S) = M_{22}$. Theorem RSLT \ [446] then implies that $S$ is surjective.

(c) Since $S$ is both injective and surjective, Theorem ILTIS \ [459] says $S$ is invertible.

C50 Contributed by Robert Beezer Statement \ [468]

Determine the kernel of $S$ first. The condition that $S(\begin{bmatrix} a & b \end{bmatrix}) = 0$ becomes $(3a + b) + (5a + 2b)x = 0 + 0x$. Equating coefficients of these polynomials yields the system

$$
3a + b = 0
5a + 2b = 0
$$

This homogeneous system has a nonsingular coefficient matrix, so the only solution is $a = 0, b = 0$ and thus

$$
K(S) = \{ \begin{bmatrix} 0 & 0 \end{bmatrix} \}
$$

By Theorem KILT \ [432], we know $S$ is injective. With $n(S) = 0$ we employ Theorem RPNDD \ [464] to find

$$
r(S) = r(S) + 0 = r(S) + n(S) = \dim(M_{12}) = 2 = \dim(P_1)
$$

Since $R(S) \subseteq P_1$ and $\dim(R(S)) = \dim(P_1)$, we can apply Theorem EDYES \ [323] to obtain the set equality $R(S) = P_1$ and therefore $S$ is surjective.

One of the two defining conditions of an invertible linear transformation is (Definition IVLT \ [456])

$$(S \circ R)(a + bx) = S(R(a + bx))$$

$$= S(\begin{bmatrix} 2a - b & -5a + 3b \end{bmatrix})$$

$$= (3(2a - b) + (-5a + 3b)) + (5(2a - b) + 2(-5a + 3b))x$$

$$= ((6a - 3b) + (-5a + 3b)) + ((10a - 5b) + (-10a + 6b))x$$

$$= a + bx$$

$$= I_{P_1}(a + bx)$$

That $(R \circ S)(\begin{bmatrix} a & b \end{bmatrix}) = I_{M_{12}}(\begin{bmatrix} a & b \end{bmatrix})$ is similar.

M30 Contributed by Robert Beezer Statement \ [468]

Suppose that $S^{-1}: M_{1,2} \mapsto P_1$ has a form given by

$$
S^{-1}(\begin{bmatrix} z & w \end{bmatrix}) = (rz + sw) + (pz + qw)x
$$

where $r, s, p, q$ are unknown scalars. Then

$$
a + bx = S^{-1}(S(a + bx))$$

$$= S^{-1}(\begin{bmatrix} 3a + b & 2a + b \end{bmatrix})$$

$$= (r(3a + b) + s(2a + b)) + (p(3a + b) + q(2a + b))x$$

$$= ((3r + 2s)a + (r + s)b) + ((3p + 2q)a + (p + q)b)x$$

Equating coefficients of these two polynomials, and then equating coefficients on $a$ and $b$, gives rise to 4 equations in 4 variables,

$$
3r + 2s = 1
$$

Version 1.04
This system has a unique solution: \( r = 1, s = -1, p = -2, q = 3 \). So the desired inverse linear transformation is

\[
S^{-1}(z \quad w) = (z - w) + (-2z + 3w)x
\]

Notice that the system of 4 equations in 4 variables could be split into two systems, each with two equations in two variables (and identical coefficient matrices). After making this split, the solution might feel like computing the inverse of a matrix (Theorem CINM [193]). Hmmmm.

**M31** Contributed by Robert Beezer

Statement [468]

We are given that \( R \) is invertible. The inverse linear transformation can be formulated by considering the pre-image of a generic element of the codomain. With injectivity and surjectivity, we know that the pre-image of any element will be a set of size one — it is this lone element that will be the output of the inverse linear transformation.

Suppose that we set \( v = \begin{bmatrix} x \\ y \end{bmatrix} \) as a generic element of the codomain, \( M_{21} \). Then if \( \begin{bmatrix} r & s \end{bmatrix} = w \in R^{-1}(v) \),

\[
\begin{bmatrix} x \\ y \end{bmatrix} = v = R(w) = \begin{bmatrix} r + 3s \\ 4r + 11s \end{bmatrix}
\]

So we obtain the system of two equations in the two variables \( r \) and \( s \),

\[
\begin{align*}
r + 3s &= x \\
4r + 11s &= y
\end{align*}
\]

With a nonsingular coefficient matrix, we can solve the system using the inverse of the coefficient matrix,

\[
\begin{align*}
r &= -11x + 3y \\
s &= 4x - y
\end{align*}
\]

So we define,

\[
R^{-1}(v) = R^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = w = \begin{bmatrix} r & s \end{bmatrix} = \begin{bmatrix} -11x + 3y \\ 4x - y \end{bmatrix}
\]

**T15** Contributed by Robert Beezer

Statement [469]

If \( T \) is surjective, then Theorem RSLT [446] says \( \mathcal{R}(T) = \mathcal{V} \), so \( r(T) = \dim(\mathcal{V}) \). In turn, the hypothesis gives \( r(T) = \dim(U) \). Then, using Theorem RPNDD [464],

\[
n(T) = (r(T) + n(T)) - r(T) = \dim(U) - \dim(U) = 0
\]

With a null space of zero dimension, \( \mathcal{K}(T) = \{0\} \), and by Theorem KILT [432] we see that \( T \) is injective. \( T \) is both injective and surjective so by Theorem ILTIS [459], \( T \) is invertible.
Chapter R
Representations

Previous work with linear transformations may have convinced you that we can convert most questions about linear transformations into questions about systems of equations or properties of subspaces of \( \mathbb{C}^m \). In this section we begin to make these vague notions precise. We have used the word “representation” prior, but it will get a heavy workout in this chapter. In many ways, everything we have studied so far was in preparation for this chapter.

Section VR
Vector Representations

We begin by establishing an invertible linear transformation between any vector space \( V \) of dimension \( m \) and \( \mathbb{C}^m \). This will allow us to “go back and forth” between the two vector spaces, no matter how abstract the definition of \( V \) might be.

**Definition VR**

**Vector Representation**

Suppose that \( V \) is a vector space with a basis \( B = \{v_1, v_2, v_3, \ldots, v_n\} \). Define a function \( \rho_B : V \mapsto \mathbb{C}^n \) as follows. For \( w \in V \), find scalars \( a_1, a_2, a_3, \ldots, a_n \) so that

\[
    w = a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_nv_n
\]

then

\[
    [\rho_B (w)]_i = a_i \quad 1 \leq i \leq n
\]

We need to show that \( \rho_B \) is really a function (since “find scalars” sounds like it could be accomplished in many ways, or perhaps not at all) and right now we want to establish that \( \rho_B \) is a linear transformation. We will wrap up both objectives in one theorem, even though the first part is working backwards to make sure that \( \rho_B \) is well-defined.

**Theorem VRLT**

**Vector Representation is a Linear Transformation**

The function \( \rho_B \) (Definition VR) is a linear transformation.

**Proof**

The definition of \( \rho_B \) (Definition VR) appears to allow considerable latitude in selecting the scalars \( a_1, a_2, a_3, \ldots, a_n \). However, since \( B \) is a basis for \( V \), Theorem VRRB says this can be done, and done uniquely. So despite appearances, \( \rho_B \) is indeed a function.

Suppose that \( x \) and \( y \) are two vectors in \( V \) and \( \alpha \in \mathbb{C} \). Then the vector space properties, Property AC and Property SC, assure us that the vectors \( x + y \) and \( \alpha x \) are also vectors in \( V \). Then provides the following sets of scalars for the four vectors \( x, y, \alpha x, \alpha y \).
$\mathbf{x} + \mathbf{y}$ and $\alpha \mathbf{x}$, and tells us that each set of scalars is the only way to express the given vector as a linear combination of the basis vectors in $B$.

$$
\mathbf{x} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \cdots + a_n \mathbf{v}_n \\
\mathbf{y} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + b_3 \mathbf{v}_3 + \cdots + b_n \mathbf{v}_n \\
\mathbf{x} + \mathbf{y} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 + \cdots + c_n \mathbf{v}_n \\
\alpha \mathbf{x} = d_1 \mathbf{v}_1 + d_2 \mathbf{v}_2 + d_3 \mathbf{v}_3 + \cdots + d_n \mathbf{v}_n
$$

Then these coefficients are related, as we now show.

$$
\mathbf{x} + \mathbf{y} = (a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \cdots + a_n \mathbf{v}_n) \\
+ (b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + b_3 \mathbf{v}_3 + \cdots + b_n \mathbf{v}_n) \\
= a_1 \mathbf{v}_1 + b_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + b_2 \mathbf{v}_2 + \cdots + a_n \mathbf{v}_n + b_n \mathbf{v}_n \\
= (a_1 + b_1) \mathbf{v}_1 + (a_2 + b_2) \mathbf{v}_2 + \cdots + (a_n + b_n) \mathbf{v}_n
$$

By the uniqueness of the expression of $\mathbf{x} + \mathbf{y}$ as a linear combination of the vectors in $B$ (Theorem VRRB 288), we conclude that $c_i = a_i + b_i$, $1 \leq i \leq n$.

Similarly, $\alpha \mathbf{x} = \alpha (a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \cdots + a_n \mathbf{v}_n)$

$$
= \alpha a_1 \mathbf{v}_1 + \alpha a_2 \mathbf{v}_2 + \alpha a_3 \mathbf{v}_3 + \cdots + \alpha a_n \mathbf{v}_n
$$

By the uniqueness of the expression of $\alpha \mathbf{x}$ as a linear combination of the vectors in $B$ (Theorem VRRB 288), we conclude that $d_i = \alpha a_i$, $1 \leq i \leq n$.

Now, for $1 \leq i \leq n$, we have

$$
[\rho_B (\mathbf{x} + \mathbf{y})]_i = c_i \\
= a_i + b_i \\
= [\rho_B (\mathbf{x})]_i + [\rho_B (\mathbf{y})]_i \\
= [\rho_B (\mathbf{x}) + \rho_B (\mathbf{y})]_i
$$

Thus the vectors $\rho_B (\mathbf{x} + \mathbf{y})$ and $\rho_B (\mathbf{x}) + \rho_B (\mathbf{y})$ are equal in each entry and Definition CVE 73 tells us that $\rho_B (\mathbf{x} + \mathbf{y}) = \rho_B (\mathbf{x}) + \rho_B (\mathbf{y})$. This is the first necessary property for $\rho_B$ to be a linear transformation (Definition LT 105).

Similarly, for $1 \leq i \leq n$, we have

$$
[\rho_B (\alpha \mathbf{x})]_i = d_i \\
= \alpha a_i \\
= \alpha [\rho_B (\mathbf{x})]_i \\
= [\alpha \rho_B (\mathbf{x})]_i
$$

and so, the vectors $\rho_B (\alpha \mathbf{x})$ and $\alpha \rho_B (\mathbf{x})$ are equal in each entry and therefore by Definition CVE 73 we have the vector equality $\rho_B (\alpha \mathbf{x}) = \alpha \rho_B (\mathbf{x})$. This establishes the second property of a linear transformation (Definition LT 105) so we can conclude that $\rho_B$ is a linear transformation.

**Example VRC4**

**Vector representation in $\mathbb{C}^4$**

Consider the vector $\mathbf{y} \in \mathbb{C}^4$

$$
\mathbf{y} = \begin{bmatrix} 6 \\ 14 \\ 6 \\ 7 \end{bmatrix}
$$
We will find several coordinate representations of \( y \) in this example. Notice that \( y \) never changes, but the representations of \( y \) do change.

One basis for \( \mathbb{C}^4 \) is

\[
B = \{ u_1, u_2, u_3, u_4 \} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \\ 6 \end{pmatrix} \right\}
\]

as can be seen by making these vectors the columns of a matrix, checking that the matrix is non-singular and applying Theorem CNMB 299. To find \( \rho_B(y) \), we need to find scalars, \( a_1, a_2, a_3, a_4 \) such that

\[
y = a_1 u_1 + a_2 u_2 + a_3 u_3 + a_4 u_4
\]

By Theorem SLSLC 82 the desired scalars are a solution to the linear system of equations with a coefficient matrix whose columns are the vectors in \( B \) and with a vector of constants \( y \). With a non-singular coefficient matrix, the solution is unique, but this is no surprise as this is the content of Theorem VRRB 288. This unique solution is

\[
a_1 = 2 \quad a_2 = -1 \quad a_3 = -3 \quad a_4 = 4
\]

Then by Definition VR 473, we have

\[
\rho_B(y) = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 4 \end{bmatrix}
\]

Suppose now that we construct a representation of \( y \) relative to another basis of \( \mathbb{C}^4 \),

\[
C = \left\{ \begin{pmatrix} -15 \\ 9 \\ -4 \\ -2 \end{pmatrix}, \begin{pmatrix} 16 \\ -14 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} -26 \\ -6 \\ -3 \end{pmatrix}, \begin{pmatrix} 14 \\ 4 \end{pmatrix} \right\}
\]

As with \( B \), it is easy to check that \( C \) is a basis. Writing \( y \) as a linear combination of the vectors in \( C \) leads to solving a system of four equations in the four unknown scalars with a non-singular coefficient matrix. The unique solution can be expressed as

\[
y = \begin{bmatrix} 6 \\ 14 \\ 6 \\ 7 \end{bmatrix} = (-28) \begin{bmatrix} -15 \\ 9 \\ -4 \\ -2 \end{bmatrix} + (-8) \begin{bmatrix} 16 \\ -14 \\ 5 \\ 2 \end{bmatrix} + 11 \begin{bmatrix} -26 \\ -6 \\ -3 \end{bmatrix} + 0 \begin{bmatrix} 14 \\ 4 \end{bmatrix}
\]

so that Definition VR 473 gives

\[
\rho_C(y) = \begin{bmatrix} -28 \\ -8 \\ 11 \\ 0 \end{bmatrix}
\]

We often perform representations relative to standard bases, but for vectors in \( \mathbb{C}^m \) its a little silly. Let’s find the vector representation of \( y \) relative to the standard basis (Theorem SUVB 294),

\[
D = \{ e_1, e_2, e_3, e_4 \}
\]

Then, without any computation, we can check that

\[
y = \begin{bmatrix} 6 \\ 14 \\ 6 \\ 7 \end{bmatrix} = 6e_1 + 14e_2 + 6e_3 + 7e_4
\]
so by Definition VR [473],

\[
\rho_D(y) = \begin{bmatrix}
6 \\
14 \\
6 \\
7 
\end{bmatrix}
\]

which is not very exciting. Notice however that the order in which we place the vectors in the basis is critical to the representation. Let’s keep the standard unit vectors as our basis, but rearrange the order we place them in the basis. So a fourth basis is

\[
E = \{e_3, e_4, e_2, e_1\}
\]

Then,

\[
y = \begin{bmatrix}
6 \\
14 \\
6 \\
7 
\end{bmatrix} = 6e_3 + 7e_4 + 14e_2 + 6e_1
\]

so by Definition VR [473],

\[
\rho_E(y) = \begin{bmatrix}
6 \\
7 \\
14 \\
6 
\end{bmatrix}
\]

So for every possible basis of \( \mathbb{C}^4 \) we could construct a different representation of \( y \). \( \Box \)

Vector representations are most interesting for vector spaces that are not \( \mathbb{C}^m \).

**Example VRP2**

**Vector representations in \( P_2 \)**

Consider the vector \( u = 15 + 10x - 6x^2 \in P_2 \) from the vector space of polynomials with degree at most 2 (Example VSP [253]). A nice basis for \( P_2 \) is

\[
B = \{1, x, x^2\}
\]

so that

\[
u = 15 + 10x - 6x^2 = 15(1) + 10(x) + (-6)(x^2)
\]

so by Definition VR [473]

\[
\rho_B(u) = \begin{bmatrix}
15 \\
10 \\
-6
\end{bmatrix}
\]

Another nice basis for \( P_2 \) is

\[
B = \{1, 1 + x, 1 + x + x^2\}
\]

so now it takes a bit of computation to determine the scalars for the representation. We want \( a_1, a_2, a_3 \) so that

\[
15 + 10x - 6x^2 = a_1(1) + a_2(1 + x) + a_3(1 + x + x^2)
\]

Performing the operations in \( P_2 \) on the right-hand side, and equating coefficients, gives the three equations in the three unknown scalars,

\[
15 = a_1 + a_2 + a_3 \\
10 = a_2 + a_3 \\
-6 = a_3
\]

The coefficient matrix of this sytem is nonsingular, leading to a unique solution (no surprise there, see Theorem VRRB [288]),

\[
a_1 = 5 \\
a_2 = 16 \\
a_3 = -6
\]
so by Definition VR \[473\]
\[
\rho_C(u) = \begin{bmatrix} 5 \\ 16 \\ -6 \end{bmatrix}
\]

While we often form vector representations relative to “nice” bases, nothing prevents us from forming representations relative to “nasty” bases. For example, the set
\[
D = \{-2 - x + 3x^2, 1 - 2x^2, 5 + 4x + x^2\}
\]
can be verified as a basis of \(P_2\) by checking linear independence with Definition LI \[280\] and then arguing that 3 vectors from \(P_2\), a vector space of dimension 3 (Theorem DF \[311\]), must also be a spanning set (Theorem G \[320\]). Now we desire scalars \(a_1, a_2, a_3\) so that
\[
15 + 10x - 6x^2 = a_1(-2 - x + 3x^2) + a_2(1 - 2x^2) + a_3(5 + 4x + x^2)
\]
Performing the operations in \(P_2\) on the right-hand side, and equating coefficients, gives the three equations in the three unknown scalars,
\[
\begin{align*}
15 &= -2a_1 + a_2 + 5a_3 \\
10 &= -a_1 + 4a_3 \\
-6 &= 3a_1 - 2a_2 + a_3
\end{align*}
\]
The coefficient matrix of this system is nonsingular, leading to a unique solution (no surprise there, see Theorem VRRB \[288\]),
\[
\begin{align*}
a_1 &= -2 \\
a_2 &= 1 \\
a_3 &= 2
\end{align*}
\]
so by Definition VR \[473\]
\[
\rho_D(u) = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}
\]

**Theorem VRI**  
**Vector Representation is Injective**  
The function \(\rho_B\) (Definition VR \[473\]) is an injective linear transformation. □

**Proof**  
We will appeal to Theorem KILT \[432\]. Suppose \(U\) is a vector space of dimension \(n\), so vector representation is of the form \(\rho_B: U \mapsto \mathbb{C}^n\). Let \(B = \{u_1, u_2, u_3, \ldots, u_n\}\) be the basis of \(U\) used in the definition of \(\rho_B\). Suppose \(u \in K(\rho_B)\). Finally, since \(B\) is a basis for \(U\), by Theorem VRRB \[288\] there are (unique) scalars, \(a_1, a_2, a_3, \ldots, a_n\) such that
\[
u = a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_nu_n
\]
Then for \(1 \leq i \leq n\)
\[
\begin{align*}
a_i &= [\rho_B(u)]_i \\
&= [0]_i \\
&= 0
\end{align*}
\]
So
\[
u = a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_nu_n
\]
\[
= 0u_1 + 0u_2 + 0u_3 + \cdots + 0u_n
\]
\[
= 0 + 0 + \cdots + 0
\]
\[
= 0
\]
Thus an arbitrary vector, \( u \), from the kernel, \( \mathcal{K}(\rho_B) \), must equal the zero vector of \( U \). So \( \mathcal{K}(\rho_B) = \{0\} \) and by Theorem KILT \[432\], \( \rho_B \) is injective.

**Theorem VRS**  
**Vector Representation is Surjective**

The function \( \rho_B \) (Definition VR \[473\]) is a surjective linear transformation.  

**Proof** We will appeal to Theorem RSLT \[446\]. Suppose \( U \) is a vector space of dimension \( n \), so vector representation is of the form \( \rho_B : U \to \mathbb{C}^n \). Let \( B = \{u_1, u_2, u_3, \ldots, u_n\} \) be the basis of \( U \) used in the definition of \( \rho_B \). Suppose \( v \in \mathbb{C}^n \). Define the vector \( u \) by

\[
\begin{align*}
  u &= [v]_1 u_1 + [v]_2 u_2 + [v]_3 u_3 + \cdots + [v]_n u_n \\
  &\quad \text{Definition VR \[473\]}
\end{align*}
\]

Then for \( 1 \leq i \leq n \)

\[
[\rho_B(u)]_i = [\rho_B([v]_1 u_1 + [v]_2 u_2 + [v]_3 u_3 + \cdots + [v]_n u_n)]_i
\]

so the entries of vectors \( \rho_B(u) \) and \( v \) are equal and Definition CVE \[73\] yields the vector equality \( \rho_B(u) = v \). This demonstrates that \( v \in \mathcal{R}(\rho_B) \), so \( \mathbb{C}^n \subseteq \mathcal{R}(\rho_B) \). Since \( \mathcal{R}(\rho_B) \subseteq \mathbb{C}^n \) by Definition RLT \[444\], we have \( \mathcal{R}(\rho_B) = \mathbb{C}^n \) and Theorem RSLT \[446\] says \( \rho_B \) is surjective.  

We will have many occasions later to employ the inverse of vector representation, so we will record the fact that vector representation is an invertible linear transformation.

**Theorem VRILT**  
**Vector Representation is an Invertible Linear Transformation**

The function \( \rho_B \) (Definition VR \[473\]) is an invertible linear transformation.

**Proof** The function \( \rho_B \) (Definition VR \[473\]) is a linear transformation (Theorem VRLT \[473\]) that is injective (Theorem VRI \[477\]) and surjective (Theorem VRS \[478\]) with domain \( V \) and codomain \( \mathbb{C}^n \). By Theorem ILTIS \[459\] we then know that \( \rho_B \) is an invertible linear transformation.

Informally, we will refer to the application of \( \rho_B \) as **coordinatizing** a vector, while the application of \( \rho_B^{-1} \) will be referred to as **un-coordinatizing** a vector.

### Subsection CVS

**Characterization of Vector Spaces**

Limiting our attention to vector spaces with finite dimension, we now describe every possible vector space. All of them. Really.

**Theorem CFDVS**  
**Characterization of Finite Dimensional Vector Spaces**

Suppose that \( V \) is a vector space with dimension \( n \). Then \( V \) is isomorphic to \( \mathbb{C}^n \).

**Proof** Since \( V \) has dimension \( n \) we can find a basis of \( V \) of size \( n \) (Definition D \[307\]) which we will call \( B \). The linear transformation \( \rho_B \) is an invertible linear transformation from \( V \) to \( \mathbb{C}^n \), so by Definition IVS \[461\], we have that \( V \) and \( \mathbb{C}^n \) are isomorphic.

**Theorem CFDVS** \[478\] is the first of several surprises in this chapter, though it might be a bit demoralizing too. It says that there really are not all that many different (finite dimensional) vector spaces, and none are really any more complicated than \( \mathbb{C}^n \). Hmmm. The following examples should make this point.
Example TIVS
Two isomorphic vector spaces
The vector space of polynomials with degree 8 or less, \( P_8 \), has dimension 9 (Theorem DP [311]). By Theorem CFDVS [478], \( P_8 \) is isomorphic to \( \mathbb{C}^9 \).

Example CVSR
Crazy vector space revealed
The crazy vector space, \( C \) of Example CVS [255], has dimension 2 by Example DC [312]. By Theorem CFDVS [478], \( C \) is isomorphic to \( \mathbb{C}^2 \). Hmmmm. Not really so crazy after all?

Example ASC
A subspace characterized
In Example DSP4 [312] we determined that a certain subspace \( W \) of \( P_4 \) has dimension 4. By Theorem CFDVS [478], \( W \) is isomorphic to \( \mathbb{C}^4 \).

Theorem IFDVS
Isomorphism of Finite Dimensional Vector Spaces
Suppose \( U \) and \( V \) are both finite-dimensional vector spaces. Then \( U \) and \( V \) are isomorphic if and only if \( \dim(U) = \dim(V) \).

\[ \text{Proof} \]
\[ (\Rightarrow) \] This is just the statement proved in Theorem IVSED [462].
\[ (\Leftarrow) \] This is the advertised converse of Theorem IVSED [462]. We will assume \( U \) and \( V \) have equal dimension and discover that they are isomorphic vector spaces. Let \( n \) be the common dimension of \( U \) and \( V \). Then by Theorem CFDVS [478] there are isomorphisms \( T: U \rightarrow \mathbb{C}^n \) and \( S: V \rightarrow \mathbb{C}^n \).

\( T \) is therefore an invertible linear transformation by Definition IVS [461]. Similarly, \( S \) is an invertible linear transformation, and so \( S^{-1} \) is an invertible linear transformation (Theorem HILT [459]). The composition of invertible linear transformations is again invertible (Theorem CIVLT [460]) so the composition of \( S^{-1} \) with \( T \) is invertible. Then \( (S^{-1} \circ T): U \rightarrow V \) is an invertible linear transformation from \( U \) to \( V \) and Definition IVS [461] says \( U \) and \( V \) are isomorphic.

Example MIVS
Multiple isomorphic vector spaces
\( \mathbb{C}^{10}, P_9, M_{2,5} \) and \( M_{5,2} \) are all vector spaces and each has dimension 10. By Theorem IFDVS [479] each is isomorphic to any other.

The subspace of \( M_{4,4} \) that contains all the symmetric matrices (Definition SYM [166]) has dimension 10, so this subspace is also isomorphic to each of the four vector spaces above.

Subsection CP
Coordinatization Principle

With \( \rho_B \) available as an invertible linear transformation, we can translate between vectors in a vector space \( U \) of dimension \( m \) and \( \mathbb{C}^n \). Furthermore, as a linear transformation, \( \rho_B \) respects the addition and scalar multiplication in \( U \), while \( \rho_B^{-1} \) respects the addition and scalar multiplication in \( \mathbb{C}^n \). Since our definitions of linear independence, spans, bases and dimension are all built up from linear combinations, we will finally be able to translate fundamental properties between abstract vector spaces (\( U \)) and concrete vector spaces (\( \mathbb{C}^n \)).

Theorem CLI
Coordinatization and Linear Independence
Suppose that \( U \) is a vector space with a basis \( B \) of size \( n \). Then \( S = \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k \} \) is a linearly independent subset of \( U \) if and only if \( R = \{ \rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \ldots, \rho_B(\mathbf{u}_k) \} \) is a linearly independent subset of \( \mathbb{C}^n \).

\[ \text{Proof} \] The linear transformation \( \rho_B \) is an isomorphism between \( U \) and \( \mathbb{C}^n \) (Theorem VRILT [478]). As an invertible linear transformation, \( \rho_B \) is an injective linear transformation (Theorem
ILTIS \[459\], and \( \rho_B^{-1} \) is also an injective linear transformation. \( \text{Theorem IILT} \ [459] \), \( \text{Theorem ILTLI} \ [433] \).

(\( \Rightarrow \)) Since \( \rho_B \) is an injective linear transformation and \( S \) is linearly independent, \( \text{Theorem ILTLI} \ [433] \) says that \( R \) is linearly independent.

(\( \Leftarrow \)) If we apply \( \rho_B^{-1} \) to each element of \( R \), we will create the set \( S \). Since we are assuming \( R \) is linearly independent and \( \rho_B^{-1} \) is injective, \( \text{Theorem ILTLI} \ [433] \) says that \( S \) is linearly independent.

\[ \text{Theorem CSS} \]

\text{Coordination and Spanning Sets}

Suppose that \( U \) is a vector space with a basis \( B \) of size \( n \). Then \( u \in \langle \{u_1, u_2, u_3, \ldots, u_k\} \rangle \) if and only if \( \rho_B(u) \in \langle \{\rho_B(u_1), \rho_B(u_2), \rho_B(u_3), \ldots, \rho_B(u_k)\} \rangle \).

\textbf{Proof} \ (\( \Rightarrow \)) Suppose \( u \in \langle \{u_1, u_2, u_3, \ldots, u_k\} \rangle \). Then there are scalars, \( a_1, a_2, a_3, \ldots, a_k \), such that

\[ u = a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_ku_k \]

Then,

\[ \rho_B(u) = \rho_B(a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_ku_k) = a_1\rho_B(u_1) + a_2\rho_B(u_2) + a_3\rho_B(u_3) + \cdots + a_k\rho_B(u_k) \]

\[ \text{Theorem LTLC} \ [413] \]

which says that \( \rho_B(u) \in \langle \{\rho_B(u_1), \rho_B(u_2), \rho_B(u_3), \ldots, \rho_B(u_k)\} \rangle \).

(\( \Leftarrow \)) Suppose that \( \rho_B(u) \in \langle \{\rho_B(u_1), \rho_B(u_2), \rho_B(u_3), \ldots, \rho_B(u_k)\} \rangle \). Then there are scalars \( b_1, b_2, b_3, \ldots, b_k \) such that

\[ \rho_B(u) = b_1\rho_B(u_1) + b_2\rho_B(u_2) + b_3\rho_B(u_3) + \cdots + b_k\rho_B(u_k) \]

Recall that \( \rho_B \) is invertible, \( \text{Theorem VRILT} \ [478] \), so

\[ u = I_U(u) = (\rho_B^{-1} \circ \rho_B)(u) = \rho_B^{-1}(\rho_B(u)) = \rho_B^{-1}(b_1\rho_B(u_1) + b_2\rho_B(u_2) + b_3\rho_B(u_3) + \cdots + b_k\rho_B(u_k)) = b_1\rho_B^{-1}(\rho_B(u_1)) + b_2\rho_B^{-1}(\rho_B(u_2)) + b_3\rho_B^{-1}(\rho_B(u_3)) + \cdots + b_k\rho_B^{-1}(\rho_B(u_k)) = b_1I_U(u_1) + b_2I_U(u_2) + b_3I_U(u_3) + \cdots + b_kI_U(u_k) = b_1u_1 + b_2u_2 + b_3u_3 + \cdots + b_ku_k \]

\[ \text{Definition IDLT} \ [456] \]

which says that \( u \in \langle \{u_1, u_2, u_3, \ldots, u_k\} \rangle \). \( \square \)

Here’s a fairly simple example that illustrates a very, very important idea.

\textbf{Example CP2}

\textbf{Coordinating in } \( P_2 \)

In \textbf{Example VRP2} \ [476] \ we needed to know that

\[ D = \{-2 - x + 3x^2, 1 - 2x^2, 5 + 4x + x^2\} \]

is a basis for \( P_2 \). With \( \text{Theorem CLI} \ [479] \) and \( \text{Theorem CSS} \ [480] \) this task is much easier. First, choose a known basis for \( P_2 \), a basis that forms vector representations easily. We will choose

\[ B = \{1, x, x^2\} \]

Now, form the subset of \( \mathbb{C}^3 \) that is the result of applying \( \rho_B \) to each element of \( D \),

\[ F = \{\rho_B(-2 - x + 3x^2), \rho_B(1 - 2x^2), \rho_B(5 + 4x + x^2)\} = \left\{ \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} \right\} \]

\[ \text{Version 1.04} \]
and ask if \( F \) is a linearly independent spanning set for \( \mathbb{C}^3 \). This is easily seen to be the case by forming a matrix \( A \) whose columns are the vectors of \( F \), row-reducing \( A \) to the identity matrix \( I_3 \), and then using the nonsingularity of \( A \) to assert that \( F \) is a basis for \( \mathbb{C}^3 \) (Theorem CNMB 299). Now, since \( F \) is a basis for \( \mathbb{C}^3 \), Theorem CLI 479 and Theorem CSS 480 tell us that \( D \) is also a basis for \( \mathbb{P}_2 \).

Example CP2 480 illustrates the broad notion that computations in abstract vector spaces can be reduced to computations in \( \mathbb{C}^m \). You may have noticed this phenomenon as you worked through examples in Chapter VS 251 or Chapter LT 405 employing vector spaces of matrices or polynomials. These computations seemed to invariably result in systems of equations or the like from Chapter SLE 2, Chapter V 72 and Chapter M 163. It is vector representation, \( \rho_B \), that allows us to make this connection formal and precise.

Knowing that vector representation allows us to translate questions about linear combinations, linear independence and spans from general vector spaces to \( \mathbb{C}^m \) allows us to prove a great many theorems about how to translate other properties. Rather than prove these theorems, each of the same style as the other, we will offer some general guidance about how to best employ Theorem VRLT 473, Theorem CLI 479 and Theorem CSS 480. This comes in the form of a “principle”: a basic truth, but most definitely not a theorem (hence, no proof).

The Coordinatization Principle  Suppose that \( U \) is a vector space with a basis \( B \) of size \( n \). Then any question about \( U \), or its elements, which ultimately depends on the vector addition or scalar multiplication in \( U \), or depends on linear independence or spanning, may be translated into the same question in \( \mathbb{C}^n \) by application of the linear transformation \( \rho_B \) to the relevant vectors. Once the question is answered in \( \mathbb{C}^n \), the answer may be translated back to \( U \) (if necessary) through application of the inverse linear transformation \( \rho_B^{-1} \).

Example CM32  

Coordinatization in \( \mathbb{M}_{32} \)  

This is a simple example of the Coordinatization Principle 481, depending only on the fact that coordinatizing is an invertible linear transformation (Theorem VRILT 478). Suppose we have a linear combination to perform in \( \mathbb{M}_{32} \), the vector space of \( 3 \times 2 \) matrices, but we are adverse to doing the operations of \( \mathbb{M}_{32} \) (Definition MA 163, Definition MSM 164). More specifically, suppose we are faced with the computation

\[
6 \begin{bmatrix} 3 & 7 \\
-2 & 4 \\
0 & -3 \\
\end{bmatrix} + 2 \begin{bmatrix} -1 & 3 \\
4 & 8 \\
-2 & 5 \\
\end{bmatrix}
\]

We choose a nice basis for \( \mathbb{M}_{32} \) (or a nasty basis if we are so inclined),

\[
B = \begin{bmatrix}
[1 & 0] \\
[0 & 0] \\
[0 & 1] \\
\end{bmatrix}, \begin{bmatrix}
[0 & 1] \\
[0 & 0] \\
[0 & 0] \\
\end{bmatrix}, \begin{bmatrix}
[0 & 0] \\
[0 & 0] \\
[0 & 1] \\
\end{bmatrix}, \begin{bmatrix}
[0 & 1] \\
[0 & 0] \\
[1 & 0] \\
\end{bmatrix}, \begin{bmatrix}
[0 & 0] \\
[0 & 1] \\
[0 & 0] \\
\end{bmatrix}, \begin{bmatrix}
[0 & 0] \\
[0 & 0] \\
[1 & 0] \\
\end{bmatrix}
\end{bmatrix}
\]

and apply \( \rho_B \) to each vector in the linear combination. This gives us a new computation, now in the vector space \( \mathbb{C}^6 \),

\[
6 \begin{bmatrix} 3 & -2 \\
0 & 7 \\
4 & -3 \\
\end{bmatrix} + 2 \begin{bmatrix} -1 \\
4 \\
-2 \\
\end{bmatrix}
\]

which we can compute with the operations of \( \mathbb{C}^6 \) (Definition CVA 73, Definition CVSM 74), to
arrive at
\[
\begin{bmatrix}
16 \\
-4 \\
-4 \\
48 \\
40 \\
-8
\end{bmatrix}
\]
We are after the result of a computation in \( M_{32} \), so we now can apply \( \rho_B^{-1} \) to obtain a \( 3 \times 2 \) matrix,
\[
\begin{bmatrix}
16 \\
0 \\
0 \\
\end{bmatrix} + \begin{bmatrix}
-4 \\
0 \\
0 \\
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix} = \begin{bmatrix}
16 \\
48 \\
-4 \\
40 \\
-4 \\
-8
\end{bmatrix}
\]
which is exactly the matrix we would have computed had we just performed the matrix operations in the first place. So this was not meant to be an easier way to compute a linear combination of two matrices, just a different way.

Subsection READ
Reading Questions

1. The vector space of \( 3 \times 5 \) matrices, \( M_{3,5} \) is isomorphic to what fundamental vector space?

2. A basis for \( \mathbb{C}^3 \) is
\[
B = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}
\]
Compute \( \rho_B \left( \begin{bmatrix} 5 \\ 8 \\ -1 \end{bmatrix} \right) \).

3. What is the first “surprise,” and why is it surprising?
Subsection EXC  Exercises

**C10**  In the vector space $\mathbb{C}^3$, compute the vector representation $\rho_B(v)$ for the basis $B$ and vector $v$ below.

$$B = \left\{ \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \right\} \quad v = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$$

Contributed by Robert Beezer  Solution 484

**C20**  Rework Example CM32 replacing the basis $B$ by the basis

$$C = \left\{ \begin{bmatrix} -14 & -9 \\ 10 & 10 \end{bmatrix}, \begin{bmatrix} -7 & -4 \\ 5 & 5 \end{bmatrix}, \begin{bmatrix} -3 & -1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -7 & -4 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ -3 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & -2 \end{bmatrix} \right\}$$

Contributed by Robert Beezer  Solution 484

**M10**  Prove that the set $S$ below is a basis for the vector space of $2 \times 2$ matrices, $M_{22}$. Do this choosing a natural basis for $M_{22}$ and coordinatizing the elements of $S$ with respect to this basis. Examine the resulting set of column vectors from $\mathbb{C}^4$ and apply the Coordinatization Principle.

$$S = \left\{ \begin{bmatrix} 33 & 99 \\ 78 & -9 \end{bmatrix}, \begin{bmatrix} -16 & -47 \\ -36 & 2 \end{bmatrix}, \begin{bmatrix} 10 & 27 \\ 17 & 3 \end{bmatrix}, \begin{bmatrix} -2 & -7 \\ -6 & 4 \end{bmatrix} \right\}$$

Contributed by Andy Zimmer
We need to express the vector $v$ as a linear combination of the vectors in $B$. Theorem VRRB tells us we will be able to do this, and do it uniquely. The vector equation

$$a_1 \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \\ 8 \end{pmatrix}$$

becomes (via Theorem SLSLC) a system of linear equations with augmented matrix,

$$\begin{pmatrix} 2 & 1 & 3 & 11 \\ -2 & 3 & 5 & 5 \\ 2 & 1 & 2 & 8 \end{pmatrix}$$

This system has the unique solution $a_1 = 2$, $a_2 = -2$, $a_3 = 3$. So by Definition VR, 

$$\rho_B(v) = \rho_B \left( \begin{pmatrix} 11 \\ 5 \\ 8 \end{pmatrix} \right) = \rho_B \left( 2 \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} + (-2) \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

The following computations replicate the computations given in Example CM32, only using the basis $C$.

$$\rho_C \left( \begin{pmatrix} 3 & 7 \\ -2 & 4 \\ 0 & -3 \end{pmatrix} \right) = \begin{pmatrix} -9 \\ 12 \\ -6 \\ 7 \\ -2 \\ -1 \end{pmatrix}$$

$$\rho_C \left( \begin{pmatrix} -1 & 3 \\ 4 & 8 \\ -2 & 5 \end{pmatrix} \right) = \begin{pmatrix} -11 \\ 34 \\ -4 \\ -1 \\ 16 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} -9 \\ 12 \\ -6 \\ 7 \\ -2 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} -11 \\ 34 \\ -4 \\ -1 \\ 16 \\ 5 \end{pmatrix} = \begin{pmatrix} -76 \\ 140 \\ -44 \\ 40 \\ 20 \\ 4 \end{pmatrix}$$

$$\rho_C^{-1} \left( \begin{pmatrix} -76 \\ 140 \\ -44 \\ 40 \\ 20 \\ 4 \end{pmatrix} \right) = \begin{pmatrix} 16 \\ 48 \\ -4 \\ 30 \\ -4 \\ -8 \end{pmatrix}$$
Section MR
Matrix Representations

We have seen that linear transformations whose domain and codomain are vector spaces of columns vectors have a close relationship with matrices (Theorem MBLT, Theorem MLCTV). In this section, we will extend the relationship between matrices and linear transformations to the setting of linear transformations between abstract vector spaces.

Definition MR
Matrix Representation
Suppose that \( T: U \rightarrow V \) is a linear transformation, \( B = \{u_1, u_2, u_3, \ldots, u_n\} \) is a basis for \( U \) of size \( n \), and \( C \) is a basis for \( V \) of size \( m \). Then the matrix representation of \( T \) relative to \( B \) and \( C \) is the \( m \times n \) matrix,

\[
M_{B,C}^T = [\rho_C(T(u_1))] \rho_C(T(u_2)) \rho_C(T(u_3)) \ldots \rho_C(T(u_n))
\]

△

Example OLTTR
One linear transformation, three representations
Consider the linear transformation

\[
S: P_3 \rightarrow M_{22}, \quad S(a + bx + cx^2 + dx^3) = \begin{bmatrix} 3a + 7b - 2c - 5d & 8a + 14b - 2c - 11d \\ -4a - 8b + 2c + 6d & 12a + 22b - 4c - 17d \end{bmatrix}
\]

First, we build a representation relative to the bases,

\[
B = \{1 + 2x + x^2 - x^3, 1 + 3x + x^2 + x^3, -1 - 2x + 2x^3, 2 + 3x + 2x^2 - 5x^3\}
\]

\[
C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & -4 \\ -2 & -4 \end{bmatrix}
\]

We evaluate \( S \) with each element of the basis for the domain, \( B \), and coordinatize the result relative to the vectors in the basis for the codomain, \( C \).

\[
\rho_C(S(1 + 2x + x^2 - x^3)) = \rho_C\left(\begin{bmatrix} 20 & 45 \\ -24 & 69 \end{bmatrix}\right)
\]

\[
= \rho_C\left(\begin{bmatrix} -90 & 37 \\ -40 & 4 \end{bmatrix}\right)
\]

\[
\rho_C(S(1 + 3x + x^2 + x^3)) = \rho_C\left(\begin{bmatrix} 17 & 37 \\ -20 & 57 \end{bmatrix}\right)
\]

\[
= \rho_C\left(\begin{bmatrix} -72 & 29 \\ -34 & 3 \end{bmatrix}\right)
\]

\[
\rho_C(S(-1 - 2x + 2x^3)) = \rho_C\left(\begin{bmatrix} -27 & -58 \\ 32 & -90 \end{bmatrix}\right)
\]

\[
= \rho_C\left(\begin{bmatrix} -114 & -46 \\ 54 & -5 \end{bmatrix}\right)
\]

\[
\rho_C(S(2 + 3x + 2x^2 - 5x^3)) = \rho_C\left(\begin{bmatrix} 48 & 109 \\ -58 & 167 \end{bmatrix}\right)
\]
\[ \rho_C \left( (-220) \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + 91 \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix} + -96 \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} + 10 \begin{bmatrix} -1 & -4 \\ -2 & -4 \end{bmatrix} \right) = \begin{bmatrix} -220 & 91 \\ -96 & 10 \end{bmatrix} \]

Thus, employing Definition MR 485

\[ M_{D,C}^S = \begin{bmatrix} -90 & -72 & 114 & -220 \\ 37 & 29 & -46 & 91 \\ -40 & -34 & 54 & -96 \\ 4 & 3 & -5 & 10 \end{bmatrix} \]

Often we use “nice” bases to build matrix representations and the work involved is much easier. Suppose we take bases

\[ D = \{1, x, x^2, x^3\} \quad E = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \]

The evaluation of \( S \) at the elements of \( D \) is easy and coordinatization relative to \( E \) can be done on sight,

\[ \rho_E (S(1)) = \rho_E \left( \begin{bmatrix} 3 & 8 \\ -4 & 12 \end{bmatrix} \right) = \rho_E \left( 3 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 8 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-4) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 12 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \]

\[ \rho_E (S(x)) = \rho_E \left( \begin{bmatrix} 7 & 14 \\ -8 & 22 \end{bmatrix} \right) = \rho_E \left( 7 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 14 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-8) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 22 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 7 \\ -8 \end{bmatrix} \]

\[ \rho_E (S(x^2)) = \rho_E \left( \begin{bmatrix} -2 & -2 \\ 2 & -4 \end{bmatrix} \right) = \rho_E \left( (-2) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-2) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-4) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \]

\[ \rho_E (S(x^3)) = \rho_E \left( \begin{bmatrix} -5 & -11 \\ 6 & -17 \end{bmatrix} \right) = \rho_E \left( (-5) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-11) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 6 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-17) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} -5 \\ 6 \end{bmatrix} \]

So the matrix representation of \( S \) relative to \( D \) and \( E \) is

\[ M_{D,E}^S = \begin{bmatrix} 3 & 7 & -2 & -5 \\ 8 & 14 & -2 & -11 \\ -4 & -8 & 2 & 6 \\ 12 & 22 & -4 & -17 \end{bmatrix} \]
One more time, but now let’s use bases

\[ F = \{1 + x - x^2 + 2x^3, -1 + 2x + 2x^3, 2 + x - 2x^2 + 3x^3, 1 + x + 2x^3\} \]

\[ G = \begin{pmatrix} 1 & 1 & -1 & 2 & 2 & 1 & 1 \\ -1 & 2 & 0 & 2 & 0 & 2 & 1 \\ 2 & 1 & -2 & 3 & 1 & 0 & 2 \end{pmatrix} \]

and evaluate \( S \) with the elements of \( F \), then coordinatize the results relative to \( G \),

\[
\rho_G \left( S \left( 1 + x - x^2 + 2x^3 \right) \right) = \rho_G \left( \begin{pmatrix} 2 & 2 \\ -2 & 4 \end{pmatrix} \right) = \rho_G \left( 2 \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \right) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}
\]

\[
\rho_G \left( S \left( -1 + 2x + 2x^3 \right) \right) = \rho_G \left( \begin{pmatrix} 1 & -2 \\ 0 & -2 \end{pmatrix} \right) = \rho_G \left( -1 \begin{pmatrix} -1 & 2 \\ 0 & 2 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}
\]

\[
\rho_G \left( S \left( 2 + x - 2x^2 + 3x^3 \right) \right) = \rho_G \left( \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix} \right) = \rho_G \left( 2 \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\]

\[
\rho_G \left( S \left( 1 + x + 2x^3 \right) \right) = \rho_G \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) = \rho_G \left( 0 \begin{pmatrix} 1 & 0 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

So we arrive at an especially economical matrix representation,

\[
M_{F,G}^S = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

We may choose to use whatever terms we want when we make a definition. Some are arbitrary, while others make sense, but only in light of subsequent theorems. Matrix representation is in the latter category. We begin with a linear transformation and produce a matrix. So what? Here’s the theorem that justifies the term “matrix representation.”

**Theorem FTMR**

**Fundamental Theorem of Matrix Representation**

Suppose that \( T: U \rightarrow V \) is a linear transformation, \( B \) is a basis for \( U \), \( C \) is a basis for \( V \) and \( M_{B,C}^T \) is the matrix representation of \( T \) relative to \( B \) and \( C \). Then, for any \( u \in U \),

\[
\rho_C \left( T \left( u \right) \right) = M_{B,C}^T \left( \rho_B \left( u \right) \right)
\]

or equivalently

\[
T \left( u \right) = \rho_C^{-1} \left( M_{B,C}^T \left( \rho_B \left( u \right) \right) \right)
\]

**Proof**  Let \( B = \{u_1, u_2, u_3, \ldots, u_n\} \) be the basis of \( U \). Since \( u \in U \), there are scalars \( a_1, a_2, a_3, \ldots, a_n \) such that

\[
u = a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_n u_n
\]
Then,
\[
M_{B,C}^T \rho_B (u) = [\rho_C (T (u_1))] \rho_C (T (u_2)) | \rho_C (T (u_3)) | \cdots | \rho_C (T (u_n))] \rho_B (u)
\]

\[
= [\rho_C (T (u_1))] \rho_C (T (u_2)) | \rho_C (T (u_3)) | \cdots | \rho_C (T (u_n))]
\]

\[
= a_1 \rho_C (T (u_1)) + a_2 \rho_C (T (u_2)) + \cdots + a_n \rho_C (T (u_n))
\]

\[
= \rho_C (a_1 T (u_1) + a_2 T (u_2) + a_3 T (u_3) + \cdots + a_n T (u_n))
\]

\[
= \rho_C (T (a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_n u_n))
\]

\[
= \rho_C (T (u))
\]

The alternative conclusion is obtained as

\[
T (u) = I_V (T (u))
\]

\[
= (\rho_C^{-1} \circ \rho_C) (T (u))
\]

\[
= \rho_C^{-1} (\rho_C (T (u)))
\]

\[
= \rho_C^{-1} (M_{B,C}^T (\rho_B (u)))
\]

This theorem says that we can apply \( T \) to \( u \) and coordinatize the result relative to \( C \) in \( V \), or we can first coordinatize \( u \) relative to \( B \) in \( U \), then multiply by the matrix representation. Either way, the result is the same. So the effect of a linear transformation can always be accomplished by a matrix-vector product (Definition MVP 173). That’s important enough to say again. The effect of a linear transformation is a matrix-vector product.

\[
\begin{array}{c}
\rho_B (u) M_{B,C}^T \rho_B (u) \xrightarrow{\rho_C} \rho_C (T (u))
\end{array}
\]

The alternative conclusion of this result might be even more striking. It says that to effect a linear transformation \( (T) \) of a vector \( (u) \), coordinatize the input (with \( \rho_B \)), do a matrix-vector product (with \( M_{B,C}^T \)), and un-coordinatize the result (with \( \rho_C^{-1} \)). So, absent some bookkeeping about vector representations, a linear transformation is a matrix.

Here’s an example to illustrate how the “action” of a linear transformation can be effected by matrix multiplication.

**Example ALTMM**

A linear transformation as matrix multiplication

In Example OLTTR 485 we found three representations of the linear transformation \( S \). In this example, we will compute a single output of \( S \) in four different ways. First “normally,” then three times over using Theorem FTMR 487.

Choose \( p(x) = 3 - x + 2x^2 - 5x^3 \), for no particular reason. Then the straightforward application of \( S \) to \( p(x) \) yields

\[
S (p(x)) = S (3 - x + 2x^2 - 5x^3)
\]

\[
= \begin{bmatrix}
3(3) + 7(-1) - 2(2) - 5(-5) & 8(3) + 14(-1) - 2(2) - 11(-5) \\
-4(3) - 8(-1) + 2(2) + 6(-5) & 12(3) + 22(-1) - 4(2) - 17(-5)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
13 & 11 \\
10 & 8
\end{bmatrix}
\]
Now use the representation of \( S \) relative to the bases \( B \) and \( C \) and Theorem FTMR [487]. Note that we will employ the following linear combination in moving from the second line to the third,

\[
3 - x + 2x^2 - 5x^3 = 48(1 + 2x + x^2 - x^3) + (-20)(1 + 3x + x^2 + x^3) + (-1)(-1 - 2x + 2x^3) + (-13)(2 + 3x + 2x^2 - 5x^3)
\]

\[
S(p(x)) = \rho_C^{-1}(M_{B,C}^S \rho_B(p(x)))
\]

\[
= \rho_C^{-1}(M_{B,C}^S(3 - x + 2x^2 - 5x^3))
\]

\[
= \rho_C^{-1}\begin{pmatrix}
M_{B,C}^S
\end{pmatrix}
\]

\[
= \rho_C^{-1}\begin{pmatrix}
-90 & -72 & 114 & -220
37 & 29 & -46 & 91
-40 & -34 & 54 & -96
4 & 3 & -5 & 10
\end{pmatrix}
\]

\[
= \rho_C^{-1}\begin{pmatrix}
-134
59
-46
7
\end{pmatrix}
\]

\[
= (-134)\begin{pmatrix}1 & 1 & 2\end{pmatrix} + 59\begin{pmatrix}2 & 3 & 5\end{pmatrix} + (-46)\begin{pmatrix}-1 & -1 & 0\end{pmatrix} + 7\begin{pmatrix}-1 & -2 & -4\end{pmatrix}
\]

\[
= \begin{pmatrix}23 & 61 & -30 & 91\end{pmatrix}
\]

Again, but now with “nice” bases like \( D \) and \( E \), and the computations are more transparent.

\[
S(p(x)) = \rho_E^{-1}(M_{D,E}^S \rho_D(p(x)))
\]

\[
= \rho_E^{-1}(M_{D,E}^S(3 - x + 2x^2 - 5x^3))
\]

\[
= \rho_E^{-1}(M_{D,E}^S(3(1) + (-1)(x) + 2(x^2) + (-5)(x^3)))
\]

\[
= \rho_E^{-1}\begin{pmatrix}
M_{D,E}^S
\end{pmatrix}
\]

\[
= \rho_E^{-1}\begin{pmatrix}
3 & 7 & -2 & -5
8 & 14 & -2 & -11
-4 & -8 & 2 & 6
12 & 22 & -4 & -17
\end{pmatrix}
\]

\[
= \rho_E^{-1}\begin{pmatrix}
23
61
-30
91
\end{pmatrix}
\]

\[
= 23\begin{pmatrix}1 & 0 & 0\end{pmatrix} + 61\begin{pmatrix}0 & 1 & 0\end{pmatrix} + (-30)\begin{pmatrix}0 & 0 & 1\end{pmatrix} + 91\begin{pmatrix}0 & 0 & 1\end{pmatrix}
\]

\[
= \begin{pmatrix}23 & 61 & -30 & 91\end{pmatrix}
\]

OK, last time, now with the bases \( F \) and \( G \). The coordinatizations will take some work this time, but the matrix-vector product (Definition MVP [173]) (which is the actual action of the linear
transformation) will be especially easy, given the diagonal nature of the matrix representation, \(M^S_{F,G}\). Here we go,

\[
S(p(x)) = \rho^{-1}_G \left( M^S_{F,G} \rho_F(p(x)) \right)
\]

\[
= \rho^{-1}_G \left( M^S_{F,G} \rho_F \left( 3 - x + 2x^2 - 5x^3 \right) \right)
\]

\[
= \rho^{-1}_G \left( M^S_{F,G} \rho_F \left( 32(1 + x - x^2 + 2x^3) - 7(-1 + 2x + 2x^3) - 17(2 + x - 2x^2 + 3x^3) - 2(1 + x + 2x^3) \right) \right)
\]

\[
= \rho^{-1}_G \left( \begin{bmatrix} 32 & -7 & -17 & -2 \end{bmatrix} \right)
\]

\[
= \rho^{-1}_G \left( \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \right)
\]

\[
= \rho^{-1}_G \left( \begin{bmatrix} 64 & -7 & -17 & 0 \\ 7 & -17 & 0 & 0 \end{bmatrix} \right)
\]

\[
= 64 \left[ \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \right] + 7 \left[ \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \right] + (-17) \left[ \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix} \right] + 0 \left[ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \right]
\]

\[
= \begin{bmatrix} 23 & 61 \\ -30 & 91 \end{bmatrix}
\]

This example is not meant to necessarily illustrate that any one of these four computations is simpler than the others. Instead, it is meant to illustrate the many different ways we can arrive at the same result, with the last three all employing a matrix representation to effect the linear transformation.

We will use Theorem FTMR \[487\] frequently in the next few sections. A typical application will feel like the linear transformation \(T\) “commutes” with a vector representation, \(\rho_C\), and as it does the transformation morphs into a matrix, \(M^T_{B,C}\), while the vector representation changes to a new basis, \(\rho_B\). Or vice-versa.

**Subsection NRFO**

New Representations from Old

In Subsection LT.NLTFO \[417\] we built new linear transformations from other linear transformations. Sums, scalar multiples and compositions. These new linear transformations will have matrix representations as well. How do the new matrix representations relate to the old matrix representations? Here are the three theorems.

**Theorem MRSLT**

Matrix Representation of a Sum of Linear Transformations

Suppose that \(T: U \rightarrow V\) and \(S: U \rightarrow V\) are linear transformations, \(B\) is a basis of \(U\) and \(C\) is a basis of \(V\). Then

\[
M^{T+S}_{B,C} = M^T_{B,C} + M^S_{B,C}
\]

**Proof** Let \(x\) be any vector in \(\mathbb{C}^n\). Define \(u \in U\) by \(u = \rho^{-1}_B(x)\), so \(x = \rho_B(u)\). Then,

\[
M^{T+S}_{B,C} x = M^{T+S}_{B,C} \rho_B(u)
\]

\[
= \rho_C((T+S)(u))
\]

Substitution

Theorem FTMR \[487\]
\[ \rho_C (T(u) + S(u)) = \rho_C (T(u)) + \rho_C (S(u)) \]
\[ = M_{B,C}^T (\rho_B (u)) + M_{B,C}^S (\rho_B (u)) \]
\[ = (M_{B,C}^T + M_{B,C}^S) \rho_B (u) \]
\[ = (M_{B,C}^T + M_{B,C}^S) x \]

Since the matrices \( M_{B,C}^{T+S} \) and \( M_{B,C}^T + M_{B,C}^S \) have equal matrix-vector products for every vector in \( \mathbb{C}^n \), by Theorem EMMVP, they are equal matrices. (Now would be a good time to double-back and study the proof of Theorem EMMVP. You did promise, didn’t you?)

**Theorem MRMLT**

**Matrix Representation of a Multiple of a Linear Transformation**

Suppose that \( T: U \rightarrow V \) is a linear transformation, \( \alpha \in \mathbb{C} \), \( B \) is a basis of \( U \) and \( C \) is a basis of \( V \). Then

\[
M_{B,C}^{\alpha T} = \alpha M_{B,C}^T
\]

**Proof** Let \( x \) be any vector in \( \mathbb{C}^n \). Define \( u \in U \) by \( u = \rho_B^{-1} (x) \), so \( x = \rho_B (u) \). Then,

\[
M_{B,C}^{\alpha T} x = M_{B,C}^{\alpha T} \rho_B (u) = \rho_C (\alpha T(u)) = \rho_C (\alpha T(u)) = \alpha (M_{B,C}^T \rho_B (u)) = (\alpha M_{B,C}^T) \rho_B (u) = (\alpha M_{B,C}^T) x
\]

Since the matrices \( M_{B,C}^{\alpha T} \) and \( \alpha M_{B,C}^T \) have equal matrix-vector products for every vector in \( \mathbb{C}^n \), by Theorem EMMVP, they are equal matrices.

The vector space of all linear transformations from \( U \) to \( V \) is now isomorphic to the vector space of all \( m \times n \) matrices.

**Theorem MRCLT**

**Matrix Representation of a Composition of Linear Transformations**

Suppose that \( T: U \rightarrow V \) and \( S: V \rightarrow W \) are linear transformations, \( B \) is a basis of \( U \), \( C \) is a basis of \( V \), and \( D \) is a basis of \( W \). Then

\[
M_{B,D}^{S \circ T} = M_{C,D}^S M_{B,C}^T
\]

**Proof** Let \( x \) be any vector in \( \mathbb{C}^n \). Define \( u \in U \) by \( u = \rho_B^{-1} (x) \), so \( x = \rho_B (u) \). Then,

\[
M_{B,D}^{S \circ T} x = M_{B,D}^{S \circ T} \rho_B (u) = \rho_D ((S \circ T)(u)) = \rho_D (S(T(u))) = \rho_D (S(T(u))) = M_{C,D}^S \rho_C (T(u)) = M_{C,D}^S (M_{B,C}^T \rho_B (u)) = (M_{C,D}^S M_{B,C}^T) \rho_B (u) = (M_{C,D}^S M_{B,C}^T) x
\]
Since the matrices $M_{B,D}^{S,T}$ and $M_{B,C}^{S,D}M_{B,C}^{T}$ have equal matrix-vector products for every vector in $\mathbb{C}^n$, by Theorem EMMVP they are equal matrices.

This is the second great surprise of introductory linear algebra. Matrices are linear transformations (functions, really), and matrix multiplication is function composition! We can form the composition of two linear transformations, then form the matrix representation of the result. Or we can form the matrix representation of each linear transformation separately, then multiply the two representations together via Definition MM. In either case, we arrive at the same result.

**Example MPMR**  
**Matrix product of matrix representations**  
Consider the two linear transformations,

$T : \mathbb{C}^2 \mapsto P_2 \quad T \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = (-a + 3b) + (2a + 4b)x + (a - 2b)x^2$

$S : P_2 \mapsto M_{22} \quad S \left( a + bx + cx^2 \right) = \begin{bmatrix} 2a + b + 2c & a + 4b - c \\ -a + 3c & 3a + b + 2c \end{bmatrix}$

and bases for $\mathbb{C}^2$, $P_2$ and $M_{22}$ (respectively),

\[B = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\} \]

\[C = \{ 1 - 2x + x^2, -1 + 3x, 2x + 3x^2 \} \]

\[D = \left\{ \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix} \right\} \]

Begin by computing the new linear transformation that is the composition of $T$ and $S$ (Definition LTC, Theorem CLTLT), $(S \circ T) : \mathbb{C}^2 \mapsto M_{22}$,

\[(S \circ T) \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = S \left( T \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) \right)
\]

\[= S ((-a + 3b) + (2a + 4b)x + (a - 2b)x^2)
\]

\[= \begin{bmatrix} 2(-a + 3b) + (2a + 4b) + 2(a - 2b) & (-a + 3b) + 4(2a + 4b) - (a - 2b) \\ -(-a + 3b) + 3(a - 2b) & 3(-a + 3b) + 2(a + 4b) + 2(a - 2b) \end{bmatrix}
\]

\[= \begin{bmatrix} 2a + 6b & 6a + 21b \\ 4a - 9b & a + 9b \end{bmatrix}
\]

Now compute the matrix representations (Definition MR) for each of these three linear transformations ($T$, $S$, $S \circ T$), relative to the appropriate bases. First for $T$,

\[\rho_C \left( T \left( \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) \right) = \rho_C \left( 10x + x^2 \right)
\]

\[= \rho_C \left( 28(1 - 2x + x^2) + 28(-1 + 3x) + (-9)(2x + 3x^2) \right) = \begin{bmatrix} 28 \\ 28 \\ -9 \end{bmatrix}
\]

\[\rho_C \left( T \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \right) = \rho_C \left( 1 + 8x \right)
\]

\[= \rho_C \left( 33(1 - 2x + x^2) + 32(-1 + 3x) + (-11)(2x + 3x^2) \right) = \begin{bmatrix} 33 \\ 32 \\ -11 \end{bmatrix}
\]

So we have the matrix representation of $T$,

\[M_{B,C}^{T} = \begin{bmatrix} 28 & 33 \\ 28 & 32 \\ -9 & -11 \end{bmatrix}
\]
Now, a representation of $S$,

$$\rho_D(S(1-2x+x^2)) = \rho_D\left(\begin{bmatrix} 2 & -8 \\ 2 & 3 \end{bmatrix}\right)$$

$$= \rho_D\left((-11)\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} + (-21)\begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix} + 0\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} + (17)\begin{bmatrix} 2 & -3 \\ 2 & 2 \end{bmatrix}\right)$$

$$= \left[\begin{array}{c} -11 \\ -21 \\ 0 \\ 17 \end{array}\right]$$

$$\rho_D(S(-1+3x)) = \rho_D\left(\begin{bmatrix} 1 & 11 \\ 1 & 0 \end{bmatrix}\right)$$

$$= \rho_D\left(26\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} + 51\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} + 0\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} + (-38)\begin{bmatrix} 2 & -3 \\ 2 & 2 \end{bmatrix}\right)$$

$$= \left[\begin{array}{c} 26 \\ 51 \\ 0 \\ -38 \end{array}\right]$$

$$\rho_D(S(2x+3x^2)) = \rho_D\left(\begin{bmatrix} 8 & 5 \\ 9 & 8 \end{bmatrix}\right)$$

$$= \rho_D\left(34\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} + 67\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} + 1\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} + (-46)\begin{bmatrix} 2 & -3 \\ 2 & 2 \end{bmatrix}\right)$$

$$= \left[\begin{array}{c} 34 \\ 67 \\ 1 \\ -46 \end{array}\right]$$

So we have the matrix representation of $S$,

$$M_{C,D}^S = \left[\begin{array}{cccc} -11 & 26 & 34 \\ -21 & 51 & 67 \\ 0 & 0 & 1 \\ 17 & -38 & -46 \end{array}\right]$$

Finally, a representation of $S \circ T$,

$$\rho_D\left((S \circ T)\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)\right) = \rho_D\left(\begin{bmatrix} 12 & 39 \\ 3 & 12 \end{bmatrix}\right)$$

$$= \rho_D\left(114\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} + 237\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} + (9)\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} + (-174)\begin{bmatrix} 2 & -3 \\ 2 & 2 \end{bmatrix}\right)$$

$$= \left[\begin{array}{c} 114 \\ 237 \\ -9 \\ -174 \end{array}\right]$$

$$\rho_D\left((S \circ T)\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)\right) = \rho_D\left(\begin{bmatrix} 10 & 33 \\ -1 & 11 \end{bmatrix}\right)$$

$$= \rho_D\left(95\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} + 202\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} + (11)\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} + (-149)\begin{bmatrix} 2 & -3 \\ 2 & 2 \end{bmatrix}\right)$$

$$= \left[\begin{array}{c} 95 \\ 202 \\ -11 \\ -149 \end{array}\right]$$
So we have the matrix representation of $S \circ T$:

$$M_{B,D}^{S \circ T} = \begin{bmatrix} 114 & 95 \\ 237 & 202 \\ -9 & -11 \\ -174 & -149 \end{bmatrix}$$

Now, we are all set to verify the conclusion of Theorem MRCLT [491],

$$M_{E,D}^{S}M_{B,C}^{T} = \begin{bmatrix} -11 & 26 & 34 \\ -21 & 51 & 67 \\ 0 & 0 & 1 \\ 17 & -38 & -46 \end{bmatrix} \begin{bmatrix} 28 \\ 33 \\ 28 \\ 32 \\ -9 \\ -11 \end{bmatrix} = \begin{bmatrix} 114 \\ 237 \\ -9 \\ -174 \end{bmatrix} = M_{B,D}^{S \circ T}$$

We have intentionally used non-standard bases. If you were to choose “nice” bases for the three vector spaces, then the result of the theorem might be rather transparent. But this would still be a worthwhile exercise — give it a go.

A diagram, similar to ones we have seen earlier, might make the importance of this theorem clearer,

$$\begin{array}{c}
S, \ T \xrightarrow{\text{Definition LTC}[419]} \begin{bmatrix} -11 & 26 & 34 \\ -21 & 51 & 67 \\ 0 & 0 & 1 \\ 17 & -38 & -46 \end{bmatrix} \begin{bmatrix} 28 \\ 33 \\ 28 \\ 32 \\ -9 \\ -11 \end{bmatrix} \\
\xrightarrow{\text{Definition MR}[485]} \begin{bmatrix} 114 \\ 237 \\ -9 \\ -174 \end{bmatrix} = M_{B,D}^{S \circ T}
\end{array}$$

One of our goals in the first part of this book is to make the definition of matrix multiplication (Definition MVP [173], Definition MM [176]) seem as natural as possible. However, many are brought up with an entry-by-entry description of matrix multiplication (Theorem ME [384]) as the definition of matrix multiplication, and then theorems about columns of matrices and linear combinations follow from that definition. With this unmotivated definition, the realization that matrix multiplication is function composition is quite remarkable. It is an interesting exercise to begin with the question, “What is the matrix representation of the composition of two linear transformations?” and then, without using any theorems about matrix multiplication, finally arrive at the entry-by-entry description of matrix multiplication. Try it yourself (Exercise MR.T80 [505]).

Subsection PMR
Properties of Matrix Representations

It will not be a surprise to discover that the kernel and range of a linear transformation are closely related to the null space and column space of the transformation’s matrix representation. Perhaps this idea has been bouncing around in your head already, even before seeing the definition of a matrix representation. However, with a formal definition of a matrix representation (Definition MR [485]), and a fundamental theorem to go with it (Theorem FTMR [487]), we can be formal about the relationship, using the idea of isomorphic vector spaces (Definition IVS [461]). Here are the twin theorems.
Theorem KNSI

Kernel and Null Space Isomorphism

Suppose that $T: U \rightarrow V$ is a linear transformation, $B$ is a basis for $U$ of size $n$, and $C$ is a basis for $V$. Then the kernel of $T$ is isomorphic to the null space of $M_{B,C}^T$,

$$\mathcal{K}(T) \cong \mathcal{N}(M_{B,C}^T)$$

Proof To establish that two vector spaces are isomorphic, we must find an isomorphism between them, an invertible linear transformation [Definition IVS [461]]. The kernel of the linear transformation $T$, $\mathcal{K}(T)$, is a subspace of $U$, while the null space of the matrix representation, $\mathcal{N}(M_{B,C}^T)$, is a subspace of $\mathbb{C}^n$. The function $\rho_B$ is defined as a function from $U$ to $\mathbb{C}^n$, but we can just as well employ the definition of $\rho_B$ as a function from $\mathcal{K}(T)$ to $\mathcal{N}(M_{B,C}^T)$.

We must first ensure that if we choose an input for $\rho_B$ from $\mathcal{K}(T)$ that then the output will be an element of $\mathcal{N}(M_{B,C}^T)$. So suppose that $u \in \mathcal{K}(T)$. Then

$$M_{B,C}^T \rho_B (u) = \rho_C (T(u)) = \rho_C (0) = 0$$

Theorem FTMR [487] Definition KLT [429] Theorem LTTZZ [408]

This says that $\rho_B (u) \in \mathcal{N}(M_{B,C}^T)$, as desired.

The restriction in the size of the domain and codomain $\rho_B$ will not affect the fact that $\rho_B$ is a linear transformation (Theorem VRLT [473]), nor will it affect the fact that $\rho_B$ is injective (Theorem VRI [477]). Something must be done though to verify that $\rho_B$ is surjective. To this end, appeal to the definition of surjective (Definition SLT [440]), and suppose that we have an element of the codomain, $x \in \mathcal{N}(M_{B,C}^T) \subseteq \mathbb{C}^n$ and we wish to find an element of the domain with $x$ as its image. We now show that the desired element of the domain is $u = \rho_B^{-1}(x)$. First, verify that $u \in \mathcal{K}(T)$,

$$T(u) = T(\rho_B^{-1}(x)) = \rho_C^{-1}(M_{B,C}^T \rho_B (\rho_B^{-1}(x))) = \rho_C^{-1}(M_{B,C}^T \rho_B (\rho_B^{-1}(x))) = \rho_C^{-1}(M_{B,C}^T \rho_B (\rho_B^{-1}(x))) = \rho_C^{-1}(\mathbf{0}_C) = 0$$


Second, verify that the proposed isomorphism, $\rho_B$, takes $u$ to $x$,

$$\rho_B (u) = \rho_B (\rho_B^{-1}(x)) = \rho_B (\rho_B^{-1}(x)) = \rho_B (\rho_B^{-1}(x)) = x$$

Substitution Definition IVLT [456] Definition IDLT [456]

With $\rho_B$ demonstrated to be an injective and surjective linear transformation from $\mathcal{K}(T)$ to $\mathcal{N}(M_{B,C}^T)$, Theorem ILTIS [459] tells us $\rho_B$ is invertible, and so by Definition IVS [461], we say $\mathcal{K}(T)$ and $\mathcal{N}(M_{B,C}^T)$ are isomorphic.

Example KVMR

Kernel via matrix representation

Consider the kernel of the linear transformation

$$T: M_{22} \rightarrow P_2, \quad T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (2a - b + c - 5d) + (a + 4b + 5b + 2d)x + (3a - 2b + c - 8d)x^2$$

Version 1.04
We will begin with a matrix representation of $T$ relative to the bases for $M_{22}$ and $P_2$ (respectively),

$$B = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ -2 & -4 \end{bmatrix} \right\}$$

$$C = \left\{ 1 + x + x^2, 2 + 3x, -1 - 2x^2 \right\}$$

Then,

$$\rho_C \left( T \left( \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \right) \right) = \rho_C \left( 4 + 2x + 6x^2 \right)$$

$$= \rho_C \left( 2(1 + x + x^2) + 0(2 + 3x) + (-2)(-1 - 2x^2) \right)$$

$$= \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

$$\rho_C \left( T \left( \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix} \right) \right) = \rho_C \left( 18 + 28x^2 \right)$$

$$= \rho_C \left( (-24)(1 + x + x^2) + 8(2 + 3x) + (-26)(-1 - 2x^2) \right)$$

$$= \begin{bmatrix} -24 \\ 8 \\ -26 \end{bmatrix}$$

$$\rho_C \left( T \left( \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \right) \right) = \rho_C \left( 10 + 5x + 15x^2 \right)$$

$$= \rho_C \left( 5(1 + x + x^2) + 0(2 + 3x) + (-5)(-1 - 2x^2) \right)$$

$$= \begin{bmatrix} 5 \\ 0 \\ -5 \end{bmatrix}$$

$$\rho_C \left( T \left( \begin{bmatrix} 2 & 5 \\ -2 & -4 \end{bmatrix} \right) \right) = \rho_C \left( 17 + 4x + 26x^2 \right)$$

$$= \rho_C \left( (-8)(1 + x + x^2) + (4)(2 + 3x) + (-17)(-1 - 2x^2) \right)$$

$$= \begin{bmatrix} -8 \\ 4 \\ -17 \end{bmatrix}$$

So the matrix representation of $T$ (relative to $B$ and $C$) is

$$M_{B,C}^T = \begin{bmatrix} 2 & -24 & 5 & -8 \\ 0 & 8 & 0 & 4 \\ -2 & -26 & -5 & -17 \end{bmatrix}$$

We know from Theorem KNSI \[495\] that the kernel of the linear transformation $T$ is isomorphic to the null space of the matrix representation $M_{B,C}^T$, and by studying the proof of Theorem KNSI \[495\] we learn that $\rho_B$ is an isomorphism between these null spaces. Rather than trying to compute the kernel of $T$ using definitions and techniques from Chapter LT \[405\] we will instead analyze the null space of $M_{B,C}^T$ using techniques from way back in Chapter V \[72\]. First row-reduce $M_{B,C}^T$,

$$\begin{bmatrix} 2 & -24 & 5 & -8 \\ 0 & 8 & 0 & 4 \\ -2 & -26 & -5 & -17 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 0 & 5/2 & 2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, by Theorem BNS \[128\], a basis for $\mathcal{N}(M_{B,C}^T)$ is

$$\left\langle \begin{bmatrix} -5/2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$
We can now convert this basis of $\mathcal{N}(M_{B,C}^T)$ into a basis of $\mathcal{K}(T)$ by applying $\rho_B^{-1}$ to each element of the basis,

$$\rho_B^{-1}\left(\begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) = \left(\begin{array}{ccc} -\frac{5}{2} & 1 & 2 \\ -1 & -1 & 3 \\ 0 & -2 & 1 \end{array}\right) + 0 \left(\begin{array}{ccc} 1 & 2 \\ -1 & -4 \end{array}\right) + 1 \left(\begin{array}{ccc} 0 & 2 \\ -2 & -4 \end{array}\right)$$

$$= \left(\begin{array}{c} -3 \\ -3 \\ 2 \end{array}\right)$$

$$\rho_B^{-1}\left(\begin{bmatrix} -\frac{2}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}\right) = \left(\begin{array}{ccc} -\frac{2}{2} & 1 & 2 \\ -1 & -1 & 3 \\ 0 & -2 & 1 \end{array}\right) + (-\frac{1}{2}) \left(\begin{array}{ccc} 1 & 2 \\ -1 & -4 \end{array}\right) + 0 \left(\begin{array}{ccc} 0 & 2 \\ -2 & -4 \end{array}\right)$$

$$= \left(\begin{array}{c} -\frac{1}{2} \\ -\frac{1}{2} \end{array}\right)$$

So the set \{\begin{bmatrix} -\frac{3}{2} \\ -3 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}\} is a basis for $\mathcal{K}(T)$. Just for fun, you might evaluate $T$ with each of these two basis vectors and verify that the output is the zero polynomial (Exercise MR.C10 [504]).

An entirely similar result applies to the range of a linear transformation and the column space of a matrix representation of the linear transformation.

**Theorem RCSI**

**Range and Column Space Isomorphism**

Suppose that $T: U \mapsto V$ is a linear transformation, $B$ is a basis for $U$ of size $n$, and $C$ is a basis for $V$ of size $m$. Then the range of $T$ is isomorphic to the column space of $M_{B,C}^T$,

$$\mathcal{R}(T) \cong \mathcal{C}(M_{B,C}^T)$$

**Proof** To establish that two vector spaces are isomorphic, we must find an isomorphism between them, an invertible linear transformation (Definition IVS [461]). The range of the linear transformation $T$, $\mathcal{R}(T)$, is a subspace of $V$, while the column space of the matrix representation, $\mathcal{C}(M_{B,C}^T)$, is a subspace of $\mathbb{C}^m$. The function $\rho_C$ is defined as a function from $V$ to $\mathbb{C}^m$, but we can just as well employ the definition of $\rho_C$ as a function from $\mathcal{R}(T)$ to $\mathcal{C}(M_{B,C}^T)$.

We must first insure that if we choose an input for $\rho_C$ from $\mathcal{R}(T)$ that then the output will be an element of $\mathcal{C}(M_{B,C}^T)$. So suppose that $v \in \mathcal{R}(T)$. Then there is a vector $u \in U$, such that $T(u) = v$. Consider

$$M_{B,C}^T \rho_B (u) = \rho_C(T(u)) \quad \text{Theorem FTMR [487]}

= \rho_C(v) \quad \text{Definition RLT [444]}$$

This says that $\rho_C(v) \in \mathcal{C}(M_{B,C}^T)$, as desired.

The restriction in the size of the domain and codomain will not affect the fact that $\rho_C$ is a linear transformation (Theorem VRLT [473]), nor will it affect the fact that $\rho_C$ is injective (Theorem VRI [477]). Something must be done though to verify that $\rho_C$ is surjective. This all gets a bit confusing, since the domain of our isomorphism is the range of the linear transformation, so think about your objects as you go. To establish that $\rho_C$ is surjective, appeal to the definition of a surjective linear transformation (Definition SLT [440]), and suppose that we have an element of the codomain, $y \in \mathcal{C}(M_{B,C}^T) \subseteq \mathbb{C}^m$ and we wish to find an element of the domain with $y$ as its image. Since
\( y \in \mathcal{C}(M_{B,C}^T), \) there exists a vector, \( x \in \mathbb{C}^n \) with \( M_{B,C}^T x = y. \) We now show that the desired element of the domain is \( v = \rho_C^{-1}(y). \) First, verify that \( v \in \mathbb{R}^T \) by applying \( T \) to \( u = \rho_B^{-1}(x), \)

\[
T(u) = T(\rho_B^{-1}(x)) \\
= \rho_C^{-1}(M_{B,C}^T(\rho_B(\rho_B^{-1}(x)))) \\
= \rho_C^{-1}(M_{B,C}^T(I_{C^n}(x))) \\
= \rho_C^{-1}(M_{B,C}^T x) \\
= \rho_C^{-1}(y) \\
= v
\]

Second, verify that the proposed isomorphism, \( \rho_C, \) takes \( v \) to \( y, \)

\[
\rho_C(v) = \rho_C(\rho_C^{-1}(y)) \\
= I_{C^n}(y) \\
= y
\]

With \( \rho_C \) demonstrated to be an injective and surjective linear transformation from \( \mathbb{R}^T \) to \( \mathcal{C}(M_{B,C}^T), \) Theorem ILTIS \[459\] tells us \( \rho_C \) is invertible, and so by Definition IVS \[461\], we say \( \mathbb{R}^T \) and \( \mathcal{C}(M_{B,C}^T) \) are isomorphic.

**Example RVMR**

**Range via matrix representation**

In this example, we will recycle the linear transformation \( T \) and the bases \( B \) and \( C \) of Example [KVMR 495] but now we will compute the range of \( T, \)

\[
T: M_{22} \mapsto P_2, \quad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (2a - b + c - 5d) + (a + 4b + 5b + 2d)x + (3a - 2b + c - 8d)x^2
\]

With bases \( B \) and \( C, \)

\[
B = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ -2 & -4 \end{bmatrix} \right\} \\
C = \{ 1 + x + x^2, 2 + 3x, -1 - 2x^2 \}
\]

we obtain the matrix representation

\[
M_{B,C}^T = \begin{bmatrix} 2 & -24 & 5 & -8 \\ 0 & 8 & 4 & 0 \\ -2 & -26 & -5 & -17 \end{bmatrix}
\]

We know from Theorem RCSI \[497\] that the range of the linear transformation \( T \) is isomorphic to the column space of the matrix representation \( M_{B,C}^T \) and by studying the proof of Theorem RCSI \[497\] we learn that \( \rho_C \) is an isomorphism between these subspaces. Notice that since the range is a subspace of the codomain, we will employ \( \rho_C \) as the isomorphism, rather than \( \rho_B, \) which was the correct choice for an isomorphism between the null spaces of Example KVMR [495].

Rather than trying to compute the range of \( T \) using definitions and techniques from Chapter [LT 405] we will instead analyze the column space of \( M_{B,C}^T \) using techniques from way back in Chapter M [163]. First row-reduce \( (M_{B,C}^T)^t, \)

\[
\begin{bmatrix} 2 & 0 & -2 \\ -24 & 8 & -26 \\ 5 & 0 & -5 \\ -8 & 4 & -17 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -25/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]
Now employ Theorem CSRST \[221\] and Theorem BRS \[220\] (there are other methods we could choose here to compute the column space, such as Theorem BCS \[214\]) to obtain the basis for $C(M^T_{B,C})$,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{25}{4} \end{bmatrix} \right\}$$

We can now convert this basis of $C(M^T_{B,C})$ into a basis of $R(T)$ by applying $\rho_C^{-1}$ to each element of the basis,

$$\rho_C^{-1} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = (1 + x + x^2) - (-1 - 2x^2) = 2 + x + 3x^2$$

$$\rho_C^{-1} \begin{bmatrix} 0 \\ 1 \\ \frac{25}{4} \end{bmatrix} = (2 + 3x) - \frac{25}{4}(-1 - 2x^2) = \frac{33}{4} + 3x + \frac{31}{2}x^2$$

So the set

$$\left\{ 2 + 3x + 3x^2, \frac{33}{4} + 3x + \frac{31}{2}x^2 \right\}$$

is a basis for $R(T)$.

Theorem KNSI \[495\] and Theorem RCSI \[497\] can be viewed as further formal evidence for the Coordinatization Principle \[481\], though they are not direct consequences.

Subsection IVLT
Invertible Linear Transformations

We have seen, both in theorems and in examples, that questions about linear transformations are often equivalent to questions about matrices. It is the matrix representation of a linear transformation that makes this idea precise. Here’s our final theorem that solidifies this connection.

**Theorem IMR**

**Invertible Matrix Representations**

Suppose that $T: U \rightarrow V$ is an invertible linear transformation, $B$ is a basis for $U$ and $C$ is a basis for $V$. Then the matrix representation of $T$ relative to $B$ and $C$, $M^T_{B,C}$ is an invertible matrix, and

$$M^{-1}_{C,B} = (M^T_{B,C})^{-1}$$

**Proof** This theorem states that the matrix representation of $T^{-1}$ can be found by finding the matrix inverse of the matrix representation of $T$ (with suitable bases in the right places). It also says that the matrix representation of $T$ is an invertible matrix. We can establish the invertibility, and precisely what the inverse is, by appealing to the definition of a matrix inverse, Definition MI \[189\]. To this end, let $B = \{u_1, u_2, u_3, \ldots, u_n\}$ and $C = \{v_1, v_2, v_3, \ldots, v_n\}$. Then

$$M^{-1}_{C,B} M^T_{B,C} = M^{-1}_{B,B}$$

$$= M^T_{B,B}$$

$$= [\rho_B(I_U(u_1))|\rho_B(I_U(u_2))| \ldots |\rho_B(I_U(u_n))]$$

$$= [\rho_B(u_1)|\rho_B(u_2)| \ldots |\rho_B(u_n)]$$

$$= [e_1|e_2|e_3| \ldots |e_n]$$

$$= I_n$$

**Theorem MRCLT \[491\]**

**Definition IVLT \[456\]**

**Definition MR \[485\]**

**Definition IDLT \[456\]**

**Definition VR \[473\]**

**Definition IM \[62\]**

Version 1.04
Subsection MR.IVLT  Invertible Linear Transformations  500

and

\[ M_B^T M_C^{-1} = M_B^T C^{-1} \]

= \[ M_C^{-1} \]

= \[ \{ \rho_C (I_V (v_1)), \rho_C (I_V (v_2)), \ldots, \rho_C (I_V (v_n)) \} \]

= \[ \{ \rho_C (v_1), \rho_C (v_2), \ldots, \rho_C (v_n) \} \]

= \[ \{ e_1, e_2, e_3, \ldots, e_n \} \]

= \[ I_n \]

So by Definition MI 189, the matrix \( M_B^T C \) has an inverse, and that inverse is \( M_C^{-1} \).

\[ \square \]

Example ILT VR
Inverse of a linear transformation via a representation

Consider the linear transformation

\[ R: P_3 \mapsto M_{22}, \quad R(a + bx + cx^2 + x^3) = \begin{bmatrix} a + b - c + 2d \\ a + b + 2d \\ -a + b + 2c - 5d \end{bmatrix} \]

If we wish to quickly find a formula for the inverse of \( R \) (presuming it exists), then choosing “nice” bases will work best. So build a matrix representation of \( R \) relative to the bases \( B \) and \( C \),

\[ B = \{ 1, x, x^2, x^3 \} \]

\[ C = \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \} \]

Then,

\[ \rho_C (R(1)) = \rho_C \left( \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \]

\[ \rho_C (R(x)) = \rho_C \left( \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \]

\[ \rho_C (R(x^2)) = \rho_C \left( \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} \]

\[ \rho_C (R(x^3)) = \rho_C \left( \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix} \]

So a representation of \( R \) is

\[ M_B^R = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 3 & -2 & 3 \\ 1 & 1 & 0 & 2 \\ -1 & 1 & 2 & -5 \end{bmatrix} \]

The matrix \( M_B^R \) is invertible (as you can check) so we know by Theorem IMR 499 that \( R \) is invertible. Furthermore,

\[ M_B^{-1} \]

\[ M_B^{-1} = \begin{bmatrix} 20 & -7 & -2 & 3 \\ -8 & 3 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ -6 & 2 & 1 & -1 \end{bmatrix} \]

Version 1.04
We can use this representation of the inverse linear transformation, in concert with Theorem FTMR 487, to determine an explicit formula for the inverse itself,

\[
R^{-1}
\begin{pmatrix}
  a & b \\
  c & d \\
\end{pmatrix}
= \rho_B^{-1}
\begin{pmatrix}
  M_{B,C}^{-1} \rho_C
  \begin{pmatrix}
    a & b \\
    c & d \\
  \end{pmatrix}
\end{pmatrix}
\]

Theorem FTMR 487

\[
= \rho_B^{-1}
\begin{pmatrix}
  (M_{B,C}^R)^{-1} \rho_C
  \begin{pmatrix}
    a & b \\
    c & d \\
  \end{pmatrix}
\end{pmatrix}
\]

Theorem IMR 499

\[
= \rho_B^{-1}
\begin{pmatrix}
  (M_{B,C}^R)^{-1}
  \begin{pmatrix}
    a & b \\
    c & d \\
  \end{pmatrix}
\end{pmatrix}
\]

Definition VR 473

\[
= \rho_B^{-1}
\begin{pmatrix}
  \begin{pmatrix}
    20 & -7 & -2 & 3 \\
    -8 & 3 & 1 & -1 \\
    -1 & 0 & 1 & 0 \\
    -6 & 2 & 1 & -1 \\
  \end{pmatrix}
  \begin{pmatrix}
    a \\
    b \\
    c \\
    d \\
  \end{pmatrix}
\end{pmatrix}
\]

Definition MI 189

\[
= \rho_B^{-1}
\begin{pmatrix}
  \begin{pmatrix}
    20a - 7b - 2c + 3d \\
    -8a + 3b + c - d \\
    -a + c \\
    -6a + 2b + c - d \\
  \end{pmatrix}
\end{pmatrix}
\]

Definition MVP 173

\[
= (20a - 7b - 2c + 3d) + (-8a + 3b + c - d)x \\
+ (-a + c)x^2 + (-6a + 2b + c - d)x^3
\]

Definition VR 473

You might look back at Example AIVLT 456, where we first witnessed the inverse of a linear transformation and recognize that the inverse \( S \) was built from using the method of this example on a matrix representation of \( T \).

**Theorem IMILT**

**Invertible Matrices, Invertible Linear Transformation**

Suppose that \( A \) is a square matrix of size \( n \) and \( T : \mathbb{C}^n \rightarrow \mathbb{C}^n \) is the linear transformation defined by \( T(x) = Ax \). Then \( A \) is invertible matrix if and only if \( T \) is an invertible linear transformation.

\[
\square
\]

**Proof**

Choose bases \( B = C = \{e_1, e_2, e_3, \ldots, e_n\} \) consisting of the standard unit vectors as a basis of \( \mathbb{C}^n \) (Theorem SUVB 204) and build a matrix representation of \( T \) relative to \( B \) and \( C \). Then

\[
\rho_C(T(e_i)) = \rho_C(Ae_i) = \rho_C(A_i)
\]

So then the matrix representation of \( T \), relative to \( B \) and \( C \), is simply \( M_{B,C}^T = A \). This is the basic observation that makes the rest of this proof go.

\((\Leftarrow)\) Suppose \( T \) is invertible. Then \( T \) is injective by Theorem ILTIS 459 and

\[
n(A) = \dim(\mathcal{N}(A)) = \dim(\mathcal{N}(M_{B,C}^T)) = \dim(\ker T) = \dim(\{0\}) = 0
\]

Then Theorem RNNM 314 tells us that \( A \) is nonsingular, and therefore \( A \) is invertible (Theorem NI 204).
Suppose $A$ is a nonsingular matrix, then $A$ is invertible (Theorem NI [204]) and has zero nullity (Theorem RNNM [314]). So

\[ n(T) = \dim(\mathcal{K}(T)) = \dim(\mathcal{N}(M_{R,C}^T)) = \dim(\mathcal{N}(A)) = \dim(\{0\}) = 0 \]

So $T$ has zero nullity, and therefore has a trivial kernel and by Theorem KILT [432] $T$ is injective. Furthermore, by Theorem RPNDD [464],

\[ r(T) = \dim(\mathbb{C}^n) - n(T) = n - 0 = n \]

So $T$ has full rank and therefore the range of $T$ is all of $\mathbb{C}^n$ and by Theorem RSLT [446] $T$ is surjective. Finally, with $T$ known to be injective and surjective, Theorem ILTIS [459] says $T$ is invertible.

This theorem looks like more work than you would imagine it to be. But by now, the connections between matrices and linear transformations should be starting to become more transparent, and you may have already recognized the invertibility of a matrix as being tantamount to the invertibility of the associated matrix representation. See Exercise MR.T60 [505] as well.

We can update the NMEx series of theorems, yet again.

**Theorem NME9**

**Nonsingular Matrix Equivalences, Round 9**

Suppose that $A$ is a square matrix of size $n$. The following are equivalent.

1. $A$ is nonsingular.

2. $A$ row-reduces to the identity matrix.

3. The null space of $A$ contains only the zero vector, $\mathcal{N}(A) = \{0\}$.

4. The linear system $\mathcal{L}\mathcal{S}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.

5. The columns of $A$ are a linearly independent set.

6. $A$ is invertible.

7. The column space of $A$ is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of $A$ are a basis for $\mathbb{C}^n$.

9. The rank of $A$ is $n$, $r(A) = n$.

10. The nullity of $A$ is zero, $n(A) = 0$.

11. The determinant of $A$ is nonzero, $\det(A) \neq 0$.

12. $\lambda = 0$ is not an eigenvalue of $A$.

13. The linear transformation $T$: $\mathbb{C}^n \mapsto \mathbb{C}^n$ defined by $T(x) = Ax$ is invertible.

**Proof** By Theorem IMILT [501], the new addition to this list is equivalent to the statement that $A$ is invertible so we can expand Theorem NMEX [379].
1. Why does Theorem FTMR\,487 deserve the moniker “fundamental”?

2. Find the matrix representation, $M^T_{B,C}$ of the linear transformation

$$T: \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 - x_2 \\ 3x_1 + 2x_2 \end{bmatrix}$$

relative to the bases

$$B = \left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \\ 2 \end{bmatrix} \right\}, \quad C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

3. What is the second “surprise,” and why is it surprising?
C10  Example KVMR[495] concludes with a basis for the kernel of the linear transformation \( T \). Compute the value of \( T \) for each of these two basis vectors. Did you get what you expected? Contributed by Robert Beezer

C20  Compute the matrix representation of \( T \) relative to the bases \( B \) and \( C \).

\[
T: P_3 \rightarrow \mathbb{C}^3, \quad T(a + bx + cx^2 + dx^3) = \begin{bmatrix} 2a - 3b + 4c - 2d \\ a + b - c + d \\ 3a + 2c - 3d \end{bmatrix}
\]

\( B = \{1, x, x^2, x^3\} \quad C = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{Bmatrix} \)

Contributed by Robert Beezer  Solution[507]

C21  Find a matrix representation of the linear transformation \( T \) relative to the bases \( B \) and \( C \).

\[
T: P_2 \rightarrow \mathbb{C}^2, \quad T(p(x)) = \begin{bmatrix} p(1) \\ p(3) \end{bmatrix}
\]

\( B = \{2 - 5x + x^2, 1 + x - x^2, x^2\} \quad C = \begin{Bmatrix} 3 \\ 4 \\ 3 \end{Bmatrix} \)

Contributed by Robert Beezer  Solution[507]

C22  Let \( S_{22} \) be the vector space of \( 2 \times 2 \) symmetric matrices. Build the matrix representation of the linear transformation \( T: P_2 \rightarrow S_{22} \) relative to the bases \( B \) and \( C \) and then use this matrix representation to compute \( T(3 + 5x - 2x^2) \).

\[
B = \{1, 1 + x, 1 + x + x^2\} \quad C = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}
\]

\[
T(a + bx + cx^2) = \begin{bmatrix} 2a - b + c & a + 3b - c \\ a + 3b - c & a - c \end{bmatrix}
\]

Contributed by Robert Beezer  Solution[507]

C25  Use a matrix representation to determine if the linear transformation \( T: P_3 \rightarrow M_{22} \) surjective.

\[
T(a + bx + cx^2 + dx^3) = \begin{bmatrix} -a + 4b + c + 2d & 4a - b + 6c - d \\ a + 5b - 2c + 2d & a + 2c + 5d \end{bmatrix}
\]

Contributed by Robert Beezer  Solution[508]

C30  Find bases for the kernel and range of the linear transformation \( S \) below.

\[
S: M_{22} \rightarrow P_2, \quad S\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = (a + 2b + 5c - 4d) + (3a - b + 8c + 2d)x + (a + b + 4c - 2d)x^2
\]

Contributed by Robert Beezer  Solution[509]

C40  Let \( S_{22} \) be the set of \( 2 \times 2 \) symmetric matrices. Verify that the linear transformation \( R \) is invertible and find \( R^{-1} \).

\[
R: S_{22} \rightarrow P_2, \quad R\begin{bmatrix} a \\ b \\ c \end{bmatrix} = (a - b) + (2a - 3b - 2c)x + (a - b + c)x^2
\]
C41  Prove that the linear transformation $S$ is invertible. Then find a formula for the inverse linear transformation, $S^{-1}$, by employing a matrix inverse. (15 points)

$$S : P_1 \mapsto M_{1,2}, \quad S(a + bx) = [3a + b \quad 2a + b]$$

C42  The linear transformation $R : M_{12} \mapsto M_{21}$ is invertible. Use a matrix representation to determine a formula for the inverse linear transformation $R^{-1} : M_{21} \mapsto M_{12}$.

$$R([a \quad b]) = \begin{bmatrix} a + 3b \\ 4a + 11b \end{bmatrix}$$

C50  Use a matrix representation to find a basis for the range of the linear transformation $L$. (15 points)

$$L : M_{22} \mapsto P_2, \quad T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = (a + 2b + 4c + d) + (3a + c - 2d)x + (-a + b + 3c + 3d)x^2$$

C51  Use a matrix representation to find a basis for the kernel of the linear transformation $L$. (15 points)

$$L : M_{22} \mapsto P_2, \quad T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = (a + 2b + 4c + d) + (3a + c - 2d)x + (-a + b + 3c + 3d)x^2$$

C52  Find a basis for the kernel of the linear transformation $T : P_2 \mapsto M_{22}$.

$$T(a + bx + cx^2) = \begin{bmatrix} a + 2b - 2c \\ -a + b - 4c \\ 2a + 2b \\ 3a + 2b + 2c \end{bmatrix}$$

M20  The linear transformation $D$ performs differentiation on polynomials. Use a matrix representation of $D$ to find the rank and nullity of $D$.

$$D : P_n \mapsto P_n, \quad D(p(x)) = p'(x)$$

T60  Create an entirely different proof of Theorem IMILT that relies on Definition IVLT to establish the invertibility of $T$, and that relies on Definition MI to establish the invertibility of $A$.

T80  Suppose that $T : U \mapsto V$ and $S : V \mapsto W$ are linear transformations, and that $B$, $C$ and $D$ are bases for $U$, $V$, and $W$. Using only Definition MR, define matrix representations for $T$ and $S$. Using these two definitions, and Definition MR, derive a matrix representation for the composition $S \circ T$ in terms of the entries of the matrices $M^T_{B,C}$ and $M^S_{C,D}$. Explain how you
would use this result to *motivate a definition* for matrix multiplication that is strikingly similar to Theorem EMP\[177\].
Contributed by Robert Beezer  Solution \[513\]
Apply Definition MR \[485\], Contributed by Robert Beezer

\[
\rho_C(T(1)) = \rho_C \left( \begin{array}{c} 2 \\ 1 \\ 3 \\ \end{array} \right) = \rho_C \left( \begin{array}{ccc} 1 & 0 & -2 \\ 1 & 0 & 3 \\ \end{array} \right) = \left( \begin{array}{c} 1 \\ -2 \\ 3 \\ \end{array} \right)
\]

\[
\rho_C(T(x)) = \rho_C \left( \begin{array}{c} -3 \\ 1 \\ 0 \\ \end{array} \right) = \rho_C \left( \begin{array}{ccc} -4 & 1 & 1 \\ 0 & 1 & 0 \\ \end{array} \right) = \left( \begin{array}{c} -4 \\ 1 \\ 0 \\ \end{array} \right)
\]

\[
\rho_C(T(x^2)) = \rho_C \left( \begin{array}{c} 4 \\ -1 \\ 2 \\ \end{array} \right) = \rho_C \left( \begin{array}{ccc} 5 & 1 & 1 \\ 0 & 1 & 0 \\ \end{array} \right) = \left( \begin{array}{c} 5 \\ -3 \\ 2 \\ \end{array} \right)
\]

\[
\rho_C(T(x^3)) = \rho_C \left( \begin{array}{c} -2 \\ 1 \\ -3 \\ \end{array} \right) = \rho_C \left( \begin{array}{ccc} -3 & 1 & 1 \\ 0 & 1 & 0 \\ \end{array} \right) = \left( \begin{array}{c} -3 \\ 4 \\ -3 \\ \end{array} \right)
\]

These four vectors are the columns of the matrix representation,

\[
M_{B,C}^T = \begin{bmatrix} 1 & -4 & 5 & -3 \\ -2 & 1 & -3 & 4 \\ 3 & 0 & 2 & -3 \end{bmatrix}
\]

Apply Definition MR \[485\], Contributed by Robert Beezer

\[
\rho_C(T(2-5x+x^2)) = \rho_C \left( \begin{array}{c} -2 \\ -4 \\ \end{array} \right) = \rho_C \left( \begin{array}{ccc} 2 & 3 & 2 \\ \frac{4}{3} & \frac{1}{2} & \frac{2}{3} \end{array} \right) = \left( \begin{array}{c} 2 \\ -4 \end{array} \right)
\]

\[
\rho_C(T(1+x-x^2)) = \rho_C \left( \begin{array}{c} 1 \\ -5 \\ \end{array} \right) = \rho_C \left( \begin{array}{ccc} 13 & 3 & 2 \\ \frac{4}{3} & \frac{1}{2} & \frac{2}{3} \end{array} \right) = \left( \begin{array}{c} 13 \\ -19 \end{array} \right)
\]

\[
\rho_C(T(x^2)) = \rho_C \left( \begin{array}{c} 1 \\ 9 \\ \end{array} \right) = \rho_C \left( \begin{array}{ccc} -15 & 3 & 2 \\ \frac{4}{3} & \frac{1}{2} & \frac{2}{3} \end{array} \right) = \left( \begin{array}{c} -15 \\ 23 \end{array} \right)
\]

So the resulting matrix representation is

\[
M_{B,C}^T = \begin{bmatrix} 2 & 13 & -15 \\ -4 & -19 & 23 \end{bmatrix}
\]

Input to \( T \) the vectors of the basis \( B \) and coordinatize the outputs relative to \( C \),

\[
\rho_C(T(1)) = \rho_C \left( \begin{array}{c} 2 \\ 1 \\ 1 \\ \end{array} \right) = \rho_C \left( \begin{array}{ccc} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right) = \left( \begin{array}{c} 2 \\ 1 \\ 1 \end{array} \right)
\]

\[
\rho_C(T(1+x)) = \rho_C \left( \begin{array}{c} 1 \\ 4 \\ 1 \\ \end{array} \right) = \rho_C \left( \begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right) = \left( \begin{array}{c} 1 \\ 4 \\ 1 \end{array} \right)
\]

\[
\rho_C(T(1+x+x^2)) = \rho_C \left( \begin{array}{c} 2 \\ 3 \\ 0 \\ \end{array} \right) = \rho_C \left( \begin{array}{ccc} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) = \left( \begin{array}{c} 2 \\ 3 \\ 0 \end{array} \right)
\]
Applying Definition MR 485 we have the matrix representation

\[
M_{B,C}^T = \begin{bmatrix}
2 & 1 & 2 \\
1 & 4 & 3 \\
1 & 1 & 0
\end{bmatrix}
\]

To compute \( T (3 + 5x - 2x^2) \) employ Theorem FTMR 487.

\[
T (3 + 5x - 2x^2) = \rho_C^{-1} (M_{B,C}^{T} \rho_B (3 + 5x - 2x^2)) \\
= \rho_C^{-1} (M_{B,C}^{T} \rho_B ((-2)(1) + 7(1 + x) + (-2)(1 + x + x^2))) \\
= \rho_C^{-1} \begin{pmatrix}
2 & 1 & 2 & -2 \\
1 & 4 & 3 & 7 \\
1 & 1 & 0 & -2
\end{pmatrix} \\
= \rho_C^{-1} \begin{pmatrix}
-1 & 20 \\
20 & 5
\end{pmatrix}
= (-1) \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} + 20 \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} + 5 \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
-1 & 20 \\
20 & 5
\end{pmatrix}
\]

You can, of course, check your answer by evaluating \( T (3 + 5x - 2x^2) \) directly.

C25 Contributed by Robert Beezer Statement 504
Choose bases \( B \) and \( C \) for the matrix representation,

\[
B = \{ 1, x, x^2, x^3 \} \\
C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]

Input to \( T \) the vectors of the basis \( B \) and coordinatize the outputs relative to \( C \),

\[
\rho_C (T (1)) = \rho_C \left( \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix} \right) = \rho_C \left( (-1) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 4 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 1 \end{bmatrix}
\]

\[
\rho_C (T (x)) = \rho_C \left( \begin{bmatrix} 4 & -1 \\ 5 & 0 \end{bmatrix} \right) = \rho_C \left( 4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 5 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 4 \\ -1 \\ 5 \\ 0 \end{bmatrix}
\]

\[
\rho_C (T (x^2)) = \rho_C \left( \begin{bmatrix} 1 & 6 \\ -2 & 2 \end{bmatrix} \right) = \rho_C \left( 1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 6 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-2) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 6 \\ -2 \\ 2 \end{bmatrix}
\]

\[
\rho_C (T (x^3)) = \rho_C \left( \begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix} \right) = \rho_C \left( 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 5 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 5 \end{bmatrix}
\]

Applying Definition MR 485 we have the matrix representation

\[
M_{B,C}^{T} = \begin{bmatrix}
-1 & 4 & 1 & 2 \\
4 & -1 & 6 & -1 \\
1 & 5 & -2 & 2 \\
1 & 0 & 2 & 5
\end{bmatrix}
\]

Version 1.04
Properties of this matrix representation will translate to properties of the linear transformation. The matrix representation is nonsingular since it row-reduces to the identity matrix (Theorem NMRRRI[62]) and therefore has a column space equal to $\mathbb{C}^4$ (Theorem CNMB[299]). The column space of the matrix representation is isomorphic to the range of the linear transformation (Theorem RCSI[497]). So the range of $T$ has dimension 4, equal to the dimension of the codomain $M_{22}$. By Theorem ROSLT[463], $T$ is surjective.

Contributed by Robert Beezer Statement[504]

These subspaces will be easiest to construct by analyzing a matrix representation of $S$. Since we can use any matrix representation, we might as well use natural bases that allow us to construct the matrix representation quickly and easily,

$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$  

$C = \{1, x, x^2\}$

then we can practically build the matrix representation on sight,

$$M_{B,C}^S = \begin{bmatrix} 1 & 2 & 5 & -4 \\ 3 & -1 & 8 & 2 \\ 1 & 1 & 4 & -2 \end{bmatrix}$$

The first step is to find bases for the null space and column space of the matrix representation.

Row-reducing the matrix representation we find,

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So by Theorem BNS[128] and Theorem BCS[214], we have

$$\mathcal{N}(M_{B,C}^S) = \left\langle \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$  

$$\mathcal{C}(M_{B,C}^S) = \left\langle \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\rangle$$

Now, the proofs of Theorem KNSI[498] and Theorem RCSI[497] tell us that we can apply $\rho_B^{-1}$ and $\rho_C^{-1}$ (respectively) to “un-coordinatize” and get bases for the kernel and range of the linear transformation $S$ itself,

$$\mathcal{K}(S) = \left\langle \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$  

$$\mathcal{R}(S) = \left\langle \{1 + 3x + x^2, 2 - x + x^2\} \right\rangle$$

Contributed by Robert Beezer Statement[504]

The analysis of $R$ will be easiest if we analyze a matrix representation of $R$. Since we can use any matrix representation, we might as well use natural bases that allow us to construct the matrix representation quickly and easily,

$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$  

$C = \{1, x, x^2\}$

then we can practically build the matrix representation on sight,

$$M_{B,C}^R = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

This matrix representation is invertible (it has a nonzero determinant of $-1$, Theorem SMZD[351], Theorem NI[204]) so Theorem IMR[499] tells us that the linear transformation $S$ is also invertible. To find a formula for $R^{-1}$ we compute,

$$R^{-1} (a + bx + cx^2) = \rho_B^{-1} \left( M_{C,B}^{R^{-1}} \rho_C (a + bx + cx^2) \right)$$  

Theorem FTMR[487]
\[
\begin{align*}
\text{Theorem IMR} \ [499] &\quad = \rho_B^{-1} \left( (M_{B,C}^R)^{-1} \rho_{BC} (a + bx + cx^2) \right) \\
\text{Definition VR} \ [473] &\quad = \rho_B^{-1} \left( (M_{B,C}^R)^{-1} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) \\
\text{Definition MI} \ [189] &\quad = \rho_B^{-1} \left( \begin{bmatrix} 5 & -1 & -2 \\ 4 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) \\
\text{Definition MVP} \ [173] &\quad = \rho_B^{-1} \left( \begin{bmatrix} 5a - b - 2c \\ 4a - b - 2c \\ -a + c \end{bmatrix} \right) \\
\text{Definition VR} \ [473] &\quad = \rho_B^{-1} \left( \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right) \\
&\quad = (a - b) + (-2a + 3b)x
\end{align*}
\]

**C41** Contributed by Robert Beezer \ Statement \ [505]

First, build a matrix representation of \( S \) (Definition MR \ [485]). We are free to choose whatever bases we wish, so we should choose ones that are easy to work with, such as

\[ B = \{1, x\} \]
\[ C = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\} \]

The resulting matrix representation is then

\[ M_{B,C}^B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \]

this matrix is invertible, since it has a nonzero determinant, so by Theorem IMR \ [499] the linear transformation \( S \) is invertible. We can use the matrix inverse and Theorem IMR \ [499] to find a formula for the inverse linear transformation,

\[ S^{-1} \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = \rho_B^{-1} \left( (M_{C,B}^{S^{-1}}) \rho_{BC} \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) \right) \]
\[ = \rho_B^{-1} \left( (M_{B,C}^S)^{-1} \rho_{BC} \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) \right) \]
\[ = \rho_B^{-1} \left( (M_{B,C}^S)^{-1} \begin{bmatrix} a \\ b \end{bmatrix} \right) \]
\[ = \rho_B^{-1} \left( \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right) \]
\[ = \rho_B^{-1} \left( \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right) \]
\[ = \rho_B^{-1} \left( \begin{bmatrix} a - b \\ -2a + 3b \end{bmatrix} \right) \]
\[ = (a - b) + (-2a + 3b)x \]

**C42** Contributed by Robert Beezer \ Statement \ [505]

Choose bases \( B \) and \( C \) for \( M_{12} \) and \( M_{21} \) (respectively),

\[ B = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\} \]
\[ C = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\} \]

The resulting matrix representation is

\[ M_{B,C}^B = \begin{bmatrix} 1 & 3 \\ 4 & 11 \end{bmatrix} \]
This matrix is invertible (its determinant is nonzero, Theorem SMZD 351), so by Theorem IMR 499, we can compute the matrix representation of \( R^{-1} \) with a matrix inverse (Theorem TTMI 191),

\[
M_{C,B}^{R^{-1}} = \begin{bmatrix} 1 & 3 \\ 4 & 11 \end{bmatrix}^{-1} = \begin{bmatrix} -11 & 3 \\ 4 & -1 \end{bmatrix}
\]

To obtain a general formula for \( R^{-1} \), use Theorem FTMR 487,

\[
R^{-1}\left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \rho_B \left( M_{C,B}^{R^{-1}} \rho_C \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) \right) = \rho_B \left( \begin{bmatrix} -11x + 3y \\ 4x - y \end{bmatrix} \right) = \begin{bmatrix} -11x + 3y \\ 4x - y \end{bmatrix}
\]

C50 Contributed by Robert Beezer Statement 505
As usual, build any matrix representation of \( L \), most likely using a “nice” bases, such as

\[
B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

\[
C = \{1, x, x^2\}
\]

Then the matrix representation (Definition MR 485) is,

\[
M_{B,C}^L = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 3 & 0 & 1 & -2 \\ -1 & 1 & 3 & 3 \end{bmatrix}
\]

Theorem RCSI 497 tells us that we can compute the column space of the matrix representation, then use the isomorphism \( \rho_C^{-1} \) to convert the column space of the matrix representation into the range of the linear transformation. So we first analyze the matrix representation,

\[
\begin{bmatrix} 1 & 2 & 4 & 1 \\ 3 & 0 & 1 & -2 \\ -1 & 1 & 3 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}
\]

With three nonzero rows in the reduced row-echelon form of the matrix, we know the column space has dimension 3. Since \( P_2 \) has dimension 3 (Theorem DP 311), the range must be all of \( P_2 \). So any basis of \( P_2 \) would suffice as a basis for the range. For instance, \( C \) itself would be a correct answer.

A more laborious approach would be to use Theorem BCS 214 and choose the first three columns of the matrix representation as a basis for the range of the matrix representation. These could then be “un-coordinatized” with \( \rho_C^{-1} \) to yield a (“not nice”) basis for \( P_2 \).

C52 Contributed by Robert Beezer Statement 505
Choose bases \( B \) and \( C \) for the matrix representation,

\[
B = \{1, x, x^2\} \quad C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

Input to \( T \) the vectors of the basis \( B \) and coordinatize the outputs relative to \( C \),

\[
\rho_C (T(1)) = \rho_C \left( \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} \right) = \rho_C \left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{3} \end{bmatrix}
\]

\[
\rho_C (T(x)) = \rho_C \left( \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} \right) = \rho_C \left( \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \frac{2}{1} \\ \frac{2}{1} \end{bmatrix}
\]
\[ \rho_C(T(x^2)) = \rho_C \left( \begin{bmatrix} -2 & 0 \\ -4 & 2 \end{bmatrix} \right) = \rho_C \left( -2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (-4) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 0 \\ -4 \\ 2 \end{bmatrix} \]

Applying [Definition MR 485] we have the matrix representation

\[ M^T_{B,C} = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 2 & 0 \\ -1 & 1 & -4 \\ 3 & 2 & 2 \end{bmatrix} \]

The null space of the matrix representation is isomorphic (via \( \rho_B \)) to the kernel of the linear transformation [Theorem KNSI 495]. So we compute the null space of the matrix representation by first row-reducing the matrix to,

\[ \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

Employing [Theorem BNS 128] we have

\[ N(M^T_{B,C}) = \langle \{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \} \rangle \]

We only need to uncoordinatize this one basis vector to get a basis for \( \mathcal{K}(T) \),

\[ \mathcal{K}(T) = \langle \{ \rho_B^{-1} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \} \rangle = \langle \{ -2 + 2x + x^2 \} \rangle \]

M20 Contributed by Robert Beezer Statement 505

Build a matrix representation [Definition MR 485] with the set

\[ B = \{1, x, x^2, \ldots, x^n\} \]

employed as a basis of both the domain and codomain. Then

\[ \rho_B(D(1)) = \rho_B(0) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \rho_B(D(x)) = \rho_B(1) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \]

\[ \rho_B(D(x^2)) = \rho_B(2x) = \begin{bmatrix} 0 \\ 2 \\ \vdots \\ 0 \end{bmatrix} \quad \rho_B(D(x^3)) = \rho_B(3x^2) = \begin{bmatrix} 0 \\ 0 \\ 3 \\ \vdots \\ 0 \end{bmatrix} \]

\[ \vdots \]
\[ \rho_B(D(x^n)) = \rho_B(nx^{n-1}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ n \\ 0 \end{bmatrix} \]

and the resulting matrix representation is

\[
M_{B,B}^D = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 2 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 3 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ldots & n \\
0 & 0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix}
\]

This \((n+1) \times (n+1)\) matrix is very close to being in reduced row-echelon form. Multiply row \(i\) by \(\frac{1}{n_i}\), for \(1 \leq i \leq n\), to convert it to reduced row-echelon form. From this we can see that matrix representation \(M_{B,B}^D\) has rank \(n\) and nullity 1. Applying \text{Theorem RCSI} \([497]\) and \text{Theorem KNSI} \([495]\) tells us that the linear transformation \(D\) will have the same values for the rank and nullity, as well.

\textbf{T80} Contributed by \text{Robert Beezer} \hspace{1cm} \textbf{Statement} \([505]\)

Suppose that \(B = \{u_1, u_2, u_3, \ldots, u_n\}, C = \{v_1, v_2, v_3, \ldots, v_n\}\) and \(D = \{w_1, w_2, w_3, \ldots, w_n\}\). For convenience, set \(M = M_{B,C}^D, m_{ij} = [M]_{ij}, 1 \leq i \leq n, 1 \leq j \leq m,\) and similarly, set \(N = M_{C,D}^B, n_{ij} = [N]_{ij}, 1 \leq i \leq p, 1 \leq j \leq n.\) We want to learn about the matrix representation of \(S \circ T: V \mapsto W\) relative to \(B\) and \(D\). We will examine a single (generic) entry of this representation.

\[
\left[ M_{B,D}^{S \circ T} \right]_{ij} = \rho_D((S \circ T)(u_j))_{i} = \rho_D(S(T(u_j)))_{i}
\]

\[
= \rho_D\left( S \left( \sum_{k=1}^{n} m_{kj}v_k \right) \right)_{i}
\]

\[
= \rho_D\left( \sum_{k=1}^{n} m_{kj}S(v_k) \right)_{i}
\]

\[
= \rho_D\left( \sum_{k=1}^{n} m_{kj} \sum_{\ell=1}^{p} n_{\ell k}w_{\ell} \right)_{i}
\]

\[
= \rho_D\left( \sum_{k=1}^{n} \sum_{\ell=1}^{p} m_{kj}n_{\ell k}w_{\ell} \right)_{i}
\]

\[
= \rho_D\left( \sum_{\ell=1}^{p} \left( \sum_{k=1}^{n} m_{kj}n_{\ell k} \right)w_{\ell} \right)_{i}
\]

\[
= \sum_{k=1}^{n} m_{kj}n_{ik}
\]

\[
= \sum_{k=1}^{n} n_{ik}m_{kj}
\]
\[
= \sum_{k=1}^{n} [M_{C,D}^S]_{ik} [M_{B,C}^T]_{kj}
\]

This formula for the entry of a matrix should remind you of Theorem EMP \[177\]. However, while the theorem presumed we knew how to multiply matrices, the solution before us never uses any understanding of matrix products. It uses the definitions of vector and matrix representations, properties of linear transformations and vector spaces. So if we began a course by first discussing vector space, and then linear transformations between vector spaces, we could carry matrix representations into a motivation for a definition of matrix multiplication that is grounded in function composition. That is worth saying again — a definition of matrix representations of linear transformations results in a matrix product being the representation of a composition of linear transformations.

This exercise is meant to explain why many authors take the formula in Theorem EMP \[177\] as their definition of matrix multiplication, and why it is a natural choice when the proper motivation is in place. If we first defined matrix multiplication in the style of Theorem EMP \[177\], then the above argument, followed by a simple application of the definition of matrix equality (Definition ME \[163\]), would yield Theorem MRCLT \[491\].
Section CB
Change of Basis

We have seen in Section MR \[485\] that a linear transformation can be represented by a matrix, once we pick bases for the domain and codomain. How does the matrix representation change if we choose different bases? Which bases lead to especially nice representations? From the infinite possibilities, what is the best possible representation? This section will begin to answer these questions. But first we need to define eigenvalues for linear transformations and the change-of-basis matrix.

Subsection EELT
Eigenvalues and Eigenvectors of Linear Transformations

We now define the notion of an eigenvalue and eigenvector of a linear transformation. It should not be too surprising, especially if you remind yourself of the close relationship between matrices and linear transformations.

Definition EELT
Eigenvalue and Eigenvector of a Linear Transformation
Suppose that $T: V \rightarrow V$ is a linear transformation. Then a nonzero vector $v \in V$ is an eigenvector of $T$ for the eigenvalue $\lambda$ if $T(v) = \lambda v$.

We will see shortly the best method for computing the eigenvalues and eigenvectors of a linear transformation, but for now, here are some examples to verify that such things really do exist.

Example ELTBM
Eigenvectors of linear transformation between matrices
Consider the linear transformation $T: M_{22} \rightarrow M_{22}$ defined by

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -17a + 11b + 8c - 11d & -57a + 35b + 24c - 33d \\ -14a + 10b + 6c - 10d & -41a + 25b + 16c - 23d \end{bmatrix}$$

and the vectors

$$x_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \quad x_4 = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$$

Then compute

$$T(x_1) = T \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} = 2x_1$$

$$T(x_2) = T \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} = 2x_2$$

$$T(x_3) = T \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ -2 & -3 \end{bmatrix} = (-1)x_3$$

$$T(x_4) = T \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -12 \\ -2 & -8 \end{bmatrix} = (-2)x_4$$

So $x_1, x_2, x_3, x_4$ are eigenvectors of $T$ with eigenvalues (respectively) $\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = -1, \lambda_4 = -2$.

Here’s another.
Example ELTBP

Eigenvectors of linear transformation between polynomials

Consider the linear transformation $R: P_2 \mapsto P_2$ defined by

$$R(a + bx + cx^2) = (15a + 8b - 4c) + (-12a - 6b + 3c)x + (24a + 14b - 7c)x^2$$

and the vectors

$w_1 = 1 - x + x^2$ \hspace{1cm} $w_2 = x + 2x^2$ \hspace{1cm} $w_3 = 1 + 4x^2$

Then compute

$$R(w_1) = R(1 - x + x^2) = 3 - 3x + 3x^2 = 3w_1$$

$$R(w_2) = R(x + 2x^2) = 0 + 0x + 0x^2 = 0w_2$$

$$R(w_3) = R(1 + 4x^2) = -1 - 4x^2 = (-1)w_3$$

So $w_1$, $w_2$, $w_3$ are eigenvectors of $R$ with eigenvalues (respectively) $\lambda_1 = 3$, $\lambda_2 = 0$, $\lambda_3 = -1$. Notice how the eigenvalue $\lambda_2 = 0$ indicates that the eigenvector $w_2$ is a non-trivial element of the kernel of $R$, and therefore $R$ is not injective (Exercise CB.T15 [536]).

Of course, these examples are meant only to illustrate the definition of eigenvectors and eigenvalues for linear transformations, and therefore beg the question, “How would I find eigenvectors?” We’ll have an answer before we finish this section. We need one more construction first.

Subsection CBM

Change-of-Basis Matrix

Given a vector space, we know we can usually find many different bases for the vector space, some nice, some nasty. If we choose a single vector from this vector space, we can build many different representations of the vector by constructing the representations relative to different bases. How are these different representations related to each other? A change-of-basis matrix answers this question.

**Definition CBM**

**Change-of-Basis Matrix**

Suppose that $V$ is a vector space, and $I_V: V \mapsto V$ is the identity linear transformation on $V$. Let $B = \{v_1, v_2, v_3, \ldots, v_n\}$ and $C$ be two bases of $V$. Then the **change-of-basis matrix** from $B$ to $C$ is the matrix representation of $I_V$ relative to $B$ and $C$,

$$C_{B,C} = M_{B,C}^{I_V} = \begin{bmatrix}
\rho_C(I_V(v_1)) & \rho_C(I_V(v_2)) & \cdots & \rho_C(I_V(v_n)) \\
\rho_C(v_1) & \rho_C(v_2) & \cdots & \rho_C(v_n)
\end{bmatrix}$$

Notice that this definition is primarily about a single vector space ($V$) and two bases of $V$ ($B$, $C$). The linear transformation ($I_V$) is necessary but not critical. As you might expect, this matrix has something to do with changing bases. Here is the theorem that gives the matrix its name (not the other way around).

**Theorem CB**

**Change-of-Basis**

Suppose that $v$ is a vector in the vector space $V$ and $B$ and $C$ are bases of $V$. Then

$$\rho_C(v) = C_{B,C}\rho_B(v)$$
Subsection CB.CBM  Change-of-Basis Matrix  517

Proof

\[ \rho_C(v) = \rho_C(I_{v}(v)) \]
\[ = M_{B,C}^I \rho_B(v) \]
\[ = C_{B,C} \rho_B(v) \]

Definition IDLT [456]
Theorem FTMR [487]
Definition CBM [516]

\[ \square \]

So the change-of-basis matrix can be used with matrix multiplication to convert a vector representation of a vector \((v)\) relative to one basis \((\rho_B(v))\) to a representation of the same vector relative to a second basis \((\rho_C(v))\).

Theorem ICBM

Inverse of Change-of-Basis Matrix

Suppose that \(V\) is a vector space, and \(B\) and \(C\) are bases of \(V\). Then the change-of-basis matrix \(C_{B,C}\) is nonsingular and

\[ C_{B,C}^{-1} = C_{C,B} \]

\[ \square \]

Proof  The linear transformation \(I_{V} : V \rightarrow V\) is invertible, and its inverse is itself, \(I_{V}\) (check this!). So by Theorem IMR [499], the matrix \(M_{B,C}^I = C_{B,C}\) is invertible. Theorem NI [204] says an invertible matrix is nonsingular.

Then

\[ C_{B,C}^{-1} = \left( M_{B,C}^I \right)^{-1} \]
\[ = M_{C,B}^I \]
\[ = M_{C,B}^I \]
\[ = C_{C,B} \]

Definition CBM [516]
Theorem IMR [499]
Definition IDLT [456]
Definition CBM [516]

\[ \square \]

Example CBP

Change of basis with polynomials

The vector space \(P_4\) (Example VSP [253]) has two nice bases (Example BP [295]),

\[ B = \{ 1, x, x^2, x^3, x^4 \} \quad C = \{ 1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3, 1 + x + x^2 + x^3 + x^4 \} \]

To build the change-of-basis matrix between \(B\) and \(C\), we must first build a vector representation of each vector in \(B\) relative to \(C\),

\[ \rho_C(1) = \rho_C((1)(1)) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \rho_C(x) = \rho_C((-1)(1) + (1)(1 + x)) = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]
\[ \rho_C (x^2) = \rho_C ((-1)(1+x) + (1)(1+x+x^2)) = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \]

\[ \rho_C (x^3) = \rho_C ((-1)(1+x+x^2) + (1)(1+x+x^2+x^3)) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \]

\[ \rho_C (x^4) = \rho_C ((-1)(1+x+x^2+x^3) + (1)(1+x+x^2+x^3+x^4)) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \]

Then we package up these vectors as the columns of a matrix,

\[ C_{B,C} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

Now, to illustrate Theorem CB [516], consider the vector \( u = 5 - 3x + 2x^2 + 8x^3 - 3x^4 \). We can build the representation of \( u \) relative to \( B \) easily,

\[ \rho_B (u) = \rho_B (5 - 3x + 2x^2 + 8x^3 - 3x^4) = \begin{bmatrix} 5 \\ -3 \\ 2 \\ 8 \\ -3 \end{bmatrix} \]

Applying Theorem CB [516], we obtain a second representation of \( u \), but now relative to \( C \),

\[ \rho_C (u) = C_{B,C} \rho_B (u) \]

\[ = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \\ 8 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ -6 \\ 11 \\ -3 \end{bmatrix} \]

We can check our work by unraveling this second representation,

\[ u = \rho_C^{-1} (\rho_C (u)) \]

\[ = \rho_C^{-1} \begin{bmatrix} 8 \\ -5 \\ -6 \\ 11 \\ -3 \end{bmatrix} \]
The change-of-basis matrix from $C$ to $B$ is actually easier to build. Grab each vector in the basis $C$ and form its representation relative to $B$

$$
\rho_B(1) = \rho_B((1)1) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\rho_B(1 + x) = \rho_B((1)1 + (1)x) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\rho_B(1 + x + x^2) = \rho_B((1)1 + (1)x + (1)x^2) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\rho_B(1 + x + x^2 + x^3) = \rho_B((1)1 + (1)x + (1)x^2 + (1)x^3) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}
$$

$$
\rho_B(1 + x + x^2 + x^3 + x^4) = \rho_B((1)1 + (1)x + (1)x^2 + (1)x^3 + (1)x^4) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

Then we package up these vectors as the columns of a matrix,

$$
C_{C,B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

We formed two representations of the vector $u$ above, so we can again provide a check on our computations by converting from the representation of $u$ relative to $C$ to the representation of $u$ relative to $B$,

$$
\rho_B(u) = C_{C,B}\rho_C(u) \quad \text{Theorem CB 516}
$$

$$
= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ -5 \\ -6 \\ 11 \\ -3 \end{bmatrix}
$$
= \begin{bmatrix} 5 \\ -3 \\ 2 \\ 8 \\ -3 \end{bmatrix}

\text{Definition MVP 173}

One more computation that is either a check on our work, or an illustration of a theorem. The two change-of-basis matrices, $C_{B,C}$ and $C_{C,B}$, should be inverses of each other, according to Theorem ICBM 517. Here we go,

$$C_{B,C}C_{C,B} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The computations of the previous example are not meant to present any labor-saving devices, but instead are meant to illustrate the utility of the change-of-basis matrix. However, you might have noticed that $C_{C,B}$ was easier to compute than $C_{B,C}$. If you needed $C_{B,C}$, then you could first compute $C_{C,B}$ and then compute its inverse, which by Theorem ICBM 517, would equal $C_{B,C}$.

Here’s another illustrative example. We have been concentrating on working with abstract vector spaces, but all of our theorems and techniques apply just as well to $\mathbb{C}^n$, the vector space of column vectors. We only need to use more complicated bases than the standard unit vectors (Theorem SUV 294) to make things interesting.

Example CBCV

Change of basis with column vectors

For the vector space $\mathbb{C}^4$ we have the two bases,

\[ B = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \right\} \quad C = \left\{ \begin{bmatrix} 1 \\ -6 \\ -4 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 8 \\ 8 \\ -1 \end{bmatrix}, \begin{bmatrix} -5 \\ 13 \\ 9 \\ -1 \end{bmatrix}, \begin{bmatrix} -7 \\ 13 \\ 9 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \\ -6 \end{bmatrix} \right\} \]

The change-of-basis matrix from $B$ to $C$ requires writing each vector of $B$ as a linear combination of the vectors in $C$,

$$\rho_C \begin{bmatrix} 1 \\ -2 \\ 1 \\ -2 \end{bmatrix} = \rho_C \begin{bmatrix} 1 \\ -6 \\ -4 \\ -1 \end{bmatrix} + (-2) \begin{bmatrix} -1 \\ 3 \\ 3 \\ -4 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ -3 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \end{bmatrix}$$

$$\rho_C \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \rho_C \begin{bmatrix} -1 \\ -6 \\ -4 \\ -1 \end{bmatrix} + (-3) \begin{bmatrix} 1 \\ -2 \\ 3 \\ -3 \end{bmatrix} + (0) \begin{bmatrix} 2 \\ -3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 3 \\ 0 \end{bmatrix}$$

$$\rho_C \begin{bmatrix} 2 \\ 3 \\ 3 \\ 3 \end{bmatrix} = \rho_C \begin{bmatrix} 2 \\ -6 \\ -4 \\ -1 \end{bmatrix} + (-3) \begin{bmatrix} -1 \\ 8 \\ 8 \\ 9 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 3 \\ -2 \end{bmatrix}$$

$$\rho_C \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = \rho_C \begin{bmatrix} 3 \\ -6 \\ -4 \\ -1 \end{bmatrix} + (-2) \begin{bmatrix} -1 \\ 8 \\ 8 \\ 9 \end{bmatrix} + (3) \begin{bmatrix} 2 \\ 3 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 4 \\ 3 \end{bmatrix}$$
Then we package these vectors up as the change-of-basis matrix,

\[
C_{B,C} = \begin{bmatrix}
1 & 2 & 1 & 2 \\
-2 & -3 & -3 & -2 \\
1 & 3 & 1 & 4 \\
-1 & 0 & -2 & 3
\end{bmatrix}
\]

Now consider a single (arbitrary) vector \( \mathbf{y} = \begin{bmatrix} 2 \\ 6 \\ -3 \\ 4 \end{bmatrix} \). First, build the vector representation of \( \mathbf{y} \) relative to \( B \). This will require writing \( \mathbf{y} \) as a linear combination of the vectors in \( B \),

\[
\rho_B(\mathbf{y}) = \rho_B \begin{bmatrix} 2 \\ 6 \\ -3 \\ 4 \end{bmatrix} = \rho_B \begin{bmatrix} -21 \\ 6 \\ 11 \\ -7 \end{bmatrix}
\]

Now, applying Theorem CB[516] we can convert the representation of \( \mathbf{y} \) relative to \( B \) into a representation relative to \( C \),

\[
\rho_C(\mathbf{y}) = C_{B,C} \rho_B(\mathbf{y}) = \begin{bmatrix} -21 \\ 6 \\ 11 \\ -7 \end{bmatrix}
\]

We could continue further with this example, perhaps by computing the representation of \( \mathbf{y} \) relative to the basis \( C \) directly as a check on our work (Exercise CB.C20[536]). Or we could choose another vector to play the role of \( \mathbf{y} \) and compute two different representations of this vector relative to the two bases \( B \) and \( C \).

\[\Box\]

Subsection MRS
Matrix Representations and Similarity

Here is the main theorem of this section. It looks a bit involved at first glance, but the proof should make you realize it is not all that complicated. In any event, we are more interested in a special case.

**Theorem MRCB**

Matrix Representation and Change of Basis

Suppose that \( T : U \mapsto V \) is a linear transformation, \( B \) and \( C \) are bases for \( U \), and \( D \) and \( E \) are bases for \( V \). Then

\[
M_{T,B,D}^{T} = C_{E,D} M_{T,C,E}^{T} C_{B,C}
\]
Proof

\[
C_{E,D}^T M_{C,E} C_{B,C} = M_{E,D}^T M_{C,E}^T M_{B,C}
\]

\[
= M_{E,D}^T M_{B,E}^{T_{ol}}
\]

\[
= M_{E,D}^T M_{B,E}^T
\]

\[
= M_{B,D}^T
\]

\[
= M_{B,D}^T
\]

Definition CBM [516]

Theorem MRCLT [491]

Definition IDLT [456]

We will be most interested in a special case of this theorem (Theorem SCB [524]), but here’s an example that illustrates the full generality of Theorem MRCB [521].

Example MRCM

Matrix representations and change-of-basis matrices

Begin with two vector spaces, \(S_2\), the subspace of \(M_{22}\) containing all \(2 \times 2\) symmetric matrices, and \(P_3\) (Example VSP [253]), the vector space of all polynomials of degree 3 or less. Then define the linear transformation \(Q: S_2 \rightarrow P_3\) by

\[
Q\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = (5a - 2b + 6c) + (3a - b + 2c)x + (a + 3b - c)x^2 + (-4a + 2b + c)x^3
\]

Here are two bases for each vector space, one nice, one nasty. First for \(S_2\),

\[
B = \left\{ \begin{bmatrix} 5 & -3 \\ -3 & -2 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ -3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \right\}
\]

\[
C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]

and then for \(P_3\),

\[
D = \{2 + x - 2x^2 + 3x^3, -1 - 2x^2 + 3x^3, -3 - x + x^3, -x^2 + x^3\} \quad E = \{1, x, x^2, x^3\}
\]

We’ll begin with a matrix representation of \(Q\) relative to \(C\) and \(E\). We first find vector representations of the elements of \(C\) relative to \(E\),

\[
\rho_E \left( Q \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \right) = \rho_E \left( 5 + 3x + x^2 - 4x^3 \right) = \begin{bmatrix} 5 \\ 3 \\ 1 \\ -4 \end{bmatrix}
\]

\[
\rho_E \left( Q \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \right) = \rho_E \left( -2 - x + 3x^2 + 2x^3 \right) = \begin{bmatrix} -2 \\ -1 \\ 3 \\ 2 \end{bmatrix}
\]

\[
\rho_E \left( Q \left( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \right) = \rho_E \left( 6 + 2x - x^2 + x^3 \right) = \begin{bmatrix} 6 \\ 2 \\ -1 \\ 1 \end{bmatrix}
\]

So

\[
M_{C,E}^Q = \begin{bmatrix}
5 & -2 & 6 \\
3 & -1 & 2 \\
1 & 3 & -1 \\
-4 & 2 & 1
\end{bmatrix}
\]
Now we construct two change-of-basis matrices. First, $C_{B,C}$ requires vector representations of the elements of $B$, relative to $C$. Since $C$ is a nice basis, this is straightforward,

\[
\rho_C \left( \begin{bmatrix} 5 & -3 \\ -3 & -2 \end{bmatrix} \right) = \rho_C \left( \begin{bmatrix} 5 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -3 \end{bmatrix}
\]

\[
\rho_C \left( \begin{bmatrix} 2 & -3 \\ -3 & 0 \end{bmatrix} \right) = \rho_C \left( \begin{bmatrix} 2 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (0) \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}
\]

\[
\rho_C \left( \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \right) = \rho_C \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (4) \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

So

\[
C_{B,C} = \begin{bmatrix} 5 & 2 & 1 \\ -3 & -3 & 2 \\ -2 & 0 & 4 \end{bmatrix}
\]

The other change-of-basis matrix we’ll compute is $C_{E,D}$. However, since $E$ is a nice basis (and $D$ is not) we’ll turn it around and instead compute $C_{D,E}$ and apply Theorem ICBM \[517\] to use an inverse to compute $C_{E,D}$.

\[
\rho_E \left( 2 + x - 2x^2 + 3x^3 \right) = \rho_E \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (1)x + (-2)x^2 + (3)x^3 \right) = \begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \end{bmatrix}
\]

\[
\rho_E \left( -1 - 2x^2 + 3x^3 \right) = \rho_E \left( \begin{bmatrix} -1 \\ 0 \end{bmatrix} + (0)x + (-2)x^2 + (3)x^3 \right) = \begin{bmatrix} -1 \\ 0 \\ -2 \\ 3 \end{bmatrix}
\]

\[
\rho_E \left( -3 - x + x^3 \right) = \rho_E \left( \begin{bmatrix} -3 \\ -1 \end{bmatrix} + (-1)x + (0)x^2 + (1)x^3 \right) = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}
\]

\[
\rho_E \left( -x^2 + x^3 \right) = \rho_E \left( \begin{bmatrix} 0 \end{bmatrix} + (0)x + (-1)x^2 + (1)x^3 \right) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}
\]

So, we can package these column vectors up as a matrix to obtain $C_{D,E}$ and then,

\[
C_{E,D} = (C_{D,E})^{-1} \quad \text{Theorem ICBM} \ [517]
\]

\[
= \begin{bmatrix} 2 & -1 & -3 & 0 \\ 1 & 0 & -1 & 0 \\ -2 & -2 & 0 & -1 \\ 3 & 3 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 & 1 & 1 \\ -2 & 5 & -1 & -1 \\ 1 & -3 & 1 & 1 \\ 2 & -6 & -1 & 0 \end{bmatrix}
\]

We are now in a position to apply Theorem MRCB \[521\]. The matrix representation of $Q$ relative to $B$ and $D$ can be obtained as follows,

\[
M^{Q}_{B,D} = C_{E,D}M^{Q}_{C,E}C_{B,C} \quad \text{Theorem MRCB} \ [521]
\]
Now check our work by computing $M_{B,D}^Q$ directly (Exercise CB.C21 [536]).

Here is a special case of the previous theorem, where we choose $U$ and $V$ to be the same vector space, so the matrix representations and the change-of-basis matrices are all square of the same size.

**Theorem SCB**

**Similarity and Change of Basis**

Suppose that $T: V \mapsto V$ is a linear transformation and $B$ and $C$ are bases of $V$. Then

$$M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$$

**Proof** In the conclusion of Theorem MRCB [521], replace $D$ by $B$, and replace $E$ by $C$,

$$M_{B,B}^T = C_{C,B} M_{C,C}^T C_{B,C}$$

Theorem MRCB [521]

$$= C_{B,C}^{-1} M_{C,C}^T C_{B,C}$$

Theorem ICBM [517]

This is the third surprise of this chapter. Theorem SCB [524] considers the special case where a linear transformation has the same vector space for the domain and codomain ($V$). We build a matrix representation of $T$ using the basis $B$ simultaneously for both the domain and codomain ($M_{B,B}^T$), and then we build a second matrix representation of $T$, now using the basis $C$ for both the domain and codomain ($M_{C,C}^T$). Then these two representations are related via a similarity transformation (Definition SIM [390]) using a change-of-basis matrix ($C_{B,C}$)!

**Example MRBE**

**Matrix representation with basis of eigenvectors**

We return to the linear transformation $T: M_{22} \mapsto M_{22}$ of Example ELTBM [515] defined by

$$T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} -17a + 11b + 8c - 11d \\ -14a + 10b + 6c - 10d \end{bmatrix} = \begin{bmatrix} -57a + 35b + 24c - 33d \\ -41a + 25b + 16c - 23d \end{bmatrix}$$

In Example ELTBM [515] we showcased four eigenvectors of $T$. We will now put these four vectors in a set,

$$B = \{ x_1, x_2, x_3, x_4 \} = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix} \right\}$$

Check that $B$ is a basis of $M_{22}$ by first establishing the linear independence of $B$ and then employing Theorem G [320] to get the spanning property easily. Here is a second set of $2 \times 2$ matrices, which also forms a basis of $M_{22}$ (Example BM [295]),

$$C = \{ y_1, y_2, y_3, y_4 \} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
We can build two matrix representations of \( T \), one relative to \( B \) and one relative to \( C \). Each is easy, but for wildly different reasons. In our computation of the matrix representation relative to \( B \) we borrow some of our work in Example ELTBM \[515\]. Here are the representations, then the explanation.

\[
\rho_B(T(x_1)) = \rho_B(2x_1) = \rho_B(2x_1 + 0x_2 + 0x_3 + 0x_4) = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
\rho_B(T(x_2)) = \rho_B(2x_2) = \rho_B(0x_1 + 2x_2 + 0x_3 + 0x_4) = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}
\]

\[
\rho_B(T(x_3)) = \rho_B((-1)x_3) = \rho_B(0x_1 + 0x_2 + (-1)x_3 + 0x_4) = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}
\]

\[
\rho_B(T(x_4)) = \rho_B((-2)x_4) = \rho_B(0x_1 + 0x_2 + 0x_3 + (-2)x_4) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}
\]

So the resulting representation is

\[
M_{B,B}^T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}
\]

Very pretty. Now for the matrix representation relative to \( C \) first compute,

\[
\rho_C(T(y_1)) = \rho_C\left(\begin{bmatrix} -17 \\ -33 \\ -14 \\ -23 \end{bmatrix}\right)
\]

\[
= \rho_C\left((-17) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (-33) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (-14) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (-23) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -17 \\ -33 \\ -14 \\ -23 \end{bmatrix}
\]

\[
\rho_C(T(y_2)) = \rho_C\left(\begin{bmatrix} 11 \\ 10 \\ 35 \\ 25 \end{bmatrix}\right)
\]

\[
= \rho_C\left(11 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 35 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 10 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 25 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 11 \\ 35 \\ 10 \\ 25 \end{bmatrix}
\]

\[
\rho_C(T(y_3)) = \rho_C\left(\begin{bmatrix} 8 \\ 6 \\ 24 \\ 16 \end{bmatrix}\right)
\]

\[
= \rho_C\left(8 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 24 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 16 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ 24 \\ 6 \\ 16 \end{bmatrix}
\]

\[
\rho_C(T(y_4)) = \rho_C\left(\begin{bmatrix} -11 \\ -10 \\ -33 \\ -23 \end{bmatrix}\right)
\]

\[
= \rho_C\left((-11) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (-33) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + (-10) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (-23) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -11 \\ -33 \\ -10 \\ -23 \end{bmatrix}
\]
So the resulting representation is

\[
M_{C,C}^T = \begin{bmatrix}
-17 & 11 & 8 & -11 \\
-57 & 35 & 24 & -33 \\
-14 & 10 & 6 & -10 \\
-41 & 25 & 16 & -23
\end{bmatrix}
\]

Not quite as pretty. The purpose of this example is to illustrate Theorem SCB [524]. This theorem says that the two matrix representations, \(M_{B,B}^T\) and \(M_{C,C}^T\), of the one linear transformation, \(T\), are related by a similarity transformation using the change-of-basis matrix \(C_{B,C}\). Let's compute this change-of-basis matrix. Notice that since \(C\) is such a nice basis, this is fairly straightforward,

\[
\rho_C(x_1) = \rho_C\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) = \rho_C\left(0\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]

\[
\rho_C(x_2) = \rho_C\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = \rho_C\left(1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
\]

\[
\rho_C(x_3) = \rho_C\left(\begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}\right) = \rho_C\left(1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}
\]

\[
\rho_C(x_4) = \rho_C\left(\begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}\right) = \rho_C\left(2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}
\]

So we have,

\[
C_{B,C} = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{bmatrix}
\]

Now, according to Theorem SCB [524] we can write,

\[
M_{B,B}^T = C_{B,C}^{-1}M_{C,C}^TC_{B,C}
\]

\[
\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{bmatrix}^{-1}\begin{bmatrix} -17 & 11 & 8 & -11 \\ -57 & 35 & 24 & -33 \\ -14 & 10 & 6 & -10 \\ -41 & 25 & 16 & -23 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{bmatrix}
\]

This should look and feel exactly like the process for diagonalizing a matrix, as was described in Section SD [390]. And it is.

We can now return to the question of computing an eigenvalue or eigenvector of a linear transformation. For a linear transformation of the form \(T: V \mapsto V\), we know that representations relative to different bases are similar matrices. We also know that similar matrices have equal characteristic polynomials by Theorem SMEE [392]. We will now show that eigenvalues of a linear transformation \(T\) are precisely the eigenvalues of \(any\) matrix representation of \(T\). Since the choice of a different matrix representation leads to a similar matrix, there will be no “new” eigenvalues obtained from this second representation. Similarly, the change-of-basis matrix can be used to show that eigenvectors obtained from one matrix representation will be precisely those obtained...
from any other representation. So we can determine the eigenvalues and eigenvectors of a linear
transformation by forming one matrix representation, using any basis we please, and analyzing the
matrix in the manner of [Chapter E 356].

**Theorem EER**

**Eigenvalues, Eigenvectors, Representations**
Suppose that \( T: V \rightarrow V \) is a linear transformation and \( B \) is a basis of \( V \). Then \( v \in V \) is
an eigenvector of \( T \) for the eigenvalue \( \lambda \) if and only if \( \rho_B(v) \) is an eigenvector of \( M_{B,B}^T \) for the
eigenvalue \( \lambda \).

**Proof** \((\Rightarrow)\) Assume that \( v \in V \) is an eigenvector of \( T \) for the eigenvalue \( \lambda \). Then
\[
M_{B,B}^T \rho_B(v) = \rho_B(T(v)) = \rho_B(\lambda v) = \lambda \rho_B(v)
\]
which by [Definition EEM 356] says that \( \rho_B(v) \) is an eigenvector of the matrix \( M_{B,B}^T \) for the
eigenvalue \( \lambda \).

\((\Leftarrow)\) Assume that \( \rho_B(v) \) is an eigenvector of \( M_{B,B}^T \) for the eigenvalue \( \lambda \). Then
\[
T(v) = \rho_B^{-1}(\rho_B(T(v))) = \rho_B^{-1}(M_{B,B}^T \rho_B(v)) = \rho_B^{-1}(\lambda \rho_B(v)) = \lambda \rho_B^{-1}(\rho_B(v)) = \lambda v
\]
which by [Definition EELT 515] says \( v \) is an eigenvector of \( T \) for the eigenvalue \( \lambda \). \(\blacksquare\)

---

**Subsection CELT**

**Computing Eigenvectors of Linear Transformations**

Knowing that the eigenvalues of a linear transformation are the eigenvalues of any representation,
no matter what the choice of the basis \( B \) might be, we could now unambiguously define items such
as the characteristic polynomial of a linear transformation, rather than a matrix. We’ll say that
again — eigenvalues, eigenvectors, and characteristic polynomials are intrinsic properties of a linear
transformation, independent of the choice of a basis used to construct a matrix representation.

As a practical matter, how does one compute the eigenvalues and eigenvectors of a linear
transformation of the form \( T: V \rightarrow V \)? Choose a nice basis \( B \) for \( V \), one where the vector
representations of the values of the linear transformations necessary for the matrix representation
are easy to compute. Construct the matrix representation relative to this basis, and find the
eigenvalues and eigenvectors of this matrix using the techniques of [Chapter E 356]. The resulting
eigenvalues of the matrix are precisely the eigenvalues of the linear transformation. The eigenvectors
of the matrix are column vectors that need to be converted to vectors in \( V \) through application of
\( \rho_B^{-1} \).

Now consider the case where the matrix representation of a linear transformation is diagonaliz-
able. The \( n \) linearly independent eigenvectors that must exist for the matrix (Theorem DC 394) can be converted (via \( \rho_B^{-1} \)) into eigenvectors of the linear transformation. A matrix representation
of the linear transformation relative to a basis of eigenvectors will be a diagonal matrix — an espe-
cially nice representation! Though we did not know it at the time, the diagonalizations of [Section
SD 390] were really finding especially pleasing matrix representations of linear transformations.

Here are some examples.
Example ELTT
Eigenvectors of a linear transformation, twice

Consider the linear transformation $S: M_{22} \rightarrow M_{22}$ defined by

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -b-c-3d & -14a-15b-13c+d \\ 18a+21b+19c+3d & -6a-7b-7c-3d \end{pmatrix}$$

To find the eigenvalues and eigenvectors of $S$ we will build a matrix representation and analyze the matrix. Since Theorem EER [527] places no restriction on the choice of the basis $B$, we may as well use a basis that is easy to work with. So set

$$B = \{ x_1, x_2, x_3, x_4 \} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Then to build the matrix representation of $S$ relative to $B$ compute,

$$\rho_B (S(x_1)) = \rho_B \left( \begin{pmatrix} 0 & -14 \\ 18 & -6 \end{pmatrix} \right) = \rho_B (0x_1 + (-14)x_2 + 18x_3 + (-6)x_4) = \begin{pmatrix} 0 \\ -14 \\ 18 \\ -6 \end{pmatrix}$$

$$\rho_B (S(x_2)) = \rho_B \left( \begin{pmatrix} -1 & -15 \\ 21 & -7 \end{pmatrix} \right) = \rho_B ((-1)x_1 + (-15)x_2 + 21x_3 + (-7)x_4) = \begin{pmatrix} -1 \\ -15 \\ 21 \\ -7 \end{pmatrix}$$

$$\rho_B (S(x_3)) = \rho_B \left( \begin{pmatrix} -1 & -13 \\ 19 & -7 \end{pmatrix} \right) = \rho_B ((-1)x_1 + (-13)x_2 + 19x_3 + (-7)x_4) = \begin{pmatrix} -1 \\ -13 \\ 19 \\ -7 \end{pmatrix}$$

$$\rho_B (S(x_4)) = \rho_B \left( \begin{pmatrix} -3 & 1 \\ 3 & -3 \end{pmatrix} \right) = \rho_B ((-3)x_1 + 1x_2 + 3x_3 + (-3)x_4) = \begin{pmatrix} -3 \\ 1 \\ 3 \\ -3 \end{pmatrix}$$

So by Definition MR [485] we have

$$M = M^S_{B,B} = \begin{pmatrix} 0 & -1 & -1 & -3 \\ -14 & -15 & -13 & 1 \\ 18 & 21 & 19 & 3 \\ -6 & -7 & -7 & -3 \end{pmatrix}$$

Now compute eigenvalues and eigenvectors of the matrix representation of $M$ with the techniques of Section EE [356]. First the characteristic polynomial,

$$p_M (x) = \text{det} (M - xI_4) = x^4 - x^3 - 10x^2 + 4x + 24 = (x - 3)(x - 2)(x + 2)^2$$

We could now make statements about the eigenvalues of $M$, but in light of Theorem EER [527] we can refer to the eigenvalues of $S$ and mildly abuse (or extend) our notation for multiplicities to write

$$\alpha_S (3) = 1 \quad \alpha_S (2) = 1 \quad \alpha_S (-2) = 2$$

Now compute the eigenvectors of $M$,

$$\lambda = 3 \quad M - 3I_4 = \begin{pmatrix} -3 & -1 & -1 & -3 \\ -14 & -18 & -13 & 1 \\ 18 & 21 & 16 & 3 \\ -6 & -7 & -7 & -6 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
\[ \mathcal{E}_M(3) = \mathcal{N}(M - 3I_4) = \left\langle \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix} \right\rangle \]

\[ \lambda = 2 \quad M - 2I_4 = \begin{bmatrix} -2 & -1 & -1 & -3 \\ -14 & -17 & -13 & 1 \\ 18 & 21 & 17 & 3 \\ -6 & -7 & -7 & -5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \mathcal{E}_M(2) = \mathcal{N}(M - 2I_4) = \left\langle \begin{bmatrix} -2 \\ 4 \\ -3 \\ 1 \end{bmatrix} \right\rangle \]

\[ \lambda = -2 \quad M - (-2)I_4 = \begin{bmatrix} 2 & -1 & -1 & -3 \\ -14 & -13 & -13 & 1 \\ 18 & 21 & 21 & 3 \\ -6 & -7 & -7 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \mathcal{E}_M(-2) = \mathcal{N}(M - (-2)I_4) = \left\langle \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle \]

According to Theorem EER [527] the eigenvectors just listed as basis vectors for the eigenspaces of \( M \) are vector representations (relative to \( B \)) of eigenvectors for \( S \). So the application of the inverse function \( \rho_B^{-1} \) will convert these column vectors into elements of the vector space \( \mathbb{M}_{22} \) (2 \( \times \) 2 matrices) that are eigenvectors of \( S \). Since \( \rho_B \) is an isomorphism (Theorem VRILT [478]), so is \( \rho_B^{-1} \). Applying the inverse function will then preserve linear independence and spanning properties, so with a sweeping application of the Coordinatization Principle [481] and some extensions of our previous notation for eigenspaces and geometric multiplicities, we can write,

\[ \rho_B^{-1} \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix} = (-1)x_1 + 3x_2 + (-3)x_3 + 1x_4 = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix} \]

\[ \rho_B^{-1} \begin{bmatrix} -2 \\ 4 \\ -3 \\ 1 \end{bmatrix} = (-2)x_1 + 4x_2 + (-3)x_3 + 1x_4 = \begin{bmatrix} -2 \\ 4 \\ -3 \\ 1 \end{bmatrix} \]

\[ \rho_B^{-1} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = 0x_1 + (-1)x_2 + 1x_3 + 0x_4 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \]

\[ \rho_B^{-1} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = 1x_1 + (-1)x_2 + 0x_3 + 1x_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \]

So

\[ \mathcal{E}_S(3) = \left\langle \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix} \right\rangle \]
\[ \mathcal{E}_S(2) = \left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} \right\} \]

\[ \mathcal{E}_S(-2) = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \]

with geometric multiplicities given by
\[ \gamma_S(3) = 1 \quad \gamma_S(2) = 1 \quad \gamma_S(-2) = 2 \]

Suppose we now decided to build another matrix representation of \( S \), only now relative to a linearly independent set of eigenvectors of \( S \), such as
\[ C = \left\{ \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \]

At this point you should have computed enough matrix representations to predict that the result of representing \( S \) relative to \( C \) will be a diagonal matrix. Computing this representation is an example of how Theorem SCB [524] generalizes the diagonalizations from Section SD [390]. For the record, here is the diagonal representation,

\[ M^S_{C,C} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \]

Our interest in this example is not necessarily building nice representations, but instead we want to demonstrate how eigenvalues and eigenvectors are an intrinsic property of a linear transformation, independent of any particular representation. To this end, we will repeat the foregoing, but replace \( B \) by another basis. We will make this basis different, but not extremely so,
\[ D = \{ y_1, y_2, y_3, y_4 \} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \]

Then to build the matrix representation of \( S \) relative to \( D \) compute,

\[ \rho_D(S(y_1)) = \rho_D \left( \begin{bmatrix} 0 & -14 \\ 18 & -6 \end{bmatrix} \right) = \rho_D (14y_1 + (-32)y_2 + 24y_3 + (-6)y_4) = \begin{bmatrix} 14 \\ -32 \\ 24 \\ -6 \end{bmatrix} \]

\[ \rho_D(S(y_2)) = \rho_D \left( \begin{bmatrix} -1 & -29 \\ 39 & -13 \end{bmatrix} \right) = \rho_D (28y_1 + (-68)y_2 + 52y_3 + (-13)y_4) = \begin{bmatrix} 28 \\ -68 \\ 52 \\ -13 \end{bmatrix} \]

\[ \rho_D(S(y_3)) = \rho_D \left( \begin{bmatrix} -2 & -42 \\ 58 & -20 \end{bmatrix} \right) = \rho_D (40y_1 + (-100)y_2 + 78y_3 + (-20)y_4) = \begin{bmatrix} 40 \\ -100 \\ 78 \\ -20 \end{bmatrix} \]

\[ \rho_D(S(y_4)) = \rho_D \left( \begin{bmatrix} -5 & -41 \\ 61 & -23 \end{bmatrix} \right) = \rho_D (36y_1 + (-102)y_2 + 84y_3 + (-23)y_4) = \begin{bmatrix} 36 \\ -102 \\ 84 \\ -23 \end{bmatrix} \]

So by Definition MR [485] we have

\[ N = M^S_{D,D} = \begin{bmatrix} 14 & 28 & 40 & 36 \\ -32 & -68 & -100 & -102 \\ 24 & 52 & 78 & 84 \\ -6 & -13 & -20 & -23 \end{bmatrix} \]
Now compute eigenvalues and eigenvectors of the matrix representation of $N$ with the techniques of Section EE [356]. First the characteristic polynomial,

$$p_N(x) = \det(N - xI_4) = x^4 - x^3 - 10x^2 + 4x + 24 = (x - 3)(x - 2)(x + 2)^2$$

Of course this is not news. We now know that $M = M_{S,B,B}$ and $N = M_{D,D}$ are similar matrices (Theorem SCB [524]). But Theorem SMEE [392] told us long ago that similar matrices have identical characteristic polynomials. Now compute eigenvectors for the matrix representation, which will be different than what we found for $M$,

$$\lambda = 3 \quad N - 3I_4 = \begin{bmatrix} 11 & 28 & 40 & 36 \\ -32 & -71 & -100 & -102 \\ 24 & 52 & 75 & 84 \\ -6 & -13 & -20 & -26 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_N(3) = \mathcal{N}(N - 3I_4) = \left\langle \begin{bmatrix} -4 \\ 6 \\ -4 \\ 1 \end{bmatrix} \right\rangle$$

$$\lambda = 2 \quad N - 2I_4 = \begin{bmatrix} 12 & 28 & 40 & 36 \\ -32 & -70 & -100 & -102 \\ 24 & 52 & 76 & 84 \\ -6 & -13 & -20 & -25 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_N(2) = \mathcal{N}(N - 2I_4) = \left\langle \begin{bmatrix} -6 \\ 7 \\ -4 \\ 1 \end{bmatrix} \right\rangle$$

$$\lambda = -2 \quad N - (-2)I_4 = \begin{bmatrix} 16 & 28 & 40 & 36 \\ -32 & -66 & -100 & -102 \\ 24 & 52 & 80 & 84 \\ -6 & -13 & -20 & -21 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_N(-2) = \mathcal{N}(N - (-2)I_4) = \left\langle \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

Employing Theorem EER [527] we can apply $\rho^{-1}_D$ to each of the basis vectors of the eigenspaces of $N$ to obtain eigenvectors for $S$ that also form bases for eigenspaces of $S$,

$$\rho^{-1}_D\begin{bmatrix} -4 \\ 6 \\ -4 \\ 1 \end{bmatrix} = (-4)y_1 + 6y_2 + (-4)y_3 + 1y_4 = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix}$$

$$\rho^{-1}_D\begin{bmatrix} -6 \\ 7 \\ -4 \\ 1 \end{bmatrix} = (-6)y_1 + 7y_2 + (-4)y_3 + 1y_4 = \begin{bmatrix} -2 \\ 4 \\ -3 \\ 1 \end{bmatrix}$$

$$\rho^{-1}_D\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} = 1y_1 + (-2)y_2 + 1y_3 + 0y_4 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$
\[ \rho_D^1 \left( \begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right) = 3y_1 + (-3)y_2 + 0y_3 + 1y_4 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \]

The eigenspaces for the eigenvalues of algebraic multiplicity 1 are exactly as before,

\[ \mathcal{E}_S(3) = \left\langle \left\{ \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix} \right\} \right\rangle \]

\[ \mathcal{E}_S(2) = \left\langle \left\{ \begin{bmatrix} -2 \\ 4 \\ -3 \\ 1 \end{bmatrix} \right\} \right\rangle \]

However, the eigenspace for \( \lambda = -2 \) would at first glance appear to be different. Here are the two eigenspaces for \( \lambda = -2 \), first the eigenspace obtained from \( M = M_{B,B}^S \), then followed by the eigenspace obtained from \( M = M_{B,D}^S \).

\[ \mathcal{E}_S(-2) = \left\langle \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \right\rangle \]

\[ \mathcal{E}_S(-2) = \left\langle \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} \right\rangle \]

Subspaces generally have many bases, and that is the situation here. With a careful proof of set equality, you can show that these two eigenspaces are equal sets. The key observation to make such a proof go is that

\[ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \]

which will establish that the second set is a subset of the first. With equal dimensions, Theorem [EDYES 323] will finish the task. So the eigenvalues of a linear transformation are independent of the matrix representation employed to compute them!

Another example, this time a bit larger and with complex eigenvalues.

**Example CELT**

**Complex eigenvectors of a linear transformation**

Consider the linear transformation \( Q : P_4 \rightarrow P_4 \) defined by

\[
Q(a + bx + cx^2 + dx^3 + ex^4)
\]

\[
= (-46a - 22b + 13c + 5d + e) + (117a + 57b - 32c - 15d - 4e)x +
\]

\[
(-69a - 29b + 21c - 7e)x^2 + (159a + 73b - 44c - 13d + 2e)x^3 +
\]

\[
(-195a - 87b + 55c + 10d - 13e)x^4
\]

Choose a simple basis to compute with, say

\[ B = \{ 1, x, x^2, x^3, x^4 \} \]

Then it should be apparent that the matrix representation of \( Q \) relative to \( B \) is

\[
M = M_{B,B}^Q = \begin{bmatrix}
-46 & -22 & 13 & 5 & 1 \\
117 & 57 & -32 & -15 & -4 \\
-69 & -29 & 21 & 0 & -7 \\
159 & 73 & -44 & -13 & 2 \\
-195 & -87 & 55 & 10 & -13
\end{bmatrix}
\]

Compute the characteristic polynomial, eigenvalues and eigenvectors according to the techniques of Section EE [356].

\[
p_Q(x) = -x^5 + 6x^4 - x^3 - 88x^2 + 252x - 208
\]
\[(x - 2)^2 (x + 4) (x - 3 - 2i)(x - 3 - 2i)\]

\[
\begin{align*}
\alpha_Q (2) = 2 & \quad \alpha_Q (-4) = 1 & \quad \alpha_Q (3 + 2i) = 1 & \quad \alpha_Q (3 - 2i) = 1
\end{align*}
\]

\[
\lambda = 2
\]

\[
M - (2)I_5 = \begin{bmatrix}
-48 & -22 & 13 & 5 & 1 \\
117 & 55 & -32 & -15 & -4 \\
-69 & -29 & 19 & 0 & -7 \\
159 & 73 & -44 & -15 & 2 \\
-195 & -87 & 55 & 10 & -15
\end{bmatrix}
\]

\[
E_M (2) = N(M - (2)I_5) = \left\{ \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{4} \\ 2 \\ 0 \end{bmatrix} \right\}
\]

\[
\lambda = -4
\]

\[
M - (-4)I_5 = \begin{bmatrix}
-42 & -22 & 13 & 5 & 1 \\
117 & 61 & -32 & -15 & -4 \\
-69 & -29 & 25 & 0 & -7 \\
159 & 73 & -44 & -9 & 2 \\
-195 & -87 & 55 & 10 & -9
\end{bmatrix}
\]

\[
E_M (-4) = N(M - (-4)I_5) = \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} \right\}
\]

\[
\lambda = 3 + 2i
\]

\[
M - (3 + 2i)I_5 = \begin{bmatrix}
-49 - 2i & -22 & 13 & 5 & 1 \\
117 & 54 - 2i & -32 & -15 & -4 \\
-69 & -29 & 18 - 2i & 0 & -7 \\
159 & 73 & -44 & -16 - 2i & 2 \\
-195 & -87 & 55 & 10 & -16 - 2i
\end{bmatrix}
\]

\[
E_M (3 + 2i) = N(M - (3 + 2i)I_5) = \left\{ \begin{bmatrix} \frac{3 - i}{4} - \frac{i}{4} \\ -\frac{i}{4} + \frac{1}{4} \\ -\frac{i}{4} + \frac{1}{4} \\ -\frac{7}{4} + \frac{i}{4} \end{bmatrix} \right\}
\]

\[
\lambda = 3 - 2i
\]

\[
M - (3 - 2i)I_5 = \begin{bmatrix}
-49 + 2i & -22 & 13 & 5 & 1 \\
117 & 54 + 2i & -32 & -15 & -4 \\
-69 & -29 & 18 + 2i & 0 & -7 \\
159 & 73 & -44 & -16 + 2i & 2 \\
-195 & -87 & 55 & 10 & -16 + 2i
\end{bmatrix}
\]

\[
E_M (3 - 2i) = N(M - (3 - 2i)I_5) = \left\{ \begin{bmatrix} \frac{3 - i}{4} + \frac{i}{4} \\ -\frac{7}{4} - \frac{i}{4} \\ -\frac{7}{4} - \frac{i}{4} \\ -\frac{7}{4} + \frac{i}{4} \end{bmatrix} \right\}
\]

Version 1.04
\[ \mathcal{E}_M(3 - 2i) = \mathcal{N}(M - (3 - 2i)I_5) = \left\langle \left\{ \begin{bmatrix} 3 + i \\ -7 - i \\ 2 + 2i \\ -7 - i \\ 4 \end{bmatrix} \right\} \right\rangle = \left\langle \left\{ \begin{bmatrix} 3 + 4i \\ 2 + 4i \\ -2 + 4i \\ -2 + 4i \\ 1 \end{bmatrix} \right\} \right\rangle \]

It is straightforward to convert each of these basis vectors for eigenspaces of \( M \) back to elements of \( P_4 \) by applying the isomorphism \( \rho_B^{-1} \).

\[ \rho_B^{-1} \begin{bmatrix} -1 \\ 5 \\ 4 \\ 2 \\ 0 \end{bmatrix} = -1 + 5x + 4x^2 + 2x^3 \]
\[ \rho_B^{-1} \begin{bmatrix} 1 \\ 5 \\ 12 \\ 0 \\ 2 \end{bmatrix} = 1 + 5x + 12x^2 + 2x^4 \]
\[ \rho_B^{-1} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \\ 1 \end{bmatrix} = -1 + 3x + x^2 + 2x^3 + x^4 \]
\[ \rho_B^{-1} \begin{bmatrix} 3 - i \\ -7 + i \\ 2 - 2i \\ -7 + i \\ 4 \end{bmatrix} = (3 - i) + (-7 + i)x + (2 - 2i)x^2 + (-7 + i)x^3 + 4x^4 \]
\[ \rho_B^{-1} \begin{bmatrix} 3 + i \\ -7 - i \\ 2 + 2i \\ -7 - i \\ 4 \end{bmatrix} = (3 + i) + (-7 - i)x + (2 + 2i)x^2 + (-7 - i)x^3 + 4x^4 \]

So we apply Theorem EER [527] and the Coordinatization Principle [481] to get the eigenspaces for \( Q \),

\[ \mathcal{E}_Q(2) = \left\langle \left\{ -1 + 5x + 4x^2 + 2x^3, 1 + 5x + 12x^2 + 2x^4 \right\} \right\rangle \]
\[ \mathcal{E}_Q(-4) = \left\langle \left\{ -1 + 3x + x^2 + 2x^3 + x^4 \right\} \right\rangle \]
\[ \mathcal{E}_Q(3 + 2i) = \left\langle \left\{ (3 - i) + (-7 + i)x + (2 - 2i)x^2 + (-7 + i)x^3 + 4x^4 \right\} \right\rangle \]
\[ \mathcal{E}_Q(3 - 2i) = \left\langle \left\{ (3 + i) + (-7 - i)x + (2 + 2i)x^2 + (-7 - i)x^3 + 4x^4 \right\} \right\rangle \]

with geometric multiplicities

\[ \gamma_Q(2) = 2 \quad \gamma_Q(-4) = 1 \quad \gamma_Q(3 + 2i) = 1 \quad \gamma_Q(3 - 2i) = 1 \]

Subsection READ
Reading Questions

1. The change-of-basis matrix is a matrix representation of which linear transformation?
2. Find the change-of-basis matrix, $C_{B,C}$, for the two bases of $\mathbb{C}^2$

\[ B = \left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \\ 2 \end{bmatrix} \right\} \quad C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \]

3. What is the third “surprise,” and why is it surprising?
Subsection EXC
Exercises

C20 In Example CBCV \[520\] we computed the vector representation of \( y \) relative to \( C \), \( \rho_C (y) \), as an example of Theorem CB \[516\]. Compute this same representation directly. In other words, apply Definition VR \[473\] rather than Theorem CB \[516\].
Contributed by Robert Beezer

C21 Perform a check on Example MRCM \[522\] by computing \( M^Q \) directly. In other words, apply Definition MR \[485\] rather than Theorem MRCB \[521\].
Contributed by Robert Beezer

Solution

C30 Find a basis for the vector space \( P_3 \) composed of eigenvectors of the linear transformation \( T \). Then find a matrix representation of \( T \) relative to this basis.

\[ T: P_3 \mapsto P_3, \quad T (a + bx + cx^2 + dx^3) = (a + c + d) + (b + c + d)x + (a + b + c)x^2 + (a + b + d)x^3 \]

Contributed by Robert Beezer

Solution

C40 Let \( S_{22} \) be the vector space of \( 2 \times 2 \) symmetric matrices. Find a basis \( B \) for \( S_{22} \) that yields a diagonal matrix representation of the linear transformation \( R \). (15 points)

\[ R: S_{22} \mapsto S_{22}, \quad R \left( \begin{bmatrix} a & b \\ b & c \end{bmatrix} \right) = \begin{bmatrix} -5a + 2b - 3c & -12a + 5b - 6c \\ -12a + 5b - 6c & 6a - 2b + 4c \end{bmatrix} \]

Contributed by Robert Beezer

Solution

C41 Let \( S_{22} \) be the vector space of \( 2 \times 2 \) symmetric matrices. Find a basis for \( S_{22} \) composed of eigenvectors of the linear transformation \( Q: S_{22} \mapsto S_{22} \). (15 points)

\[ Q \left( \begin{bmatrix} a & b \\ b & c \end{bmatrix} \right) = \begin{bmatrix} 25a + 18b + 30c & -16a - 11b - 20c \\ -16a - 11b - 20c & -11a - 9b - 12c \end{bmatrix} \]

Contributed by Robert Beezer

Solution

T10 Suppose that \( T: V \mapsto V \) is an invertible linear transformation with a nonzero eigenvalue \( \lambda \). Prove that \( \frac{1}{\lambda} \) is an eigenvalue of \( T^{-1} \).

Contributed by Robert Beezer

T15 Suppose that \( V \) is a vector space and \( T: V \mapsto V \) is a linear transformation. Prove that \( T \) is injective if and only if \( \lambda = 0 \) is not an eigenvalue of \( T \).

Contributed by Robert Beezer
Apply Definition MR \[536\],

\[
\rho_D \left( Q \left( \begin{bmatrix} 5 & -3 \\ -3 & -2 \end{bmatrix} \right) \right) = \rho_D \left( 19 + 14x - 2x^2 - 28x^3 \right)
\]

\[
= \rho_D \left( (-39)(2 + x - 2x^2 + 3x^3) + 62(-1 - 2x^2 + 3x^3) + (-53)(-3 - x + 3x^3) + (-44)(-x^2 + x^3) \right)
\]

\[
= \begin{bmatrix} -39 \\ 62 \\ -53 \\ -44 \end{bmatrix}
\]

\[
\rho_D \left( Q \left( \begin{bmatrix} 2 & -3 \\ -3 & 0 \end{bmatrix} \right) \right) = \rho_D \left( 16 + 9x - 7x^2 - 14x^3 \right)
\]

\[
= \rho_D \left( (-23)(2 + x - 2x^2 + 3x^3) + (34)(-1 - 2x^2 + 3x^3) + (-32)(-3 - x + 3x^3) + (-15)(-x^2 + x^3) \right)
\]

\[
= \begin{bmatrix} -23 \\ 34 \\ -32 \\ -15 \end{bmatrix}
\]

\[
\rho_D \left( Q \left( \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \right) \right) = \rho_D \left( 25 + 9x + 3x^2 + 4x^3 \right)
\]

\[
= \rho_D \left( (14)(2 + x - 2x^2 + 3x^3) + (-12)(-1 - 2x^2 + 3x^3) + 5(-3 - x + 3x^3) + (-7)(-x^2 + x^3) \right)
\]

\[
= \begin{bmatrix} 14 \\ -12 \\ 5 \\ -7 \end{bmatrix}
\]

These three vectors are the columns of the matrix representation,

\[
Q \begin{bmatrix} -39 & -23 & 14 \\ 62 & 34 & -12 \\ -53 & -32 & 5 \\ -44 & -15 & -7 \end{bmatrix}
\]

which coincides with the result obtained in Example MRCM \[522\].

C30 Contributed by Robert Beezer Statement \[536\]

With the domain and codomain being identical, we will build a matrix representation using the same basis for both the domain and codomain. The eigenvalues of the matrix representation will be the eigenvalues of the linear transformation, and we can obtain the eigenvectors of the linear transformation by un-coordinatizing (Theorem EER \[527\]). Since the method does not depend on which basis we choose, we can choose a natural basis for ease of computation, say,

\[
B = \{1, x, x^2, x^3\}
\]

The matrix representation is then,

\[
Q \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}
\]
The eigenvalues and eigenvectors of this matrix were computed in Example ESMS4 \[367\]. A basis for \( \mathbb{C}^4 \), composed of eigenvectors of the matrix representation is,

\[
C = \begin{bmatrix}
1 & -1 & 0 & -1 \\
1 & 1 & 0 & 1 \\
1 & 0 & -1 & 1 \\
1 & 0 & 1 & 1
\end{bmatrix}
\]

Applying \( \rho_B^{-1} \) to each vector of this set, yields a basis of \( P_3 \) composed of eigenvectors of \( T \),

\[
D = \{1 + x + x^2 + x^3, -1 + x, -x^2 + x^3, -1 - x + x^2 + x^3\}
\]

The matrix representation of \( T \) relative to the basis \( D \) will be a diagonal matrix with the corresponding eigenvalues along the diagonal, so in this case we get

\[
M^T_{D,D} = \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

C40 Contributed by Robert Beezer Statement \[536\]

Begin with a matrix representation of \( R \), any matrix representation, but use the same basis for both instances of \( S_{22} \). We’ll choose a basis that makes it easy to compute vector representations in \( S_{22} \).

\[
B = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Then the resulting matrix representation of \( R \) (Definition MR \[485\]) is

\[
M^R_{B,B} = \begin{bmatrix}
-5 & 2 & -3 \\
-12 & 5 & -6 \\
6 & -2 & 4
\end{bmatrix}
\]

Now, compute the eigenvalues and eigenvectors of this matrix, with the goal of diagonalizing the matrix (Theorem DC \[394\]),

\[
\lambda = 2 \quad \mathcal{E}_{M^R_{B,B}}(2) = \langle \begin{bmatrix}
-1 \\
-2 \\
1
\end{bmatrix} \rangle
\]

\[
\lambda = 1 \quad \mathcal{E}_{M^R_{B,B}}(1) = \langle \begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
0 \\
2 \\
0
\end{bmatrix} \rangle
\]

The three vectors that occur as basis elements for these eigenspaces will together form a linearly independent set (check this!). So these column vectors may be employed in a matrix that will diagonalize the matrix representation. If we “un-coordinatize” these three column vectors relative to the basis \( B \), we will find three linearly independent elements of \( S_{22} \) that are eigenvectors of the linear transformation \( R \) (Theorem EER \[527\]). A matrix representation relative to this basis of eigenvectors will be diagonal, with the eigenvalues (\( \lambda = 2, 1 \)) as the diagonal elements. Here we go,

\[
\rho_B^{-1} \begin{bmatrix}
-1 \\
-2 \\
1
\end{bmatrix} = (-1) \begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0
\end{bmatrix} + (-2) \begin{bmatrix}
0 & 1 \\
1 & 0 \\
0 & 1
\end{bmatrix} + 1 \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & -1
\end{bmatrix} = \begin{bmatrix}
-1 & -2 \\
-2 & 1
\end{bmatrix}
\]

\[
\rho_B^{-1} \begin{bmatrix}
0 \\
2
\end{bmatrix} = (-1) \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix} + 0 \begin{bmatrix}
0 & 1 \\
1 & 0 \\
0 & 1
\end{bmatrix} + 2 \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & -1
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
0 & 2
\end{bmatrix}
\]
\[
\rho_B^{-1} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}
\]

So the requested basis of \( S_{22} \), yielding a diagonal matrix representation of \( R \), is

\[
\left\{ \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix} \right\}
\]

C41 Contributed by Robert Beezer Statement 536

Use a single basis for both the domain and codomain, since they are equal.

\[
B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]

The matrix representation of \( Q \) relative to \( B \) is

\[
M = M_Q^{B,B} = \begin{bmatrix} 25 & 18 & 30 \\ -16 & -11 & -20 \\ -11 & -9 & -12 \end{bmatrix}
\]

We can analyze this matrix with the techniques of Section EE 356 and then apply Theorem EER 527. The eigenvalues of this matrix are \( \lambda = -2, 1, 3 \) with eigenspaces

\[
\mathcal{E}_M(-2) = \left\langle \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix} \right\rangle, \quad \mathcal{E}_M(1) = \left\langle \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\rangle, \quad \mathcal{E}_M(3) = \left\langle \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\rangle
\]

Because the three eigenvalues are distinct, the three basis vectors from the three eigenspaces for a linearly independent set (Theorem EDELI 378). Theorem EER 527 says we can uncoordinatize these eigenvectors to obtain eigenvectors of \( Q \). By Theorem ILTLI 433 the resulting set will remain linearly independent. Set

\[
C = \left\{ \rho_B^{-1} \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}, \rho_B^{-1} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \rho_B^{-1} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}
\]

Then \( C \) is a linearly independent set of size 3 in the vector space \( M_{22} \), which has dimension 3 as well. By Theorem G 320, \( C \) is a basis of \( M_{22} \).

T10 Contributed by Robert Beezer Statement 536

Let \( \mathbf{v} \) be an eigenvector of \( T \) for the eigenvalue \( \lambda \). Then,

\[
T^{-1}(\mathbf{v}) = \frac{1}{\lambda} \lambda T^{-1}(\mathbf{v}) = \frac{1}{\lambda} T^{-1}(\lambda \mathbf{v}) = \frac{1}{\lambda} T^{-1}(T(\mathbf{v})) = \frac{1}{\lambda} I_V(\mathbf{v}) = \frac{1}{\lambda} \mathbf{v}
\]

which says that \( \frac{1}{\lambda} \) is an eigenvalue of \( T^{-1} \) with eigenvector \( \mathbf{v} \). Note that it is possible to prove that any eigenvalue of an invertible linear transformation is never zero. So the hypothesis that \( \lambda \) be nonzero is just a convenience for this problem.
Section OD
Orthonormal Diagonalization

This Section Under Construction
Theorems & definitions complete, needs examples

We have seen in Section SD \[390\] that under the right conditions a square matrix is similar to a diagonal matrix. We recognize now, via Theorem SCB \[524\], that a similarity transformation is a change of basis on a matrix representation. So we can now discuss the choice of a basis used to build a matrix representation, and decide if some bases are better than others for this purpose. This will be the tone of this section. We will also see that every matrix has a reasonably useful matrix representation, and we will discover a new class of diagonalizable linear transformations. First we need some basic facts about triangular matrices.

Subsection TM
Triangular Matrices

An upper, or lower, triangular matrix is exactly what it sounds like it should be, but here are the two relevant definitions.

Definition UTM
Upper Triangular Matrix
The \( n \times n \) square matrix \( A \) is upper triangular if \([A]_{ij} = 0\) whenever \( i > j \).

Definition LTM
Lower Triangular Matrix
The \( n \times n \) square matrix \( A \) is lower triangular if \([A]_{ij} = 0\) whenever \( i < j \).

Obviously, properties of a lower triangular matrices will have analogues for upper triangular matrices. Rather than stating two very similar theorems, we will say that matrices are “triangular of the same type” as a convenient shorthand to cover both possibilities and then give a proof for just one type.

Theorem PTMT
Product of Triangular Matrices is Triangular
Suppose that \( A \) and \( B \) are square matrices of size \( n \) that are triangular of the same type. Then \( AB \) is also triangular of that type.

Proof We prove this for lower triangular matrices and leave the proof for upper triangular matrices to you. Suppose that \( A \) and \( B \) are both lower triangular. We need only establish that certain entries of the product \( AB \) are zero. Suppose that \( i < j \), then

\[
[AB]_{ij} = \sum_{k=1}^{n} [A]_{ik} [B]_{kj} \\
= \sum_{k=1}^{j-1} [A]_{ik} [B]_{kj} + \sum_{k=j}^{n} [A]_{ik} [B]_{kj} \\
= \sum_{k=1}^{j-1} [A]_{ik} + \sum_{k=j}^{n} [A]_{ik} [B]_{kj} \\
= \sum_{k=1}^{j-1} [A]_{ik} + \sum_{k=j}^{n} 0 [B]_{kj} \\
= \sum_{k=1}^{j-1} [A]_{ik} + \sum_{k=j}^{n} 0 [B]_{kj}
\]

Theorem EMP \[177\]
Property AACN \[612\]
Definition LTM \[540\]
\[
\sum_{k=1}^{j-1} 0 + \sum_{k=j}^{n} 0 = 0
\]

Since \([AB]_{ij} = 0\) whenever \(i < j\), by Definition LTM 540, \(AB\) is lower triangular.

The inverse of a triangular matrix is triangular, of the same type.

**Theorem ITMT**

**Inverse of a Triangular Matrix is Triangular**

Suppose that \(A\) is a nonsingular matrix of size \(n\) that is triangular. Then the inverse of \(A\), \(A^{-1}\), is triangular of the same type. Furthermore, the diagonal entries of \(A^{-1}\) are the reciprocals of the corresponding diagonal entries of \(A\). More precisely, \([A^{-1}]_{ii} = [A]_{ii}^{-1}\). □

**Proof** We give the proof for the case when \(A\) is lower triangular, and leave the case when \(A\) is upper triangular for you. Consider the process for computing the inverse of a matrix that is outlined in the proof of Theorem CINM 193. We augment \(A\) with the size \(n\) identity matrix, \(I_n\), and row-reduce the \(n \times 2n\) matrix to reduced row-echelon form via the algorithm in Theorem REMEF 27. The proof involves tracking the peculiarities of this process in the case of a lower triangular matrix. Let \(M = [A \mid I_n]\).

First, none of the diagonal elements of \(A\) are zero. By repeated expansion about the first row, the determinant of a lower triangular matrix can be seen to be the product of the diagonal entries (Theorem DER 339). If just one of these diagonal elements was zero, then the determinant of \(A\) is zero and \(A\) is singular by Theorem SMZD 351. Slightly violating the exact algorithm for row reduction we can form a matrix, \(M'\), that is row-equivalent to \(M\), by multiplying row \(i\) by the nonzero scalar \([A]_{ii}^{-1}\), for \(1 \leq i \leq n\). This sets \([M']_{ii} = 1\) and \([M']_{i,n+1} = [A]_{ii}^{-1}\), and leaves every zero entry of \(M\) unchanged.

Let \(M_j\) denote the matrix obtained from \(M'\) after converting column \(j\) to a pivot column. We can convert column \(j\) of \(M_{j-1}\) into a pivot column with a set of \(n - j - 1\) row operations of the form \(\alpha R_j + R_k\) with \(j + 1 \leq k \leq n\). The key observation here is that we add multiples of row \(j\) only to higher-numbered rows. This means that none of the entries in rows 1 through \(j - 1\) is changed, and since row \(j\) has zeros in columns \(j + 1\) through \(n\), none of the entries in rows \(j + 1\) through \(n\) is changed in columns \(j + 1\) through \(n\). The first \(n\) columns of \(M'\) form a lower triangular matrix with 1’s on the diagonal. In its conversion to the identity matrix through this sequence of row operations, it remains lower triangular with 1’s on the diagonal.

What happens in columns \(n + 1\) through \(2n\) of \(M'\)? These columns began in \(M\) as the identity matrix, and in \(M'\) each diagonal entry was scaled to a reciprocal of the corresponding diagonal entry of \(A\). Notice that trivially, these final \(n\) columns of \(M'\) form a lower triangular matrix. Just as we argued for the first \(n\) columns, the row operations that convert \(M_{j-1}\) into \(M_j\) will preserve the lower triangular form in the final \(n\) columns and preserve the exact values of the diagonal entries. By Theorem CINM 193, the final \(n\) columns of \(M_n\) is the inverse of \(A\), and this matrix has the necessary properties advertised in the conclusion of this theorem.

**Subsection OD.UTMR**

**Upper Triangular Matrix Representation**

Not every matrix is diagonalizable, but every linear transformation has a matrix representation that is an upper triangular matrix, and the basis that achieves this representation is especially pleasing. Here’s the theorem.

**Theorem UTMR**

**Upper Triangular Matrix Representation**

Suppose that \(T: V \rightarrow V\) is a linear transformation. Then there is a basis \(B\) for \(V\) such that the
matrix representation of $T$ relative to $B$, $M^T_{B,B}$, is an upper triangular matrix. Each diagonal entry is an eigenvalue of $T$, and if $\lambda$ is an eigenvalue of $T$, then $\lambda$ occurs $\alpha_T(\lambda)$ times on the diagonal. □

**Proof** We begin with a proof by induction (Technique I [626]) of the first statement in the conclusion of the theorem. We use induction on the dimension of $V$ to show that if $T: V \mapsto V$ is a linear transformation, then there is a basis $B$ for $V$ such that the matrix representation of $T$ relative to $B$, $M^T_{B,B}$, is an upper triangular matrix.

To start suppose that $\dim(V) = 1$. Choose any nonzero vector $v \in V$ and realize that $V = \langle \{v\} \rangle$. Then we can describe $T$ completely by $T(v) = \beta v$ for some $\beta \in \mathbb{C}$ (Theorem LTDB [413]). This description of $T$ also gives us a matrix representation relative to the basis $B = \{v\}$ as the $1 \times 1$ matrix with lone entry equal to $\beta$. And this matrix representation is upper triangular (Definition UTM [540]).

For the induction step let $\dim(V) = m$, and assume the theorem is true for every linear transformation defined on a vector space of dimension less than $m$. By Theorem EMHE [359] (suitably converted to the setting of a linear transformation), $T$ has at least one eigenvalue, and we denote this eigenvalue as $\lambda$. (We will remark later about how critical this step is.) We now consider properties of the linear transformation $T - \lambda I_V: V \mapsto V$.

Let $x$ be an eigenvector of $T$ for $\lambda$. By definition $x \neq 0$. Then

\[
(T - \lambda I_V)(x) = T(x) - \lambda I_V(x) = T(x) - \lambda x = \lambda x - \lambda x = 0
\]

So $T - \lambda I_V$ is not injective, as it has a nontrivial kernel (Theorem KILT [432]). With an application of Theorem RPNDD [464] we bound the rank of $T - \lambda I_V$,

\[
r(T - \lambda I_V) = \dim(V) - n(T - \lambda I_V) \leq m - 1
\]

Define $W$ to be the subspace of $V$ that is the range of $T - \lambda I_V$, $W = R(T - \lambda I_V)$. We define a new linear transformation $S$, on $W$,

\[
S: W \mapsto W
\]

\[
S(w) = T(w)
\]

This does not look we have accomplished much, since the action of $S$ is identical to the action of $T$. For our purposes this will be a good thing. What is different is the domain and codomain. $S$ is defined on $W$, a vector space with dimension less than $m$, and so is susceptible to our induction hypothesis. Verifying that $S$ is really a linear transformation is almost entirely routine, with one exception. Employing $T$ in our definition of $S$ raises the possibility that the outputs of $S$ will not be contained within $W$ (but instead will lie inside $V$, but outside $W$). To examine this possibility, suppose that $w \in W$.

\[
S(w) = T(w) = T(w) + 0 = T(w) + (\lambda I_V(w) - \lambda I_V(w)) = (T(w) - \lambda I_V(w)) + \lambda I_V(w) = (T(w) - \lambda I_V(w)) + \lambda w = (T - \lambda I_V)(w) + \lambda w
\]

Since $W$ is the range of $T - \lambda I_V$, $(T - \lambda I_V)(w) \in W$. And by Property AC [251], $\lambda w \in W$. Finally, applying Property SC [251], we see by closure that the sum is in $W$ and so we conclude that $S(w) \in W$. This argument convinces us that it is legitimate to define $S$ as we did with $W$ as the codomain.

$S$ is a linear transformation defined on a vector space with dimension less than $m$, so we can apply the induction hypothesis and conclude that $W$ has a basis, $C = \{w_1, w_2, w_3, \ldots, w_k\}$, such that the matrix representation of $S$ relative to $C$ is an upper triangular matrix.
By Theorem DSFOS [327] there exists a second subspace of $V$, which we will call $U$, so that $V$ is a direct sum of $W$ and $U$, $V = W \oplus U$. Choose a basis $D = \{u_1, u_2, u_3, \ldots, u_k\}$ for $U$. So $m = k + \ell$ by Theorem DSD [329], and $B = C \cup D$ is basis for $V$ by Theorem DSLI [328] and Theorem G [320]. $B$ is the basis we desire. What does a matrix representation of $T$ look like, relative to $B$?

Since the definition of $T$ and $S$ agree on $W$, the first $k$ columns of $M_{B,B}^T$ will have the upper triangular matrix representation of $S$ in the first $k$ rows. The remaining $\ell = m - k$ rows of these first $k$ columns will be all zeros since the outputs of $T$ on $C$ are all contained in $W$. The situation for $T$ on $D$ is not quite as pretty, but it is close.

For $1 \leq i \leq \ell$, consider

$$
\rho_B(T(u_i)) = \rho_B(T(u_i) + 0) = \rho_B(T(u_i) + (\lambda I_V(u_i) - \lambda I_V(u_i))) = \rho_B((T(u_i) - \lambda I_V(u_i)) + \lambda I_V(u_i)) = \rho_B((T - \lambda I)(u_i) + \lambda I(u_i)) = \rho_B(a_1w_1 + a_2w_2 + a_3w_3 + \cdots + a_kw_k + \lambda u_i)
$$

In the penultimate step of this proof, we have rewritten an element of the range of $T - \lambda I_V$ as a linear combination of the basis vectors, $C$, for the range of $T - \lambda I_V$, $W$, using the scalars $a_1, a_2, a_3, \ldots, a_k$. If we incorporate these $\ell$ column vectors into the matrix representation $M_{B,B}^T$ we find $\ell$ occurences of $\lambda$ on the diagonal, and any nonzero entries lying only in the first $k$ rows. Together with the $k \times k$ upper triangular representation in the upper left-hand corner, the entire matrix representation is now clearly upper triangular. This completes the induction step, so for any linear transformation there is a basis that creates an upper triangular matrix representation.

We have one more statement in the conclusion of the theorem to verify. The eigenvalues of $T$, and their multiplicities, can be computed with the techniques of Chapter E [356] relative to any matrix representation (Theorem EER [527]). We take this approach with our upper triangular matrix representation $M_{B,B}^T$. Let $d_i$ be the diagonal entry of $M_{B,B}^T$ in row $i$ and column $i$. Then the characteristic polynomial, computed as a determinant (Definition CP [363]) with repeated expansions about the first column, is

$$
p_{M_{B,B}^T}(x) = (d_1 - x)(d_2 - x)(d_3 - x)\cdots(d_m - x)
$$

The roots of the polynomial equation $p_{M_{B,B}^T}(x) = 0$ are the eigenvalues of the linear transformation (Theorem EMRCP [363]). So each diagonal entry is an eigenvalue, and is repeated on the diagonal exactly $\alpha_T(\lambda)$ times (Definition AME [366]).

A key step in this proof was the construction of the subspace $W$ with dimension strictly less than that of $V$. This required an eigenvalue/eigenvector pair, which was guaranteed to us by Theorem EMHE [359]. Digging deeper, the proof of Theorem EMHE [359] requires that we can factor polynomials completely, into linear factors. This will not always happen if our set of scalars
is the reals, \( \mathbb{R} \). So this is our final explanation of our choice of the complex numbers, \( \mathbb{C} \), as our set of scalars. In \( \mathbb{C} \) polynomials factor completely, so every matrix has at least one eigenvalue, and an inductive argument will get us to upper triangular matrix representations.

In the case of linear transformations defined on \( \mathbb{C}^m \), we can use the inner product (Definition IP \[152\]) profitably to fine-tune the basis that yields an upper triangular matrix representation. Recall that the adjoint of matrix \( A \) (Definition A \[207\]) is written as \( A^* \).

**Theorem OBUTR**

**Orthonormal Basis for Upper Triangular Representation**

Suppose that \( A \) is a square matrix. Then there is a unitary matrix \( U \), and an upper triangular matrix \( T \), such that

\[ U^* A U = T \]

and \( T \) has the eigenvalues of \( A \) as the entries of the diagonal. □

**Proof** This theorem is a statement about matrices and similarity. We can convert it to a statement about linear transformations, matrix representations and bases (Theorem SCB \[524\]). Suppose that \( A \) is an \( n \times n \) matrix, and define the linear transformation \( S: \mathbb{C}^n \to \mathbb{C}^n \) by \( S(x) = Ax \). Then Theorem UTMR \[582\] gives us a basis \( B = \{v_1, v_2, v_3, \ldots, v_n\} \) for \( \mathbb{C}^n \) such that a matrix representation of \( S \) relative to \( B \), \( M_{S,B,B} \), is upper triangular.

Now convert the basis \( B \) into an orthogonal basis, \( C \), by an application of the Gram-Schmidt procedure (Theorem GSPCV \[158\]). This is a messy business computationally, but here we have an excellent illustration of the power of the Gram-Schmidt procedure. We need only be sure that \( B \) is linearly independent and spans \( \mathbb{C}^n \), and then we know that \( C \) is linearly independent, spans \( \mathbb{C}^n \) and is also an orthogonal set. We will now consider the matrix representation of \( S \) relative to \( C \) (rather than \( B \)). Write the new basis as \( C = \{y_1, y_2, y_3, \ldots, y_n\} \). The application of the Gram-Schmidt procedure creates each vector of \( C \), say \( y_j \), as the difference of \( v_j \) and a linear combination of \( y_1, y_2, y_3, \ldots, y_{j-1} \). We are not concerned here with the actual values of the scalars in this linear combination, so we will write

\[ y_j = v_j - \sum_{k=1}^{j-1} b_{jk} y_k \]

where the \( b_{jk} \) are shorthand for the scalars. The equation above is in a form useful for creating the basis \( C \) from \( B \). To better understand the relationship between \( B \) and \( C \) convert it to read

\[ v_j = y_j + \sum_{k=1}^{j-1} b_{jk} y_k \]

In this form, we recognize that the change-of-basis matrix \( C_{B,C} = M_{I_{\mathbb{C}^n}}^{S_{B,C}} \) (Definition CBM \[516\]) is an upper triangular matrix. By Theorem SCB \[524\] we have

\[ M_{C,C}^S = C_{B,C} M_{S,B,B} C_{B,C}^{-1} \]

The inverse of an upper triangular matrix is upper triangular (Theorem ITMT \[541\]), and the product of two upper triangular matrices is again upper triangular (Theorem PTMT \[540\]). So \( M_{C,C}^S \) is an upper triangular matrix.

Now, multiply each vector of \( C \) by a nonzero scalar, so that the result has norm 1. In this way we create a new basis \( D \) which is an orthonormal set (Definition ONS \[160\]). Note that the change-of-basis matrix \( C_{C,D} \) is a diagonal matrix with nonzero entries equal to the norms of the vectors in \( C \).

Now we can convert our results into the language of matrices. Let \( E \) be the basis of \( \mathbb{C}^n \) formed with the standard unit vectors (Definition SUV \[190\]). Then the matrix representation of \( S \) relative to \( E \) is simply \( A, A = M_{E,E}^S \). The change-of-basis matrix \( C_{D,E} \) has columns that are simply the
vectors in \( D \), the orthonormal basis. As such, \( \text{Theorem CUMOS} \) tells us that \( C_{D,E} \) is a unitary matrix, and by \( \text{Definition UM} \) has an inverse equal to its adjoint. Write \( U = C_{D,E} \). We have

\[
U^*AU = U^{-1}AU
\]

Theorem UMI

\[
= C_{D,E}^{-1}M_{E,E}^S C_{D,E}
\]

Theorem SCB

\[
= M_{D,D}^S
\]

Theorem SCB

The inverse of a diagonal matrix is also a diagonal matrix, and so this final expression is the product of three upper triangular matrices, and so is again upper triangular (Theorem PTMT). Thus the desired upper triangular matrix, \( T \), is the matrix representation of \( S \) relative to the orthonormal basis \( D, M_{D,D}^S \).

Subsection NM
Normal Matrices

Normal matrices comprise a broad class of interesting matrices, many of which we have met already. But they are most interesting since they define exactly which matrices we can diagonalize via a unitary matrix. This is the upcoming \( \text{Theorem OD} \). Here’s the definition.

Definition NRML
Normal Matrix
The square matrix \( A \) is normal if \( A^*A = AA^* \).

So a normal matrix commutes with its adjoint. Part of the beauty of this definition is that it includes many other types of matrices. A diagonal matrix will commute with its adjoint, since the adjoint is again diagonal and the entries are just conjugates of the entries of the original diagonal matrix. A Hermitian (self-adjoint) matrix (Definition HM) will trivially commute with its adjoint, since the two matrices are the same. A real, symmetric matrix is Hermitian, so these matrices are also normal. A unitary matrix (Definition UM) has its adjoint as its inverse, and inverses commute (Theorem OSIS), so unitary matrices are normal. Another class of normal matrices is the skew-symmetric matrices. However, these broad descriptions still do not capture all of the normal matrices, as the next example shows.

Example ANM
A normal matrix
Let

\[
A = \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
\]

Then

\[
\begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} = \begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}
\]

so we see by \( \text{Definition NRML} \) that \( A \) is normal. However, \( A \) is not symmetric (hence, as a real matrix, not Hermetian), not unitary, and not skew-symmetric.

Subsection OD
Orthonormal Diagonalization

A diagonal matrix is very easy to work with in matrix multiplication (Example HPDM) and an orthonormal basis also has many advantages (Theorem COB). How about converting a
matrix to a diagonal matrix through a similarity transformation using a unitary matrix (i.e. build a diagonal matrix representation with an orthonormal matrix)? That’d be fantastic! When can we do this? We can always accomplish this feat when the matrix is normal, and normal matrices are the only ones that behave this way. Here’s the theorem.

**Theorem OD**

**Orthonormal Diagonalization**

Suppose that $A$ is a square matrix. Then there is a unitary matrix $U$ and a diagonal matrix $D$ such that $U^*AU = D$ if and only if $A$ is a normal matrix. □

**Proof**  (⇒) Suppose there is a unitary matrix $U$ that diagonalizes $A$, resulting in $D$, i.e. $U^*AU = D$. We check the normality of $A$,

\[
A^*A = I_nA^*IA_n
\]

\[
= UU^*A^*UU^*AUU^*
\]

\[
= UU^*A^*UDU^*
\]

\[
= UU^*A^*(U^*)^*DU^*
\]

\[
= U(U^*AU)^*DU^*
\]

\[
= UD^*DU^*
\]

\[
= U(U^*AU)^tDU^*
\]

\[
= UDD^*U^*
\]

\[
= UD(U^*AU)^*U^*
\]

\[
= UDU^*A^*(U^*)^*U^*
\]

\[
= UDU^*A^*UU^*
\]

\[
= UU^*AU^*A^*UU^*
\]

\[
= I_nAI_n,A^*I_n
\]

\[
= AA^*
\]

So by [Definition NRML 545], $A$ is a normal matrix.

(⇐) For the converse, suppose that $A$ is a normal matrix. Whether or not $A$ is normal, [Theorem OBUTR 544] provides a unitary matrix $U$ and an upper triangular matrix $T$ such that $U^*AU = T$. With the added condition that $A$ is normal, we will determine that the entries of $T$ above the diagonal must be all zero. Here we go. First we show that $T$ is a normal matrix.

\[
T^*T = (U^*AU)^*U^*AU
\]

\[
= U^*A^*(U^*)^*U^*AU
\]

\[
= U^*A^*UU^*AU
\]

\[
= U^*A^*I_nAU
\]

\[
= U^*A^*AU
\]

\[
= U^*AA^*U
\]

\[
= U^*AI_nA^*U
\]

\[
= U^*AUU^*A^*U
\]

\[
= U^*AUU^*A^*(U^*)^*
\]

\[
= U^*AU(U^*AU)^*
\]

\[
= TT^*
\]

So by [Definition NRML 545], $T$ is a normal matrix.
We can translate the normality of \( T \) into the statement \( TT^* - T^*T = 0 \). We now establish an equality we will use repeatedly. For \( 1 \leq i \leq n \),

\[
0 = [0]_{ii} = [TT^* - T^*T]_{ii} = [TT^*]_{ii} - [T^*T]_{ii} = \sum_{k=1}^{n} [T]_{ik} [T^*]_{ki} - \sum_{k=1}^{n} [T^*]_{ik} [T]_{ki} = \sum_{k=1}^{n} |[T]_{ik}|^2 - \sum_{k=1}^{i} |[T]_{ki}|^2 = \sum_{k=1}^{n} |[T]_{ik}|^2 - \sum_{k=1}^{i} |[T]_{ki}|^2
\]

Definition ZM 166

Definition NRML 545

Definition MA 163

Theorem EMP 177

Definition A 207

Definition UTM 540

Definition MCN 614

To conclude, we use the above equality repeatedly, beginning with \( i = 1 \), and discover, row by row, that the entries above the diagonal of \( T \) are all zero. The key observation is that a sum of squares can only equal zero when each term of the sum is zero. For \( i = 1 \) we have

\[
0 = \sum_{k=1}^{n} |[T]_{1k}|^2 - \sum_{k=1}^{1} |[T]_{k1}|^2 = \sum_{k=2}^{n} |[T]_{1k}|^2
\]

which forces the conclusions

\[
[T]_{12} = 0 \quad [T]_{13} = 0 \quad [T]_{14} = 0 \quad \cdots \quad [T]_{1n} = 0
\]

For \( i = 2 \) we use the same equality, but also incorporate the portion of the above conclusions that says \( [T]_{12} = 0 \),

\[
0 = \sum_{k=2}^{n} |[T]_{2k}|^2 - \sum_{k=1}^{2} |[T]_{k2}|^2 = \sum_{k=2}^{n} |[T]_{2k}|^2 - \sum_{k=2}^{2} |[T]_{k2}|^2 = \sum_{k=3}^{n} |[T]_{2k}|^2
\]

which forces the conclusions

\[
[T]_{23} = 0 \quad [T]_{24} = 0 \quad [T]_{25} = 0 \quad \cdots \quad [T]_{2n} = 0
\]

We can repeat this process for the subsequent values of \( i = 3, 4, 5, \ldots, n - 1 \). Notice that it is critical we do this in order, since we need to employ portions of each of the previous conclusions about rows having zero entries in order to successfully get the same conclusion for later rows. Eventually, we conclude that all of the nondiagonal entries of \( T \) are zero, so the extra assumption of normality forces \( T \) to be diagonal.

\[\square\]
Section NLT
Nilpotent Linear Transformations

Draft: This Section Complete, But Subject To Change

We have seen that some matrices are diagonalizable and some are not. Some authors refer to a non-diagonalizable matrix as defective, but we will study them carefully anyway. Examples of such matrices include Example EMMS4, Example HMEM5, and Example CEMS6. Each of these matrices has at least one eigenvalue with geometric multiplicity strictly less than its geometric multiplicity, and therefore Theorem DMFE tells us these matrices are not diagonalizable.

Given a square matrix $A$, it is likely similar to many, many other matrices. Of all these possibilities, which is the best? “Best” is a subjective term, but we might agree that a diagonal matrix is certainly a very nice choice. Unfortunately, as we have seen, this will not always be possible. What form of a matrix is “next-best”? Our goal, which will take us several sections to reach, is to show that every matrix is similar to a matrix that is “nearly-diagonal” (Section JCF). More precisely, every matrix is similar to a matrix with elements on the diagonal, and zeros and ones on the diagonal just above the main diagonal (the “super diagonal”), with zeros everywhere else. In the language of equivalence relations (see Theorem SER), we are determining a systematic representative for each equivalence class. Such a representative for a set of similar matrices is called a canonical form.

We have just discussed the determination of a canonical form as a question about matrices. However, we know that every square matrix creates a natural linear transformation (Theorem MBLT) and every linear transformation with identical domain and codomain has a square matrix representation for each choice of a basis, with a change of basis creating a similarity transformation (Theorem SCB). So we will state, and prove, theorems using the language of linear transformations on abstract vector spaces, while most of our examples will work with square matrices. You can, and should, mentally translate between the two settings frequently and easily.

Subsection NLT
Nilpotent Linear Transformations

We will discover that nilpotent linear transformations are the essential obstacle in a non-diagonalizable linear transformation. So we will study them carefully first, both as an object of inherent mathematical interest, but also as the object at the heart of the argument that leads to a pleasing canonical form for any linear transformation. Once we understand these linear transformations thoroughly, we will be able to easily analyze the structure of any linear transformation.

Definition NLT
Nilpotent Linear Transformation
Suppose that $T: V \rightarrow V$ is a linear transformation such that there is an integer $p > 0$ such that $T^p(v) = 0$ for every $v \in V$. The smallest $p$ for which this condition is met is called the index of $T$.

Of course, the linear transformation $T$ defined by $T(v) = 0$ will qualify as nilpotent of index 1. But are there others?

Example NM64
Nilpotent matrix, size 6, index 4
Recall that our definitions and theorems are being stated for linear transformations on abstract vector spaces, while our examples will work with square matrices (and use the same terms interchangeably). In this case, to demonstrate the existence of nontrivial nilpotent linear transformations, we
desire a matrix such that some power of the matrix is the zero matrix. Consider

\[ A = \begin{bmatrix}
-3 & 3 & -2 & 5 & 0 & -5 \\
-3 & 5 & -3 & 4 & 3 & -9 \\
-3 & 4 & -2 & 6 & -4 & -3 \\
-3 & 3 & -2 & 5 & 0 & -5 \\
-3 & 3 & -2 & 4 & 2 & -6 \\
-2 & 3 & -2 & 2 & 4 & -7
\end{bmatrix} \]

and compute powers of \( A \),

\[ A^2 = \begin{bmatrix}
1 & -2 & 1 & 0 & -3 & 4 \\
0 & -2 & 1 & 1 & -3 & 4 \\
3 & 0 & 0 & -3 & 0 & 0 \\
1 & -2 & 1 & 0 & -3 & 4 \\
0 & -2 & 1 & 1 & -3 & 4 \\
-1 & -2 & 1 & 2 & -3 & 4
\end{bmatrix} \]

\[ A^3 = \begin{bmatrix}
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0
\end{bmatrix} \]

\[ A^4 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \]

Thus we can say that \( A \) is nilpotent of index 4.

Because it will presage some upcoming theorems, we will record some extra information about the eigenvalues and eigenvectors of \( A \) here. \( A \) has just one eigenvalue, \( \lambda = 0 \), with algebraic multiplicity 6 and geometric multiplicity 2. The eigenspace for this eigenvalue is

\[ E_A(0) = \langle \begin{bmatrix}
2 \\
2 \\
5 \\
2 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
-1 \\
-1 \\
-5 \\
-1 \\
0 \\
1
\end{bmatrix} \rangle \]

If there were degrees of singularity, we might say this matrix was very singular, since zero is an eigenvalue with maximum algebraic multiplicity \( \text{Theorem SMZE [379], Theorem ME [384]} \). Notice too that \( A \) is “far” from being diagonalizable \( \text{Theorem DMFE [396]} \).

Another example.

**Example NM62**

**Nilpotent matrix, size 6, index 2**

Consider the matrix

\[ B = \begin{bmatrix}
-1 & 1 & -1 & 4 & -3 & -1 \\
1 & 1 & -1 & 2 & -3 & -1 \\
-9 & 10 & -5 & 9 & 5 & -15 \\
-1 & 1 & -1 & 4 & -3 & -1 \\
1 & -1 & 0 & 2 & -4 & 2 \\
4 & -3 & 1 & -1 & -5 & 5
\end{bmatrix} \]
and compute the second power of $B$,

$$B^2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

So $B$ is nilpotent of index 2. Again, the only eigenvalue of $B$ is zero, with algebraic multiplicity 6. The geometric multiplicity of the eigenvalue is 3, as seen in the eigenspace,

$$\mathcal{E}_B(0) = \left\langle \begin{bmatrix} 1 \\ 3 \\ 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ -7 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

Again, Theorem DMFE [396] tells us that $B$ is far from being diagonalizable.

On a first encounter with the definition of a nilpotent matrix, you might wonder if such a thing was possible at all. That a high power of a nonzero object could be zero is so very different from our experience with scalars that it seems very unnatural. Hopefully the two previous examples were somewhat surprising. But we have seen that matrix algebra does not always behave the way we expect (Example MMNC [177]), and we also now recognize matrix products not just as arithmetic, but as function composition (Theorem MRCLT [491]). We will now turn to some examples of nilpotent matrices which might be more transparent.

**Definition JB**

**Jordan Block**

Given the scalar $\lambda \in \mathbb{C}$, the Jordan block $J_n(\lambda)$ is the $n \times n$ matrix defined by

$$[J_n(\lambda)]_{ij} = \begin{cases} 
\lambda & i = j \\
1 & j = i + 1 \\
0 & \text{otherwise}
\end{cases}$$

(This definition contains Notation JB.)

**Example JB4**

**Jordan block, size 4**

A simple example of a Jordan block,

$$J_4(5) = \begin{bmatrix}
5 & 1 & 0 & 0 \\
0 & 5 & 1 & 0 \\
0 & 0 & 5 & 1 \\
0 & 0 & 0 & 5
\end{bmatrix}$$

We will return to general Jordan blocks later, but in this section we are just interested in Jordan blocks where $\lambda = 0$. Here’s an example of why we are specializing in these matrices now.

**Example NJB5**

**Nilpotent Jordan block, size 5**
Consider

\[
J_5(0) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and compute powers,

\[
(J_5(0))^2 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
(J_5(0))^3 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
(J_5(0))^4 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
(J_5(0))^5 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

So \(J_5(0)\) is nilpotent of index 5. As before, we record some information about the eigenvalues and eigenvectors of this matrix. The only eigenvalue is zero, with algebraic multiplicity 5, the maximum possible (Theorem ME [384]). The geometric multiplicity of this eigenvalue is just 1, the minimum possible (Theorem ME [384]), as seen in the eigenspace,

\[
E_{J_5(0)}(0) = \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

There should not be any real surprises in this example. We can watch the ones in the powers of \(J_5(0)\) slowly march off to the upper-right hand corner of the powers. In some vague way, the eigenvalues and eigenvectors of this matrix are equally extreme.

We can form combinations of Jordan blocks to build a variety of nilpotent matrices. Simply place Jordan blocks on the diagonal of a matrix with zeros everywhere else, to create a **block diagonal** matrix.

**Example NM83**

Nilpotent matrix, size 8, index 3
Consider the matrix

\[
C = \begin{bmatrix}
J_3(0) & 0 & 0 \\
0 & J_3(0) & 0 \\
0 & 0 & J_2(0)
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and compute powers,

\[
C^2 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
C^3 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

So \(C\) is nilpotent of index 3. You should notice how block diagonal matrices behave in products (much like diagonal matrices) and that it was the largest Jordan block that determined the index of this combination. All eight eigenvalues are zero, and each of the three Jordan blocks contributes one eigenvector to a basis for the eigenspace, resulting in zero having a geometric multiplicity of 3.

It would appear that nilpotent matrices only have zero as an eigenvalue, so the algebraic multiplicity will be the maximum possible. However, by creating block diagonal matrices with Jordan blocks on the diagonal you should be able to attain any desired geometric multiplicity for this lone eigenvalue. Likewise, the size of the largest Jordan block employed will determine the index of the matrix. So nilpotent matrices with various combinations of index and geometric multiplicities are easy to manufacture. The predictable properties of block diagonal matrices in matrix products and eigenvector computations, along with the next theorem, make this possible. You might find Example NJB5 [550] a useful companion to this proof.

**Theorem NJB**

**Nilpotent Jordan Blocks**

The Jordan block \(J_n(0)\) is nilpotent of index \(n\). □

**Proof** While not phrased as an if-then statement, the statement in the theorem is understood to mean that if we have a specific matrix \((J_n(0))\) then we need to establish it is nilpotent of a specified index. The first column of \(J_n(0)\) is the zero vector, and the remaining \(n-1\) columns are the standard unit vectors \(e_i\), \(1 \leq i \leq n-1\) (Definition SUV [190]), which are also the first \(n-1\) columns of the size \(n\) identity matrix \(I_n\). As shorthand, write \(J = J_n(0)\).

\[
J = [0 | e_1 | e_2 | e_3 | \ldots | e_{n-1}]
\]
We will use the definition of matrix multiplication (Definition MM [176]), together with a proof by induction (Technique I [626]), to study the powers of $J$. Our claim is that

$$J^k = [0 \mid 0 \mid \ldots \mid 0 \mid e_1 \mid e_2 \mid \ldots \mid e_{n-k}]$$

for $1 \leq k \leq n$. For the base case, $k = 1$, and the definition of $J^1 = J_n(0)$ establishes the claim. For the induction step, first note that $Je_1 = 0$ and $Je_i = e_{i-1}$ for $2 \leq i \leq n$. Then, assuming the claim is true for $k$, we examine the $k + 1$ case,

$$J^{k+1} = JJ^k = [0 \mid 0 \mid \ldots \mid 0 \mid e_1 \mid e_2 \mid \ldots \mid e_{n-k}]$$

This concludes the induction. So $J^k$ has a nonzero entry (a one) in row $n-k$ and column $n$, for $1 \leq k \leq n-1$, and is therefore a nonzero matrix. However, $J^n = [0 \mid 0 \mid \ldots \mid 0] = O$. By Definition NLT [548], $J$ is nilpotent of index $n$. ■

Subsection PNLT

Properties of Nilpotent Linear Transformations

In this subsection we collect some basic properties of nilpotent linear transformations. After studying the examples in the previous section, some of these will be no surprise.

**Theorem ENLT**

**Eigenvalues of Nilpotent Linear Transformations**

Suppose that $T: V \mapsto V$ is a linear transformation and $\lambda$ is an eigenvalue of $T$. Then $\lambda = 0$. □

**Proof** Let $x$ be an eigenvector of $T$ for the eigenvalue $\lambda$, and suppose that $T$ is nilpotent with index $p$. Then

$$0 = T^p (x) \quad \text{Definition NLT [548]}$$

$$\lambda^p x \quad \text{Theorem EOMP [380]}$$

Because $x$ is an eigenvector, it is nonzero, and therefore Theorem SMEZV [259] tells us that $\lambda^p = 0$ and so $\lambda = 0$. ■

Paraphrasing, all of the eigenvalues of a nilpotent linear transformation are zero. So in particular, the characteristic polynomial of a nilpotent linear transformation, $T$, on a vector space of dimension $n$, is simply $p_T(x) = x^n$.

The next theorem is not critical for what follows, but it will explain our interest in nilpotent linear transformations. More specifically, it is the first step in backing up the assertion that nilpotent linear transformations are the essential obstacle in a non-diagonalizable linear transformation. While it is not obvious from the statement of the theorem, it says that a nilpotent linear transformation is not diagonalizable, unless it is trivially so.

**Theorem DNLT**

**Diagonalizable Nilpotent Linear Transformations**

Suppose the linear transformation $T: V \mapsto V$ is nilpotent. Then $T$ is diagonalizable if and only if $T$ is the zero linear transformation. □

**Proof** We start with the easy direction. Let $n = \dim (V)$.

$(\Leftarrow)$ The linear transformation $Z: V \mapsto V$ defined by $Z(v) = 0$ for all $v \in V$ is nilpotent of index $p = 1$ and a matrix representation relative to any basis of $V$ is the $n \times n$ zero matrix,
Quite obviously, the zero matrix is a diagonal matrix (Definition DIM [393]) and hence $Z$ is diagonalizable (Definition DZM [393]).

($\Rightarrow$) Assume now that $T$ is diagonalizable, so $\gamma_T(\lambda) = \alpha_T(\lambda)$ for every eigenvalue $\lambda$ (Theorem DMFE [396]). By Theorem ENLT [553], $T$ has only one eigenvalue (zero), which therefore must have algebraic multiplicity $n$ (Theorem NEM [383]). So the geometric multiplicity of zero will be $n$ as well, $\gamma_T(0) = n$.

Let $B$ be a basis for the eigenspace $E_T(0)$. Then $B$ is a linearly independent subset of $V$ of size $n$, and by Theorem G [320] will be a basis for $V$. For any $x \in B$ we have

$$T(x) = 0x = 0$$

Definition EM [364]

Theorem ZSSM [258]

So $T$ is identically zero on a basis for $B$, and since the action of a linear transformation on a basis determines all of the values of the linear transformation (Theorem LTDB [413]), it is easy to see that $T(v) = 0$ for every $v \in V$. ■

So, other than one trivial case (the zero matrix), every nilpotent linear transformation is not diagonalizable. It remains to see what is so “essential” about this broad class of non-diagonalizable linear transformations. For this we now turn to a discussion of kernels of powers of nilpotent linear transformations, beginning with a result about general linear transformations that may not necessarily be nilpotent.

**Theorem KPLT**

**Kernels of Powers of Linear Transformations**

Suppose $T: V \rightarrow V$ is a linear transformation, where dim $(V) = n$. Then there is an integer $m$, $0 \leq m \leq n$, such that

$$\{0\} = \mathcal{K}(T^0) \subset \mathcal{K}(T^1) \subset \mathcal{K}(T^2) \subset \cdots \subset \mathcal{K}(T^m) = \mathcal{K}(T^{m+1}) = \mathcal{K}(T^{m+2}) = \cdots$$

**Proof**  There are several items to verify in the conclusion as stated. First, we show that $\mathcal{K}(T^k) \subset \mathcal{K}(T^{k+1})$ for any $k$. Choose $z \in \mathcal{K}(T^k)$. Then

$$T^{k+1}(z) = T \left( T^k(z) \right)$$

Definition LTC [419]

$$= T(0)$$

Definition KLT [429]

$$= 0$$

Theorem LTTZ [408]

So by Definition KLT [429], $z \in \mathcal{K}(T^{k+1})$ and by Definition SSET [615] we have $\mathcal{K}(T^k) \subset \mathcal{K}(T^{k+1})$.

Second, we demonstrate the existence of a power $m$ where consecutive powers result in equal kernels. A by-product will be the condition that $m$ can be chosen so that $m \leq n$. To the contrary, suppose that

$$\{0\} = \mathcal{K}(T^0) \subset \mathcal{K}(T^1) \subset \mathcal{K}(T^2) \subset \cdots \subset \mathcal{K}(T^{n-1}) \subset \mathcal{K}(T^n) \subset \mathcal{K}(T^{n+1}) \subset \cdots$$

Since $\mathcal{K}(T^k) \subset \mathcal{K}(T^{k+1})$, Theorem PSSD [323] implies that dim $(\mathcal{K}(T^{k+1})) \geq$ dim $(\mathcal{K}(T^{k})) + 1$. Repeated application of this observation yields

$$\text{dim } (\mathcal{K}(T^{n+1})) \geq \text{dim } (\mathcal{K}(T^n)) + 1$$

$$\geq \text{dim } (\mathcal{K}(T^{n-1})) + 2$$

$$\vdots$$

$$\geq \text{dim } (\mathcal{K}(T^0)) + (n + 1)$$

$$= \text{dim } (\{0\}) + n + 1$$

$$= n + 1$$
Thus, $\mathcal{K}(T^{n+1})$ has a basis of size at least $n + 1$, which is a linearly independent set of size greater than $n$ in the vector space $V$ of dimension $n$. This contradicts \textbf{Theorem G} [320].

This contradiction yields the existence of an integer $k$ such that $\mathcal{K}(T^k) \neq \mathcal{K}(T^{k+1})$, so we can define $m$ to be smallest such integer with this property. From the argument above about dimensions resulting from a strictly increasing chain of subspaces, it should be clear that $m \leq n$.

It remains to show that once two consecutive kernels are equal, then all of the remaining kernels are equal. More formally, if $\mathcal{K}(T^m) = \mathcal{K}(T^{m+1})$, then $\mathcal{K}(T^m) = \mathcal{K}(T^{m+j})$ for all $j \geq 1$. We will give a proof by induction on $j$ (Technique I [626]). The base case ($j = 1$) is precisely our defining property for $m$.

In the induction step, we assume that $\mathcal{K}(T^m) = \mathcal{K}(T^{m+j})$ and endeavor to show that $\mathcal{K}(T^m) = \mathcal{K}(T^{m+j+1})$. At the outset of this proof we established that $\mathcal{K}(T^m) \subseteq \mathcal{K}(T^{m+j+1})$. So Definition SE [616] requires only that we establish the subset inclusion in the opposite direction. To wit, choose $z \in \mathcal{K}(T^{m+j+1})$. Then

$$0 = T^{m+j+1}(z) = T^{m+j}(T(z)) = T^m(T(z)) = T^{m+1}(z) = T^m(z)$$

So by Definition KLT [429], $z \in \mathcal{K}(T^m)$ as desired. $lacksquare$

We now specialize \textbf{Theorem KPLT} [554] to the case of nilpotent linear transformations, which buys us just a bit more precision in the conclusion.

\textbf{Theorem KPNLT}

\textbf{Kernels of Powers of Nilpotent Linear Transformations}

Suppose $T$: $V \mapsto V$ is a nilpotent linear transformation with index $p$ and $\dim(V) = n$. Then $0 \leq p \leq n$ and

$$\{0\} = \mathcal{K}(T^0) \subseteq \mathcal{K}(T^1) \subseteq \mathcal{K}(T^2) \subseteq \cdots \subseteq \mathcal{K}(T^p) = \mathcal{K}(T^{p+1}) = \cdots = V$$

\textbf{Proof} Since $T^p = 0$ it follows that $T^{p+j} = 0$ for all $j \geq 0$ and thus $\mathcal{K}(T^{p+j}) = V$ for $j \geq 0$. So the value of $m$ guaranteed by \textbf{Theorem KPLT} [554] is at most $p$. The only remaining aspect of our conclusion that does not follow from \textbf{Theorem KPLT} [554] is that $m = p$. To see this we must show that $\mathcal{K}(T^k) \subseteq \mathcal{K}(T^{k+1})$ for $0 \leq k \leq p - 1$. If $\mathcal{K}(T^k) = \mathcal{K}(T^{k+1})$ for some $k < p$, then $\mathcal{K}(T^k) = \mathcal{K}(T^p) = V$. This implies that $T^k = 0$, violating the fact that $T$ has index $p$. So the smallest value of $m$ is indeed $p$, and we learn that $p < n$. $lacksquare$

The structure of the kernels of powers of nilpotent linear transformations will be crucial to what follows. But immediately we can see a practical benefit. Suppose we are confronted with the question of whether or not an $n \times n$ matrix, $A$, is nilpotent or not. If we don’t quickly find a low power that equals the zero matrix, when do we stop trying higher and higher powers? \textbf{Theorem KPNLT} [555] gives us the answer: if we don’t see a zero matrix by the time we finish computing $A^n$, then it is not going to ever happen. We’ll now take a look at one example of \textbf{Theorem KPNLT} [555] in action.

\textbf{Example KPNLT}

\textbf{Kernels of powers of a nilpotent linear transformation}

We will recycle the nilpotent matrix $A$ of index 4 from \textbf{Example NM64} [548]. We now know that would have only needed to look at the first 6 powers of $A$ if the matrix had not been nilpotent. We list bases for the null spaces of the powers of $A$. (Notice how we are using null spaces for matrices...
interchangeably with kernels of linear transformations, see Theorem KNSI \([495]\) for justification.

With the exception of some convenience scaling of the basis vectors in \(\mathcal{N}(A^2)\) these are exactly the basis vectors described in Theorem BNS \([128]\). We can see that the dimension of \(\mathcal{N}(A)\) equals the geometric multiplicity of the zero eigenvalue. Why is this not an accident? We can see the dimensions of the kernels consistently increasing, and we can see that \(\mathcal{N}(A^4) = \mathbb{C}^6\). But Theorem KPNLT \([553]\) says a little more. Each successive kernel should be a superset of the previous one. We ought to be able to begin with a basis of \(\mathcal{N}(A)\) and extend it to a basis of \(\mathcal{N}(A^2)\). Then we should be able to extend a basis of \(\mathcal{N}(A^2)\) into a basis of \(\mathcal{N}(A^3)\), all with repeated applications of Theorem ELIS \([320]\). Verify the following,

\[
\mathcal{N}(A) = \mathcal{N}(\begin{pmatrix}
-3 & 3 & -2 & 5 & 0 & -5 \\
-3 & 5 & -3 & 4 & 3 & -9 \\
-3 & 4 & -2 & 6 & -4 & -3 \\
-3 & 3 & -2 & 5 & 0 & -5 \\
-3 & 3 & -2 & 4 & 2 & -6 \\
-2 & 3 & -2 & 2 & 4 & -7
\end{pmatrix}) = \left\langle \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\rangle
\]

\[
\mathcal{N}(A^2) = \mathcal{N}(\begin{pmatrix}
1 & -2 & 1 & 0 & -3 & 4 \\
0 & -2 & 1 & 1 & -3 & 4 \\
3 & 0 & 0 & -3 & 0 & 0 \\
1 & -2 & 1 & 0 & -3 & 4 \\
0 & -2 & 1 & 1 & -3 & 4 \\
-1 & -2 & 1 & 2 & -3 & 4
\end{pmatrix}) = \left\langle \begin{pmatrix} 0 \\ 2 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle
\]

\[
\mathcal{N}(A^3) = \mathcal{N}(\begin{pmatrix}
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle
\]

\[
\mathcal{N}(A^4) = \mathcal{N}(\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle
\]
Subsection PD.DS

OK, here we go. as a direct sum. Now would be a good time to review the results on direct sums collected in V. K is a proper subspace of each lying in between K. Example, n = 0, n = n(T), and can be used in practice. As we begin, the basis vectors will not be in the proper order, so for n = n(T), so we can think of n as “how much bigger” K(T) is than K(Ti). In particular, Theorem KPNLT 555 implies that si > 0 for 1 ≤ i ≤ p.

Proof. We will explicitly construct the desired basis, so the proof is constructive (Technique C 621), and can be used in practice. As we begin, the basis vectors will not be in the proper order, but we will rearrange them at the end of the proof. For convenience, define n = n(T), so for example, n = 0, n = n(T), and = dim(V). Define si = ni - ni-1 for 1 ≤ i ≤ p, so we can think of si as “how much bigger” K(T) is than K(Ti). In particular, Theorem KPNLT 555 implies that si > 0 for 1 ≤ i ≤ p.

We are going to build a set of vectors zi, 1 ≤ i ≤ p, 1 ≤ j ≤ si. Each zi,j will be an element of K(Ti) and not an element of K(Tj-1). In total, we will obtain a linearly independent set of vectors that form a basis of V. We construct this set in pieces, starting at the “wrong” end. Our procedure will build a series of subspaces, Z, each lying in between K(Ti-1) and K(Ti), having bases zi,j, 1 ≤ j ≤ si, and which together equal V as a direct sum. Now would be a good time to review the results on direct sums collected in Subsection PD.DS 325. OK, we go here.

We build the subspace Zp first (this is what we meant by “starting at the wrong end”). K(Tp) is a proper subspace of K(Tp) = V (Theorem KPNLT 555). Theorem DSFOS 327 says that there

\[ \mathcal{N}(A^2) = \left\langle \begin{bmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \\ 5 & -5 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle \]

\[ \mathcal{N}(A^4) = \left\langle \begin{bmatrix} 2 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 5 & -5 & 0 & 0 \\ 2 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \right\rangle \]

Do not be concerned at the moment about how these bases were constructed since we are not describing the applications of Theorem ELIS 320 here. Do verify carefully for each alleged basis that, (1) it is a superset of the basis for the previous kernel, (2) the basis vectors really are members of the kernel of the right power of A, (3) the basis is a linearly independent set, (4) the size of the basis is equal to the size of the basis found previously for each kernel. With these verifications, Theorem G 320 will tell us that we have successfully demonstrated what Theorem KPNLT 555 guarantees.

Subsection CFNLT
Canonical Form for Nilpotent Linear Transformations

Our main purpose in this section is to find a basis so that a nilpotent linear transformation will have a pleasing, nearly-diagonal matrix representation. Of course, we will not have a definition for “pleasing,” nor for “nearly-diagonal.” But the short answer is that our preferred matrix representation will be built up from Jordan blocks, Jn(0). Here’s the theorem. You will find Example CFNLT 561 helpful as you study this proof, since it uses the same notation, and is large enough to (barely) illustrate the full generality of the theorem (see ).

Theorem CFNLT
Canonical Form for Nilpotent Linear Transformations

Suppose that T: V → V is a nilpotent linear transformation of index p. Then there is a basis for V so that the matrix representation, MTB, is block diagonal with each block being a Jordan block, Jn(0). The size of the largest block is the index p, and the total number of blocks is the nullity of T, n(T).

Proof. We will explicitly construct the desired basis, so the proof is constructive (Technique C 621), and can be used in practice. As we begin, the basis vectors will not be in the proper order, but we will rearrange them at the end of the proof. For convenience, define ni = n(Ti), so for example, n = 0, n = n(T), and = dim(V). Define si = ni - ni-1 for 1 ≤ i ≤ p, so we can think of si as “how much bigger” K(Ti) is than K(Ti-1). In particular, Theorem KPNLT 555 implies that si > 0 for 1 ≤ i ≤ p.

We are going to build a set of vectors zi,j, 1 ≤ i ≤ p, 1 ≤ j ≤ si. Each zi,j will be an element of K(Ti) and not an element of K(Ti-1). In total, we will obtain a linearly independent set of vectors that form a basis of V. We construct this set in pieces, starting at the “wrong” end. Our procedure will build a series of subspaces, Z, each lying in between K(Ti-1) and K(Ti), having bases zi,j, 1 ≤ j ≤ si, and which together equal V as a direct sum. Now would be a good time to review the results on direct sums collected in Subsection PD.DS 325. OK, we go here.

We build the subspace Zp first (this is what we meant by “starting at the wrong end”). K(Tp) is a proper subspace of K(Tp) = V (Theorem KPNLT 555). Theorem DSFOS 327 says that there
is a subspace of $V$ that will pair with the subspace $\mathcal{K}(T^{p-1})$ to form a direct sum of $V$. Call this subspace $Z_p$, and choose vectors $z_{p,j}, 1 \leq j \leq s_p$ as a basis of $Z_p$, which we will denote as $B_p$. Note that we have a fair amount of freedom in how to choose these first basis vectors. Several observations will be useful in the next step. First $V = \mathcal{K}(T^{p-1}) \oplus Z_p$. The basis $B_p = \{ z_{p,1}, z_{p,2}, z_{p,3}, \ldots, z_{p,s_p} \}$ is linearly independent. For $1 \leq j \leq s_t, z_{p,j} \in \mathcal{K}(T^p) = V$. Since the two subspaces of a direct sum have no nonzero vectors in common (Theorem DSZI [328]), for $1 \leq j \leq s_t, z_{p,j} \notin \mathcal{K}(T^{p-1})$. That was comparably easy.

If obtaining $Z_p$ was easy, getting $Z_{p-1}$ will be harder. We will repeat the next step $p - 1$ times, and so will do it carefully the first time. Eventually, $Z_{p-1}$ will have dimension $s_{p-1}$. However, the first $s_p$ vectors of a basis are straightforward. Define $z_{p-1,j} = T(z_{p,j}), 1 \leq j \leq s_p$. Notice that we have no choice in creating these vectors, they are a consequence of our choices for $z_{p,j}$. In retrospect (i.e. on a second reading of this proof), you will recognize this as the key step in realizing a matrix representation of a nilpotent linear transformation with Jordan blocks. We need to know that this set of vectors in linearly independent, so start with a relation of linear dependence (Definition RLD 280), and massage it,

$$0 = a_1 z_{p-1,1} + a_2 z_{p-1,2} + a_3 z_{p-1,3} + \cdots + a_{s_p} z_{p-1,s_p}$$

$$= a_1 T(z_{p,1}) + a_2 T(z_{p,2}) + a_3 T(z_{p,3}) + \cdots + a_{s_p} T(z_{p,s_p})$$

$$= T(a_1 z_{p,1} + a_2 z_{p,2} + a_3 z_{p,3} + \cdots + a_{s_p} z_{p,s_p})$$

Theorem LTLC 413

Define $x = a_1 z_{p,1} + a_2 z_{p,2} + a_3 z_{p,3} + \cdots + a_{s_p} z_{p,s_p}$. The statement just above means that $x \in \mathcal{K}(T) \subseteq \mathcal{K}(T^{p-1})$ (Definition RLT 429, Theorem KPNLT 555). As defined, $x$ is a linear combination of the basis vectors $B_p$, and therefore $x \in Z_p$. Thus $x \in \mathcal{K}(T^{p-1}) \cap Z_p$ (Definition SI 617). Because $V = \mathcal{K}(T^{p-1}) \oplus Z_p$, Theorem DSZI 328 tells us that $x = 0$. Now we recognize the definition of $x$ as a relation of linear dependence on the linearly independent set $B_p$, and therefore $a_1 = a_2 = \cdots = a_{s_p} = 0$ (Definition LI 280). This establishes the linear independence of $z_{p-1,j}, 1 \leq j \leq s_p$ (Definition LI 280).

We also need to know where the vectors $z_{p-1,j}, 1 \leq j \leq s_p$ live. First we demonstrate that they are members of $\mathcal{K}(T^{p-1})$,

$$T^{p-1}(z_{p-1,j}) = T^{p-1}(T(z_{p,j})) = T^p(z_{p,j}) = 0$$

So $z_{p-1,j} \in \mathcal{K}(T^{p-1}), 1 \leq j \leq s_p$. However, we now show that these vectors are not elements of $\mathcal{K}(T^{p-2})$. Suppose to the contrary (Technique CD 623) that $z_{p-1,j} \in \mathcal{K}(T^{p-2})$. Then

$$0 = T^{p-2}(z_{p-1,j}) = T^{p-2}(T(z_{p,j})) = T^{p-1}(z_{p,j})$$

which contradicts the earlier statement that $z_{p,j} \notin \mathcal{K}(T^{p-1})$. So $z_{p-1,j} \notin \mathcal{K}(T^{p-2}), 1 \leq j \leq s_p$.

Now choose a basis $C_{p-2} = \{ u_1, u_2, u_3, \ldots, u_{n_{p-2}} \}$ for $\mathcal{K}(T^{p-2})$. We want to extend this basis by adding in the $z_{p-1,j}$ to span a subspace of $\mathcal{K}(T^{p-1})$. But first we want to know that this set is linearly independent. Let $a_k, 1 \leq k \leq n_{p-2}$ and $b_j, 1 \leq j \leq s_p$ be the scalars in a relation of linear dependence,

$$0 = a_1 u_1 + a_2 u_2 + \cdots + a_{n_{p-2}} u_{n_{p-2}} + b_1 z_{p-1,1} + b_2 z_{p-1,2} + \cdots + b_{s_p} z_{p-1,s_p}$$

Then,

$$0 = T^{p-2}(0)$$
Define \( y = b_1z_{p,1} + b_2z_{p,2} + \cdots + b_s z_{p,s,p} \). The statement just above means that \( y \in K(T^{p-1}) \) (Definition KLT [429]). As defined, \( y \) is a linear combination of the basis vectors \( B_p \), and therefore \( y \in Z_p \). Thus \( y \in K(T^{p-1}) \cap Z_p \). Because \( V = K(T^{p-1}) \oplus Z_p \) (Theorem DSZI [328]) tells us that \( y = 0 \). Now we recognize the definition of \( y \) as a relation of linear dependence on the linearly independent set \( B_p \), and therefore \( b_1 = b_2 = \cdots = b_s = 0 \) (Definition LI [280]). Return to the full relation of linear dependence with both sets of scalars (the \( a_i \) and \( b_j \)). Now that we know that \( b_j = 0 \) for \( 1 \leq j \leq s_p \), this relation of linear dependence simplifies to a relation of linear dependence on just the basis \( C_{p-1} \). Therefore, \( a_i = 0 \), \( 1 \leq i \leq n_{p-1} \) and we have the desired linear independence.

Define a new subspace of \( K(T^{p-1}) \) as

\[ Q_{p-1} = \langle \{ u_1, u_2, u_3, \ldots, u_{n_{p-1}}, z_{p-1,1}, z_{p-1,2}, z_{p-1,3}, \ldots, z_{p-1,s_p} \} \rangle \]

By Theorem DSFOS [327] there exists a subspace of \( K(T^{p-1}) \) which will pair with \( Q_{p-1} \) to form a direct sum. Call this subspace \( R_{p-1} \), so by definition, \( K(T^{p-1}) = Q_{p-1} \oplus R_{p-1} \). We are interested in the dimension of \( R_{p-1} \). Note first, that since the spanning set of \( Q_{p-1} \) is linearly independent, \( \dim (Q_{p-1}) = n_{p-2} + s_p \). Then

\[
\dim (R_{p-1}) = \dim (K(T^{p-1})) - \dim (Q_{p-1}) \quad \text{Theorem DSD [329]}
\]

\[
= n_{p-1} - (n_{p-2} + s_p)
\]

\[
= (n_{p-1} - n_{p-2}) - s_p
\]

\[
= s_{p-1} - s_p
\]

Notice that if \( s_{p-1} = s_p \), then \( R_{p-1} \) is trivial. Now choose a basis of \( R_{p-1} \), and denote these \( s_{p-1} - s_p \) vectors as \( z_{p-1,s_p+1}, z_{p-1,s_p+2}, z_{p-1,s_p+3}, \ldots, z_{p-1,s_{p-1}} \). This is another occasion to notice that we have some freedom in this choice.

We now have \( K(T^{p-1}) = Q_{p-1} \oplus R_{p-1} \), and we have bases for each of the two subspaces. The union of these two bases will therefore be a linearly independent set in \( K(T^{p-1}) \) with size

\[
(n_{p-2} + s_p) + (s_{p-1} - s_p) = n_{p-2} + s_{p-1}
\]

\[
= n_{p-2} + n_{p-1} - n_{p-2}
\]

\[
= n_{p-1} = \dim (K(T^{p-1}))
\]

So, by Theorem G [320], the following set is a basis of \( K(T^{p-1}) \),

\[
\{ u_1, u_2, u_3, \ldots, u_{n_{p-2}}, z_{p-1,1}, z_{p-1,2}, \ldots, z_{p-1,s_p}, z_{p-1,s_p+1}, z_{p-1,s_p+2}, \ldots, z_{p-1,s_{p-1}} \}
\]

We built up this basis in three parts, we will now split it in half. Define the subspace \( Z_{p-1} \) by

\[
Z_{p-1} = \langle B_{p-1} \rangle = \langle \{ z_{p-1,1}, z_{p-1,2}, \ldots, z_{p-1,s_{p-1}} \} \rangle
\]

where we have implicitly denoted the basis as \( B_{p-1} \). Then Theorem DSFB [326] allows us to split up the basis for \( K(T^{p-1}) \) as \( C_{p-1} \cup B_{p-1} \) and write

\[
K(T^{p-1}) = K(T^{p-2}) \oplus Z_{p-1}
\]
Whew! This is a good place to recap what we have achieved. The vectors \( z_{i,j} \) form bases for the subspaces \( Z_i \) and right now

\[
V = K(T^{p-1}) \oplus Z_p = K(T^{p-2}) \oplus Z_{p-1} \oplus Z_p
\]

The key feature of this decomposition of \( V \) is that the first \( s_p \) vectors in the basis for \( Z_{p-1} \) are outputs of the linear transformation \( T \) using the basis vectors of \( Z_p \) as inputs.

Now we want to further decompose \( K(T^{p-2}) \) into \( K(T^{p-3}) \) and \( Z_{p-2} \). The procedure is the same as above, so we will only sketch the key steps. Checking the details proceeds in the same manner as above. Technically, we could have set up the preceding as the induction step in a proof by induction, but this probably would make the proof harder to understand.

Hit each element of \( B_{p-1} \) with \( T \), to create vectors \( z_{p-2,j}, 1 \leq j \leq s_{p-1} \). These vectors form a linearly independent set, and each is an element of \( K(T^{p-2}) \), but not an element of \( K(T^{p-3}) \). Grab a basis \( C_{p-3} \) of \( K(T^{p-3}) \) and tack on the newly-created vectors \( z_{p-2,j}, 1 \leq j \leq s_{p-1} \). This expanded set is linearly independent, and we can define a subspace \( Q_{p-2} \) using it as a basis. Theorem DSFOS gives us a subspace \( R_{p-2} \) such that \( K(T^{p-2}) = Q_{p-2} \oplus R_{p-2} \). Vectors \( z_{p-2,j}, s_{p-1} + 1 \leq j \leq s_{p-2} \) are chosen as a basis for \( R_{p-2} \) once the relevant dimensions have been verified. The union of \( C_{p-3} \) and \( z_{p-2,j}, 1 \leq j \leq s_{p-2} \) then form a basis of \( K(T^{p-2}) \), which can be split into two parts to yield the decomposition

\[
K(T^{p-2}) = K(T^{p-3}) \oplus Z_{p-2}
\]

Here \( Z_{p-2} \) is the subspace of \( K(T^{p-2}) \) with basis \( B_{p-2} = \{ z_{p-2,j} \mid 1 \leq j \leq s_{p-2} \} \). Finally,

\[
V = K(T^{p-1}) \oplus Z_p = K(T^{p-2}) \oplus Z_{p-1} \oplus Z_p = K(T^{p-3}) \oplus Z_{p-2} \oplus Z_{p-1} \oplus Z_p
\]

Again, the key feature of this decomposition is that the first vectors in the basis of \( Z_{p-2} \) are outputs of \( T \) using vectors from the basis \( Z_{p-1} \) as inputs (and in turn, some of these inputs are outputs of \( T \) derived from inputs in \( Z_p \)).

Now assume we repeat this procedure until we decompose \( K(T^2) \) into subspaces \( K(T) \) and \( Z_2 \). Finally, decompose \( K(T) \) into subspaces \( K(T^0) = K(I_n) = \{ 0 \} \) and \( Z_1 \), so that we recognize the vectors \( z_{1,j}, 1 \leq j \leq s_1 = n_1 \) as elements of \( K(T) \). The set

\[
B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_p = \{ z_{i,j} \mid 1 \leq i \leq p, 1 \leq j \leq s_i \}
\]

is linearly independent by Theorem DSLI and has size

\[
\sum_{i=1}^{p} s_i = \sum_{i=1}^{p} n_i - n_{i-1} = n_p - n_0 = \dim (V)
\]

So by Theorem G, \( B \) is a basis of \( V \). We desire a matrix representation of \( T \) relative to \( B \) (Definition MR), but first we will reorder the elements of \( B \). The following display lists the elements of \( B \) in the desired order, when read across the rows right-to-left in the usual way. Notice that we arrived at these vectors column-by-column, beginning on the right.

\[
\begin{align*}
z_{1,1} & \quad z_{2,1} & \quad z_{3,1} & \cdots & \quad z_{d,1} \\
z_{1,2} & \quad z_{2,2} & \quad z_{3,2} & \cdots & \quad z_{d,2} \\
\vdots & & \vdots & & \vdots \\
z_{1,s_d} & \quad z_{2,s_d} & \quad z_{3,s_d} & \cdots & \quad z_{d,s_d} \\
z_{1,s_d+1} & \quad z_{2,s_d+1} & \quad z_{3,s_d+1} & \cdots & \quad z_{d,s_d} \\
\vdots & & \vdots & & \vdots \\
z_{1,s_3} & \quad z_{2,s_3} & \quad z_{3,s_3} & & \\
\vdots & & \vdots & & 
\end{align*}
\]
It is difficult to layout this table with the notation we have been using, nor would it be especially useful to invent some notation to overcome the difficulty. (One approach would be to define something like the inverse of the nonincreasing function, \( i \rightarrow s_i \).) Do notice that there are \( s_1 = n_1 \) rows and \( d \) columns. Column \( i \) is the basis \( B_i \). The vectors in the first column are elements of \( \mathcal{K}(T) \). Each row is the same length, or shorter, than the one above it. If we apply \( T \) to any vector in the table, other than those in the first column, the output is the preceding vector in the row.

Now contemplate the matrix representation of \( T \) relative to \( B \) as we read across the rows of the table above. In the first row, \( T(z_{1,1}) = 0 \), so the first column of the representation is the zero column. Next, \( T(z_{2,1}) = z_{1,1} \), so the second column of the representation is a vector with a single one in the first entry, and zeros elsewhere. Next, \( T(z_{4,1}) = z_{2,1} \), so column 3 of the representation is a zero, then a one, then all zeros. Continuing in this vein, we obtain the first \( d \) columns of the representation, which is the Jordan block \( J_d(0) \) followed by rows of zeros.

When we apply \( T \) to the basis vectors of the second row, what happens? Applying \( T \) to the first vector, the result is the zero vector, so the representation gets a zero column. Applying \( T \) to the second vector in the row, the output is simply the first vector in that row, making the next column of the representation all zeros plus a lone one, sitting just above the diagonal. Continuing, we create a Jordan block, sitting on the diagonal of the matrix representation. It is not possible in general to state the size of this block, but since the second row is no longer than the first, it cannot have size larger than \( d \).

Since there are as many rows as the dimension of \( \mathcal{K}(T) \), the representation contains as many Jordan blocks as the nullity of \( T \), \( n(T) \). Each successive block is smaller than the preceding one, with the first, and largest, having size \( d \). The blocks are Jordan blocks since the basis vectors \( z_{i,j} \) were often defined as the result of applying \( T \) to other elements of the basis already determined, and then we rearranged the basis into an order that placed outputs of \( T \) just before their inputs, excepting the start of each row, which was an element of \( \mathcal{K}(T) \).

The proof of Theorem CFNLT \[557\] is constructive (Technique C \[621\]), so we can use it to create bases of nilpotent linear transformations with pleasing matrix representations. Recall that Theorem DNLT \[553\] told us that nilpotent linear transformations are almost never diagonalizable, so this is progress. As we have hinted before, with a nice representation of nilpotent matrices, it will not be difficult to build up representations of other non-diagonalizable matrices. Here is the promised example which illustrates the previous theorem. It is a useful companion to your study of the proof of Theorem CFNLT \[557\].

Example CFNLT
Canonical form for a nilpotent linear transformation
The \( 6 \times 6 \) matrix, \( A \), of Example NM64 \[548\] is nilpotent of index \( p = 4 \). If we define the linear transformation \( T : \mathbb{C}^6 \rightarrow \mathbb{C}^6 \) by \( T(x) = Ax \), then \( T \) is nilpotent of index 4 and we can seek a basis of \( \mathbb{C}^6 \) that yields a matrix representation with Jordan blocks on the diagonal. The nullity of \( T \) is 2, so from Theorem CFNLT \[557\] we can expect the largest Jordan block to be \( J_4(0) \), and there will be just two blocks. This only leaves enough room for the second block to have size 2.

We will recycle the bases for the null spaces of the powers of \( A \) from Example KPNLT \[555\] rather than recomputing them here. We will also use the same notation used in the proof of Theorem CFNLT \[557\].

To begin, \( s_4 = n_4 - n_3 = 6 - 5 = 1 \), so we need one vector of \( \mathcal{K}(T^4) = \mathbb{C}^6 \), that is not in \( \mathcal{K}(T^3) \), to be a basis for \( Z_4 \). We have a lot of latitude in this choice, and we have not described any sure-fire method for constructing a vector outside of a subspace. Looking at the basis for \( \mathcal{K}(T^3) \) we see that if a vector is in this subspace, and has a nonzero value in the first entry, then it must
also have a nonzero value in the fourth entry. So the vector
\[
\mathbf{z}_{4,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]
will not be an element of \( K(T^3) \) (notice that many other choices could be made here, so our basis will not be unique). This completes the determination of \( Z_p = Z_4 \).

Next, \( s_3 = n_3 - n_2 = 5 - 4 = 1 \), so we again need just a single basis vector for \( Z_3 \). We start by evaluating \( T \) with each basis vector of \( Z_4 \),
\[
\mathbf{z}_{3,1} = T(\mathbf{z}_{4,1}) = A\mathbf{z}_{4,1} = \begin{bmatrix} -3 \\ -3 \\ -3 \\ -3 \\ -3 \\ -2 \end{bmatrix}
\]
Since \( s_3 = s_4 \), the subspace \( R_3 \) is trivial, and there is nothing left to do, \( \mathbf{z}_{3,1} \) is the lone basis vector of \( Z_3 \).

Now \( s_2 = n_2 - n_1 = 4 - 2 = 2 \), so the construction of \( Z_2 \) will not be as simple as the construction of \( Z_3 \). We first apply \( T \) to the basis vector of \( Z_2 \),
\[
\mathbf{z}_{2,1} = T(\mathbf{z}_{3,1}) = A\mathbf{z}_{4,1} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \\ 0 \\ -1 \end{bmatrix}
\]
The two basis vectors of \( K(T^1) \), together with \( \mathbf{z}_{2,1} \), form a basis for \( Q_2 \). Because \( \dim(K(T^2)) - \dim(Q_2) = 4 - 3 = 1 \) we need only find a single basis vector for \( R_2 \). This vector must be an element of \( K(T^2) \), but not an element of \( Q_2 \). Again, there is a variety of vectors that fit this description, and we have no precise algorithm for finding them. Since they are plentiful, they are not too hard to find. We add up the four basis vectors of \( K(T^2) \), ensuring an element of \( K(T^2) \). Then we check to see if the vector is a linear combination of three vectors: the two basis vectors of \( K(T^1) \) and \( \mathbf{z}_{2,1} \). Having passed the tests, we have chosen
\[
\mathbf{z}_{2,2} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}
\]
Thus, \( Z_2 = \langle \{ \mathbf{z}_{2,1}, \mathbf{z}_{2,2} \} \rangle \).

Lastly, \( s_1 = n_1 - n_0 = 2 - 0 = 2 \). Since \( s_2 = s_1 \), we again have a trivial \( R_1 \) and need only complete our basis by evaluating the basis vectors of \( Z_2 \) with \( T \),
\[
\mathbf{z}_{1,1} = T(\mathbf{z}_{2,1}) = A\mathbf{z}_{2,1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}
\]
\[ z_{1,2} = T(z_{2,2}) = A z_{2,2} = \begin{bmatrix} -2 \\ -2 \\ -5 \\ -2 \\ -1 \\ 0 \end{bmatrix} \]

Now we reorder these vectors as the desired basis,

\[ B = \{z_{1,1}, z_{2,1}, z_{2,2}, z_{3,1}, z_{4,1}, z_{1,2}, z_{2,2} \} \]

We now apply [Definition MR 485](#) to build a matrix representation of \( T \) relative to \( B \),

\[ \rho_B(T(z_{1,1})) = \rho_B(A z_{1,1}) = \rho_B(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \rho_B(T(z_{2,1})) = \rho_B(A z_{2,1}) = \rho_B(z_{1,1}) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \rho_B(T(z_{3,1})) = \rho_B(A z_{3,1}) = \rho_B(z_{2,1}) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \rho_B(T(z_{4,1})) = \rho_B(A z_{4,1}) = \rho_B(z_{3,1}) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \rho_B(T(z_{1,2})) = \rho_B(A z_{1,2}) = \rho_B(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \rho_B(T(z_{2,2})) = \rho_B(A z_{2,2}) = \rho_B(z_{1,2}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \]
Installing these vectors as the columns of the matrix representation we have

\[
M_{B,B}^T = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

which is a block diagonal matrix with Jordan blocks \(J_4(0)\) and \(J_2(0)\). If we constructed the matrix \(S\) having the vectors of \(B\) as columns, then Theorem SCB \[524\] tells us that a similarity transformation with \(S\) relates the original matrix representation of \(T\) with the matrix representation consisting of Jordan blocks, i.e. \(S^{-1}AS = M_{B,B}^T\).

Notice that constructing interesting examples of matrix representations requires domains with dimensions bigger than just two or three. Going forward we will see several more big examples.
We have seen in Section NLT that nilpotent linear transformations are almost never diagonalizable (Theorem DNLT), yet have matrix representations that are very nearly diagonal (Theorem CFNLT). Our goal in this section, and the next (Section JCF), is to obtain a matrix representation of any linear transformation that is very nearly diagonal. A key step in reaching this goal is an understanding of invariant subspaces, and a particular type of invariant subspace that contains vectors known as “generalized eigenvectors.”

As is often the case, we start with a definition.

**Definition IS**
**Invariant Subspace**
Suppose that $T: V \rightarrow V$ is a linear transformation and $W$ is a subspace of $V$. Suppose further that $T(w) \in W$ for every $w \in W$. Then $W$ is an **invariant subspace** of $V$ relative to $T$.

We do not have any special notation for an invariant subspace, so it is important to recognize that an invariant subspace is always relative to both a superspace ($V$) and a linear transformation ($T$), which will sometimes not be mentioned, yet will be clear from the context. Note also that the linear transformation involved must have an equal domain and codomain — the definition would not make much sense if our outputs were not of the same type as our inputs.

As usual, we begin with an example that demonstrates the existence of invariant subspaces. We will return later to understand how this example was constructed, but for now, just understand how we check the existence of the invariant subspaces.

**Example TIS**
**Two invariant subspaces**
Consider the linear transformation $T: \mathbb{C}^4 \rightarrow \mathbb{C}^4$ defined by $T(x) = Ax$ where $A$ is given by

$$A = \begin{bmatrix} -8 & 6 & -15 & 9 \\ -8 & 14 & -10 & 18 \\ 1 & 1 & 3 & 0 \\ 3 & -8 & 2 & -11 \end{bmatrix}$$

Define (with zero motivation),

$$w_1 = \begin{bmatrix} -7 \\ -2 \\ 3 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

and set $W = \langle\{w_1, w_2\}\rangle$. We verify that $W$ is an invariant subspace of $\mathbb{C}^4$ with respect to $T$. By the definition of $W$, any vector chosen from $W$ can be written as a linear combination of $w_1$ and $w_2$. Suppose that $w \in W$, and then check the details of the following verification,

$$T(w) = T(a_1w_1 + a_2w_2) = a_1T(w_1) + a_2T(w_2)$$

**Definition SS**
**Invariant Subspace**
Suppose that $T: V \rightarrow V$ is a linear transformation and $W$ is a subspace of $V$. Suppose further that $T(w) \in W$ for every $w \in W$. Then $W$ is an **invariant subspace** of $V$ relative to $T$.

We do not have any special notation for an invariant subspace, so it is important to recognize that an invariant subspace is always relative to both a superspace ($V$) and a linear transformation ($T$), which will sometimes not be mentioned, yet will be clear from the context. Note also that the linear transformation involved must have an equal domain and codomain — the definition would not make much sense if our outputs were not of the same type as our inputs.

As usual, we begin with an example that demonstrates the existence of invariant subspaces. We will return later to understand how this example was constructed, but for now, just understand how we check the existence of the invariant subspaces.

**Example TIS**
**Two invariant subspaces**
Consider the linear transformation $T: \mathbb{C}^4 \rightarrow \mathbb{C}^4$ defined by $T(x) = Ax$ where $A$ is given by

$$A = \begin{bmatrix} -8 & 6 & -15 & 9 \\ -8 & 14 & -10 & 18 \\ 1 & 1 & 3 & 0 \\ 3 & -8 & 2 & -11 \end{bmatrix}$$

Define (with zero motivation),

$$w_1 = \begin{bmatrix} -7 \\ -2 \\ 3 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

and set $W = \langle\{w_1, w_2\}\rangle$. We verify that $W$ is an invariant subspace of $\mathbb{C}^4$ with respect to $T$. By the definition of $W$, any vector chosen from $W$ can be written as a linear combination of $w_1$ and $w_2$. Suppose that $w \in W$, and then check the details of the following verification,

$$T(w) = T(a_1w_1 + a_2w_2) = a_1T(w_1) + a_2T(w_2)$$

**Definition SS**
**Invariant Subspace**
Suppose that $T: V \rightarrow V$ is a linear transformation and $W$ is a subspace of $V$. Suppose further that $T(w) \in W$ for every $w \in W$. Then $W$ is an **invariant subspace** of $V$ relative to $T$.

We do not have any special notation for an invariant subspace, so it is important to recognize that an invariant subspace is always relative to both a superspace ($V$) and a linear transformation ($T$), which will sometimes not be mentioned, yet will be clear from the context. Note also that the linear transformation involved must have an equal domain and codomain — the definition would not make much sense if our outputs were not of the same type as our inputs.

As usual, we begin with an example that demonstrates the existence of invariant subspaces. We will return later to understand how this example was constructed, but for now, just understand how we check the existence of the invariant subspaces.
Subsection IS.IS  Invariant Subspaces 566

\[
\begin{bmatrix}
-1 \\
-2 \\
0 \\
1
\end{bmatrix} + a_2
\begin{bmatrix}
5 \\
-2 \\
-3 \\
2
\end{bmatrix}
= a_1 w_2 + a_2 ((-1)w_1 + 2w_2)
= (-a_2)w_1 + (a_1 + 2a_2)w_2
\in W
\]

Definition SS 270

So, by Definition IS 565, \( W \) is an invariant subspace of \( \mathbb{C}^4 \) relative to \( T \). In an entirely similar manner we construct another invariant subspace of \( T \).

With zero motivation, define

\[
x_1 = \begin{bmatrix}
-3 \\
-1 \\
1 \\
0
\end{bmatrix} \\
x_2 = \begin{bmatrix}
0 \\
-1 \\
0 \\
1
\end{bmatrix}
\]

and set \( X = \langle \{ x_1, x_2 \} \rangle \). We verify that \( X \) is an invariant subspace of \( \mathbb{C}^4 \) with respect to \( T \). By the definition of \( X \), any vector chosen from \( X \) can be written as a linear combination of \( x_1 \) and \( x_2 \). Suppose that \( x \in X \), and then check the details of the following verification,

\[
T(x) = T(b_1x_1 + b_2x_2)
= b_1T(x_1) + b_2T(x_2)
= b_1 \begin{bmatrix}
3 \\
0 \\
-1 \\
1
\end{bmatrix} + b_2 \begin{bmatrix}
3 \\
4 \\
-1 \\
-3
\end{bmatrix}
= b_1 ((-1)x_1 + x_2) + b_2 ((-1)x_1 + (-3)x_2)
= (-b_1 - b_2)x_1 + (b_1 - 3b_2)x_2
\in X
\]

Definition SS 270

So, by Definition IS 565, \( X \) is an invariant subspace of \( \mathbb{C}^4 \) relative to \( T \).

There is a bit of magic in each of these verifications where the two outputs of \( T \) happen to equal linear combinations of the two inputs. But this is the essential nature of an invariant subspace. We’ll have a peek under the hood later, and it won’t look so magical after all.

As a hint of things to come, verify that \( B = \{ w_1, w_2, x_1, x_2 \} \) is a basis of \( \mathbb{C}^4 \). Splitting this basis in half, Theorem DSFB 326, tells us that \( \mathbb{C}^4 = W \oplus X \). To see why a decomposition of a vector space into a direct sum of invariant subspaces might be interesting, construct the matrix representation of \( T \) relative to \( B, M_B^T \). Hmmmmmm.

Example TIS 565 is a bit mysterious at this stage. Do we know any other examples of invariant subspaces? Yes, as it turns out, we have already seen quite a few. We’ll give some examples now, and in more general situations, describe broad classes of invariant subspaces with theorems. First up is eigenspaces.

Theorem EIS

Eigenspaces are Invariant Subspaces

Suppose that \( T: V \rightarrow V \) is a linear transformation with eigenvalue \( \lambda \) and associated eigenspace \( \mathcal{E}_T (\lambda) \). Let \( W \) be any subspace of \( \mathcal{E}_T (\lambda) \). Then \( W \) is an invariant subspace of \( V \) relative to \( T \). □

Proof  Choose \( w \in W \). Then

\[
T(w) = \lambda w
\in W
\]

Definition EELT 515

Property SC 251

Version 1.04
So by Definition IS \([565]\), \(W\) is an invariant subspace of \(V\) relative to \(T\).

**Theorem EIS** \([566]\) is general enough to determine that an entire eigenspace is an invariant subspace, or that simply the span of a single eigenvector is an invariant subspace. It is not always the case that any subspace of an invariant subspace is again an invariant subspace, but eigenspaces do have this property. Here is an example of the theorem, which also allows us to very quickly build several invariant (4x4, 2 evs, 1 2x2 jordan, 1 2x2 diag).

**Example EIS**

**Eigenspaces as invariant subspaces**

Define the linear transformation \(S : M_{22} \mapsto M_{22}\) by

\[
S \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} -2a + 19b - 33c + 21d \\ -2a + 9b - 13c + 9d \\ -3a + 16b - 24c + 15d \\ -a + 4b - 6c + 5d \end{bmatrix}
\]

Build a matrix representation of \(S\) relative to the standard basis (Definition MR \([485]\), Example BM \([295]\)) and compute eigenvalues and eigenspaces of \(S\) with the computational techniques of Chapter E \([356]\) in concert with Theorem EER \([527]\). Then

\[
\mathcal{E}_S(1) = \left\langle \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \right\rangle \\
\mathcal{E}_S(2) = \left\langle \begin{bmatrix} 6 \\ 3 \\ 1 \\ 0 \\ -9 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -6 \\ -3 \end{bmatrix} \right\rangle
\]

So by Theorem EIS \([566]\), both \(\mathcal{E}_S(1)\) and \(\mathcal{E}_S(2)\) are invariant subspaces of \(M_{22}\) relative to \(S\).

However, Theorem EIS \([566]\) provides even more invariant subspaces. Since \(\mathcal{E}_S(1)\) has dimension 1, it has no interesting subspaces, however \(\mathcal{E}_S(2)\) has dimension 2 and has a plethora of subspaces. For example, set

\[
u = 2 \begin{bmatrix} 6 \\ 3 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -9 \\ -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ 2 \\ 3 \end{bmatrix}
\]

and define \(U = \langle \{\nu\} \rangle\). Then since \(U\) is a subspace of \(\mathcal{E}_S(2)\), Theorem EIS \([566]\) says that \(U\) is an invariant subspace of \(M_{22}\) (or we could check this claim directly based simply on the fact that \(\nu\) is an eigenvector of \(S\)).

For every linear transformation there are some obvious, trivial invariant subspaces. Suppose that \(T : V \mapsto V\) is a linear transformation. Then simply because \(T\) is a function (Definition LT \([405]\)), the subspace \(V\) is an invariant subspace of \(T\). In only a minor twist on this theme, the range of \(T\), \(R(T)\), is an invariant subspace of \(T\) by Definition RLT \([444]\). Finally, Theorem LTTZZ \([408]\) provides the justification for claiming that \(\{0\}\) is an invariant subspace of \(T\).

That the trivial subspace is always an invariant subspace is a special case of the next theorem. As an easy exercise before reading the next theorem, prove that the kernel of a linear transformation (Definition KLT \([429]\)), \(K(T)\), is an invariant subspace. We’ll wait.

**Theorem KPIS**

**Kernels of Powers are Invariant Subspaces**

Suppose that \(T : V \mapsto V\) is a linear transformation. Then \(K(T^k)\) is an invariant subspace of \(V\).

**Proof** Suppose that \(z \in K(T^k)\). Then

\[
T^k (T(z)) = T^{k+1}(z) \\
= T(T^k(z)) \\
= T(0) \\
= 0
\]

So by Definition KLT \([429]\) we see that \(T(z) \in K(T^k)\). Thus \(K(T^k)\) is an invariant subspace of \(V\) relative to \(T\) (Definition IS \([565]\)).

Two interesting special cases of Theorem KPIS \([567]\) occur when choose \(k = 0\) and \(k = 1\). Rather than give an example of this theorem, we will refer you back to Example KPNLT \([555]\).
where we work with null spaces of the first four powers of a nilpotent matrix. By Theorem KPIS each of these null spaces is an invariant subspace of the associated linear transformation.

Here’s one more example of invariant subspaces we have encountered previously.

**Example ISJB**

**Invariant subspaces and Jordan blocks**

Refer back to Example CFNLT. We decomposed the vector space \( \mathbb{C}^6 \) into a direct sum of the subspaces \( Z_1, Z_2, Z_3, Z_4 \). The union of the basis vectors for these subspaces is a basis of \( \mathbb{C}^6 \), which we reordered prior to building a matrix representation of the linear transformation \( T \). A principal reason for this reordering was to create invariant subspaces (though it was not obvious then).

Define

\[
X_1 = \langle \{ z_{1,1}, z_{2,1}, z_{3,1}, z_{4,1} \} \rangle = \langle \begin{pmatrix}
1 & 1 & -3 & 1 \\
1 & 0 & -3 & 0 \\
0 & 3 & -3 & 0 \\
1 & 1 & -3 & 0 \\
1 & 0 & -2 & 0
\end{pmatrix}
\rangle
\]

\[
X_2 = \langle \{ z_{1,2}, z_{2,2} \} \rangle = \langle \begin{pmatrix}
-2 & 2 \\
-2 & 1 \\
-5 & 2 \\
-2 & 2 \\
-1 & 2 \\
0 & 1
\end{pmatrix}
\rangle
\]

Recall from the proof of Theorem CFNLT or the computations in Example CFNLT that first elements of \( X_1 \) and \( X_2 \) are in the kernel of \( T, K(T) \), and each element of \( X_1 \) and \( X_2 \) is the output of \( T \) when evaluated with the subsequent element of the set. This was by design, and it is this feature of these basis vectors that leads to the nearly diagonal matrix representation with Jordan blocks. However, we also recognize now that this property of these basis vectors allow us to conclude easily that \( X_1 \) and \( X_2 \) are invariant subspaces of \( \mathbb{C}^6 \) relative to \( T \).

Furthermore, \( \mathbb{C}^6 = X_1 \oplus X_2 \) (Theorem DSFB). So the domain of \( T \) is the direct sum of invariant subspaces and the resulting matrix representation has a block diagonal form. Hmmmmm.

**Subsection GEE**

**Generalized Eigenvectors and Eigenspaces**

We now define a new type of invariant subspace and explore its key properties. This generalization of eigenvalues and eigenspaces will allow us to move from diagonal matrix representations of diagonalizable matrices to nearly diagonal matrix representations of arbitrary matrices. Here are the definitions.

**Definition GEV**

**Generalized Eigenvector**
Suppose that \( T: \mathcal{V} \rightarrow \mathcal{V} \) is a linear transformation. Suppose further that for \( x \neq 0 \), \( (T - \lambda I_V)^k (x) = 0 \) for some \( k > 0 \). Then \( x \) is a **generalized eigenvector** of \( T \) with eigenvalue \( \lambda \).

**Definition GES**

**Generalized Eigenspace**
Suppose that \( T: \mathcal{V} \rightarrow \mathcal{V} \) is a linear transformation. Define the **generalized eigenspace** of \( T \) for \( \lambda \) as

\[
\mathcal{G}_T (\lambda) = \left\{ x \mid (T - \lambda I_V)^k (x) = 0 \text{ for some } k \geq 0 \right\}
\]
(This definition contains Notation GES.) △

So the generalized eigenspace is composed of generalized eigenvectors, plus the zero vector. As the name implies, the generalized eigenspace is a subspace of \( V \). But more topically, it is an invariant subspace of \( V \) relative to \( T \).

**Theorem GESIS**

**Generalized Eigenspace is an Invariant Subspace**

Suppose that \( T : V \rightarrow V \) is a linear transformation. Then the generalized eigenspace \( G_T(\lambda) \) is an invariant subspace of \( V \) relative to \( T \).

**Proof**

First we establish that \( G_T(\lambda) \) is a subspace of \( V \). First \( (T - \lambda I_V)^1(0) = 0 \) by Theorem LTTZZ 408, so \( 0 \in G_T(\lambda) \).

Suppose that \( x, y \in G_T(\lambda) \). Then there are integers \( k, \ell \) such that \( (T - \lambda I_V)^k(x) = 0 \) and \( (T - \lambda I_V)^\ell(y) = 0 \). Set \( m = k + \ell \),

\[
(T - \lambda I_V)^m(x + y) = (T - \lambda I_V)^m(x) + (T - \lambda I_V)^m(y) = (T - \lambda I_V)^{k + \ell}(x) + (T - \lambda I_V)^{k + \ell}(y) = (T - \lambda I_V)^k((T - \lambda I_V)^\ell(y)) = (T - \lambda I_V)^k(0) + (T - \lambda I_V)^k(0) = 0 + 0 = 0
\]

So \( x + y \in G_T(\lambda) \).

Suppose that \( x \in G_T(\lambda) \) and \( \alpha \in \mathbb{C} \). Then there is an integer \( k \) such that \( (T - \lambda I_V)^k(x) = 0 \).

\[
(T - \lambda I_V)^k(\alpha x) = \alpha (T - \lambda I_V)^k(x) = \alpha 0 = 0
\]

So \( \alpha x \in G_T(\lambda) \). By Theorem TSS 265, \( G_T(\lambda) \) is a subspace of \( V \).

Now we show that \( G_T(\lambda) \) is invariant relative to \( T \). Suppose that \( x \in G_T(\lambda) \). Then there is an integer \( k \) such that \( (T - \lambda I_V)^k(x) = 0 \). Recognize also that \( (T - \lambda I_V)^k(\alpha x) = 0 \) and \( (T - \lambda I_V)^k(y) = 0 \). Set \( m = k + \ell \),

\[
(T - \lambda I_V)^m(x) = (T - \lambda I_V)^m(x) + (T - \lambda I_V)^m(y) = (T - \lambda I_V)^{k + \ell}(x) + (T - \lambda I_V)^{k + \ell}(y) = (T - \lambda I_V)^k((T - \lambda I_V)^\ell(y)) = (T - \lambda I_V)^k(0) + (T - \lambda I_V)^k(0) = 0 + 0 = 0
\]

This qualifies \( T(x) \) for membership in \( G_T(\lambda) \), so by Definition GES 568, \( G_T(\lambda) \) is invariant relative to \( T \).

Before we compute some generalized eigenspaces, we state and prove one theorem that will make it much easier to create a generalized eigenspace, since it will allow us to use tools we already know well, and will remove some of the ambiguity of the clause “for some \( k \)” in the definition.

**Theorem GEK**

**Generalized Eigenspace as a Kernel**

Suppose that \( T : V \rightarrow V \) is a linear transformation, \( \dim(V) = n \), and \( \lambda \) is an eigenvalue of \( T \). Then \( G_T(\lambda) = \mathcal{K}((T - \lambda I_V)^n) \).

**Proof**

The conclusion of this theorem is a set equality, so we will apply Definition SE 616 by establishing two set inclusions. First, suppose that \( x \in G_T(\lambda) \). Then there is an integer \( k \) such
that \((T - \lambda I_V)^k (x) = 0\). This is equivalent to the statement that \(x \in \mathcal{K}\left((T - \lambda I_V)^k\right)\). No matter what the value of \(k\) is, Theorem KPLT \[554\] gives
\[
x \in \mathcal{K}\left((T - \lambda I_V)^k\right) \subseteq \mathcal{K}\left((T - \lambda I_V)^n\right)
\]
So, \(G_T(\lambda) \subseteq \mathcal{K}\left((T - \lambda I_V)^n\right)\). For the opposite inclusion, suppose \(y \in \mathcal{K}\left((T - \lambda I_V)^n\right)\). Then \((T - \lambda I_V)^n (y) = 0\), so \(y \in G_T(\lambda)\) and thus \(\mathcal{K}\left((T - \lambda I_V)^n\right) \subseteq G_T(\lambda)\). By Definition SE \[616\] we have the desired equality of sets.

Theorem GEK \[569\] allows us to compute generalized eigenspaces as a single kernel (or null space of a matrix representation) with tools like Theorem KNSI \[495\] and Theorem BNS \[128\]. Also, we do not need to consider all possible powers \(k\) and can simply consider the case where \(k = n\). It is worth noting that the “regular” eigenspace is a subspace of the generalized eigenspace since
\[
E_T(\lambda) = \mathcal{K}\left((T - \lambda I_V)^1\right) \subseteq \mathcal{K}\left((T - \lambda I_V)^n\right) = G_T(\lambda)
\]
where the subset inclusion is a consequence of Theorem KPLT \[554\]. Also, there is no such thing as a “generalized eigenvalue.” If \(\lambda\) is not an eigenvalue of \(T\), then the kernel of \(T - \lambda I_V\) is trivial and therefore subsequent powers of \(T - \lambda I_V\) also have trivial kernels (Theorem KPLT \[554\]). So the generalized eigenspace of a scalar that is not already an eigenvalue would be trivial. Alright, we know enough now to compute some generalized eigenspaces. We will record some information about algebraic and geometric multiplicities of eigenvalues (Definition AME \[366\], Definition GME \[366\]) as we go, since these observations will be of interest in light of some future theorems.

Example GE4
Generalized eigenspaces, dimension 4 domain
In Example TIS \[565\] we presented two invariant subspaces of \(\mathbb{C}^4\). There was some mystery about just how these were constructed, but we can now reveal that they are generalized eigenspaces. Example TIS \[565\] featured \(T: \mathbb{C}^4 \mapsto \mathbb{C}^4\) defined by \(T(x) = Ax\) with \(A\) given by
\[
A = \begin{bmatrix}
-8 & 6 & -15 & 9 \\
-8 & 14 & -10 & 18 \\
1 & 1 & 3 & 0 \\
3 & -8 & 2 & -11
\end{bmatrix}
\]

A matrix representation of \(T\) relative to the standard basis (Definition SUV \[190\]) will equal \(A\). So we can analyze \(A\) with the techniques of Chapter E \[356\]. Doing so, we find two eigenvalues, \(\lambda = 1, -2\), with multiplicities,
\[
\alpha_T(1) = 2 \quad \alpha_T(-2) = 2 \quad \gamma_T(1) = 1 \quad \gamma_T(-2) = 1
\]

To apply Theorem GEK \[569\] we subtract each eigenvalue from the diagonal entries of \(A\), raise the result to the power \(\text{dim} (\mathbb{C}^4) = 4\), and compute a basis for the null space.
\[
\lambda = -2 \quad (A - (-2)I_4)^4 = \begin{bmatrix}
648 & -1215 & 729 & -1215 \\
-324 & 486 & -486 & 486 \\
-405 & 729 & -486 & 729 \\
297 & -486 & 405 & -486
\end{bmatrix}
\rightarrow \text{RREF} \begin{bmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
G_T(-2) = \left\langle \left\{ \begin{bmatrix}
-3 \\
-1 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
-1 \\
0 \\
1
\end{bmatrix}\right\}\rightangle
\]
\[ \lambda = 1 \quad (A - (1)I_4)^4 = \begin{bmatrix} 81 & -405 & -81 & -729 \\ -108 & -189 & -378 & -486 \\ -27 & 135 & 27 & 243 \\ 135 & 54 & 351 & 243 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & \frac{7}{3} & 1 \\ 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \mathcal{G}_T(1) = \left\langle \begin{bmatrix} -7 \\ -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\rangle \]

In Example TIS [565] we concluded that these two invariant subspaces formed a direct sum of \( \mathbb{C}^4 \), only at that time, they were called \( X \) and \( W \). Now we can write

\[ \mathbb{C}^4 = \mathcal{G}_T(1) \oplus \mathcal{G}_T(-2) \]

This is no accident. Notice that the dimension of each of these invariant subspaces is equal to the algebraic multiplicity of the associated eigenvalue. Not an accident either. (See the upcoming Theorem GESD [584].)

Example GE6

Generalized eigenspaces, dimension 6 domain

Define the linear transformation \( S: \mathbb{C}^6 \to \mathbb{C}^6 \) by \( S(\mathbf{x}) = B\mathbf{x} \) where

\[ \begin{bmatrix} 2 & -4 & 25 & -54 & 90 & -37 \\ 2 & -3 & 4 & -16 & 26 & -8 \\ 2 & -3 & 4 & -15 & 24 & -7 \\ 10 & -18 & 6 & -36 & 51 & -2 \\ 8 & -14 & 6 & -21 & 28 & 4 \\ 5 & -7 & -6 & -7 & 8 & 7 \end{bmatrix} \]

Then \( B \) will be the matrix representation of \( S \) relative to the standard basis (Definition SUV [190]) and we can use the techniques of Chapter E [356] applied to \( B \) in order to find the eigenvalues of \( S \).

\[ \alpha_S(3) = 2 \quad \gamma_S(3) = 1 \]
\[ \alpha_S(-1) = 4 \quad \gamma_S(-1) = 2 \]

To find the generalized eigenspaces of \( S \) we need to subtract an eigenvalue from the diagonal elements of \( B \), raise the result to the power \( \dim(\mathbb{C}^6) = 6 \) and compute the null space. Here are the results for the two eigenvalues of \( S \),

\[ \lambda = 3 \quad (B - 3I_6)^6 = \begin{bmatrix} 64000 & -152576 & -59904 & 26112 & -95744 & 133632 \\ 15872 & -39936 & -11776 & 8704 & -29184 & 36352 \\ 12032 & -30208 & -9984 & 6400 & -20736 & 26368 \\ -1536 & 11264 & -23040 & 17920 & -17920 & -1536 \\ -9728 & 27648 & -6656 & 9728 & -1536 & -17920 \\ -7936 & 17920 & 5888 & 1792 & 4352 & -14080 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 & -4 & 5 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
\[ G_S(3) = \left\langle \begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ -1 \\ -1 \\ 0 \end{bmatrix} \right\rangle \]

\[ \lambda = -1 \quad (B - (-1)I_6)^6 = \begin{bmatrix} 6144 & -16384 & 18432 & -36864 & 57344 & -18432 \\ 4096 & -8192 & 4096 & -16384 & 24576 & -4096 \\ 4096 & -8192 & 4096 & -16384 & 24576 & -4096 \\ 18432 & -32768 & 6144 & -61440 & 90112 & -6144 \\ 14336 & -24576 & 2048 & -45056 & 65536 & -2048 \\ 10240 & -16384 & -2048 & -28672 & 40960 & 2048 \end{bmatrix} \]

\[ RREF \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & -5 & 2 & -4 & 5 \\ 0 & 1 & -3 & 3 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ G_S(-1) = \left\langle \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\rangle \]

If we take the union of the two bases for these two invariant subspaces we obtain the set

\[ C = \{ v_1, v_2, v_3, v_4, v_5, v_6 \} = \left\{ \begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \end{bmatrix} \right\} \]

You can check that this set is linearly independent (right now we have no guarantee this will happen). Once this is verified, we have a linearly independent set of size 6 inside a vector space of dimension 6, so by Theorem G [320], the set \( C \) is a basis for \( \mathbb{C}^6 \). This is enough to apply Theorem DSFB [326] and conclude that

\[ \mathbb{C}^6 = G_S(3) \oplus G_S(-1) \]

This is no accident. Notice that the dimension of each of these invariant subspaces is equal to the algebraic multiplicity of the associated eigenvalue. Not an accident either. (See the upcoming Theorem GESD [58].)

Subsection RLT
Restrictions of Linear Transformations

Generalized eigenspaces will prove to be an important type of invariant subspace. A second reason for our interest in invariant subspaces is they provide us with another method for creating new linear transformations from old ones.

Version 1.04
By Theorem GESIS [569], we know the generalized eigenspace as the invariant subspace for the construction of the restriction. Furthermore, we will use a matrix representation for the restriction. In order to gain some experience with restrictions of linear transformations, we construct one.

**Example LTRGE**

### Linear transformation restriction on generalized eigenspace

In order to gain some experience with restrictions of linear transformations, we construct one and then also construct a matrix representation for the restriction. Furthermore, we will use a generalized eigenspace as an invariant subspace for the construction of the restriction.

Consider the linear transformation $T: \mathbb{C}^5 \to \mathbb{C}^5$ defined by $T(x) = Ax$, where

$$A = \begin{bmatrix} -22 & -24 & -24 & -24 & -46 \\ 3 & 2 & 6 & 0 & 11 \\ -12 & -16 & -6 & -14 & -17 \\ 6 & 8 & 4 & 10 & 8 \\ 11 & 14 & 8 & 13 & 18 \end{bmatrix}$$

One of the eigenvalues of $A$ is $\lambda = 2$, with geometric multiplicity $\gamma_T(2) = 1$, and algebraic multiplicity $\alpha_T(2) = 3$. We get the generalized eigenspace in the usual manner,

$$W = \mathcal{G}_T(2) = \mathcal{K}\left((T - 2I_{\mathbb{C}^5})^3\right) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \langle \{w_1, w_2, w_3\} \rangle$$

By Theorem GESIS [569], we know $W$ is invariant relative to $T$, so we can employ Definition LTR [572] to form the restriction, $T|_W: W \to W$.

To better understand exactly what a restriction is (and isn’t), we’ll form a matrix representation of $T|_W$. This will also be a skill we will use in subsequent examples. For a basis of $W$ we will use $C = \{w_1, w_2, w_3\}$. Notice that $\dim(W) = 3$, so our matrix representation will be a square matrix of size 3. Applying Definition MR [485], we compute

$$\rho_C(T(w_1)) = \rho_C(Aw_1) = \rho_C\left( \begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) = \rho_C\left( \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + \rho_C\left( \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) + \rho_C\left( \begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho_C(T(w_2)) = \rho_C(Aw_2) = \rho_C\left( \begin{bmatrix} 0 \\ -2 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right) = \rho_C\left( \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) + \rho_C\left( \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + \rho_C\left( \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$
\[
\rho_C(T(w_3)) = \rho_C(Aw_3) = \rho_C \begin{pmatrix} -6 \\ 3 \\ -1 \\ 0 \\ 2 \end{pmatrix} = \rho_C \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} + 0 + 2 \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}
\]

So the matrix representation of \(T|_W\) relative to \(C\) is

\[
M_{C,C}^{T|_W} = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}
\]

The question arises: how do we use a \(3 \times 3\) matrix to compute with vectors from \(C^5\)? To answer this question, consider the randomly chosen vector

\[
w = \begin{pmatrix} -4 \\ 4 \\ 4 \\ -2 \\ -1 \end{pmatrix}
\]

First check that \(w \in G_T(2)\). There are two ways to do this, first verify that

\[
(T - 2I_{C^5})^5(w) = (A - 2I_5)^5w = 0
\]

meeting Definition GES 568 (with \(k = 5\)). Or, express \(w\) as a linear combination of the basis \(C\) for \(W\), to wit, \(w = 4w_1 - 2w_2 - w_3\). Now compute \(T|_W(w)\) directly using Definition LTR 572,

\[
T|_W(w) = T(w) = Aw = \begin{pmatrix} -10 \\ 9 \\ 5 \\ -4 \\ 0 \end{pmatrix}
\]

It was necessary to verify that \(w \in G_T(2)\), and if we trust our work so far, then this output will also be an element of \(W\), but it would be wise to check this anyway (using either of the methods we used for \(w\)). We’ll wait.

Now we will repeat this sample computation, but instead using the matrix representation of \(T|_W\) relative to \(C\).

\[
T|_W(w) = \rho_C^{-1} \left( M_{C,C}^{T|_W} \rho_C(w) \right) \quad \text{Theorem FTMR 487}
\]

\[
= \rho_C^{-1} \left( M_{C,C}^{T|_W} \rho_C(4w_1 - 2w_2 - w_3) \right)
\]

\[
= \rho_C^{-1} \begin{pmatrix} 2 & 2 & -1 \\ 0 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix} \quad \text{Definition VR 473}
\]

\[
= \rho_C^{-1} \begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix} \quad \text{Definition MVP 173}
\]

\[
= 5w_1 - 4w_2 + 0w_3 \quad \text{Definition VR 473}
\]

\[
= 5 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + (-4) \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} -10 \\ 9 \\ 5 \\ -4 \\ 0 \end{pmatrix}
\]

Version 1.04
which matches the previous computation. Notice how the “action” of $T|_W$ is accomplished by a $3 \times 3$ matrix multiplying a column vector of size $3$. If you would like more practice with these sorts of computations, mimic the above using the other eigenvalue of $T$, which is $\lambda = -2$. The generalized eigenspace has dimension 2, so the matrix representation of the restriction to the generalized eigenspace will be a $2 \times 2$ matrix.

Suppose that $T: V \mapsto V$ is a linear transformation and we can find a decomposition of $V$ as a direct sum, say $V = U_1 \oplus U_2 \oplus U_3 \oplus \cdots \oplus U_m$ where each $U_i$ is an invariant subspace of $V$ relative to $T$. Then, for any $v \in V$ there is a unique decomposition $v = u_1 + u_2 + u_3 + \cdots + u_m$ with $u_i \in U_i$, $1 \leq i \leq m$ and furthermore

$$T(v) = T(u_1 + u_2 + u_3 + \cdots + u_m) = T(u_1) + T(u_2) + T(u_3) + \cdots + T(u_m) = T|_{U_1}(u_1) + T|_{U_2}(u_2) + T|_{U_3}(u_3) + \cdots + T|_{U_m}(u_m)$$

So in a very real sense, we obtain a decomposition of the linear transformation $T$ into the restrictions $T|_{U_i}$, $1 \leq i \leq m$. If we wanted to be more careful, we could extend each restriction to a linear transformation defined on $V$ by setting the output of $T|_{U_i}$ to be the zero vector for inputs outside of $U_i$. Then $T$ would be exactly equal to the sum (Definition LTA) of these extended restrictions. However, the irony of extending our restrictions is more than we could handle right now.

Our real interest is in the matrix representation of a linear transformation when the domain decomposes as a direct sum of invariant subspaces. Consider forming a basis $B$ of $V$ as the union of bases $B_i$ from the individual $U_i$, i.e. $B = \bigcup_{i=1}^{m} B_i$. Now form the matrix representation of $T$ relative to $B$. The result will be block diagonal, where each block is the matrix representation of a restriction $T|_{U_i}$ relative to a basis $B_i$. Though we did not have the definitions to describe it then, this is exactly what was going on in the latter portion of the proof of Theorem CFNLT. Two examples should help to clarify these ideas.

**Example ISMR4**

**Invariant subspaces, matrix representation, dimension 4 domain** Similar to Example TIS and Example GE4 describe a basis of $C^4$ which is derived from bases for two invariant subspaces (both generalized eigenspaces). In this example we will construct a matrix representation of the linear transformation $T$ relative to this basis. Recycling the notation from Example TIS, we work with the basis,

$$B = \{w_1, w_2, x_1, x_2\} = \left\{ \begin{bmatrix} -7 \\ -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Now we compute the matrix representation of $T$ relative to $B$, borrowing some computations from Example TIS.

$$\rho_B(T(w_1)) = \rho_B \left( \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right) = \rho_B((0)w_1 + (1)w_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho_B(T(w_2)) = \rho_B \left( \begin{bmatrix} 5 \\ -2 \\ -3 \\ 2 \end{bmatrix} \right) = \rho_B((-1)w_1 + (2)w_2) = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$
\[
\rho_B(T(x_1)) = \rho_B \left( \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right) = \rho_B \left( (-1)x_1 + (1)x_2 \right) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}
\]

\[
\rho_B(T(x_2)) = \rho_B \left( \begin{bmatrix} 3 \\ 4 \\ -1 \\ -3 \end{bmatrix} \right) = \rho_B \left( (-1)x_1 + (-3)x_2 \right) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -3 \end{bmatrix}
\]

Applying Definition MR \[485\], we have

\[
M_{B,B}^T = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}
\]

The interesting feature of this representation is the two \(2 \times 2\) blocks on the diagonal that arise from the decomposition of \(\mathbb{C}^4\) into a direct sum (of generalized eigenspaces). Or maybe the interesting feature of this matrix is the two \(2 \times 2\) submatrices in the “other” corners that are all zero. You decide.

Example ISMR6

Invariant subspaces, matrix representation, dimension 6 domain

In Example GE6 \[571\] we computed the generalized eigenspaces of the linear transformation \(S: \mathbb{C}^6 \rightarrow \mathbb{C}^6\) by \(S(x) = Bx\) where

\[
\begin{bmatrix} 2 & -4 & 25 & -54 & 90 & -37 \\ 2 & -3 & 4 & -16 & 26 & -8 \\ 2 & -3 & 4 & -15 & 24 & -7 \\ 10 & -18 & 6 & -36 & 51 & -2 \\ 8 & -14 & 0 & -21 & 28 & 4 \\ 5 & -7 & -6 & -7 & 8 & 7 \end{bmatrix}
\]

From this we found the basis

\[
C = \{v_1, v_2, v_3, v_4, v_5, v_6\}
\]

\[
= \left\{ \begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

of \(\mathbb{C}^6\) where \(\{v_1, v_2\}\) is a basis of \(G_S(3)\) and \(\{v_3, v_4, v_5, v_6\}\) is a basis of \(G_S(-1)\). We can employ \(C\) in the construction of a matrix representation of \(S\) (Definition MR \[485\]). Here are the computations,

\[
\rho_C(S(v_1)) = \rho_C \left( \begin{bmatrix} 11 \\ 3 \\ 4 \\ 1 \end{bmatrix} \right) = \rho_C (4v_1 + 1v_2) = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
\rho_C(S(v_2)) = \rho_C \left( \begin{bmatrix} -14 \\ -3 \\ -3 \\ -1 \\ 2 \end{bmatrix} \right) = \rho_C ((-1)v_1 + 2v_2) = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]
\[ \rho_C (S(v_3)) = \rho_C \begin{pmatrix} 2 \ 3 \\ 5 \\ 2 \\ -2 \\ -2 \end{pmatrix} = \rho_C (5v_3 + 2v_4 + (-2)v_5 + (-2)v_6) = \begin{pmatrix} 0 \\ 5 \\ 2 \\ -2 \\ -2 \end{pmatrix} \]

\[ \rho_C (S(v_4)) = \rho_C \begin{pmatrix} -46 \\ -11 \\ -10 \\ -2 \\ 5 \\ 4 \end{pmatrix} = \rho_C ((-10)v_3 + (-2)v_4 + 5v_5 + 4v_6) = \begin{pmatrix} 0 \\ 0 \\ -10 \\ -2 \\ 4 \end{pmatrix} \]

\[ \rho_C (S(v_5)) = \rho_C \begin{pmatrix} 78 \\ 19 \\ 17 \\ 1 \\ -10 \\ -7 \end{pmatrix} = \rho_C (17v_3 + 1v_4 + (-10)v_5 + (-7)v_6) = \begin{pmatrix} 0 \\ 0 \\ 17 \\ 1 \\ -10 \\ -7 \end{pmatrix} \]

\[ \rho_C (S(v_6)) = \rho_C \begin{pmatrix} -35 \\ -9 \\ -8 \\ 2 \\ 6 \\ 3 \end{pmatrix} = \rho_C ((-8)v_3 + 2v_4 + 6v_5 + 3v_6) = \begin{pmatrix} 0 \\ 0 \\ -8 \\ 2 \\ 6 \\ 3 \end{pmatrix} \]

These column vectors are the columns of the matrix representation, so we obtain

\[ M_{SC,C}^S = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -10 & 17 & -8 \\ 0 & 0 & 2 & -2 & 1 & 2 \\ 0 & 0 & -2 & 5 & -10 & 6 \\ 0 & 0 & -2 & 4 & -7 & 3 \end{bmatrix} \]

As before, the key feature of this representation is the \(2 \times 2\) and \(4 \times 4\) blocks on the diagonal. We will discover in the final theorem of this section (Theorem RGEN [578]) that we already understand these blocks fairly well. For now, we recognize them as arising from generalized eigenspaces and suspect that their sizes are equal to the algebraic multiplicities of the eigenvalues.

The paragraph prior to these last two examples is worth repeating. A basis derived from a direct sum decomposition into invariant subspaces will provide a matrix representation of a linear transformation with a block diagonal form.

Diagonalizing a linear transformation is the most extreme example of decomposing a vector space into invariant subspaces. When a linear transformation is diagonalizable, then there is a basis composed of eigenvectors (Theorem DC [394]). Each of these basis vectors can be used individually as the lone element of a spanning set for an invariant subspace (Theorem EIS [566]). So the domain decomposes into a direct sum of one-dimensional invariant subspaces (Theorem DSFB [326]). The corresponding matrix representation is then block diagonal with all the blocks of size 1, i.e. the matrix is diagonal. Section NLT [548], Section IS [565] and Section JCF [582] are all devoted to generalizing this extreme situation when there are not enough eigenvectors available to make such a complete decomposition and arrive at such an elegant matrix representation.

One last theorem will roll up much of this section and Section NLT [548] into one nice, neat package.

**Theorem RGEN**
Restriction to Generalized Eigenspace is Nilpotent
Suppose $T: V \mapsto V$ is a linear transformation with eigenvalue $\lambda$. Then the linear transformation $T|_{G_T(\lambda)} - \lambda I_{G_T(\lambda)}$ is nilpotent.

**Proof**  Notice first that every subspace of $V$ is invariant with respect to $I_V$, so $I_{G_T(\lambda)} = I_V|_{G_T(\lambda)}$. Let $n = \dim(V)$ and choose $v \in G_T(\lambda)$. Then

$$
(T|_{G_T(\lambda)} - \lambda I_{G_T(\lambda)})^n(v) = (T - \lambda I_V)^n(v) = 0 \quad \text{Definition LTR 572}
$$

So by **Definition NLT 548**, $T|_{G_T(\lambda)} - \lambda I_{G_T(\lambda)}$ is nilpotent. □

The proof of Theorem RGEN 578 indicates that the index of the nilpotent linear transformation is less than or equal to the dimension of $V$. In practice, it will be less than or equal to the dimension of the domain of the linear transformation, $G_T(\lambda)$. In any event, the exact value of this index will be of some interest, so we define it now. Notice that this is a property of the eigenvalue $\lambda$, similar to the algebraic and geometric multiplicities (Definition AME 366, Definition GME 366).

**Definition IE**
Index of an Eigenvalue
Suppose $T: V \mapsto V$ is a linear transformation with eigenvalue $\lambda$. Then the index of $\lambda$, $\iota_T(\lambda)$, is the index of the nilpotent linear transformation $T|_{G_T(\lambda)} - \lambda I_{G_T(\lambda)}$.

(This definition contains Notation IE.) △

**Example GENR6**
Generalized eigenspaces and nilpotent restrictions, dimension 6 domain
In Example GE6 571 we computed the generalized eigenspaces of the linear transformation $S: \mathbb{C}^6 \mapsto \mathbb{C}^6$ defined by $S(x) = Bx$ where

$$
\begin{bmatrix}
2 & -4 & 25 & -54 & 90 & -37 \\
2 & -3 & 4 & -16 & 26 & -8 \\
2 & -3 & 4 & -15 & 24 & -7 \\
10 & -18 & 6 & -36 & 51 & -2 \\
8 & -14 & 0 & -21 & 28 & 4 \\
5 & -7 & -6 & -7 & 8 & 7
\end{bmatrix}
$$

The generalized eigenspace, $G_S(3)$, has dimension 2, while $G_S(-1)$, has dimension 4. We’ll investigate each thoroughly in turn, with the intent being to illustrate Theorem RGEN 578. Much of our computations will be repeats of those done in Example ISMR6 576.

For $U = G_S(3)$ we compute a matrix representation of $S|_U$ using the basis found in Example GE6 571.

$$B = \{u_1, u_2\} = \begin{bmatrix}
4 \\
1 \\
1 \\
2 \\
1 \\
0
\end{bmatrix}, \quad \begin{bmatrix}
-5 \\
-1 \\
-1 \\
0 \\
1
\end{bmatrix}
$$

Since $B$ has size 2, we obtain a $2 \times 2$ matrix representation (Definition MR 485) from

$$
\rho_B(S|_U(u_1)) = \rho_B \begin{bmatrix}
11 \\
3 \\
3 \\
7 \\
4 \\
1
\end{bmatrix} = \rho_B(4u_1 + u_2) = \begin{bmatrix}
4 \\
1
\end{bmatrix}
$$
\[ \rho_B (S|_U (u_2)) = \rho_B \begin{pmatrix} -4 \\ -3 \\ -3 \\ -4 \\ -1 \\ 2 \end{pmatrix} = \rho_B ((-1)u_1 + 2u_2) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \]

Thus

\[ M = M^{S|_U}_{U,U} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \]

Now we can illustrate Theorem RGEN with powers of the matrix representation (rather than the restriction itself),

\[ M - 3I_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad (M - 3I_2)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

So \(M - 3I_2\) is a nilpotent matrix of index 2 (meaning that \(S|_U - 3I_U\) is a nilpotent linear transformation of index 2) and according to Definition IE we say \(t_S (3) = 2\).

For \(W = G_S (-1)\) we compute a matrix representation of \(S|_W\) using the basis found in Example GE6.

\[ C = \{w_1, w_2, w_3, w_4\} = \bigg\{ \begin{bmatrix} 5 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -4 \\ 0 \\ 1 \end{bmatrix} \bigg\} \]

Since \(C\) has size 4, we obtain a 4 × 4 matrix representation from

\[ \rho_C (S|_W (w_1)) = \rho_C \begin{pmatrix} 23 \\ 5 \\ 2 \\ -2 \end{pmatrix} = \rho_C (5w_1 + 2w_2 + (-2)w_3 + (-2)w_4) = \begin{bmatrix} 5 \\ -2 \\ -2 \end{bmatrix} \]

\[ \rho_C (S|_W (w_2)) = \rho_C \begin{pmatrix} 46 \\ -11 \\ -10 \\ 5 \\ 4 \end{pmatrix} = \rho_C ((-10)w_1 + (-2)w_2 + 5w_3 + 4w_4) = \begin{bmatrix} -10 \\ -2 \\ 5 \\ 4 \end{bmatrix} \]

\[ \rho_C (S|_W (w_3)) = \rho_C \begin{pmatrix} 78 \\ 19 \\ 17 \\ 1 \\ -10 \\ -7 \end{pmatrix} = \rho_C (17w_1 + w_2 + (-10)w_3 + (-7)w_4) = \begin{bmatrix} 17 \\ 1 \\ -10 \\ -7 \end{bmatrix} \]

\[ \rho_C (S|_W (w_4)) = \rho_C \begin{pmatrix} -35 \\ -9 \\ -8 \\ 2 \\ 6 \\ 3 \end{pmatrix} = \rho_C ((-8)w_1 + 2w_2 + 6w_3 + 3w_4) = \begin{bmatrix} -8 \\ 2 \\ 6 \\ 3 \end{bmatrix} \]
Thus

\[
N = M_{W,W}^{S|W} = \begin{bmatrix}
5 & -10 & 17 & -8 \\
2 & -2 & 1 & 2 \\
-2 & 5 & -10 & 6 \\
-2 & 4 & -7 & 3 \\
\end{bmatrix}
\]

Now we can illustrate Theorem RGEN \[578\] with powers of the matrix representation (rather than the restriction itself),

\[
N - (-1)I_4 = \begin{bmatrix}
6 & -10 & 17 & -8 \\
2 & -1 & 1 & 2 \\
-2 & 5 & -9 & 6 \\
-2 & 4 & -7 & 4 \\
\end{bmatrix}
\]

\[
(N - (-1)I_4)^2 = \begin{bmatrix}
-2 & 3 & -5 & 2 \\
4 & -6 & 10 & -4 \\
4 & -6 & 10 & -4 \\
2 & -3 & 5 & -2 \\
\end{bmatrix}
\]

\[
(N - (-1)I_4)^3 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

So \(N - (-1)I_4\) is a nilpotent matrix of index 3 (meaning that \(S|W - (-1)I_W\) is a nilpotent linear transformation of index 3) and according to Definition IE \[578\] we say \(\iota_S(-1) = 3\).

Notice that if we were to take the union of the two bases of the generalized eigenspaces, we would have a basis for \(\mathbb{C}^6\). Then a matrix representation of \(S\) relative to this basis would be the same block diagonal matrix we found in Example ISMR6 \[576\], only we now understand each of these blocks as being very close to being a nilpotent matrix.

Invariant subspaces, and restrictions of linear transformations, are topics you will see again and again if you continue with further study of linear algebra. Our reasons for discussing them now is to arrive at a nice matrix representation of the restriction of a linear transformation to one of its generalized eigenspaces. Here’s the theorem.

**Theorem MRRGE**

**Matrix Representation of a Restriction to a Generalized Eigenspace**

Suppose that \(T : V \mapsto V\) is a linear transformation with eigenvalue \(\lambda\). Then there is a basis of the the generalized eigenspace \(\mathcal{G}_T(\lambda)\) such that the restriction \(T|_{\mathcal{G}_T(\lambda)} : \mathcal{G}_T(\lambda) \mapsto \mathcal{G}_T(\lambda)\) has a matrix representation that is block diagonal where each block is a Jordan block of the form \(J_n(\lambda)\).

**Proof** Theorem RGEN \[578\] tells us that \(T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}\) is a nilpotent linear transformation. Theorem CFNLTI \[557\] tells us that a nilpotent linear transformation has a basis for its domain that yields a matrix representation that is block diagonal where the blocks are Jordan blocks of the form \(J_n(0)\). Let \(B\) be a basis of \(\mathcal{G}_T(\lambda)\) that yields such a matrix representation for \(T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}\).

By Definition LTA \[417\], we can write

\[
T|_{\mathcal{G}_T(\lambda)} = (T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}) + \lambda I_{\mathcal{G}_T(\lambda)}
\]

The matrix representation of \(\lambda I_{\mathcal{G}_T(\lambda)}\) relative to the basis \(B\) is then simply the diagonal matrix \(\lambda I_m\), where \(m = \dim(\mathcal{G}_T(\lambda))\). By Theorem MRSLT \[490\] we have the rather unwieldy expression,

\[
M_{B,B}^{T|_{\mathcal{G}_T(\lambda)}} = M_{B,B}^{(T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}) + \lambda I_{\mathcal{G}_T(\lambda)}}
\]

\[
= M_{B,B}^{T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)}} + M_{B,B}^{I_{\mathcal{G}_T(\lambda)}}
\]
The first of these matrix representations has Jordan blocks with zero in every diagonal entry, while the second matrix representation has \( \lambda \) in every diagonal entry. The result of adding the two representations is to convert the Jordan blocks from the form \( J_n(0) \) to the form \( J_n(\lambda) \). ■

Of course, Theorem CFNLT [557] provides some extra information on the sizes of the Jordan blocks in a representation and we could carry over this information to Theorem MRRGE [580], but will save that for a subsequent application of this result.
Section JCF
Jordan Canonical Form

This Section Under Construction
(Just Lacking JCF Examples)

We have seen in Section IS [565] that generalized eigenspaces are invariant subspaces that in every instance have led to a direct sum decomposition of the domain of the associated linear transformation. This allows us to create a block diagonal matrix representation [Example ISMR4 575, Example ISMR6 576]. We also know from Theorem RGEN [578] that the restriction of a linear transformation to a generalized eigenspace is almost a nilpotent linear transformation. Of course, we understand nilpotent linear transformations very well from Section NLT [548] and we have carefully determined a nice matrix representation for them.

So here is the game plan for the final push. Prove that the domain of a linear transformation always decomposes into a direct sum of generalized eigenspaces. We have unravelled Theorem RGEN [578] at Theorem MRRGE [580] so that we can formulate the matrix representations of the restrictions on the generalized eigenspaces using our storehouse of results about nilpotent linear transformations. Arrive at a matrix representation of any linear transformation that is block diagonal with each block being a Jordan block.

We will be strictly theoretical at first, proving two major theorems without any explanatory examples, so hang on. Then we can state our main result and move on to several interesting examples.

Subsection UTMR
Upper Triangular Matrix Representation

Our theorems in this section will each assert that certain bases exist, but what we are really after is the matrix representation that arises from the basis (this is the style of Theorem CFNLT [557]).

Theorem UTMR
Upper Triangular Matrix Representation

Suppose that \( T : V \rightarrow V \) is a linear transformation. Then there is a basis, \( B \), for \( V \) such that the matrix representation of \( T \) relative to \( B \), \( M_{B,B}^T \), is an upper triangular matrix. Each diagonal entry is an eigenvalue of \( T \), and if \( \lambda \) is an eigenvalue of \( T \), then \( \lambda \) occurs \( \alpha_T(\lambda) \) times on the diagonal. \( \square \)

**Proof** We will establish this result using induction of the dimension of \( V \) (Technique I 626). To start suppose that \( \dim(V) = 1 \). Choose any nonzero vector \( v \in V \), and then realize that \( V = \langle \{v\} \rangle \). Subsequently, we can describe \( T \) completely by \( T(v) = \beta v \) for some \( \beta \in \mathbb{C} \). Thus, we recognize \( \beta \) as one eigenvalue of \( T \), and there are no others (Theorem ME 384). And \( \alpha_T(\lambda) = 1 \). Our description of \( T \) also gives us a matrix representation relative to \( \{v\} \) as the \( 1 \times 1 \) matrix with lone entry equal to \( \beta \). This representation is upper triangular and the diagonal entry is an eigenvalue of \( T \), occurring \( \alpha_T(\beta) \) times.

For the induction step let \( \dim(V) = m \), and assume the theorem is true for every linear transformation defined on a vector space of dimension less than \( m \). By Theorem EMHE 359 (suitably converted to the setting of a linear transformation), \( T \) has at least one eigenvalue, denote this eigenvalue as \( \lambda \). (We will remark later about how critical this step is.) We now consider properties of the linear transformation \( T - \lambda I_V : V \rightarrow V \).

Let \( x \) be an eigenvector of \( T \) for \( \lambda \). By definition \( x \neq 0 \). Then

\[
T - \lambda I_V(x) = T(x) - \lambda I_V(x) = T(x) - \lambda x = \lambda x - \lambda x
\]

\( \square \)
So \( T - \lambda I_V \) is not injective \((\text{Theorem KILT} \ [132])\). With an argument on dimensions using \(\text{Theorem RPND} \ [164]\) we can conclude that \( T \) does not have full rank, \( r(T) < m \). Define \( W \) to be the subspace of \( V \) that is the range of \( T - \lambda I_V, W = \mathcal{R}(T - \lambda I_V) \). The range of a linear transformation is always invariant with respect to the linear transformation, but we want to establish that \( W \) is also invariant with respect to just \( T \). To this end, suppose that \( w \in W \),

\[
T(w) = T(w) + 0
\]

\[
= T(w) + (\lambda I_V(w) - \lambda I_V(w)) \quad \text{Property Z} [251]
\]

\[
= (T(w) - \lambda I_V(w)) + \lambda w
\]

\[
= (T(w) - \lambda I_V(w)) + \lambda w
\]

\[
= T - \lambda I_V(w) + \lambda w \quad \text{Definition IDLT} [456]
\]

Since \( W \) is the range of \( T - \lambda I_V, T - \lambda I_V(w) \in W \). And by \(\text{Property SC} [251], \lambda w \in W \). Finally, applying \(\text{Property AC} [251]\) we see that \( T(w) \in W \) and conclude that \( W \) is invariant relative to \( T \) \((\text{Definition IS} [563])\).

Since \( W \) is invariant relative to \( T \) we can consider the restriction \( T|_W : W \to W \) \((\text{Definition LTR} [572])\). \( T|_W \) is a linear transformation defined on a vector space with dimension less than \( m \), so we can apply the induction hypothesis and conclude that \( W \) has a basis, \( C = \{w_1, w_2, w_3, \ldots, w_k\} \), such that the matrix representation of \( T|_W \) relative to \( C \) is an upper triangular matrix.

By \(\text{Theorem DSFOS} [327]\), there exists a second subspace of \( V \), which we will call \( U \), so that \( V \) is a direct sum of \( W \) and \( U \), \( V = W \oplus U \). Choose a basis \( D = \{u_1, u_2, u_3, \ldots, u_k\} \) for \( U \). So \( m = k + \ell \) by \(\text{Theorem DSD} [329]\), and \( B = C \cup D \) is basis for \( V \) by \(\text{Theorem DSLI} [328]\) and \(\text{Theorem G} [320]\). \( B \) is the basis we desire. What does a matrix representation of \( T \) look like, relative to \( B \)?

Since \( W \) is invariant relative to \( T \), the first \( k \) columns of \( M_{B,B}^T \) will have the upper triangular matrix representation of \( T|_W \) using the basis \( C, \text{matrixrep} T|_W CC \), in the first \( k \) rows. The remaining \( \ell = m - k \) rows will be all zeros. The situation for \( T \) on \( D \) is not quite as pretty, but it is close.

For \( 1 \leq i \leq \ell \), consider

\[
\rho_B(T(u_i)) = \rho_B(T(u_i) + 0)
\]

\[
= \rho_B(T(u_i) + (\lambda I_V(u_i) - \lambda I_V(u_i)))
\]

\[
= \rho_B((T(u_i) - \lambda I_V(u_i)) + \lambda u_i)
\]

\[
= \rho_B(T - \lambda I_V(u_i) + \lambda u_i)
\]

\[
= \rho_B(a_1w_1 + a_2w_2 + a_3w_3 + \cdots + a_kw_k + \lambda u_i)
\]

\[
= \begin{bmatrix}
    a_1 \\
    a_2 \\
    \vdots \\
    a_k \\
    0 \\
    \vdots \\
    0 \\
    \lambda \\
    0 \\
    \vdots \\
    0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    0 \\
    0 \\
    \vdots \\
    0 \\
    0 \\
    \vdots \\
    0 \\
\end{bmatrix}
\]

In the penultimate step of this proof, we have rewritten an element of the range of \( T - \lambda I_V \) as a linear combination of the basis vectors for \( W \) in \( C \), using the scalars \( a_1, a_2, a_3, \ldots, a_\ell \). If we
incorporate these \( \ell \) column vectors into the matrix representation \( M_{B,B}^T \) we find \( \ell \) occurences of \( \lambda \) on the diagonal, and any nonzero entries lying in the first \( k \) rows. Together with the \( k \times k \) upper triangular representation in the upper left-hand corner, the entire matrix representation is now clearly upper triangular. This completes the induction step, so for any linear transformation there is a basis that creates a diagonal matrix representation.

We have one more statement in the conclusion of the theorem to verify. The eigenvalues, and their multiplicities, of \( T \) can be computed with the techniques of Chapter E \( [356] \) relative to any matrix representation (Theorem EER \( [527] \)). We take this approach with our upper triangular matrix representation \( M_{B,B}^T \). Let \( d_i \) be the diagonal entry of \( M_{B,B}^T \) in row and column \( i \). Then the characteristic polynomial, computed as a determinant (Definition CP \( [363] \)), is

\[
p_{M_{B,B}^T}(x) = (x - d_1)(x - d_2)(x - d_3)\cdots(x - d_m)
\]

So each diagonal entry is an eigenvalue (Theorem EMRCP \( [363] \), and is repeated exactly \( \alpha_T(\lambda) \) times (Definition AME \( [366] \)).

A key step in this proof was the construction of the subspace \( W \) with dimension strictly less than that of \( V \). This required an eigenvalue/eigenvector pair, which was guaranteed to us by Theorem EMHE \( [359] \). Digging deeper, the proof of Theorem EMHE \( [359] \) requires that we can factor polynomials completely, into linear factors. This will not always happen if our set of scalars is the reals, \( \mathbb{R} \). So this is our final explanation of our choice of the complex numbers, \( \mathbb{C} \), as our set of scalars. In \( \mathbb{C} \) polynomials factor completely, so every matrix has at least one eigenvalue, and an inductive argument will get us to upper triangular matrix representations.

The complex numbers are an example of an **algebraically closed field** which contains the roots of any polynomial created with coefficients from itself. If we had chosen to use the reals as our set of scalars, then we would arrive at matrix decompositions known as **rational canonical form** where the diagonal blocks are derived from certain relevant polynomials associated with the linear transformation. The theory is similar, but not identical to, what we have done here.

**Subsection GESD**
**Generalized Eigenspace Decomposition**

We now massage the basis from Theorem UTMR \( [582] \) so that it yields an upper triangular representation that is also block diagonal. The subspaces associated with each block will be generalized eigenspaces, so the most general result will be a decomposition of the domain of a linear transformation into a direct sum of generalized eigenspaces.

**Theorem GESD**
**Generalized Eigenspace Decomposition**
Suppose that \( T(V) \) is a linear transformation with distinct eigenvalues \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m \). Then

\[
V = G_T(\lambda_1) \oplus G_T(\lambda_2) \oplus G_T(\lambda_3) \oplus \cdots \oplus G_T(\lambda_m)
\]

**Proof** Suppose that \( \dim(V) = n \) and the \( n \) (not necessarily distinct) eigenvalues of \( T \) are scalarlist\( \rho n \). We begin with a basis of \( V \) that yields an upper triangular matrix representation, as guaranteed by Theorem UTMR \( [582] \), \( B = \{x_1, x_2, x_3, \ldots, x_n\} \). Since the matrix representation is upper triangular, and the eigenvalues of the linear transformation are the diagonal elements we can choose this basis so that there are then scalars \( a_{ij}, 1 \leq j \leq n, 1 \leq i \leq j - 1 \) such that

\[
T(x_j) = \sum_{i=1}^{j-1} a_{ij}x_i + \rho_j x_j
\]
We now define a new basis for \( V \) which is just a slight variation in the basis \( B \). Choose any \( k \) and \( \ell \) such that \( 1 \leq k < \ell \leq n \) and \( \rho_k \neq \rho_\ell \). Define the scalar \( \alpha = a_{kl}/(\rho_\ell - \rho_k) \). The new basis is \( C = \{ y_1, y_2, y_3, \ldots, y_n \} \) where

\[
y_j = x_j, \quad j \neq \ell, \quad 1 \leq j \leq n \quad \quad y_\ell = x_\ell + \alpha x_k
\]

We now compute the values of the linear transformation \( T \) with inputs from \( C \), noting carefully the changed scalars in the linear combinations of \( C \) describing the outputs. These changes will translate to minor changes in the matrix representation built using the basis \( C \). There are three cases to consider, depending on which column of the matrix representation we are examining. First, assume \( j < \ell \). Then

\[
T(y_j) = (T(x_j)
= \sum_{i=1}^{j-1} a_{ij}x_i + \rho_jx_j
= \sum_{i=1}^{j-1} a_{ij}y_i + \rho_jy_j
\]

That seems a bit pointless. The first \( \ell - 1 \) columns of the matrix representations of \( T \) relative to \( B \) and \( C \) are identical. OK, if that was too easy, here’s the main act. Assume \( j = \ell \). Then

\[
T(y_\ell) = T(x_\ell + \alpha x_k)
= T(x_\ell) + \alpha T(x_k)
= \left( \sum_{i=1}^{\ell-1} a_{i\ell}x_i + \rho_\ell x_\ell \right) + \alpha \left( \sum_{i=1}^{k-1} a_{ik}x_i + \rho_k x_k \right)
= \sum_{i=1}^{\ell-1} a_{i\ell}x_i + \sum_{i=1}^{k-1} \alpha a_{ik}x_i + \alpha \rho_k x_k + \rho_\ell x_\ell
= \sum_{i=1}^{\ell-1} a_{i\ell}x_i + \sum_{i=1}^{k-1} \alpha a_{ik}x_i + a_{k\ell}x_k + \alpha \rho_k x_k + \rho_\ell x_\ell
= \sum_{i=1}^{\ell-1} a_{i\ell}x_i + \sum_{i \neq k}^{k-1} \alpha a_{ik}x_i + a_{k\ell}x_k + \alpha \rho_k x_k + \rho_\ell x_\ell
= \sum_{i=1}^{\ell-1} a_{i\ell}x_i + \sum_{i \neq k}^{k-1} \alpha a_{ik}x_i + \alpha a_{k\ell} x_k + \alpha \rho_k x_k - \rho_\ell \alpha x_k + \rho_\ell \alpha x_k \rho_\ell x_\ell
= \sum_{i=1}^{\ell-1} a_{i\ell}x_i + \sum_{i \neq k}^{k-1} \alpha a_{ik}x_i + (a_{k\ell} + \alpha \rho_k - \rho_\ell \alpha) x_k + \rho_\ell (\alpha x_k + x_\ell)
= \sum_{i=1}^{\ell-1} a_{i\ell}x_i + \sum_{i \neq k}^{k-1} \alpha a_{ik}x_i + (a_{k\ell} + \alpha (\rho_k - \rho_\ell)) x_k + \rho_\ell (x_\ell + \alpha x_k)
= \sum_{i=1}^{\ell-1} a_{i\ell}y_i + \sum_{i \neq k}^{k-1} \alpha a_{ik}y_i + (a_{k\ell} + \alpha (\rho_k - \rho_\ell)) y_k + \rho_\ell y_\ell
\]

So how different are the matrix representations relative to \( B \) and \( C \) in column \( \ell \)? For \( i > k \), the coefficient of \( y_i \) is \( a_{ij} \), as in the representation relative to \( B \). It is a different story for \( i \leq k \), where the coefficients of \( y_i \) may be very different. We are especially interested in the coefficient of \( y_k \). In
fact, this whole first part of this proof is about this particular entry of the matrix representation. The coefficient of $y_k$ is
\[
a_{k\ell} + \alpha (\rho_k - \rho_{\ell}) = a_{k\ell} + \frac{a_{k\ell}}{\rho_{\ell} - \rho_k} (\rho_k - \rho_{\ell}) = a_{k\ell} + (-1)a_{k\ell} = 0
\]
If the definition of $\alpha$ was a mystery, then no more. In the matrix representation of $T$ relative to $C$, the entry in column $\ell$, row $k$ is a zero. Nice. The only price we pay is that other entries in column $\ell$, specifically rows 1 through $k - 1$, may also change in a way we can’t control.

One more case to consider. Assume $j > \ell$. Then
\[
T(y_j) = T(x_j) = \sum_{i=1}^{j-1} a_{ij} x_i + \rho_j x_j = \sum_{i=1}^{j-1} a_{ij} x_i + a_{\ell j} x_\ell + a_{k j} x_k + \rho_j x_j
\]
\[
= \sum_{i=1}^{j-1} a_{ij} x_i + a_{\ell j} x_\ell + \alpha a_{\ell j} x_\ell - \alpha a_{\ell j} x_k + a_{k j} x_k + \rho_j x_j
\]
\[
= \sum_{i=1}^{j-1} a_{ij} x_i + a_{\ell j} (x_\ell + \alpha x_k) + (a_{k j} - \alpha a_{\ell j}) x_k + \rho_j x_j
\]
\[
= \sum_{i=1}^{j-1} a_{ij} y_i + a_{\ell j} y_\ell + (a_{k j} - \alpha a_{\ell j}) y_k + \rho_j y_j
\]
As before, we ask: how different are the matrix representations relative to $B$ and $C$ in column $j$? Only $y_k$ has a coefficient different from the corresponding coefficient when the basis is $B$. So in the matrix representations, the only entries to change are in row $k$, for columns $\ell + 1$ through $n$.

What have we accomplished? With a change of basis, we can place a zero in a desired entry (row $k$, column $\ell$) of the matrix representation, leaving most of the entries untouched. The only entries to possibly change are above the new zero entry, or to the right of the new zero entry. Suppose we repeat this procedure, starting by “zeroing out” the entry above the diagonal in the second column and first row. Then we move right to the third column, and zero out the element just above the diagonal in the second row. Next we zero out the element in the third column and first row. Then tackle the fourth column, work upwards from the diagonal, zeroing out entries as we go. Entries above, and to the right will repeatedly change, but newly created zeros will never get wrecked, since they are below, or just to the left of the entry we are working on. Similarly the values on the diagonal do not change either. This entire argument can be retooled in the language of change-of-basis matrices and similarity transformations, and this is the approach taken by Noble in his *Applied Linear Algebra*. It is interesting to concoct the change-of-basis matrix between the matrices $B$ and $C$ and compute the inverse.

Perhaps you have noticed that we have to be just a bit more careful than the previous paragraph suggests. The definition of $\alpha$ has a denominator that cannot be zero, which restricts our maneuvers to zeroing out entries in row $k$ and column $\ell$ only when $\rho_k \neq \rho_{\ell}$. So we do not necessarily arrive at a diagonal matrix. More carefully we can write
\[
T(y_j) = \sum_{i=1}^{j-1} b_{ij} y_i + \rho_j y_j
\]
where the \( b_{ij} \) are our new coefficients after repeated changes, the \( y_j \) are the new basis vectors, and the condition \( "i : \rho_i = \rho_j" \) means that we only have terms in the sum involving vectors whose final coefficients are identical diagonal values (the eigenvalues). Now reorder the basis vectors carefully. Group together vectors that have equal diagonal entries in the matrix representation, but within each group preserve the order of the precursor basis. This grouping will create a block diagonal structure for the matrix representation, while otherwise preserving the order of the basis will retain the upper triangular form of the representation. So we can arrive at a basis that yields a matrix representation that is upper triangular and block diagonal, with the diagonal entries of each block all equal to a common eigenvalue of the linear transformation.

More carefully, employing the distinct eigenvalues of \( T, \lambda_i, 1 \leq i \leq m \), we can assert there is a set of basis vectors for \( V, u_{ij}, 1 \leq i \leq m, 1 \leq j \leq \alpha_T (\lambda_i) \), such that

\[
T(u_{ij}) = \sum_{k=1}^{j-1} b_{ijk} u_{ik} + \lambda_i u_{ij}
\]

So the subspace \( U_i = \{ u_{ij} | 1 \leq j \leq \alpha_T (\lambda_i) \} \), \( 1 \leq i \leq m \) is an invariant subspace of \( V \) relative to \( T \) and the restriction \( T|_{U_i} \) has an upper triangular matrix representation relative to the basis \( \{ u_{ij} | 1 \leq j \leq \alpha_T (\lambda_i) \} \) where the diagonal entries are all equal to \( \lambda_i \). Notice too that with this definition,

\[
V = U_1 \oplus U_2 \oplus U_3 \oplus \cdots \oplus U_m
\]

Whew. This is a good place to take a break, grab a cup of coffee, use the toilet, or go for a short stroll, before we show that \( U_i \) is a subspace of the generalized eigenspace \( G_T (\lambda_i) \). This will follow if we can prove that each of the basis vectors for \( U_i \) is a generalized eigenvector of \( T \) for \( \lambda_i \) (Definition [GEV 568]). We need some power of \( T - \lambda_i I_V \) that takes \( u_{ij} \) to the zero vector. We prove by induction on \( j \) (Technique 626) the claim that \( (T - \lambda_i I_V)^j (u_{ij}) = 0 \). For \( j = 1 \) we have,

\[
(T - \lambda_i I_V) (u_{ij}) = T (u_{ij}) - \lambda_i I_V (u_{ij}) = T (u_{ij}) - \lambda_i u_{ij} = \lambda_i u_{ij} - \lambda_i u_{ij} = 0
\]

For the induction step, assume that if \( k < j \), then \( (T - \lambda_i I_V)^k \) takes \( u_{ik} \) to the zero vector. Then

\[
(T - \lambda_i I_V)^j (u_{ij}) = (T - \lambda_i I_V)^{j-1} ((T - \lambda_i I_V) (u_{ij})) = (T - \lambda_i I_V)^{j-1} (T (u_{ij}) - \lambda_i I_V (u_{ij})) = (T - \lambda_i I_V)^{j-1} (T (u_{ij}) - \lambda_i u_{ij}) = (T - \lambda_i I_V)^{j-1} \left( \sum_{k=1}^{j-1} b_{ijk} u_{ik} + \lambda_i u_{ij} - \lambda_i u_{ij} \right)
\]

\[
= (T - \lambda_i I_V)^{j-1} \left( \sum_{k=1}^{j-1} b_{ijk} u_{ik} \right) = \sum_{k=1}^{j-1} b_{ijk} (T - \lambda_i I_V)^{j-1} (u_{ik}) = \sum_{k=1}^{j-1} b_{ijk} (T - \lambda_i I_V)^{j-1-k} \left( (T - \lambda_i I_V)^k (u_{ik}) \right) = \sum_{k=1}^{j-1} b_{ijk} (T - \lambda_i I_V)^{j-1-k} (0) = \sum_{k=1}^{j-1} b_{ijk} 0
\]
This completes the induction step. Since every vector of the spanning set for \( U_i \) is an element of the subspace \( G_T(\lambda_i) \), Properties AC [251] and SC [251] allow us to conclude that \( U_i \subseteq G_T(\lambda_i) \). Then by Definition 264, \( U_i \) is a subspace of \( G_T(\lambda_i) \). Notice that this inductive proof could be interpreted to say that every element of \( U_i \) is a generalized eigenvector of \( T \) for \( \lambda_i \), and the algebraic multiplicity of \( \lambda_i \) is a sufficiently high power to demonstrate this via the definition for each vector.

We are now prepared for our final argument in this long proof. We wish to establish that the \( U \) that \( \lambda \) is a vector of hypothesis to \( T \) is a vector space with dimension \( m \). With Theorem PSSD [323] we see that \( \dim (V) \) the lone eigenvalue of \( \lambda \) is greater than or equal to the dimension of the generalized eigenspace \( G_T(\lambda) \). We want to show that \( \dim (\lambda) \)

\[
\#_{T}(\lambda) \geq \dim (G_T(\lambda)) = \dim (K((T - \lambda)^m)) = n ((T - \lambda)^m)
\]

For the base case, \( \dim V = 1 \). Every matrix representation of \( T \) is an upper triangular matrix with the lone eigenvalue of \( T \), \( \lambda \), as the diagonal entry. So \( \#_{T}(\lambda) = 1 \). The generalized eigenspace of \( \lambda \) is not trivial (since by Theorem GEK [569] it equals the regular eigenspace), and is a subspace of \( V \). With Theorem PSSD [323] we see that \( \dim (G_T(\lambda)) = 1 \).

Now for the induction step, assume the claim is true for any linear transformation defined on a vector space with dimension \( m - 1 \) or less. Suppose that \( B = \{v_1, v_2, v_3, \ldots, v_m\} \) is a basis for \( V \) that yields a diagonal matrix representation for \( T \) with diagonal entries \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m \). Then \( U = \langle (v_1, v_2, v_3, \ldots, v_{m-1}) \rangle \) is a subspace of \( V \) that is invariant relative to \( T \). The restriction \( T|_U : U \rightarrow U \) is then a linear transformation defined on \( U \), a vector space of dimension \( m - 1 \). A matrix representation of \( T|_U \) relative to the basis \( C = \{v_1, v_2, v_3, \ldots, v_{m-1}\} \) will be an upper triangular matrix with diagonal entries \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{m-1} \). We can therefore apply the induction hypothesis to \( T|_U \) and its representation relative to \( C \).

Suppose that \( \lambda \) is any eigenvalue of \( T \). Then suppose that \( v \in K((T - \lambda I)^m) \). As an element of \( V \), we can write \( v \) as a linear combination of the basis elements of \( B \), or more compactly, there is a vector \( u \in U \) and a scalar \( \alpha \) such that \( v = u + \alpha v_m \). Then,

\[
\alpha \ (\lambda_m - \lambda)^m v_m = 0
\]

The final expression in this string of equalities is an element of \( U \) since \( U \) is invariant relative to both \( T \) and \( I_V \). The expression at the beginning is a scalar multiple of \( v_m \), and as such cannot be a nonzero element of \( U \) without violating the linear independence of \( B \). So

\[
\alpha \ (\lambda_m - \lambda)^m v_m = 0
\]
The vector \( v_m \) is nonzero since \( B \) is linearly independent, so \( \text{Theorem SMEZV} \) \( 259 \) tells us that \( \alpha (\lambda_m - \lambda)^m = 0 \). From the properties of scalar multiplication, we are confronted with two possibilities.

Our first case is that \( \lambda \neq \lambda_m \). Notice then that \( \lambda \) occurs the same number of times along the diagonal in the representations of \( T|_U \) and \( T \). Now \( \alpha = 0 \) and \( v = u + \alpha v_m = u \). Since \( v \) was chosen as an arbitrary element of \( \mathcal{K}((T - \lambda I_V)^m) \), \( \text{Definition SSET} \) \( 615 \) says that \( \mathcal{K}((T - \lambda I_V)^m) \subseteq U \). It is always the case that \( \mathcal{K}((T|_U - \lambda I_U)^m) \subseteq \mathcal{K}((T - \lambda I_V)^m) \). However, we can also see that in this case, the opposite set inclusion is true as well. By \( \text{Definition SE} \) \( 616 \) we have \( \mathcal{K}((T|_U - \lambda I_U)^m) = \mathcal{K}((T - \lambda I_V)^m) \). Then

\[
\#_T(\lambda) = \#_{T|_U}(\lambda) \\
\geq \dim (\mathcal{G}_{T|_U}(\lambda)) \\
= \dim \left( \mathcal{K}\left((T|_U - \lambda I_U)^{m-1}\right) \right) \quad \text{Induction Hypothesis} \\
= \dim (\mathcal{K}((T|_U - \lambda I_U)^m)) \\
= \dim (\mathcal{K}((T - \lambda I_V)^m)) \\
= \dim (\mathcal{G}_T(\lambda)) \quad \text{Theorem GEK} \quad 569
\]

The second case is that \( \lambda = \lambda_m \). Notice then that \( \lambda \) occurs one more time along the diagonal in the representation of \( T \) compared to the representation of \( T|_U \). Then

\[
(T|_U - \lambda I_U)^m (u) = (T - \lambda I_V)^m (u) \\
= (T - \lambda I_V)^m (u) + 0 \\
= (T - \lambda I_V)^m (u) + (T - \lambda I_V)^m (v_m) \\
= (T - \lambda I_V)^m (u + \alpha v_m) \\
= (T - \lambda I_V)^m (v) \\
= 0 \
\]

So \( u \in \mathcal{K}(T|_U - \lambda I_U) \). The vector \( v \) is an arbitrary member of \( \mathcal{K}((T - \lambda I_V)^m) \) and is also equal to an element of \( \mathcal{K}(T|_U - \lambda I_U) \) (\( u \)) plus a scalar multiple of the vector \( v_m \). This observation yields

\[
\dim (\mathcal{K}((T - \lambda I_V)^m)) \leq \dim (\mathcal{K}(T|_U - \lambda I_U)) + 1
\]

Now count eigenvalues on the diagonal,

\[
\#_T(\lambda) = \#_{T|_U}(\lambda) + 1 \\
\geq \dim (\mathcal{G}_{T|_U}(\lambda)) + 1 \\
= \dim \left( \mathcal{K}\left((T|_U - \lambda I_U)^{m-1}\right) \right) + 1 \\
= \dim (\mathcal{K}((T|_U - \lambda I_U)^m)) + 1 \\
\geq \dim (\mathcal{K}((T - \lambda I_V)^m)) \\
= \dim (\mathcal{G}_T(\lambda)) \quad \text{Theorem GEK} \quad 569
\]

In \( \text{Theorem UTMR} \) \( 582 \) we constructed an upper triangular matrix representation of \( T \) where each eigenvalue occurred \( \alpha_T(\lambda) \) times on the diagonal. So

\[
\alpha_T(\lambda_i) = \#_T(\lambda_i) \quad \text{Theorem UTMR} \quad 582 \\
\geq \dim (\mathcal{G}_T(\lambda_i)) \\
\geq \dim (U_i) \quad \text{Theorem PSSD} \quad 323
\]
Thus, \( \dim (G_T(\lambda_i)) = \alpha_T(\lambda_i) \) and by Theorem EDYES \[323\], \( U_i = G_T(\lambda_i) \) and we can write

\[
V = U_1 \oplus U_2 \oplus U_3 \oplus \cdots \oplus U_m \\
= G_T(\lambda_1) \oplus G_T(\lambda_2) \oplus G_T(\lambda_3) \oplus \cdots \oplus G_T(\lambda_m)
\]

Besides a nice decomposition into invariant subspaces, this proof has a bonus for us.

**Theorem DGES**

**Dimension of Generalized Eigenspaces**

Suppose \( T: V \mapsto V \) is a linear transformation with eigenvalue \( \lambda \). Then the dimension of the generalized eigenspace for \( \lambda \) is the algebraic multiplicity of \( \lambda \), \( \dim (G_T(\lambda_i)) = \alpha_T(\lambda_i) \).

**Proof** At the very end of the proof of Theorem GESD \[584\], we obtain the inequalities

\[
\alpha_T(\lambda_i) \leq \dim (G_T(\lambda_i)) \leq \alpha_T(\lambda_i)
\]

which establishes the desired equality.

**Subsection JCF**

**Jordan Canonical Form**

Now we are in a position to define what we (and others) regard as an especially nice matrix representation. The word “canonical” has at its root, the word “canon,” which has various meanings. One is the set of laws established by a church council. Another is a set of writings that are authentic, important or representative. Here we take to to mean the accepted, or best, representative among a variety of choices. Every linear transformation admits a variety of representations, and will declare one as the best. Hopefully you will agree.

**Definition JCF**

**Jordan Canonical Form**

A square matrix is in **Jordan canonical form** if it meets the following requirements:

1. The matrix is block diagonal.
2. Each block is a Jordan block.
3. If \( \rho < \lambda \) then the block \( J_k(\rho) \) occupies rows with indices greater than the indices of the rows occupied by \( J_\ell(\rho) \).
4. If \( \rho = \lambda \) and \( \ell < k \), then the block \( J_k(\lambda) \) occupies rows with indices greater than the indices of the rows occupied by \( J_\ell(\lambda) \).

**Theorem JCFLT**

**Jordan Canonical Form for a Linear Transformation**

Suppose \( T: V \mapsto V \) is a linear transformation. Then there is a basis \( B \) for \( V \) such that the matrix representation of \( T \) with the following properties:

1. The matrix representation is in Jordan canonical form.
2. If \( J_k(\lambda) \) is one of the Jordan blocks, then \( \lambda \) is an eigenvalue of \( T \).
3. For a fixed value of \( \lambda \), the largest block of the form \( J_k(\lambda) \) has size equal to the index of \( \lambda \), \( \nu_T(\lambda) \).

4. For a fixed value of \( \lambda \), the number of blocks of the form \( J_k(\lambda) \) is the geometric multiplicity of \( \lambda \), \( \gamma_T(\lambda) \).

5. For a fixed value of \( \lambda \), the number of rows occupied by blocks of the form \( J_k(\lambda) \) is the algebraic multiplicity of \( \lambda \), \( \alpha_T(\lambda) \).

\[ \square \]

**Proof** This theorem is really just the consequence of applying to \( T \), consecutively Theorem GESD \[584\], Theorem MRRGE \[580\] and Theorem CFNLT \[557\].

Theorem GESD \[584\] gives us a decomposition of \( V \) into generalized eigenspaces, one for each distinct eigenvalue. Since these generalized eigenspaces are invariant relative to \( T \), this provides a block diagonal matrix representation where each block is the matrix representation of the restriction of \( T \) to the generalized eigenspace.

Restricting \( T \) to a generalized eigenspace results in a “nearly nilpotent” linear transformation, as stated more precisely in Theorem RGEN \[578\]. We unravel Theorem RGEN \[578\] in the proof of Theorem MRRGE \[580\] so that we can apply Theorem CFNLT \[557\] about representations of nilpotent linear transformations.

We know the dimension of a generalized eigenspace is the algebraic multiplicity of the eigenvalue (Theorem DGES \[590\]), so the blocks associated with the generalized eigenspaces are square with a size equal to the algebraic multiplicity. In refining the basis for this block, and producing Jordan blocks the results of Theorem CFNLT \[557\] apply. The total number of blocks will be the nullity of \( T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)} \), which is the geometric multiplicity of \( \lambda \) as an eigenvalue of \( T \) (Definition GME \[366\]). The largest of the Jordan blocks will have size equal to the index of the nilpotent linear transformation \( T|_{\mathcal{G}_T(\lambda)} - \lambda I_{\mathcal{G}_T(\lambda)} \), which is exactly the definition of the index of the eigenvalue \( \lambda \) (Definition IE \[578\]).

\[ \square \]

Before we do some examples of this result, notice how close Jordan canonical form is to a diagonal matrix. Or, equivalently, notice how close we have come to diagonalizing a matrix (Definition DZM \[393\]). We have a matrix representation which has diagonal entries that are the eigenvalues of a matrix. Each occurs on the diagonal as many times as the algebraic multiplicity. However, when the geometric multiplicity is strictly less than the algebraic multiplicity, we have some entries in the representation just above the diagonal (the “superdiagonal”). Furthermore, we have some idea how often this happens if we know the geometric multiplicity and the index of the eigenvalue.

We now recognize just how simple a diagonalizable linear transformation really is. For each eigenvalue, the generalized eigenspace is just the regular eigenspace, and it decomposes into a direct sum of one-dimensional subspaces, each spanned by a different eigenvector chosen from a basis of eigenvectors for the eigenspace.

Some authors create matrix representations of nilpotent linear transformations where the Jordan block has the ones just below the diagonal (the “subdiagonal”). No matter, it is really the same, just different. We have also defined Jordan canonical form to place blocks for the larger eigenvalues earlier, and for blocks with the same eigenvalue, we place the bigger ones earlier. This is fairly standard, but there is no reason we couldn’t order the blocks differently. It’d be the same, just different. The reason for choosing some ordering is to be assured that there is just one canonical matrix representation for each linear transformation.

**Example JCF10**

Jordan canonical form, size 10
Suppose that $T : \mathbb{C}^{10} \rightarrow \mathbb{C}^{10}$ is the linear transformation defined by $T(x) = Ax$ where

$$A = \begin{bmatrix}
-6 & 9 & -7 & -5 & 5 & 12 & -22 & 14 & 8 & 21 \\
-3 & 5 & -3 & -1 & 2 & 7 & -12 & 9 & 1 & 12 \\
8 & -9 & 8 & 6 & 0 & -14 & 25 & -13 & -4 & -26 \\
-7 & 9 & -7 & -5 & 0 & 13 & -23 & 13 & 2 & 24 \\
0 & -1 & 0 & -1 & -3 & -2 & 3 & -4 & -2 & -3 \\
3 & 2 & 1 & 2 & 9 & -1 & 1 & 5 & 5 & -5 \\
-1 & 3 & -3 & -2 & 4 & 3 & -6 & 4 & 4 & 3 \\
3 & -4 & 3 & 2 & 1 & -5 & 9 & -5 & 1 & -9 \\
0 & 2 & 0 & 0 & 2 & 2 & -4 & 4 & 2 & 4 \\
-4 & 4 & -5 & -4 & -1 & 6 & -11 & 4 & 1 & 10
\end{bmatrix}$$

We’ll find a basis for $\mathbb{C}^{10}$ that will yield a matrix representation of $T$ in Jordan canonical form.

First we find the eigenvalues, and their multiplicities, with the techniques of Chapter E [356].

- $\lambda = 2$ \quad $\alpha_T(2) = 2$ \quad $\gamma_T(2) = 2$
- $\lambda = 0$ \quad $\alpha_T(0) = 3$ \quad $\gamma_T(-1) = 2$
- $\lambda = -1$ \quad $\alpha_T(-1) = 5$ \quad $\gamma_T(-1) = 2$

For each eigenvalue, we can compute a generalized eigenspace. By Theorem GESD [584] we know that $\mathbb{C}^{10}$ will decompose into a direct sum of these eigenspaces, and we can restrict $T$ to each part of this decomposition. At this stage we know that the Jordan canonical form will be block diagonal with blocks of size 2, 3, and 5, since the dimensions of the generalized eigenspaces are equal to the algebraic multiplicities of the eigenvalues (Theorem DGES [590]). The geometric multiplicities tell us how many Jordan blocks occupy each of the three larger blocks, but we will discuss this as we analyze each eigenvalue. We do not yet know the index of each eigenvalue (though we can easily infer it for $\lambda = 2$) and even if we did have this information, it only determines the size of the largest Jordan block (per eigenvalue). We will press ahead, considering each eigenvalue one at a time.

The eigenvalue $\lambda = 2$ has “full” geometric multiplicity, and is not an impediment to diagonalizing $T$. We will treat it in full generality anyway. First we compute the generalized eigenspace. Since Theorem GEK [569] says that $\mathcal{G}_T(2) = \mathcal{K}\left((T - 2I_{\mathbb{C}^{10}})^{10}\right)$ we can compute this generalized eigenspace as a null space derived from the matrix $A$.

\begin{equation}
(A - 2I_{10})^{10} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{equation}

$$\mathcal{G}_T(2) = \mathcal{K}\left((A - 2I_{10})^{10}\right) = \left\{ \begin{bmatrix}
2 & 1 & -1 & 1 & -1 & 2 & -1 & 0 & 0 & 0 \\
1 & -1 & 2 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 1 & -1 & -2 & 0 & -1 & 0 & 0 & 0 & 0
\end{bmatrix} \right\}$$
The restriction of \( T \) to \( \mathcal{G}_T(2) \) relative to the two basis vectors above has a matrix representation that is a \( 2 \times 2 \) diagonal matrix with the eigenvalue \( \lambda = 2 \) as the diagonal entries. So these two vectors will be the first two vectors in our basis for \( \mathbb{C}^{10} \),

\[
\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \\ -1 \\ 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}
\]

Notice that it was not strictly necessary to compute the 10-th power of \( A - 2I_{10} \). With \( \alpha_T(2) = \gamma_T(2) \) the null space of the matrix \( A - 2I_{10} \) contains all of the generalized eigenvectors of \( T \) for the eigenvalue \( \lambda = 2 \). But there was no harm in computing the 10-th power either. This discussion is equivalent to the observation that the linear transformation \( T|_{\mathcal{G}_T(2)}: \mathcal{G}_T(2) \rightarrow \mathcal{G}_T(2) \) is nilpotent of index 1. In other words, \( \nu_T(2) = 1 \).

The eigenvalue \( \lambda = 0 \) will not be quite as simple, since the geometric multiplicity is strictly less than the geometric multiplicity. As before, we first compute the generalized eigenspace. Since Theorem GEK \[569\] says that \( \mathcal{G}_T(0) = \mathcal{K}\left((A - 0I_{10})^{10}\right) \) we can compute this generalized eigenspace as a null space derived from the matrix \( A \),

\[
(A - 0I_{10})^{10} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\mathcal{G}_T(0) = \mathcal{K}\left((A - 0I_{10})^{10}\right) = \left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix} \right\rangle = \langle F \rangle
\]

So \( \dim(\mathcal{G}_T(0)) = 3 = \alpha_T(0) \), as expected. We will use these three basis vectors for the generalized eigenspace to construct a matrix representation of \( T|_{\mathcal{G}_T(0)} \), where \( F \) is being defined implicitly as
the basis of $G_T(0)$. We construct this representation as usual, applying Definition MR 485.

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
-1 \\
0 \\
0 \\
-1
\end{pmatrix}
= \begin{pmatrix}
0 \\
-1 \\
0 \\
-1
\end{pmatrix}
\]

By Theorem RGEN 578 we can obtain a nilpotent matrix from this matrix representation by subtracting the eigenvalue from the diagonal elements, and then we can apply Theorem CFNLT 557 to $M - (0)I_3$. First check that $(M - (0)I_3)^2 = \mathcal{O}$, so we know that the index of $M - (0)I_3$ as a nilpotent matrix, and that therefore $\lambda = 0$ is an eigenvalue of $T$ with index 2, $\nu_T (0) = 2$. To determine a basis of $\mathbb{C}^3$ that converts $M - (0)I_3$ to canonical form, we need the null spaces of the powers of $M - (0)I_3$. For convenience, set $N = M - (0)I_3$.

\[
N(N^1) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle
\]

\[
N(N^2) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle = \mathbb{C}^3
\]

Then we choose a vector from $N(N^2)$ that is not an element of $N(N^1)$. Any vector with unequal
first two entries will fit the bill, say

\[ z_{2,1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

where we are employing the notation in Theorem CFNLT \[557\]. The next step is to multiply this vector by \( N \) to get part of the basis for \( \mathcal{N}(N^1) \),

\[ z_{1,1} = Nz_{2,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \]

We need a vector to pair with \( z_{1,1} \) that will make a basis for the two-dimensional subspace \( \mathcal{N}(N^1) \).

Examining the basis for \( \mathcal{N}(N^1) \) we see that a vector with its first two entries equal will do the job.

\[ z_{1,2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \]

Reordering, we find the basis,

\[ C = \{ z_{1,1}, z_{2,1}, z_{1,2} \} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \]

From this basis, we can get a matrix representation of \( N \) (when viewed as a linear transformation) relative to the basis \( C \) for \( \mathbb{C}^3 \),

\[ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} J_2(0) & 0 \\ 0 & J_1(0) \end{bmatrix} \]

Now we add back the eigenvalue \( \lambda = 0 \) to the representation of \( N \) to obtain a representation for \( M \). Of course, with an eigenvalue of zero, the change is not apparent, so we won’t display the same matrix again. This is the second block of the Jordan canonical form for \( T \). However, the three vectors in \( C \) will not suffice as basis vectors for the domain of \( T \) — they have the wrong size! The vectors in \( C \) are vectors in the domain of a linear transformation defined by the matrix \( M \). But \( M \) was a matrix representation of \( T|_{\mathcal{G}_T(0)} - 0I_{\mathcal{G}_T(0)} \) relative to the basis \( F \) for \( \mathcal{G}_T(0) \). We need to “uncoordinatize” each of the basis vectors in \( C \) to produce a linear combination of vectors in \( F \) that will be an element of the generalized eigenspace \( \mathcal{G}_T(0) \). These will be the next three vectors of our final answer, a basis for \( \mathbb{C}^{10} \) that has a pleasing matrix representation.

\[ \mathbf{v}_3 = \rho_F^{-1} \left( \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) = 0 + 0 + (-1) = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \]
\[ v_4 = \rho_F^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]

\[ v_5 = \rho_F^{-1} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \]

Five down, five to go. Basis vectors, that is. \( \lambda = -1 \) is the smallest eigenvalue, but it will require the most computation. First we compute the generalized eigenspace. Since Theorem GEK [569] says that \( \mathcal{G}_T (-1) = \mathcal{K} \left( (T - (-1)I_{C_{10}})^{10} \right) \) we can compute this generalized eigenspace as a null space derived from the matrix \( A \),

\[ (A - (-1)I_{10})^{10} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ \mathcal{G}_T (-1) = \mathcal{K} \left( (A - (-1)I_{10})^{10} \right) = \begin{Bmatrix} [-1] & [-1] & [1] & [-1] & [-1] \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Bmatrix} = \langle F \rangle \]

So \( \dim (\mathcal{G}_T (-1)) = 5 = \alpha_T (-1) \), as expected. We will use these five basis vectors for the generalized eigenspace to construct a matrix representation of \( T|_{\mathcal{G}_T (-1)} \), where \( F \) is being recycled and defined now implicitly as the basis of \( \mathcal{G}_T (-1) \). We construct this representation as usual, applying...
Definition MR 485.

\[
\rho_F \left( T|_{\mathcal{G}_T(-1)} - (-1)I_{\mathcal{G}_T(-1)} \right) = \rho_F \left( \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
= \rho_F \left( \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + 0 \left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) + (-2) \left( \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) + 0 \left( \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + (-1) \left( \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}
\]

\[
\rho_F \left( T|_{\mathcal{G}_T(-1)} - (-1)I_{\mathcal{G}_T(-1)} \right) = \rho_F \left( \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -5 \\ -3 \\ 3 \\ -1 \\ 2 \\ 4 \\ 0 \\ 0 \\ 3 \end{bmatrix}
\]

\[
= \rho_F \left( \begin{bmatrix} 7 \\ 1 \\ -5 \\ 3 \\ -1 \\ 2 \\ 4 \\ 0 \\ 0 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} -5 \\ -1 \\ -4 \\ 0 \\ 3 \end{bmatrix}
\]

\[
\rho_F \left( T|_{\mathcal{G}_T(-1)} - (-1)I_{\mathcal{G}_T(-1)} \right) = \rho_F \left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
= \rho_F \left( \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

Version 1.04
So we have the matrix representation of the restriction of \( T \) (again recycling and redefining the
matrix $M$)

$$M = M_{F,F}^{T[2]}(-1) = \begin{bmatrix} 0 & -5 & -1 & 2 & 6 \\ 0 & -1 & 0 & -1 & -1 \\ -2 & 4 & 1 & -1 & -6 \\ 0 & 0 & 0 & 1 & 2 \\ -1 & 3 & 1 & -2 & -6 \end{bmatrix}$$

By Theorem RGEN [578] we can obtain a nilpotent matrix from this matrix representation by subtracting the eigenvalue from the diagonal elements, and then we can apply Theorem CFNLT [557] to $M - (-1)I_5$. First check that $(M - (-1)I_5)^3 = O$, so we know that the index of $M - (-1)I_5$ as a nilpotent matrix, and that therefore $\lambda = -1$ is an eigenvalue of $T$ with index 3, $\nu_T(-1) = 3$.

To determine a basis of $\mathbb{C}^5$ that converts $M - (-1)I_5$ to canonical form, we need the null spaces of the powers of $M - (-1)I_5$. Again, for convenience, set $N = M - (-1)I_5$.

$$\mathcal{N}(N^1) = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ -2 \\ 2 \end{bmatrix} \rangle$$

$$\mathcal{N}(N^2) = \langle \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rangle$$

$$\mathcal{N}(N^3) = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rangle = \mathbb{C}^5$$

Then we choose a vector from $\mathcal{N}(N^3)$ that is not an element of $\mathcal{N}(N^2)$. The sum of the four basis vectors for $\mathcal{N}(N^2)$ sum to a vector with all five entries equal to 1. We will mess with the first entry to create a vector not in $\mathcal{N}(N^2)$,

$$z_{3,1} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

where we are employing the notation in Theorem CFNLT [557]. The next step is to multiply this vector by $N$ to get a portion of the basis for $\mathcal{N}(N^2)$,

$$z_{2,1} = Nz_{3,1} = \begin{bmatrix} 1 & -5 & -1 & 2 & 6 \\ 0 & 0 & 0 & -1 & -1 \\ -2 & 4 & 2 & -1 & -6 \\ 0 & 0 & 0 & 2 & 2 \\ -1 & 3 & 1 & -2 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \\ 4 \\ -3 \end{bmatrix}$$

We have a basis for the two-dimensional subspace $\mathcal{N}(N^1)$ and we can add to that the vector $z_{2,1}$ and we have three of four basis vectors for $\mathcal{N}(N^2)$. These three vectors span the subspace we call $Q_2$. We need a fourth vector outside of $Q_2$ to complete a basis of the four-dimensional subspace.
\( N(N^2) \). Check that the vector

\[
\mathbf{z}_{2,2} = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}
\]

is an element of \( N(N^2) \) that lies outside of the subspace \( Q_2 \). This vector was constructed by getting a nice basis for \( Q_2 \) and forming a linear combination of this basis that specifies three of the five entries of the result. Of the remaining two entries, one was changed to move the vector outside of \( Q_2 \) and this was followed by a change to the remaining entry to place the vector into \( N(N^2) \). The vector \( \mathbf{z}_{2,2} \) is the lone basis vector for the subspace we call \( R_2 \).

The remaining two basis vectors are easy to come by. They are the result of applying \( N \) to each of the two most recently determined basis vectors,

\[
\begin{align*}
\mathbf{z}_{1,1} &= N\mathbf{z}_{2,1} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \\
\mathbf{z}_{1,2} &= N\mathbf{z}_{2,2} = \begin{bmatrix} 3 \\ -2 \\ -3 \\ 4 \\ -4 \end{bmatrix}
\end{align*}
\]

Now we reorder these basis vectors, to arrive at the basis

\[
C = \{ \mathbf{z}_{1,1}, \mathbf{z}_{2,1}, \mathbf{z}_{3,1}, \mathbf{z}_{1,2}, \mathbf{z}_{2,2} \} = \left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \\ -4 \\ 1 \end{bmatrix} \right\}
\]

A matrix representation of \( N \) relative to \( C \) is

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
J_3(0) & O \\
O & J_2(0)
\end{bmatrix}
\]

To obtain a matrix representation of \( M \), we add back in the matrix \((-1)I_5\), placing the eigenvalue back along the diagonal, and slightly modifying the Jordan blocks,

\[
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & -1
\end{bmatrix} = \begin{bmatrix}
J_3(-1) & O \\
O & J_2(-1)
\end{bmatrix}
\]

The basis \( C \) yields a pleasant matrix representation for the restriction of the linear transformation \( T - (-1)I \) to the generalized eigenspace \( \mathcal{G}_T(-1) \). However, we must remember that these vectors in \( \mathbb{C}^5 \) are representations of vectors in \( \mathbb{C}^{10} \) relative to the basis \( F \). Each needs to be “un-coordinatized”
before joining our final basis. Here we go,

$$v_6 = \rho_F^{-1} \begin{pmatrix} 3 \\ -1 \\ 0 \\ 2 \\ -2 \end{pmatrix} = 3 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (-2) \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v_7 = \rho_F^{-1} \begin{pmatrix} 2 \\ -2 \\ -1 \\ 4 \\ -3 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (-2) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} + (-3) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ -2 \end{pmatrix}$$

$$v_8 = \rho_F^{-1} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$v_9 = \rho_F^{-1} \begin{pmatrix} 3 \\ -2 \\ -3 \\ 4 \\ -4 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (-2) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + (-3) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + (-4) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$v_{10} = \rho_F^{-1} \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$
To summarize, we list the entire basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_{10}\}$,

\[
\begin{align*}
\mathbf{v}_1 &= \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, & \mathbf{v}_2 &= \begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & \mathbf{v}_3 &= \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, & \mathbf{v}_4 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, & \mathbf{v}_5 &= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\mathbf{v}_6 &= \begin{bmatrix} -2 \\ -1 \\ -3 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & \mathbf{v}_7 &= \begin{bmatrix} -2 \\ -2 \\ -3 \\ -2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, & \mathbf{v}_8 &= \begin{bmatrix} -2 \\ 0 \\ 0 \\ -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, & \mathbf{v}_9 &= \begin{bmatrix} -4 \\ -2 \\ 1 \\ -1 \\ 4 \\ 4 \\ 1 \\ 0 \end{bmatrix}, & \mathbf{v}_{10} &= \begin{bmatrix} -3 \\ -2 \\ -3 \\ -2 \\ 3 \\ 3 \\ 3 \\ 0 \end{bmatrix} \\
\end{align*}
\]
Theorem CHT

Cayley-Hamilton Theorem

Suppose $A$ is a square matrix with characteristic polynomial $p_A(x)$. Then $p_A(A) = O$. □

Proof

Suppose $B$ and $C$ are similar matrices via the matrix $S$, $B = S^{-1}CS$, and $q(x)$ is any polynomial. Then $q(B)$ is similar to $q(C)$ via $S$, $q(B) = S^{-1}q(C)S$. (See Example HPDM 399 for hints on how to convince yourself of this.)

By Theorem JCFLT 590 and Theorem SCB 524, we know $A$ is similar to a matrix, $J$, in Jordan canonical form. Suppose $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_m$ are the distinct eigenvalues of $A$ (and are therefore the eigenvalues and diagonal entries of $J$). Then by Theorem EMRCF 363 and Definition AME 366, we can factor the characteristic polynomial as

$$p_A(x) = (x - \lambda_1)^{\alpha_A(\lambda_1)}(x - \lambda_2)^{\alpha_A(\lambda_2)}(x - \lambda_3)^{\alpha_A(\lambda_3)}\cdots(x - \lambda_m)^{\alpha_A(\lambda_m)}$$

On substituting the matrix $J$ we have

$$p_A(J) = (J - \lambda_1I)^{\alpha_A(\lambda_1)}(J - \lambda_2I)^{\alpha_A(\lambda_2)}(J - \lambda_3I)^{\alpha_A(\lambda_3)}\cdots(J - \lambda_mI)^{\alpha_A(\lambda_m)}$$

The matrix $J - \lambda_kI$ will be block diagonal, and the block arising from the generalized eigenspace for $\lambda_k$ will have zeros along the diagonal. Suitably adjusted for matrices (rather than linear transformations), Theorem RGEN 578 tells us this matrix is nilpotent. Since the size of this nilpotent matrix is equal to the algebraic multiplicity of $\lambda_k$, the power $(J - \lambda_kI)^{\alpha_A(\lambda_k)}$ will be a zero matrix (Theorem KPNLT 555) in the location of this block.

Repeating this argument for each of the $m$ eigenvalues will place a zero block in some term of the product at every location on the diagonal. The entire product will then be zero blocks on the diagonal, and zero off the diagonal. In other words, it will be the zero matrix. Since $A$ and $J$ are similar, $p_A(A) = p_A(J) = O$. ■
Appendix CN
Computation Notes

Section MMA
Mathematica

Computation Note ME.MMA
Matrix Entry

Matrices are input as lists of lists, since a list is a basic data structure in Mathematica. A matrix is a list of rows, with each row entered as a list. Mathematica uses braces (([], [])) to delimit lists. So the input

\[ a = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}\} \]

would create a 3 × 4 matrix named \( a \) that is equal to

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12
\end{pmatrix}
\]

To display a matrix named \( a \) “nicely” in Mathematica, type \( \text{MatrixForm}[a] \), and the output will be displayed with rows and columns. If you just type \( a \), then you will get a list of lists, like how you input the matrix in the first place.

Computation Note RR.MMA
Row Reduce

If \( a \) is the name of a matrix in Mathematica, then the command \( \text{RowReduce}[a] \) will output the reduced row-echelon form of the matrix.

Computation Note LS.MMA
Linear Solve

Mathematica will solve a linear system of equations using the \( \text{LinearSolve[]} \) command. The inputs are a matrix with the coefficients of the variables (but not the column of constants), and a list containing the constant terms of each equation. This will look a bit odd, since the lists in the
matrix are rows, but the column of constants is also input as a list and so looks like a row rather than a column. The result will be a single solution (even if there are infinitely many), reported as a list, or the statement that there is no solution. When there are infinitely many, the single solution reported is exactly that solution used in the proof of **Theorem RCLS** [45], where the free variables are all set to zero, and the dependent variables come along with values from the final column of the row-reduced matrix.

As an example, **Archetype A** [634] is

\[
\begin{align*}
  x_1 - x_2 + 2x_3 &= 1 \\
  2x_1 + x_2 + x_3 &= 8 \\
  x_1 + x_2 &= 5
\end{align*}
\]

To ask *Mathematica* for a solution, enter

\[
\text{LinearSolve} \left[ \{\{1, -1, 2\}, \{2, 1, 1\}, \{1, 1, 0\}\}, \{1, 8, 5\} \right]
\]

and you will get back the single solution

\[\{3, 2, 0\}\]

We will see later how to coax *Mathematica* into giving us infinitely many solutions for this system (**Computation VFSS.MMA** [606]).

**Computation Note VLC.MMA**

**Vector Linear Combinations**

Contributed by Robert Beezer

Vectors in *Mathematica* are represented as lists, written and displayed horizontally. For example, the vector

\[
v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}
\]

would be entered and named via the command

\[
v = \{1, 2, 3, 4\}
\]

Vector addition and scalar multiplication are then very natural. If \(u\) and \(v\) are two lists of equal length, then

\[
2u + (-3)v
\]

will compute the correct vector and return it as a list. If \(u\) and \(v\) have different sizes, then *Mathematica* will complain about “objects of unequal length.”

**Computation Note NS.MMA**

**Null Space**

Given a matrix \(A\), *Mathematica* will compute a set of column vectors whose span is the null space of the matrix with the \texttt{NullSpace[]} command. Perhaps not coincidentally, this set is exactly \(\{z_j | 1 \leq j \leq n - r\}\). However, *Mathematica* prefers to output the vectors in the opposite order than one we have chosen. Here’s a small example.

Begin with the \(3 \times 4\) matrix \(A\), and its row-reduced version \(B\),

\[
A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 4 & 1 & -2 \\ -1 & 1 & -5 & 3 \end{bmatrix} \quad \text{RREF} \quad B = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]
We could extract entries from $B$ to build the vectors $z_1$ and $z_2$ according to Theorem SSNS [107] and describe $N(A)$ as a span of the set $\{z_1, z_2\}$. Instead, if $a$ has been set to $A$, then executing the command `NullSpace[a]` yields the list of lists (column vectors),

$$\{\{2, -1, 0, 1\}, \{-3, 2, 1, 0\}\}$$

Notice how our $z_1$ is second in the list. To “correct” this we can use a list-processing command from Mathematica, `Reverse[]`, as follows,

```
Reverse[NullSpace[a]]
```

and receive the output in our preferred order. Give it a try yourself.

**Computation Note VFSS.MMA**

**Vector Form of Solution Set**

Suppose that $A$ is an $m \times n$ matrix and $b \in \mathbb{C}^m$ is a column vector. We might wish to find all of the solutions to the linear system $LS(A, b)$. Mathematica’s `LinearSolve[A, b]` will return at most one solution (Computation LS.MMA [604]). However, when the system is consistent, then this one solution reported is exactly the vector $c$, described in the statement of Theorem VFSLS [88].

The vectors $u_j$, $1 \leq j \leq n - r$ of Theorem VFSLS [88] are exactly the output of Mathematica’s `NullSpace[]` command, though Mathematica lists them in the opposite order from the order we have chosen. These are the same vectors listed as $z_j$, $1 \leq j \leq n - r$ in Theorem SSNS [107]. With $c$ produced from the `LinearSolve[]` command, and the $u_j$ coming from the `NullSpace[]` command we can use Mathematica’s symbolic manipulation commands to create an expression that describes all of the solutions.

Begin with the system $LS(A, b)$. Row-reduce $A$ (Computation RR.MMA [604]) and identify the free variables by determining the non-pivot columns. Suppose, for the sake of argument, that we have the three free variables $x_3$, $x_7$ and $x_8$. Then the following command will build an expression for an arbitrary solution:

```
LinearSolve[A, b]+{x8, x7, x3}.NullSpace[A]
```

Be sure to include the “dot” right before the `NullSpace[]` command — it has the effect of creating a linear combination of the vectors in the null space, using scalars that are symbols reminiscent of the variables.

A concrete example should help here. Suppose we want a solution set for the linear system with coefficient matrix $A$ and vector of constants $b$,

$$A = \begin{bmatrix} 1 & 2 & 3 & -5 & 1 & -1 & 2 \\ 2 & 4 & 0 & 8 & -4 & 1 & -8 \\ 3 & 6 & 4 & 0 & -2 & 5 & 7 \end{bmatrix} \quad b = \begin{bmatrix} 8 \\ 1 \\ -5 \end{bmatrix}$$

If we were to apply Theorem VFSLS [88], we would extract the components of $c$ and $u_j$ from the row-reduced version of the augmented matrix of the system (obtained with Mathematica, Computation RR.MMA [604]),

$$\begin{bmatrix} [1] & 2 & 0 & 4 & -2 & 0 & -5 & 2 \\ 0 & 0 & [1] & -3 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & [1] & 2 & -3 \end{bmatrix}$$

Instead, we will use this augmented matrix in reduced row-echelon form only to identify the free variables. In this example, we locate the non-pivot columns and see that $x_2, x_4, x_5$ and $x_7$ are
free. If we have set \( a \) to the coefficient matrix and \( b \) to the vector of constants, then we execute the Mathematica command,

\[
\text{LinearSolve}[a, b] + \{x_7, x_5, x_4, x_2\}.\text{NullSpace}[a]
\]

As output we obtain the column vector (list),

\[
\begin{bmatrix}
2 - 2 x_2 & -4 x_4 & +2 x_5 & +5 x_7 \\
x_2 & 1 + 3 x_4 & -x_5 & -3 x_7 \\
x_4 & x_5 & -3 - 2 x_7 & x_7
\end{bmatrix}
\]

Computation Note GSP.MMA

Gram-Schmidt Procedure

Mathematica has a built-in routine that will do the Gram-Schmidt procedure (Theorem GSPCV [158]). The input is a set of vectors, which must be linearly independent. This is written as a list, containing lists that are the vectors. Let \( a \) be such a list of lists, containing the vectors \( v_i \), \( 1 \leq i \leq p \) from the statement of the theorem. You will need to first load the right Mathematica package — execute `<<LinearAlgebra'Orthogonalization'` to make this happen. Then execute `GramSchmidt[a]`. The output will be another list of lists containing the vectors \( u_i \), \( 1 \leq i \leq p \) from the statement of the theorem. Mathematica will complain if you do not provide a linearly independent set as input (try it!).

An example. Suppose our linearly independent set (check this!) is

\[
S = \left\{ \begin{bmatrix}
-1 \\
4 \\
1 \\
0 \\
3
\end{bmatrix}, \begin{bmatrix}
0 \\
3 \\
0 \\
3 \\
-3
\end{bmatrix}, \begin{bmatrix}
-1 \\
2 \\
1 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
-2 \\
1 \\
-3 \\
-1 \\
4
\end{bmatrix}, \begin{bmatrix}
1 \\
6 \\
1 \\
4 \\
6
\end{bmatrix} \right\}
\]

The output of the `GramSchmidt[]` command will be the set,

\[
T = \left\{ \begin{bmatrix}
12 \sqrt{15} \\
12 \sqrt{15} \\
3 \sqrt{3} \\
3 \sqrt{3} \\
1 \sqrt{3}
\end{bmatrix}, \begin{bmatrix}
2 \sqrt{120423} \\
3 \sqrt{120423} \\
6 \sqrt{120423} \\
6 \sqrt{120423} \\
3 \sqrt{120423}
\end{bmatrix}, \begin{bmatrix}
-37 \\
\frac{29}{4 \sqrt{685}} \\
\frac{3}{4 \sqrt{685}} \\
\frac{79}{4 \sqrt{685}} \\
\frac{1 \sqrt{685}}{4 \sqrt{685}}
\end{bmatrix}, \begin{bmatrix}
337 \\
337 \\
337 \\
337 \\
337
\end{bmatrix}, \begin{bmatrix}
23 \\
\frac{21}{\sqrt{879}} \\
\frac{24}{\sqrt{879}} \\
\frac{21}{\sqrt{879}} \\
\frac{21}{\sqrt{879}}
\end{bmatrix} \right\}
\]

Ugly, but true. At this stage, you might just as well be encouraged to think of the Gram-Schmidt procedure as a computational black box, linearly independent set in, orthogonal span-preserving set out.

To check that the output set is orthogonal, we can easily check the orthogonality of individual pairs of vectors. Suppose the output was set equal to \( b \) (say via \( b=\text{GramSchmidt}[a] \)). We can extract the individual vectors of \( c \) as “parts” with syntax like \( c[[3]] \), which would return the third vector in the set. When our vectors have only real number entries, we can accomplish an innerproduct with a “dot.” So, for example, you should discover that \( c[[3]].c[[5]] \) will return zero. Try it yourself with another pair of vectors.
Computation Note TM.MMA
Transpose of a Matrix

Contributed by Robert Beezer

Suppose \( \mathbf{a} \) is the name of a matrix stored in Mathematica. Then \( \text{Transpose}[\mathbf{a}] \) will create the transpose of \( \mathbf{a} \).

Computation Note MM.MMA
Matrix Multiplication

If \( \mathbf{A} \) and \( \mathbf{B} \) are matrices defined in Mathematica, then \( \mathbf{A}.\mathbf{B} \) will return the product of the two matrices (notice the dot between the matrices). If \( \mathbf{A} \) is a matrix and \( \mathbf{v} \) is a vector, then \( \mathbf{A}\mathbf{v} \) will return the vector that is the matrix-vector product of \( \mathbf{A} \) and \( \mathbf{v} \). In every case the sizes of the matrices and vectors need to be correct.

Some examples:

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\begin{bmatrix}
5 & 6 & 7 \\
8 & 9 & 10
\end{bmatrix}
= 
\begin{bmatrix}
21 & 24 & 27 \\
47 & 54 & 61
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\begin{bmatrix}
5 \\
6
\end{bmatrix}
= 
\begin{bmatrix}
17 \\
39
\end{bmatrix}
\]

Understanding the difference between the last two examples will go a long way to explaining how some Mathematica constructs work.

Computation Note MI.MMA
Matrix Inverse

If \( \mathbf{A} \) is a matrix defined in Mathematica, then \( \text{Inverse}[\mathbf{A}] \) will return the inverse of \( \mathbf{A} \), should it exist. In the case where \( \mathbf{A} \) does not have an inverse Mathematica will tell you the matrix is singular (see Theorem NI 204).

Section TI86
Texas Instruments 86

Computation Note ME.TI86
Matrix Entry

On the TI-86, press the \text{MATRX} key (Yellow-7). Press the second menu key over, \text{F2}, to bring up the \text{EDIT} screen. Give your matrix a name, one letter or many, then press \text{ENTER}. You can then change the size of the matrix (rows, then columns) and begin editing individual entries (which are initially zero). \text{ENTER} will move you from entry to entry, or the \text{down arrow} key will move you to the next row. A menu gives you extra options for editing.

Matrices may also be entered on the home screen as follows. Use brackets ([ , ]) to enclose rows with elements separated by commas. Group rows, in order, into a final set of brackets (with no commas between rows). This can then be stored in a name with the \text{STO} key. So, for example,

\[
\begin{bmatrix}
1, 2, 3, 4 & 5, 6, 7, 8 & 9, 10, 11, 12
\end{bmatrix}
\rightarrow \mathbf{A}
\]
will create a matrix named \( A \) that is equal to
\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12
\end{bmatrix}
\]

**Computation Note RR.TI86**

**Row Reduce**

If \( A \) is the name of a matrix stored in the TI-86, then the command \texttt{rref} \( A \) will return the reduced row-echelon form of the matrix. This command can also be found by pressing the \texttt{MATRX} key, then \texttt{F4} for \texttt{OPS}, and finally, \texttt{F5} for \texttt{rref}.

Note that this command will not work for a matrix with more rows than columns. (Ed. Not sure just why this is!) A work-around is to pad the matrix with extra columns of zeros until the matrix is square.

**Computation Note VLC.TI86**

**Vector Linear Combinations**

Contributed by Robert Beezer

Vector operations on the TI-86 can be accessed via the \texttt{VECTR} key, which is \texttt{Yellow-8}. The \texttt{EDIT} tool appears when the \texttt{F2} key is pressed. After providing a name and giving a “dimension” (the size) then you can enter the individual entries, one at a time. Vectors can also be entered on the home screen using brackets ( \([\ ,\ ]\) ). To create the vector
\[
\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}
\]

use brackets and the store key ( \texttt{STO} ),

\([1, 2, 3, 4] \rightarrow \mathbf{v}\)

Vector addition and scalar multiplication are then very natural. If \( \mathbf{u} \) and \( \mathbf{v} \) are two vectors of equal size, then
\[
2 \times \mathbf{u} + (-3) \times \mathbf{v}
\]

will compute the correct vector and display the result as a vector.

**Computation Note TM.TI86**

**Transpose of a Matrix**

Contributed by Eric Fickenscher

Suppose \( A \) is the name of a matrix stored in the TI-86. Use the command \( A^\text{T} \) to transpose \( A \). This command can be found by pressing the \texttt{MATRX} key, then \texttt{F3} for \texttt{MATH}, then \texttt{F2} for \texttt{T}.

Version 1.04
Section TI83
Texas Instruments 83

Computation Note ME.TI83
Matrix Entry

Contributed by Douglas Phelps
On the TI-83, press the `MATRX` key. Press the right arrow key twice so that `EDIT` is highlighted. Move the cursor down so that it is over the desired letter of the matrix and press `ENTER`. For example, let’s call our matrix $B$, so press the down arrow once and press `ENTER`. To enter a $2 \times 3$ matrix, press `2 ENTER 3 ENTER`. To create the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

press `1 ENTER 2 ENTER 3 ENTER 4 ENTER 5 ENTER 6 ENTER`.

Computation Note RR.TI83
Row Reduce

Contributed by Douglas Phelps
Suppose $B$ is the name of a matrix stored in the TI-83. Press the `MATRX` key. Press the right arrow key once so that `MATH` is highlighted. Press the down arrow eleven times so that `rref` is highlighted, then press `ENTER`. to choose the matrix $B$, press `MATRX`, then the down arrow once followed by `ENTER`. Supply a right parenthesis `(` and press `ENTER`. Note that this command will not work for a matrix with more rows than columns. (Ed. Not sure just why this is!) A work-around is to pad the matrix with extra columns of zeros until the matrix is square.

Computation Note VLC.TI83
Vector Linear Combinations

Contributed by Douglas Phelps
Entering a vector on the TI-83 is the same process as entering a matrix. You press `4 ENTER 3 ENTER` for a $4 \times 3$ matrix. Likewise, you press `4 ENTER 1 ENTER` for a vector of size 4. To multiply a vector by 8, press the number 8, then press the `MATRX` key, then scroll down to the letter you named your vector ($A$, $B$, $C$, etc) and press `ENTER`. To add vectors $A$ and $B$ for example, press the `MATRX` key, then `ENTER`. Then press the `+` key. Then press the `MATRX` key, then the down arrow once, then `ENTER`. $[A] + [B]$ will appear on the screen. Press `ENTER`.
Appendix P
Preliminaries

This appendix contains important ideas about complex numbers, sets, and the logic and techniques of forming proofs. It is not meant to be read straight through, but you should head here when you need to review these ideas.

We choose to expand the set of scalars from the real numbers, \( \mathbb{R} \), to the set of complex numbers, \( \mathbb{C} \). So basic operations with complex numbers (like addition and division) will be necessary. This can be safely postponed until your arrival in Section O \[151\], and a refresher before Chapter E \[356\] would be a good idea as well.

Sets are extremely important in all of mathematics, but maybe you have not had much exposure to the basic operations. Check out Section SET \[615\]. The text will send you here frequently as well. Visit often.

This book is as much about doing mathematics as it is about linear algebra. The “Proof Techniques” are vignettes about logic, types of theorems, structure of proofs, or just plain old-fashioned advice about how to do advanced mathematics. The text will frequently point to one of these techniques in advance of their first use, and for specific instructions there will be additional references. If you find constructing proofs difficult (we all did once), then head back here and browse through the advice for second or third readings.

Section CNO
Complex Number Operations

In this section we review of the basics of working with complex numbers.

Subsection CNA
Arithmetic with complex numbers

A complex number is a linear combination of 1 and \( i = \sqrt{-1} \), typically written in the form \( a + bi \). Complex numbers can be added, subtracted, multiplied and divided, just like we are used to doing with real numbers, including the restriction on division by zero. We will not define these operations carefully, but instead illustrate with examples.

Example ACN
Arithmetic of complex numbers

\[
(2 + 5i) + (6 - 4i) = (2 + 6) + (5 + (-4))i = 8 + i
\]
\[
(2 + 5i) - (6 - 4i) = (2 - 6) + (5 - (-4))i = -4 + 9i
\]
\[
(2 + 5i)(6 - 4i) = (2)(6) + (5i)(6) + (2)(-4i) + (5i)(-4i) = 12 + 30i - 8i - 20i^2
\]

611
\[= 12 + 22i - 20(-1) = 32 + 22i\]

Division takes just a bit more care. We multiply the denominator by a complex number chosen to produce a real number and then we can produce a complex number as a result.

\[
\frac{2 + 5i}{6 - 4i} \cdot \frac{6 + 4i}{6 + 4i} = \frac{-8 + 38i}{52} = \frac{-8}{52} + \frac{38}{52}i = \frac{-2}{13} + \frac{19}{26}i
\]

\[\therefore\]

In this example, we used \(6 + 4i\) to convert the denominator in the fraction to a real number. This number is known as the conjugate, which we define in the next section. We will often exploit the basic properties of complex number addition, subtraction, multiplication and division, so we will carefully define the two basic operations, together with a definition of equality, and then collect nine basic properties in a theorem.

**Definition CNE**

**Complex Number Equality**
The complex numbers \(\alpha = a + bi\) and \(\beta = c + di\) are equal, denoted \(\alpha = \beta\), if \(a = c\) and \(b = d\).

(This definition contains Notation CNE.)

**Definition CNA**

**Complex Number Addition**
The sum of the complex numbers \(\alpha = a + bi\) and \(\beta = c + di\), denoted \(\alpha + \beta\), is \((a + c) + (b + d)i\).

(This definition contains Notation CNA.)

**Definition CNM**

**Complex Number Multiplication**
The product of the complex numbers \(\alpha = a + bi\) and \(\beta = c + di\), denoted \(\alpha \beta\), is \((ac - bd) + (ad + bc)i\).

(This definition contains Notation CNM.)

**Theorem PCNA**

**Properties of Complex Number Arithmetic**
The operations of addition and multiplication of complex numbers have the following properties.

- **ACCN** Additive Commutativity, Complex Numbers
  For any \(\alpha, \beta \in \mathbb{C}\), \(\alpha + \beta = \beta + \alpha\).

- **MCCN** Multiplicative Commutativity, Complex Numbers
  For any \(\alpha, \beta \in \mathbb{C}\), \(\alpha \beta = \beta \alpha\).

- **AACN** Additive Associativity, Complex Numbers
  For any \(\alpha, \beta, \gamma \in \mathbb{C}\), \(\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma\).

- **MACN** Multiplicative Associativity, Complex Numbers
  For any \(\alpha, \beta, \gamma \in \mathbb{C}\), \(\alpha (\beta \gamma) = (\alpha \beta) \gamma\).

- **DCN** Distributivity, Complex Numbers
  For any \(\alpha, \beta, \gamma \in \mathbb{C}\), \(\alpha (\beta + \gamma) = \alpha \beta + \alpha \gamma\).

- **ZCN** Zero, Complex Numbers
  There is a complex number \(0 = 0 + 0i\) so that for any \(\alpha \in \mathbb{C}\), \(0 + \alpha = \alpha\).

- **OCN** One, Complex Numbers
  There is a complex number \(1 = 1 + 0i\) so that for any \(\alpha \in \mathbb{C}\), \(1 \alpha = \alpha\).

- **AICN** Additive Inverse, Complex Numbers
  For every \(\alpha \in \mathbb{C}\) there exists \(-\alpha \in \mathbb{C}\) so that \(\alpha + (-\alpha) = 0\).
• **MICN** Multiplicative Inverse, Complex Numbers
  
  For every $\alpha \in \mathbb{C}$, $\alpha \neq 0$ there exists $\frac{1}{\alpha} \in \mathbb{C}$ so that $\alpha \left( \frac{1}{\alpha} \right) = 1$.

**Proof** We could derive each of these properties of complex numbers with a proof that builds on the identical properties of the real numbers. The only proof that might be at all interesting would be to show [Property MICN][613] since we would need to trot out a conjugate. For this property, and especially for the others, we might be tempted to construct proofs of the identical properties for the reals. This would take us way too far afield, so we will draw a line in the sand right here and just agree that these nine fundamental behaviors are true. OK?

Mostly we have stated these nine properties carefully so that we can make reference to them later in other proofs. So we will be linking back here often.

### Subsection CCN
**Conjugates of Complex Numbers**

#### Definition CCN
**Conjugate of a Complex Number**

The conjugate of the complex number $c = a + bi \in \mathbb{C}$ is the complex number $\overline{c} = a - bi$.

(This definition contains Notation CCN.)

#### Example CSCN
**Conjugate of some complex numbers**

\[
\begin{align*}
2 + 3i &= 2 - 3i \\
5 - 4i &= 5 + 4i \\
-3 + 0i &= -3 + 0i \\
0 + 0i &= 0 + 0i
\end{align*}
\]

Notice how the conjugate of a real number leaves the number unchanged. The conjugate enjoys some basic properties that are useful when we work with linear expressions involving addition and multiplication.

#### Theorem CCRA
**Complex Conjugation Respects Addition**

Suppose that $c$ and $d$ are complex numbers. Then $\overline{c + d} = \overline{c} + \overline{d}$.

**Proof** Let $c = a + bi$ and $d = r + si$. Then

\[
\overline{c + d} = \overline{(a + r) + (b + s)i} = (a + r) - (b + s)i = (a - bi) + (r - si) = \overline{c} + \overline{d}
\]

#### Theorem CCRM
**Complex Conjugation Respects Multiplication**

Suppose that $c$ and $d$ are complex numbers. Then $\overline{cd} = \overline{c} \overline{d}$.

**Proof** Let $c = a + bi$ and $d = r + si$. Then

\[
\begin{align*}
\overline{cd} &= (ar - bs) + (as + br)i \\
&= (ar - bs) - (as + br)i \\
&= (ar - (-b)(r - si)) + (a(-s) + (-b)r)i \\
&= (a - bi)(r - si) = \overline{c} \overline{d}
\end{align*}
\]

#### Theorem CCT
**Complex Conjugation Twice**

Suppose that $c$ is a complex number. Then $\overline{\overline{c}} = c$.

**Proof** Let $c = a + bi$. Then

\[
\overline{\overline{c}} = \overline{\overline{a - bi}} = a - (-bi) = a + bi = c
\]
Subsection MCN
Modulus of a Complex Number

We define one more operation with complex numbers that may be new to you.

Definition MCN
Modulus of a Complex Number
The modulus of the complex number \( c = a + bi \in \mathbb{C} \), is the nonnegative real number

\[
|c| = \sqrt{cc} = \sqrt{a^2 + b^2}.
\]

Example MSCN
Modulus of some complex numbers

\[
|2 + 3i| = \sqrt{13} \quad |5 - 4i| = \sqrt{41} \quad |-3 + 0i| = 3 \quad |0 + 0i| = 0
\]

The modulus can be interpreted as a version of the absolute value for complex numbers, as is suggested by the notation employed. You can see this in how \(|-3| = |-3 + 0i| = 3\). Notice too how the modulus of the complex zero, \(0 + 0i\), has value 0.
Section SET
Sets

**Definition SET**
**Set**
A set is an unordered collection of objects. If $S$ is a set and $x$ is an object that is in the set $S$, we write $x \in S$. If $x$ is not in $S$, then we write $x \notin S$. We refer to the objects in a set as its elements.

(This definition contains Notation SETM.)

Hard to get much more basic than that. Notice that the objects in a set can be anything, and there is no notion of order among the elements of the set. A set can be finite as well as infinite. A set can contain other sets as its objects. At a primitive level, a set is just a way to break up some class of objects into two groupings: those objects in the set, and those objects not in the set.

**Example SETM**
**Set membership**
From the set of all possible symbols, construct the following set of three symbols,

$$S = \{■, ♦, ★\}$$

Then the statement $■ \in S$ is true, while the statement $▲ \in S$ is false. However, then the statement $▲ \notin S$ is true.

A portion of a set is known as a subset. Notice how the following definition uses an implication (if whenever... then...). Note too how the definition of a subset relies on the definition of a set through the idea of set membership.

**Definition SSET**
**Subset**
If $S$ and $T$ are two sets, then $S$ is a subset of $T$, written $S \subseteq T$ if whenever $x \in S$ then $x \in T$.

(This definition contains Notation SSET.)

If we want to disallow the possibility that $S$ is the same as $T$, we use the notation $S \subset T$ and we say that $S$ is a proper subset of $T$. We’ll do an example, but first we’ll define a special set.

**Definition ES**
**Empty Set**
The empty set is the set with no elements. Its is denoted by $\emptyset$.

(This definition contains Notation ES.)

**Example SSET**
**Subset**
If $S = \{■, ♦, ★\}$, $T = \{★, ♦\}$, $R = \{▲, ★\}$, then

$$T \subseteq S \quad R \not\subseteq T \quad \emptyset \subseteq S$$

$$T \subset S \quad S \subseteq S \quad S \not\subset S$$

What does it mean for two sets to be equal? They must be the same. Well, that explanation is not really too helpful, is it? How about: If $A \subseteq B$ and $B \subseteq A$, then $A$ equals $B$. This gives us something to work with, if $A$ is a subset of $B$, and vice versa, then they must really be the same set. We will now make the symbol “=” do double-duty and extend its use to statements like $A = B$, where $A$ and $B$ are sets. Here’s the definition, which we will reference often.
Definition SE
Set Equality
Two sets, $S$ and $T$, are equal, if $S \subseteq T$ and $T \subseteq S$. In this case, we write $S = T$.
(This definition contains Notation SE.)

Sets are typically written inside of braces, as $\{\}$, as we have seen above. However, when sets have more than a few elements, a description will typically have two components. The first is a description of the general type of objects contained in a set, while the second is some sort of restriction on the properties the objects have. Every object in the set must be of the type described in the first part and it must satisfy the restrictions in the second part. Conversely, any object of the proper type for the first part, that also meets the conditions of the second part, will be in the set. These two parts are set off from each other somehow, often with a vertical bar (|) or a colon (:).

I like to think of sets as clubs. The first part is some description of the type of people who might belong to the club, the basic objects. For example, a bicycle club would describe its members as being people who like to ride bicycles. The second part is like a membership committee, it restricts the people who are allowed in the club. Continuing with our bicycle club analogy, we might decide to limit ourselves to “serious” riders and only have members who can document having ridden 100 kilometers or more in a single day at least one time.

The restrictions on membership can migrate around some between the first and second part, and there may be several ways to describe the same set of objects. Here’s a more mathematical example, employing the set of all integers, $\mathbb{Z}$, to describe the set of even integers.

$$E = \{x \in \mathbb{Z} \mid x \text{ is an even number}\}$$
$$= \{x \in \mathbb{Z} \mid 2 \text{ divides } x \text{ evenly}\}$$
$$= \{2k \mid k \in \mathbb{Z}\}$$

Notice how this set tells us that its objects are integer numbers (not, say, matrices or functions, for example) and just those that are even. So we can write that $10 \in E$, while $17 \notin E$ once we check the membership criteria. We also recognize the question

$$\begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & 3 \end{bmatrix} \in E?$$

as being simply ridiculous.

Subsection SC
Set Cardinality

On occasion, we will be interested in the number of elements in a finite set. Here’s the definition and the associated notation.

Definition C
Cardinality
Suppose $S$ is a finite set. Then the number of elements in $S$ is called the **cardinality** or **size** of $S$, and is denoted $|S|$.
(This definition contains Notation C.)

Example CS
Cardinality and Size
If $S = \{\clubsuit, \star, \blacksquare\}$, then $|S| = 3$. 
In this subsection we define and illustrate the three most common basic ways to manipulate sets to create other sets. Since much of linear algebra is about sets, we will use these often.

**Definition SU**

**Set Union**

Suppose $S$ and $T$ are sets. Then the union of $S$ and $T$, denoted $S \cup T$, is the set whose elements are those that are elements of $S$ or of $T$, or both. More formally,

$x \in S \cup T$ if and only if $x \in S$ or $x \in T$

(This definition contains Notation SU.)

Notice that the use of the word “or” in this definition is meant to be non-exclusive. That is, it allows for $x$ to be an element of both $S$ and $T$ and still qualify for membership in $S \cup T$.

**Example SU**

**Set union**

If $S = \{\diamondsuit, \spadesuit, \clubsuit\}$ and $T = \{\spadesuit, \heartsuit, \triangleup\}$ then $S \cup T = \{\diamondsuit, \spadesuit, \clubsuit, \heartsuit, \triangleup\}$.

**Definition SI**

**Set Intersection**

Suppose $S$ and $T$ are sets. Then the intersection of $S$ and $T$, denoted $S \cap T$, is the set whose elements are only those that are elements of $S$ and of $T$. More formally,

$x \in S \cap T$ if and only if $x \in S$ and $x \in T$

(This definition contains Notation SI.)

**Example SI**

**Set intersection**

If $S = \{\diamondsuit, \spadesuit, \clubsuit\}$ and $T = \{\spadesuit, \heartsuit, \triangleup\}$ then $S \cap T = \{\spadesuit, \heartsuit\}$.

The union and intersection of sets are operations that begin with two sets and produce a third, new, set. Our final operation is the set complement, which we usually think of as an operation that takes a single set and creates a second, new, set. However, if you study the definition carefully, you will see that it needs to be computed relative to some “universal” set.

**Definition SC**

**Set Complement**

Suppose $S$ is a set that is a subset of a universal set $U$. Then the complement of $S$, denoted $\overline{S}$, is the set whose elements are those that are elements of $U$ and not elements of $S$. More formally,

$x \in \overline{S}$ if and only if $x \in U$ and $x \notin S$

(This definition contains Notation SC.)

Notice that there is nothing at all special about the universal set. This is simply a term that suggests that $U$ contains all of the possible objects we are considering. Often this set will be clear from the context, and we won’t think much about it, nor reference it in our notation. In other cases (rarely in our work in this course) the exact nature of the universal set must be made explicit, and reference to it will possibly be carried through in our choice of notation.
Example SC
Set complement
If $U = \{\Diamond, \star, \blacksquare, \blacklozenge\}$ and $S = \{\Diamond, \star, \blacksquare\}$ then $\overline{S} = \{\blacklozenge\}$.

There are many more natural operations that can be performed on sets, such as an exclusive-or and the symmetric difference. Many of these can be defined in terms of the union, intersection and complement. We will not have much need of them in this course, and so we will not give precise descriptions here in this preliminary section.

There is also an interesting variety of basic results that describe the interplay of these operations with each other. We mention just two as an example, these are known as DeMorgan’s Laws.

$$(S \cup T) = S \cap T$$
$$(S \cap T) = S \cup T$$

Besides having an appealing symmetry, we mention these two facts, since constructing the proofs of each is a useful exercise that will require a solid understanding of all but one of the definitions presented in this section. Give it a try.
Section PT
Proof Techniques

In this section we collect many short essays designed to help you understand how to read, understand and construct proofs. Some are very factual, while others consist of advice. They appear in the order that they are first needed (or advisable) in the text, and are meant to be self-contained. So you should not think of reading through this section in one sitting as you begin this course. But be sure to head back here for a first reading whenever the text suggests it. Also think about returning to browse at various points during the course, and especially as you struggle with becoming an accomplished mathematician who is comfortable with the difficult process of designing new proofs.

Proof Technique D
Definitions

A definition is a made-up term, used as a kind of shortcut for some typically more complicated idea. For example, we say a whole number is even as a shortcut for saying that when we divide the number by two we get a remainder of zero. With a precise definition, we can answer certain questions unambiguously. For example, did you ever wonder if zero was an even number? Now the answer should be clear since we have a precise definition of what we mean by the term even.

A single term might have several possible definitions. For example, we could say that the whole number \(n\) is even if there is another whole number \(k\) such that \(n = 2k\). We say this is an equivalent definition since it categorizes even numbers the same way our first definition does.

Definitions are like two-way streets — we can use a definition to replace something rather complicated by its definition (if it fits) and we can replace a definition by its more complicated description. A definition is usually written as some form of an implication, such as “If something-nice-happens, then blatzo.” However, this also means that “If blatzo, then something-nice-happens,” even though this may not be formally stated. This is what we mean when we say a definition is a two-way street — it is really two implications, going in opposite “directions.”

Anybody (including you) can make up a definition, so long as it is unambiguous, but the real test of a definition’s utility is whether or not it is useful for describing interesting or frequent situations.

We will talk about theorems later (and especially equivalences). For now, be sure not to confuse the notion of a definition with that of a theorem.

In this book, we will display every new definition carefully set-off from the text, and the term being defined will be written thus: definition. Additionally, there is a full list of all the definitions, in order of their appearance located at the front of the book (Definitions viii). Finally, the acronym for each definition can be found in the index (Index ??). Definitions are critical to doing mathematics and proving theorems, so we’ve given you lots of ways to locate a definition should you forget its... uh, well, ... definition.

Can you formulate a precise definition for what it means for a number to be odd? (Don’t just say it is the opposite of even. Act as if you don’t have a definition for even yet.) Can you formulate your definition a second, equivalent, way? Can you employ your definition to test an odd and an even number for “odd-ness”?
Proof Technique T
Theorems

Higher mathematics is about understanding theorems. Reading them, understanding them, applying them, proving them. We are ready to prove our first momentarily. Every theorem is a shortcut — we prove something in general, and then whenever we find a specific instance covered by the theorem we can immediately say that we know something else about the situation by applying the theorem. In many cases, this new information can be gained with much less effort than if we did not know the theorem.

The first step in understanding a theorem is to realize that the statement of every theorem can be rewritten using statements of the form “If something-happens, then something-else-happens.” The “something-happens” part is the hypothesis and the “something-else-happens” is the conclusion. To understand a theorem, it helps to rewrite its statement using this construction. To apply a theorem, we verify that “something-happens” in a particular instance and immediately conclude that “something-else-happens.” To prove a theorem, we must argue based on the assumption that the hypothesis is true, and arrive through the process of logic that the conclusion must then also be true.

Proof Technique L
Language

Like any science, the language of math must be understood before further study can continue.

Erin Wilson, Student
September, 2004

Mathematics is a language. It is a way to express complicated ideas clearly, precisely, and unambiguously. Because of this, it can be difficult to read. Read slowly, and have pencil and paper at hand. It will usually be necessary to read something several times. While reading can be difficult, it is even harder to speak mathematics, and so that is the topic of this technique.

“Natural” language, in the present case English, is fraught with ambiguity. Consider the possible meanings of the sentence: The fish is ready to eat. One fish, or two fish? Are the fish hungry, or will the fish be eaten? (See Exercise SSLE.M10 [17], Exercise SSLE.M11 [17], Exercise SSLE.M12 [17], Exercise SSLE.M13 [18].) In your daily interactions with others, give some thought to how many mis-understandings arise from the ambiguity of pronouns, modifiers and objects.

I am going to suggest a simple modification to the way you use language that will make it much, much easier to become proficient at speaking mathematics and eventually it will become second nature. Think of it as a training aid or practice drill you might use when learning to become skilled at a sport.

First, eliminate pronouns from your vocabulary when discussing linear algebra, in class or with your colleagues. Do not use: it, that, those, their or similar sources of confusion. This is the single easiest step you can take to make your oral expression of mathematics clearer to others, and in turn, it will greatly help your own understanding.

Now rid yourself of the word “thing” (or variants like “something”). When you are tempted to use this word realize that there is some object you want to discuss, and we likely have a definition for that object (see the discussion at Technique D [619]). Always “think about your objects” and many aspects of the study of mathematics will get easier. Ask yourself: “Am I working with a set, a number, a function, an operation, a differential equation, or what?” Knowing what an object is will allow you to narrow down the procedures you may apply to it. If you have studied an object-oriented computer programming language, then perhaps this advice will be even clearer, since you know that a compiler will often complain with an error message if you confuse your objects.

Erin Wilson, Student
September, 2004

Mathematics is a language. It is a way to express complicated ideas clearly, precisely, and unambiguously. Because of this, it can be difficult to read. Read slowly, and have pencil and paper at hand. It will usually be necessary to read something several times. While reading can be difficult, it is even harder to speak mathematics, and so that is the topic of this technique.

“Natural” language, in the present case English, is fraught with ambiguity. Consider the possible meanings of the sentence: The fish is ready to eat. One fish, or two fish? Are the fish hungry, or will the fish be eaten? (See Exercise SSLE.M10 [17], Exercise SSLE.M11 [17], Exercise SSLE.M12 [17], Exercise SSLE.M13 [18].) In your daily interactions with others, give some thought to how many mis-understandings arise from the ambiguity of pronouns, modifiers and objects.

I am going to suggest a simple modification to the way you use language that will make it much, much easier to become proficient at speaking mathematics and eventually it will become second nature. Think of it as a training aid or practice drill you might use when learning to become skilled at a sport.

First, eliminate pronouns from your vocabulary when discussing linear algebra, in class or with your colleagues. Do not use: it, that, those, their or similar sources of confusion. This is the single easiest step you can take to make your oral expression of mathematics clearer to others, and in turn, it will greatly help your own understanding.

Now rid yourself of the word “thing” (or variants like “something”). When you are tempted to use this word realize that there is some object you want to discuss, and we likely have a definition for that object (see the discussion at Technique D [619]). Always “think about your objects” and many aspects of the study of mathematics will get easier. Ask yourself: “Am I working with a set, a number, a function, an operation, a differential equation, or what?” Knowing what an object is will allow you to narrow down the procedures you may apply to it. If you have studied an object-oriented computer programming language, then perhaps this advice will be even clearer, since you know that a compiler will often complain with an error message if you confuse your objects.
Third, eliminate the verb “works” (as in “the equation works”) from your vocabulary. This term is used as a substitute when we are not sure just what we are trying to accomplish. Usually we are trying to say that some object fulfills some condition. The condition might even have a definition associated with it, making it even easier to describe.

Last, speak sloooowly and thoughtfully as you try to get by without all these lazy words. It is hard at first, but you will get better with practice. Especially in class, when the pressure is on and all eyes are on you, don’t succumb to the temptation to use these weak words. Slow down, we’d all rather wait for a slow, well-formed question or answer than a fast, sloppy, incomprehensible one.

You will find the improvement in your ability to speak clearly about complicated ideas will greatly improve your ability to think clearly about complicated ideas. And I believe that you cannot think clearly about complicated ideas if you cannot formulate questions or answers clearly in the correct language. This is as applicable to the study of law, economics or philosophy as it is to the study of science or mathematics.

So when you come to class, check your pronouns at the door, along with other weak words. And when studying with friends, you might make a game of catching one another using pronouns, “thing,” or “works.” I know I’ll be calling you on it!

Proof Technique GS
Getting Started

“I don’t know how to get started!” is often the lament of the novice proof-builder. Here are a few pieces of advice.

1. As mentioned in Technique T [620], rewrite the statement of the theorem in an “if-then” form. This will simplify identifying the hypothesis and conclusion, which are referenced in the next few items.

2. Ask yourself what kind of statement you are trying to prove. This is always part of your conclusion. Are you being asked to conclude that two numbers are equal, that a function is differentiable or a set is a subset of another? You cannot bring other techniques to bear if you do not know what type of conclusion you have.

3. Write down reformulations of your hypotheses. Interpret and translate each definition properly.

4. Write your hypothesis at the top of a sheet of paper and your conclusion at the bottom. See if you can formulate a statement that precedes the conclusion and also implies it. Work down from your hypothesis, and up from your conclusion, and see if you can meet in the middle. When you are finished, rewrite the proof nicely, from hypothesis to conclusion, with verifiable implications giving each subsequent statement.

5. As you work through your proof, think about what kinds of objects your symbols represent. For example, suppose $A$ is a set and $f(x)$ is a real-valued function. Then the expression $A + f$ might make no sense if we have not defined what it means to “add” a set to a function, so we can stop at that point and adjust accordingly. On the other hand we might understand $2f$ to be the function whose rule is described by $(2f)(x) = 2f(x)$. “Think about your objects” means to always verify that your objects and operations are compatible.

Proof Technique C
Constructive Proofs

Conclusions of proofs come in a variety of types. Often a theorem will simply assert that something exists. The best way, but not the only way, to show something exists is to actually build it. Such
Proof Technique PT.E Equivalences

When a theorem uses the phrase “if and only if” (or the abbreviation “iff”) it is a shorthand way of saying that two if-then statements are true. So if a theorem says “P if and only if Q,” then it is true that “if P, then Q” while it is also true that “if Q, then P.” For example, it may be a theorem that “I wear bright yellow knee-high plastic boots if and only if it is raining.” This means that I *never* forget to wear my super-duper yellow boots when it is raining and I wouldn’t be seen in such silly boots *unless* it was raining. You never have one without the other. I’ve got my boots on and it is raining or I don’t have my boots on and it is dry.

The upshot for proving such theorems is that it is like a 2-for-1 sale, we get to do two proofs. Assume P and conclude Q, then start over and assume Q and conclude P. For this reason, “if and only if” is sometimes abbreviated by $\iff$, while proofs indicate which of the two implications is being proved by prefacing each with $\Rightarrow$ or $\Leftarrow$. A carefully written proof will remind the reader which statement is being used as the hypothesis, a quicker version will let the reader deduce it from the direction of the arrow. Tradition dictates we do the “easy” half first, but that’s hard for a student to know until you’ve finished doing both halves! Oh well, if you rewrite your proofs (a good habit), you can then choose to put the easy half first.

Theorems of this type are called “equivalences” or “characterizations,” and they are some of the most pleasing results in mathematics. They say that two objects, or two situations, are really the same. You don’t have one without the other, like rain and my yellow boots. The more different P and Q seem to be, the more pleasing it is to discover they are really equivalent. And if P describes a very mysterious solution or involves a tough computation, while Q is transparent or involves easy computations, then we’ve found a great shortcut for better understanding or faster computation. Remember that every theorem really is a shortcut in some form. You will also discover that if proving $P \Rightarrow Q$ is very easy, then proving $Q \Rightarrow P$ is likely to be proportionately harder. Sometimes the two halves are about equally hard. And in rare cases, you can string together a whole sequence of other equivalences to form the one you’re after and you don’t even need to do two halves. In this case, the argument of one half is just the argument of the other half, but in reverse.

One last thing about equivalences. If you see a statement of a theorem that says two things are “equivalent,” translate it first into an “if and only if” statement.

Proof Technique N Negation

When we construct the contrapositive of a theorem (Technique CP 623), we need to negate the two statements in the implication. And when we construct a proof by contradiction (Technique CD 623), we need to negate the conclusion of the theorem. One way to construct a converse (Technique CV 623) is to simultaneously negate the hypothesis and conclusion of an implication (but remember that this is not guaranteed to be a true statement). So we often have the need to negate statements, and in some situations it can be tricky.

If a statement says that a set is empty, then its negation is the statement that the set is nonempty. That’s straightforward. Suppose a statement says “something-happens” for all $i$, or every $i$, or any $i$. Then the negation is that “something-doesn’t-happen” for at least one value of $i$. If a statement says that there exists at least one “thing,” then the negation is the statement that there is no “thing.” If a statement says that a “thing” is unique, then the negation is that there is
zero, or more than one, of the “thing.”

We are not covering all of the possibilities, but we wish to make the point that logical qualifiers like “there exists” or “for every” must be handled with care when negating statements. Studying the proofs which employ contradiction (as listed in Technique CD [623]) is a good first step towards understanding the range of possibilities.

**Proof Technique CP**

Contrapositives

The **contrapositive** of an implication $P \Rightarrow Q$ is the implication $\neg(Q) \Rightarrow \neg(P)$, where “not” means the logical negation, or opposite. An implication is true if and only if its contrapositive is true. In symbols, $(P \Rightarrow Q) \iff (\neg(Q) \Rightarrow \neg(P))$ is a theorem. Such statements about logic, that are always true, are known as **tautologies**.

For example, it is a theorem that “if a vehicle is a fire truck, then it has big tires and has a siren.” (Yes, I’m sure you can conjure up a counterexample, but play along with me anyway.) The contrapositive is “if a vehicle does not have big tires or does not have a siren, then it is not a fire truck.” Notice how the “and” became an “or” when we negated the conclusion of the original theorem.

It will frequently happen that it is easier to construct a proof of the contrapositive than of the original implication. If you are having difficulty formulating a proof of some implication, see if the contrapositive is easier for you. The trick is to construct the negation of complicated statements accurately. More on that later.

**Proof Technique CV**

Converses

The **converse** of the implication $P \Rightarrow Q$ is the implication $Q \Rightarrow P$. There is no guarantee that the truth of these two statements are related. In particular, if an implication has been proven to be a theorem, then do not try to use its converse too, as if it were a theorem. Sometimes the converse is true (and we have an equivalence, see Technique E [622]). But more likely the converse is false, especially if it wasn’t included in the statement of the original theorem.

For example, we have the theorem, “if a vehicle is a fire truck, then it has big tires and has a siren.” The converse is false. The statement that “if a vehicle has big tires and a siren, then it is a fire truck” is false. A police vehicle for use on a sandy public beach would have big tires and a siren, yet is not equipped to fight fires.

We bring this up now, because Theorem CSRN [46] has a tempting converse. Does this theorem say that if $r < n$, then the system is consistent? Definitely not, as Archetype E [651] has $r = 3 < 4 = n$, yet is inconsistent. This example is then said to be a **counterexample** to the converse. Whenever you think a theorem that is an implication might actually be an equivalence, it is good to hunt around for a counterexample that shows the converse to be false (the archetypes, Appendix A [630], can be a good hunting ground).

**Proof Technique CD**

**Contradiction**

Another proof technique is known as “proof by contradiction” and it can be a powerful (and satisfying) approach. Simply put, suppose you wish to prove the implication, “If $A$, then $B$.” As usual, we assume that $A$ is true, but we also make the additional assumption that $B$ is false. If our original implication is true, then these twin assumptions should lead us to a logical inconsistency. In practice we assume the negation of $B$ to be true (see Technique N [622]). So we argue from the
assumptions $A$ and not($B$) looking for some obviously false conclusion such as $1 = 6$, or a set is simultaneously empty and nonempty, or a matrix is both nonsingular and singular.

You should be careful about formulating proofs that look like proofs by contradiction, but really aren’t. This happens when you assume $A$ and not($B$) and proceed to give a “normal” and direct proof that $B$ is true by only using the assumption that $A$ is true. Your last step is to then claim that $B$ is true and you then appeal to the assumption that not($B$) is true, thus getting the desired contradiction. Instead, you could have avoided the overhead of a proof by contradiction and just run with the direct proof. This stylistic flaw is known, quite graphically, as “setting up the strawman to knock him down.”

Here is a simple example of a proof by contradiction. There are direct proofs that are just about as easy, but this will demonstrate the point, while narrowly avoiding knocking down the straw man.

**Theorem:** If $a$ and $b$ are odd integers, then their product, $ab$, is odd.

**Proof:** To begin a proof by contradiction, assume the hypothesis, that $a$ and $b$ are odd. Also assume the negation of the conclusion, in this case, that $ab$ is even. Then there are integers, $j$, $k$, $\ell$ so that $a = 2j + 1$, $b = 2k + 1$, $ab = 2\ell$. Then

\[
0 = ab - ab = (2j + 1)(2k + 1) - (2\ell) = 4jk + 2j + 2k - 2\ell + 1 = 2(2jk + j + k - \ell) + 1
\]

Notice how we used both our hypothesis and the negation of the conclusion in the second line. Now divide the integer on each end of this string of equalities by 2. On the left we get a remainder of 0, while on the right we see that the remainder will be 1. Both remainders cannot be correct, so this is our desired contradiction. Thus, the conclusion (that $ab$ is odd) is true.

Again, we do not offer this example as the best proof of this fact about even and odd numbers, but rather it is a simple illustration of a proof by contradiction. You can find examples of proofs by contradiction in Theorem NMUS [64], Theorem NPNT [202], Theorem RREFU [96], Theorem TTMI [191], Theorem GSPCV [158], Theorem ELIS [320], Theorem EDYES [323], Theorem EMHE [359], Theorem EDELI [378], and Theorem DMFE [396], in addition to several examples and solutions to exercises.

**Proof Technique U**

**Uniqueness**

A theorem will sometimes claim that some object, having some desirable property, is unique. In other words, there should be only one such object. To prove this, a standard technique is to assume there are two such objects and proceed to analyze the consequences. The end result may be a contradiction (Technique CD [623]), or the conclusion that the two allegedly different objects really are equal.

**Proof Technique ME**

**Multiple Equivalences**

A very specialized form of a theorem begins with the statement “The following are equivalent. . . ,” which is then followed by a list of statements. Informally, this lead-in sometimes gets abbreviated by “TFAE.” This formulation means that any two of the statements on the list can be connected with an “if and only if” to form a theorem. So if the list has $n$ statements then, there are \[\frac{n(n-1)}{2}\] possible equivalences that can be constructed (and are claimed to be true).
Suppose a theorem of this form has statements denoted as $A, B, C, \ldots Z$. To prove the entire theorem, we can prove $A \Rightarrow B$, $B \Rightarrow C$, $C \Rightarrow D$, \ldots, $Y \Rightarrow Z$ and finally, $Z \Rightarrow A$. This circular chain of $n$ equivalences would allow us, logically, if not practically, to form any one of the $\frac{n(n-1)}{2}$ possible equivalences by chasing the equivalences around the circle as far as required.

**Proof Technique PI**

**Proving Identities**

Many theorems have conclusions that say two objects are equal. Perhaps one object is hard to compute or understand, while the other is easy to compute or understand. This would make for a pleasing theorem. Whether the result is pleasing or not, we take the same approach to formulate a proof. Sometimes we need to employ specialized notions of equality, such as Definition SE [616] or Definition CVE [73], but in other cases we can string together a list of equalities.

The wrong way to prove an identity is to begin by writing it down and then beating on it until it reduces to an obvious identity. The first flaw is that you would be writing down the statement you wish to prove, as if you already believed it to be true. But more dangerous is the possibility that some of your maneuvers are not reversible. Here’s an example. Let’s prove that $3 = -3$.

\[
3 = -3 \\
3^2 = (-3)^2 \\
9 = 9 \\
0 = 0
\]

(This is a bad start)

Square both sides

Subtract 9 from both sides

So because $0 = 0$ is a true statement, does it follow that $3 = -3$ is a true statement? Nope. Of course, we didn’t really expect a legitimate proof of $3 = -3$, but this attempt should illustrate the dangers of this (incorrect) approach.

What you have just seen in the proof of Theorem VSPCV [75], and what you will see consistently throughout this text, is proofs of the following form. To prove that $A = D$ we write

\[
A = B \\
= C \\
= D
\]

Theorem, Definition or Hypothesis justifying $A = B$

Theorem, Definition or Hypothesis justifying $B = C$

Theorem, Definition or Hypothesis justifying $C = D$

In your scratch work exploring possible approaches to proving a theorem you may massage a variety of expressions, sometimes making connections to various bits and pieces, while some parts get abandoned. Once you see a line of attack, rewrite your proof carefully mimicking this style.

**Proof Technique DC**

**Decompositions**

Much of your mathematical upbringing, especially once you began a study of algebra, revolved around simplifying expressions — combining like terms, obtaining common denominators so as to add fractions, factoring in order to solve polynomial equations. However, as often as not, we will do the opposite. Many theorems and techniques will revolve around taking some object and “decomposing” it into some combination of other objects, ostensibly in a more complicated fashion. When we say something can “be written as” something else, we mean that the one object can be decomposed into some combination of other objects. This may seem unnatural at first, but results of this type will give us insight into the structure of the original object by exposing its inner workings. An appropriate analogy might be stripping the wallboards away from the interior of a building to expose the structural members supporting the whole building.

This is a major shift in thinking, so come back here often, especially when we say “can be written as”, or “can be expressed as,” or “can be decomposed as.”
Proof Technique I
Induction

"Induction" or "mathematical induction" is a framework for proving statements that are indexed by integers. In other words, suppose you have a statement to prove that is really multiple statements, one for \( n = 1 \), another for \( n = 2 \), a third for \( n = 3 \), and so on. If there is enough similarity between the statements, then you can use a script (the framework) to prove them all at once.

For example, consider the theorem

**Theorem** \( 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2} \) for \( n \geq 1 \).

This is shorthand for the many statements \( 1 = \frac{1(1+1)}{2} \), \( 1 + 2 = \frac{2(2+1)}{2} \), \( 1 + 2 + 3 = \frac{3(3+1)}{2} \), \( 1 + 2 + 3 + 4 = \frac{4(4+1)}{2} \), and so on. Forever. You can do the calculations in each of these statements and verify that all four are true. We might not be surprised to learn that the fifth statement is true as well (go ahead and check). However, do we think the theorem is true for \( n = 872 \)? Or \( n = 1, 234, 529 \)?

To see that these questions are not so ridiculous, consider the following example from Rotman’s *Journey into Mathematics*. The statement "\( n^2 - n + 41 \) is prime" is true for integers \( 1 \leq n \leq 40 \) (check a few). However, when we check \( n = 41 \) we find \( 41^2 - 41 + 41 = 41^2 \), which is not prime.

So how do we prove infinitely many statements all at once? More formally, let’s denote our statements as \( P(n) \). Then, if we can prove the two assertions

1. \( P(1) \) is true.
2. If \( P(k) \) is true, then \( P(k + 1) \) is true.

then it follows that \( P(n) \) is true for all \( n \geq 1 \). To understand this, I liken the process to climbing an infinitely long ladder with equally spaced rungs. Confronted with such a ladder, suppose I tell you that you are able to step up onto the first rung, and if you are on any particular rung, then you are capable of stepping up to the next rung. It follows that you can climb the ladder as far up as you wish. The first formal assertion above is akin to stepping onto the first rung, and the second formal assertion is akin to assuming that if you are on any one rung then you can always reach the next rung.

In practice, establishing that \( P(1) \) is true is called the “base case” and in most cases is straightforward. Establishing that \( P(k) \Rightarrow P(k + 1) \) is referred to as the “induction step,” or in this book (and elsewhere) we will typically refer to the assumption of \( P(k) \) as the “induction hypothesis.” This is perhaps the most mysterious part of a proof by induction, since it looks like you are assuming \( P(k) \) what you are trying to prove \( (P(n)) \). Sometimes it is even worse, since as you get more comfortable with induction, we often don’t bother to use a different letter \( (k) \) for the index \( (n) \) in the induction step. Notice that the second formal assertion never says that \( P(k) \) is true, it simply says that if \( P(k) \) were true, what might logically follow. We can establish statements like “If I lived on the moon, then I could pole-vault over a bar 12 meters high.” This may be a true statement, but it does not say we live on the moon, and indeed we may never live there.

Enough generalities. Let’s work an example and prove the theorem above about sums of integers. Formally, our statement is \( P(n) : 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2} \).

**Proof:** Base Case. \( P(1) \) is the statement \( 1 = \frac{1(1+1)}{2} \), which we see simplifies to the true statement \( 1 = 1 \).

Induction Step: We will assume \( P(k) \) is true, and will try to prove \( P(k + 1) \). Given what we want to accomplish, it is natural to begin by examining the sum of the first \( k + 1 \) integers.

\[
1 + 2 + 3 + \cdots + (k + 1) = (1 + 2 + 3 + \cdots + k) + (k + 1)
\]
We then recognize the two ends of this chain of equalities as \( P(k+1) \). So, by mathematical induction, the theorem is true for all \( n \).

How do you recognize when to use induction? The first clue is a statement that is really many statements, one for each integer. The second clue would be that you begin a more standard proof and you find yourself using words like “and so on” (as above!) or lots of ellipses (dots) to establish patterns that you are convinced continue on and on forever. However, there are many minor instances where induction might be warranted but we don’t bother.

Induction is important enough, and used often enough, that it appears in various variations. The base case sometimes begins with \( n = 0 \), or perhaps an integer greater than \( n \). Some formulate the induction step as \( P(k-1) \Rightarrow P(k) \). There is also a “strong form” of induction where we assume all of \( P(1), P(2), P(3), \ldots, P(k) \) as a hypothesis for showing the conclusion \( P(k+1) \).

You can find examples of induction in the proofs of Theorem GSPCV [158], Theorem DER [339], Theorem DT [340], Theorem DIM [349], Theorem EOMP [380], Theorem DCP [383], and Theorem KPLT [554].

Proof Technique P
Practice

Here is a technique used by many practicing mathematicians when they are teaching themselves new mathematics. As they read a textbook, monograph or research article, they attempt to prove each new theorem themselves, before reading the proof. Often the proofs can be very difficult, so it is wise not to spend too much time on each. Maybe limit your losses and try each proof for 10 or 15 minutes. Even if the proof is not found, it is time well-spent. You become more familiar with the definitions involved, and the hypothesis and conclusion of the theorem. When you do work through the proof, it might make more sense, and you will gain added insight about just how to construct a proof.

Proof Technique LC
Lemmas and Corollaries

Theorems often go by different titles. Two of the most popular being “lemma” and “corollary.” Before we describe the fine distinctions, be aware that lemmas, corollaries, propositions, claims and facts are all just theorems. And every theorem can be rephrased as an “if-then” statement, or perhaps a pair of “if-then” statements expressed as an equivalence (Technique E [622]).

A lemma is a theorem that is not too interesting in its own right, but is important for proving other theorems. It might be a generalization or abstraction of a key step of several different proofs. For this reason you often hear the phrase “technical lemma” though some might argue that the adjective “technical” is redundant.

A corollary is a theorem that follows very easily from another theorem. For this reason, corollaries frequently do not have proofs. You are expected to easily and quickly see how a previous theorem implies the corollary.

A proposition or fact is really just a codeword for a theorem. A claim might be similar, but some authors like to use claims within a proof to organize key steps. In a similar manner, some long proofs are organized as a series of lemmas.
In order to not confuse the novice, we have just called all our theorems theorems. It is also an organizational convenience. With only theorems and definitions, the theoretical backbone of the course is laid bare in the two lists of Definitions and Theorems.
Appendix A
Archetypes

The American Heritage Dictionary of the English Language (Third Edition) gives two definitions of the word “archetype”: 1. An original model or type after which other similar things are patterned; a prototype; and 2. An ideal example of a type; quintessence.

Either use might apply here. Our archetypes are typical examples of systems of equations, matrices and linear transformations. They have been designed to demonstrate the range of possibilities, allowing you to compare and contrast them. Several are of a size and complexity that is usually not presented in a textbook, but should do a better job of being “typical.”

We have made frequent reference to many of these throughout the text, such as the frequent comparisons between Archetype A [634] and Archetype B [638]. Some we have left for you to investigate, such as Archetype J [671], which parallels Archetype I [667].

How should you use the archetypes? First, consult the description of each one as it is mentioned in the text. See how other facts about the example might illuminate whatever property or construction is being described in the example. Second, each property has a short description that usually includes references to the relevant theorems. Perform the computations and understand the connections to the listed theorems. Third, each property has a small checkbox in front of it. Use the archetypes like a workbook and chart your progress by “checking-off” those properties that you understand.

The next page has a chart that summarizes some (but not all) of the properties described for each archetype. Notice that while there are several types of objects, there are fundamental connections between them. That some lines of the table do double-duty is meant to convey some of these connections. Consult this table when you wish to quickly find an example of a certain phenomenon.
## Archetype Facts

S=System of Equations, M=Matrix, L=Linear Transformation  
U=Unique solution, I=Infinitely many solutions, N=No solutions  
Y=Yes, N=No, X=Impossible, blank=Not Applicable
Archetype A

Summary  Linear system of three equations, three unknowns. Singular coefficient matrix with dimension 1 null space. Integer eigenvalues and a degenerate eigenspace for coefficient matrix.

A system of linear equations (Definition SLE [9]):

\[
\begin{align*}
  x_1 - x_2 + 2x_3 &= 1 \\
  2x_1 + x_2 + x_3 &= 8 \\
  x_1 + x_2 &= 5
\end{align*}
\]

Some solutions to the system of linear equations (not necessarily exhaustive):

\[
\begin{align*}
  x_1 &= 2, \quad x_2 = 3, \quad x_3 = 1 \\
  x_1 &= 3, \quad x_2 = 2, \quad x_3 = 0
\end{align*}
\]

Augmented matrix of the linear system of equations (Definition AM [24]):

\[
\begin{bmatrix}
  1 & -1 & 2 & 1 \\
  2 & 1 & 1 & 8 \\
  1 & 1 & 0 & 5
\end{bmatrix}
\]

Matrix in reduced row-echelon form, row-equivalent to augmented matrix:

\[
\begin{bmatrix}
  1 & 0 & 1 & 3 \\
  0 & 1 & -1 & 2 \\
  0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix (Notation RREFA [27]):

\[
r = 2 \quad D = \{1, 2\} \quad F = \{3, 4\}
\]

Vector form of the solution set to the system of equations (Theorem VFSLS [88]). Notice the relationship between the free variables and the set \( F \) above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set \( F \) for the larger examples.

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}
\]

Given a system of equations we can always build a new, related, homogeneous system (Definition HS [52]) by converting the constant terms to zeros and retaining the coefficients of the variables.
Properties of this new system will have precise relationships with various properties of the original system.

\[
\begin{align*}
   x_1 - x_2 + 2x_3 &= 0 \\
   2x_1 + x_2 + x_3 &= 0 \\
   x_1 + x_2 &= 0
\end{align*}
\]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):

\[
\begin{align*}
   x_1 &= 0, \quad x_2 = 0, \quad x_3 = 0 \\
   x_1 &= -1, \quad x_2 = 1, \quad x_3 = 1 \\
   x_1 &= -5, \quad x_2 = 5, \quad x_3 = 5
\end{align*}
\]

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

\[
\begin{bmatrix}
   1 & 0 & 1 & 0 \\
   0 & 1 & -1 & 0 \\
   0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system (Notation RREFA 27). Notice the slight variation for the same analysis of the original system only when the original system was consistent:

\[
r = 2 \quad D = \{1, 2\} \quad F = \{3, 4\}
\]

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations.

\[
\begin{bmatrix}
   1 & -1 & 2 \\
   2 & 1 & 1 \\
   1 & 1 & 0
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:

\[
\begin{bmatrix}
   1 & 0 & 1 \\
   0 & 1 & -1 \\
   0 & 0 & 0
\end{bmatrix}
\]

Analysis of the row-reduced matrix (Notation RREFA 27):

\[
r = 2 \quad D = \{1, 2\} \quad F = \{3\}
\]
Matrix (coefficient matrix) is nonsingular or singular? (Theorem NMRRI [63]) at the same time, examine the size of the set $F$ above. Notice that this property does not apply to matrices that are not square.

Singular.

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS [107], Theorem BNS [128]). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VFSLS [88]) to see these vectors arise.

\[
\langle \begin{bmatrix}
-1 \\
1 \\
1
\end{bmatrix} \rangle
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set $D$ above. (Theorem BCS [214])

\[
\langle \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rangle
\]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix $L$ is computed as described in Definition EEF [234]. This is followed by the column space described by a set of linearly independent vectors that span the null space of $L$, computed as according to Theorem FS [237] and Theorem BNS [128]. When $r = m$, the matrix $L$ has no rows and the column space is all of $\mathbb{C}^m$.

\[
L = \begin{bmatrix} 1 & -2 & 3 \\
-3 & 2 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

\[
\langle \begin{bmatrix} 1 \\ 0 \\ 1/3 \end{bmatrix} , \begin{bmatrix} 0 \\ 0 \\
2/3 \end{bmatrix} \rangle
\]

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS [220])
\[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \] \\]

- Inverse matrix, if it exists. The inverse is not defined for matrices that are not square, and if the matrix is square, then the matrix must be nonsingular. (Definition MI [189], Theorem NI [204])

- Subspace dimensions associated with the matrix. (Definition NOM [312], Definition ROM [313]) Verify Theorem RPNC [314]
  
  Matrix columns: 3  
  Rank: 2  
  Nullity: 1

- Determinant of the matrix, which is only defined for square matrices. The matrix is nonsingular if and only if the determinant is nonzero (Theorem SMZD [351]). (Product of all eigenvalues?)
  Determinant = 0

- Eigenvalues, and bases for eigenspaces. (Definition EEM [356], Definition EM [364])
  \begin{align*}
  \lambda &= 0 \\
  \mathcal{E}_A(0) &= \left\langle \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\rangle \\
  \lambda &= 2 \\
  \mathcal{E}_A(2) &= \left\langle \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\rangle
  \end{align*}

- Geometric and algebraic multiplicities. (Definition GME [366], Definition AME [366])
  \begin{align*}
  \gamma_A(0) &= 1 \\
  \alpha_A(0) &= 2 \\
  \gamma_A(2) &= 1 \\
  \alpha_A(2) &= 1
  \end{align*}

- Diagonalizable? (Definition DZM [393])
  No, \( \gamma_A(0) \neq \alpha_B(0) \), Theorem DMFE [396].
Archetype B


- A system of linear equations (Definition SLE 9):
  \[-7x_1 - 6x_2 - 12x_3 = -33\]
  \[5x_1 + 5x_2 + 7x_3 = 24\]
  \[x_1 + 4x_3 = 5\]

- Some solutions to the system of linear equations (not necessarily exhaustive):
  \[x_1 = -3, \ x_2 = 5, \ x_3 = 2\]

- Augmented matrix of the linear system of equations (Definition AM 24):
  \[
  \begin{bmatrix}
  -7 & -6 & -12 & -33 \\
  5 & 5 & 7 & 24 \\
  1 & 0 & 4 & 5 \\
  \end{bmatrix}
  \]

- Matrix in reduced row-echelon form, row-equivalent to augmented matrix:
  \[
  \begin{bmatrix}
  1 & 0 & 0 & -3 \\
  0 & 1 & 0 & 5 \\
  0 & 0 & 1 & 2 \\
  \end{bmatrix}
  \]

- Analysis of the augmented matrix (Notation RREFA 27):
  \[r = 3\]
  \[D = \{1, 2, 3\}\]
  \[F = \{4\}\]

- Vector form of the solution set to the system of equations (Theorem VFSLS 88). Notice the relationship between the free variables and the set \(F\) above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set \(F\) for the larger examples.
  \[
  \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  \end{bmatrix} =
  \begin{bmatrix}
  -3 \\
  5 \\
  2 \\
  \end{bmatrix}
  \]

- Given a system of equations we can always build a new, related, homogeneous system (Definition HS 52) by converting the constant terms to zeros and retaining the coefficients of the variables.
Properties of this new system will have precise relationships with various properties of the original system.

\[-11x_1 + 2x_2 - 14x_3 = 0\]
\[23x_1 - 6x_2 + 33x_3 = 0\]
\[14x_1 - 2x_2 + 17x_3 = 0\]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):
\[x_1 = 0, \quad x_2 = 0, \quad x_3 = 0\]

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system (Notation RREFA [27]). Notice the slight variation for the same analysis of the original system only when the original system was consistent:

\[r = 3\]
\[D = \{1, 2, 3\}\]
\[F = \{4\}\]

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations.

\[
\begin{bmatrix}
-7 & -6 & -12 \\
5 & 5 & 7 \\
1 & 0 & 4
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Analysis of the row-reduced matrix (Notation RREFA [27]):

\[r = 3\]
\[D = \{1, 2, 3\}\]
\[F = \{\}\]

Matrix (coefficient matrix) is nonsingular or singular? (Theorem NMRRI [62]) at the same time, examine the size of the set \(F\) above. Notice that this property does not apply to matrices that are not square.
Nonsingular.

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS [107], Theorem BNS [128]). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VFSLS [88]) to see these vectors arise.

\[
\langle{}\rangle
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \( D \) above. (Theorem BCS [214])

\[
\left\langle\begin{bmatrix}
-7 \\
5 \\
1
\end{bmatrix},
\begin{bmatrix}
-6 \\
5 \\
0
\end{bmatrix},
\begin{bmatrix}
-12 \\
7 \\
4
\end{bmatrix}\right\rangle
\]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \( L \) is computed as described in Definition EEF [234]. This is followed by the column space described by a set of linearly independent vectors that span the null space of \( L \), computed as according to Theorem FS [237] and Theorem BNS [128]. When \( r = m \), the matrix \( L \) has no rows and the column space is all of \( \mathbb{C}^m \).

\[
L = \left\langle\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}\right\rangle
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST [221] and Theorem BRS [220], and in the style of Example CSROI [221], this yields a linearly independent set of vectors that span the column space.

\[
\left\langle\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}\right\rangle
\]

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS [220])

\[
\left\langle\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}\right\rangle
\]

Inverse matrix, if it exists. The inverse is not defined for matrices that are not square, and if the matrix is square, then the matrix must be nonsingular. (Definition MI [189], Theorem NI [204])
Subspace dimensions associated with the matrix. \textbf{Verify Theorem RPNC [314]}

Matrix columns: 3  
Rank: 3  
Nullity: 0

Determinant of the matrix, which is only defined for square matrices. The matrix is nonsingular if and only if the determinant is nonzero \textbf{Theorem SMZD [351]). (Product of all eigenvalues?)}

Determinant = \(-2\)

Eigenvalues, and bases for eigenspaces. \textbf{Definition EEM [356] Definition EM [364]}

\begin{align*}
\lambda &= -1 & \mathcal{E}_B (-1) &= \left\langle \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix} \right\rangle \\
\lambda &= 1 & \mathcal{E}_B (1) &= \left\langle \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\rangle \\
\lambda &= 2 & \mathcal{E}_B (2) &= \left\langle \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\rangle
\end{align*}

Geometric and algebraic multiplicities. \textbf{Definition GME [366] Definition AME [366]}

\begin{align*}
\gamma_B (-1) &= 1 & \alpha_B (-1) &= 1 \\
\gamma_B (1) &= 1 & \alpha_B (1) &= 1 \\
\gamma_B (2) &= 1 & \alpha_B (2) &= 1
\end{align*}

Diagonalizable? \textbf{Definition DZM [393]}

Yes, distinct eigenvalues, \textbf{Theorem DED [398].}

The diagonalization. \textbf{Theorem DC [394]}

\[
\begin{bmatrix}
-1 & -1 & -1 \\
2 & 3 & 1 \\
-1 & -2 & 1
\end{bmatrix}
\begin{bmatrix}
-7 & -6 & -12 \\
5 & 5 & 7 \\
1 & 0 & 4
\end{bmatrix}
\begin{bmatrix}
-5 & -3 & -2 \\
3 & 2 & 1 \\
1 & 1 & 1
\end{bmatrix}
= 
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]
Archetype C

**Summary**  System with three equations, four variables. Consistent. Null space of coefficient matrix has dimension 1.

- A system of linear equations (Definition SLE [9]):
  
  \[
  \begin{align*}
  2x_1 - 3x_2 + x_3 - 6x_4 &= -7 \\
  4x_1 + x_2 + 2x_3 + 9x_4 &= -7 \\
  3x_1 + x_2 + x_3 + 8x_4 &= -8 
  \end{align*}
  \]

- Some solutions to the system of linear equations (not necessarily exhaustive):
  
  \[
  \begin{align*}
  x_1 &= -7, & x_2 &= -2, & x_3 &= 7, & x_4 &= 1 \\
  x_1 &= -1, & x_2 &= -7, & x_3 &= 4, & x_4 &= -2 
  \end{align*}
  \]

- Augmented matrix of the linear system of equations (Definition AM [24]):
  
  \[
  \begin{bmatrix}
  2 & -3 & 1 & -6 & -7 \\
  4 & 1 & 2 & 9 & -7 \\
  3 & 1 & 1 & 8 & -8 
  \end{bmatrix}
  \]

- Matrix in reduced row-echelon form, row-equivalent to augmented matrix:
  
  \[
  \begin{bmatrix}
  1 & 0 & 0 & 2 & -5 \\
  0 & 1 & 0 & 3 & 1 \\
  0 & 0 & 1 & -1 & 6 
  \end{bmatrix}
  \]

- Analysis of the augmented matrix (Notation RREFA [27]):
  
  \[
  r = 3 \\
  D = \{1, 2, 3\} \\
  F = \{4, 5\}
  \]

- Vector form of the solution set to the system of equations (Theorem VFSLS [88]). Notice the relationship between the free variables and the set \( F \) above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set \( F \) for the larger examples.
  
  \[
  \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 
  \end{bmatrix} = \begin{bmatrix}
  -5 \\
  1 \\
  6 \\
  0 
  \end{bmatrix} + x_4 \begin{bmatrix}
  -2 \\
  -3 \\
  1 \\
  1 
  \end{bmatrix}
  \]

- Given a system of equations we can always build a new, related, homogeneous system (Definition HS [52]) by converting the constant terms to zeros and retaining the coefficients of the variables.
Properties of this new system will have precise relationships with various properties of the original system.

\[ 2x_1 - 3x_2 + x_3 - 6x_4 = 0 \]
\[ 4x_1 + x_2 + 2x_3 + 9x_4 = 0 \]
\[ 3x_1 + x_2 + x_3 + 8x_4 = 0 \]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):
\[ x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0 \]
\[ x_1 = -2, \quad x_2 = -3, \quad x_3 = 1, \quad x_4 = 1 \]
\[ x_1 = -4, \quad x_2 = -6, \quad x_3 = 2, \quad x_4 = 2 \]

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:
\[
\begin{bmatrix}
1 & 0 & 0 & 2 & 0 \\
0 & 1 & 0 & 3 & 0 \\
0 & 0 & 1 & -1 & 0 \\
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system (Notation RREFA [27]). Notice the slight variation for the same analysis of the original system only when the original system was consistent:
\[ r = 3 \quad D = \{1, 2, 3\} \quad F = \{4, 5\} \]

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations.
\[
\begin{bmatrix}
2 & -3 & 1 & -6 \\
4 & 1 & 2 & 9 \\
3 & 1 & 1 & 8 \\
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:
\[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & -1 \\
\end{bmatrix}
\]

Analysis of the row-reduced matrix (Notation RREFA [27]):
\[ r = 3 \quad D = \{1, 2, 3\} \quad F = \{4\} \]
This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS 107, Theorem BNS 128). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VFSLS 88) to see these vectors arise.

\[
\left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \\ 1 \end{bmatrix} \right\}
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \( D \) above. (Theorem BCS 214)

\[
\left\{ \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}
\]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \( L \) is computed as described in Definition EEF 234. This is followed by the column space described by a set of linearly independent vectors that span the null space of \( L \), computed as according to Theorem FS 237 and Theorem BNS 128. When \( r = m \), the matrix \( L \) has no rows and the column space is all of \( \mathbb{C}^m \).

\[
L = \left[ \right]
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST 221 and Theorem BRS 220, and in the style of Example CSROI 221, this yields a linearly independent set of vectors that span the column space.

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS 220)
\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
2 & 3 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
1 \\
1 \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
1 \\
0 \\
1 \\
\end{bmatrix}
\]

Subspace dimensions associated with the matrix. (Definition NOM 312, Definition ROM 313) Verify Theorem RPNC 314

Matrix columns: 4  
Rank: 3  
Nullity: 1
Archetype D

**Summary**  System with three equations, four variables. Consistent. Null space of coefficient matrix has dimension 2. Coefficient matrix identical to that of Archetype E, vector of constants is different.

- **A system of linear equations** (Definition SLE [9]):

  \[
  \begin{align*}
  2x_1 + x_2 + 7x_3 - 7x_4 &= 8 \\
  -3x_1 + 4x_2 - 5x_3 - 6x_4 &= -12 \\
  x_1 + x_2 + 4x_3 - 5x_4 &= 4 
  \end{align*}
  \]

- **Some solutions to the system of linear equations** (not necessarily exhaustive):

  \[
  \begin{align*}
  x_1 &= 0, & x_2 &= 1, & x_3 &= 2, & x_4 &= 1 \\
  x_1 &= 4, & x_2 &= 0, & x_3 &= 0, & x_4 &= 0 \\
  x_1 &= 7, & x_2 &= 8, & x_3 &= 1, & x_4 &= 3
  \end{align*}
  \]

- **Augmented matrix of the linear system of equations** (Definition AM [24]):

  \[
  \begin{bmatrix}
  2 & 1 & 7 & -7 & 8 \\
  -3 & 4 & -5 & -6 & -12 \\
  1 & 1 & 4 & -5 & 4
  \end{bmatrix}
  \]

- **Matrix in reduced row-echelon form, row-equivalent to augmented matrix**:

  \[
  \begin{bmatrix}
  1 & 0 & 3 & -2 & 4 \\
  0 & 1 & 1 & -3 & 0 \\
  0 & 0 & 0 & 0 & 0
  \end{bmatrix}
  \]

- **Analysis of the augmented matrix** (Notation RREFA [27]):

  \[
  r = 2 \quad D = \{1, 2\} \quad F = \{3, 4, 5\}
  \]

- **Vector form of the solution set to the system of equations** (Theorem VFSLS [88]). Notice the relationship between the free variables and the set \(F\) above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set \(F\) for the larger examples.
Archetype D 648

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\
\end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \\
\end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \\
\end{bmatrix}
\]

Given a system of equations we can always build a new, related, homogeneous system (Definition HS [52]) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system.

\[
\begin{align*}
2x_1 + x_2 + 7x_3 - 7x_4 &= 0 \\
-3x_1 + 4x_2 - 5x_3 - 6x_4 &= 0 \\
x_1 + x_2 + 4x_3 - 5x_4 &= 0
\end{align*}
\]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):

\begin{align*}
x_1 &= 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0 \\
x_1 &= -3, \quad x_2 = -1, \quad x_3 = 1, \quad x_4 = 0 \\
x_1 &= 2, \quad x_2 = 3, \quad x_3 = 0, \quad x_4 = 1 \\
x_1 &= -1, \quad x_2 = 2, \quad x_3 = 1, \quad x_4 = 1
\end{align*}

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

\[
\begin{bmatrix}
1 & 0 & 3 & -2 & 0 \\
0 & 1 & 1 & -3 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system (Notation RREF [27]). Notice the slight variation for the same analysis of the original system only when the original system was consistent:

\[r = 2\quad D = \{1, 2\} \quad F = \{3, 4, 5\}\]

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations.

\[
\begin{bmatrix}
2 & 1 & 7 & -7 \\
-3 & 4 & -5 & -6 \\
1 & 1 & 4 & -5
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:

\[
\begin{bmatrix}
1 & 0 & 3 & -2 \\
0 & 1 & 1 & -3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Version 1.04
Analysis of the row-reduced matrix (Notation RREFA 27):

\[ r = 2 \quad D = \{1, 2\} \quad F = \{3, 4\} \]

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS 107, Theorem BNS 128). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VFSLS 88) to see these vectors arise.

\[ \langle \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \rangle \]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \( D \) above. (Theorem BCS 213)

\[ \langle \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \rangle \]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \( L \) is computed as described in Definition EEF 234. This is followed by the column space described by a set of linearly independent vectors that span the null space of \( L \), computed as according to Theorem FS 237 and Theorem BNS 128. When \( r = m \), the matrix \( L \) has no rows and the column space is all of \( \mathbb{C}^m \).

\[ L = \begin{bmatrix} 1 & 1/7 & -11/7 \end{bmatrix} \]

\[ \langle \begin{bmatrix} 1/7 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/7 \\ 1 \\ 0 \end{bmatrix} \rangle \]

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST 221 and Theorem BRS 220, and in the style of Example CSROI 221, this yields a linearly independent set of vectors that span the column space.

\[ \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/7 \\ 1/11 \end{bmatrix} \rangle \]

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS 220)
\[
\langle \begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -3 \end{bmatrix} \rangle
\]

Subspace dimensions associated with the matrix. (Definition NOM 312, Definition ROM 313) Verify Theorem RPNC 314

Matrix columns: 4  
Rank: 2  
Nullity: 2
Archetype E

**Summary** System with three equations, four variables. Inconsistent. Null space of coefficient matrix has dimension 2. Coefficient matrix identical to that of Archetype D, constant vector is different.

- A system of linear equations (Definition SLE [9]):
  
  \[
  \begin{align*}
  2x_1 + x_2 + 7x_3 - 7x_4 &= 2 \\
  -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 3 \\
  x_1 + x_2 + 4x_3 - 5x_4 &= 2
  \end{align*}
  \]

- Some solutions to the system of linear equations (not necessarily exhaustive):
  
  None. (Why?)

- Augmented matrix of the linear system of equations (Definition AM [24]):
  
  \[
  \begin{bmatrix}
  2 & 1 & 7 & -7 & 2 \\
  -3 & 4 & -5 & -6 & 3 \\
  1 & 1 & 4 & -5 & 2
  \end{bmatrix}
  \]

- Matrix in reduced row-echelon form, row-equivalent to augmented matrix:
  
  \[
  \begin{bmatrix}
  1 & 0 & 3 & -2 & 0 \\
  0 & 1 & 1 & -3 & 0 \\
  0 & 0 & 0 & 0 & 1
  \end{bmatrix}
  \]

- Analysis of the augmented matrix (Notation RREFA [27]):
  
  \[ r = 3 \quad D = \{1, 2, 5\} \quad F = \{3, 4\} \]

- Vector form of the solution set to the system of equations (Theorem VFSLS [88]). Notice the relationship between the free variables and the set \( F \) above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set \( F \) for the larger examples.

  Inconsistent system, no solutions exist.

- Given a system of equations we can always build a new, related, homogeneous system (Definition HS [52]) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system.

  \[
  2x_1 + x_2 + 7x_3 - 7x_4 = 0
  \]
\[-3x_1 + 4x_2 - 5x_3 - 6x_4 = 0 \\
x_1 + x_2 + 4x_3 - 5x_4 = 0 \]

- Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):
  \[x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0 \]
  \[x_1 = 4, \quad x_2 = 13, \quad x_3 = 2, \quad x_4 = 5 \]

- Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:
  \[
  \begin{bmatrix}
  1 & 0 & 3 & -2 & 0 \\
  0 & 1 & 1 & -3 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  \end{bmatrix}
  \]

- Analysis of the augmented matrix for the homogenous system (Notation RREFA [27]). Notice the slight variation for the same analysis of the original system only when the original system was consistent:
  \[r = 2 \quad D = \{1, 2\} \quad F = \{3, 4, 5\} \]

- Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations.
  \[
  \begin{bmatrix}
  2 & 1 & 7 & -7 \\
  -3 & 4 & -5 & -6 \\
  1 & 1 & 4 & -5 \\
  \end{bmatrix}
  \]

- Matrix brought to reduced row-echelon form:
  \[
  \begin{bmatrix}
  1 & 0 & 3 & -2 \\
  0 & 1 & 1 & -3 \\
  0 & 0 & 0 & 0 \\
  \end{bmatrix}
  \]

- Analysis of the row-reduced matrix (Notation RREFA [27]):
  \[r = 2 \quad D = \{1, 2\} \quad F = \{3, 4\} \]

- This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS [107], Theorem BNS [128]). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VFSLS [88]) to see these vectors arise.
Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \( D \) above. \( \text{(Theorem BCS 214)} \)

\[
\langle \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \rangle
\]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \( L \) is computed as described in Definition EEF 234. This is followed by the column space described by a set of linearly independent vectors that span the null space of \( L \), computed as according to Theorem FS 237 and Theorem BNS 128. When \( r = m \), the matrix \( L \) has no rows and the column space is all of \( \mathbb{C}^m \).

\[
L = \begin{bmatrix} 1 & \frac{1}{7} & -\frac{11}{7} \\ -1 & 1 & 0 \\ 1 & 3 & 0 \end{bmatrix}
\]

\[
\langle \begin{bmatrix} \frac{11}{7} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{7} \\ 1 \\ 0 \end{bmatrix} \rangle
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST 221 and Theorem BRS 220, and in the style of Example CSROI 221, this yields a linearly independent set of vectors that span the column space.

\[
\langle \begin{bmatrix} 1 \\ 0 \\ \frac{7}{11} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 11 \end{bmatrix} \rangle
\]

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. \( \text{(Theorem BRS 220)} \)

\[
\langle \begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -3 \end{bmatrix} \rangle
\]

Subspace dimensions associated with the matrix. \( \text{(Definition NOM 312, Definition ROM 313)} \) Verify Theorem RPNC 314

Matrix columns: 4  
Rank: 2  
Nullity: 2
Archetype F

Summary  System with four equations, four variables. Nonsingular coefficient matrix. Integer eigenvalues, one has “high” multiplicity.

- A system of linear equations (Definition SLE[9]):
  \[
  \begin{align*}
  33x_1 - 16x_2 + 10x_3 - 2x_4 &= -27 \\
  99x_1 - 47x_2 + 27x_3 - 7x_4 &= -77 \\
  78x_1 - 36x_2 + 17x_3 - 6x_4 &= -52 \\
  -9x_1 + 2x_2 + 3x_3 + 4x_4 &= 5
  \end{align*}
  \]

- Some solutions to the system of linear equations (not necessarily exhaustive):
  \[x_1 = 1, \ x_2 = 2, \ x_3 = -2, \ x_4 = 4\]

- Augmented matrix of the linear system of equations (Definition AM[24]):
  \[
  \begin{bmatrix}
  33 & -16 & 10 & -2 & -27 \\
  99 & -47 & 27 & -7 & -77 \\
  78 & -36 & 17 & -6 & -52 \\
  -9 & 2 & 3 & 4 & 5
  \end{bmatrix}
  \]

- Matrix in reduced row-echelon form, row-equivalent to augmented matrix:
  \[
  \begin{bmatrix}
  1 & 0 & 0 & 0 & 1 \\
  0 & 1 & 0 & 0 & 2 \\
  0 & 0 & 1 & 0 & -2 \\
  0 & 0 & 0 & 1 & 4
  \end{bmatrix}
  \]

- Analysis of the augmented matrix (Notation RREFA[27]):
  \[r = 4, \ D = \{1, 2, 3, 4\}, \ F = \{5\}\]

- Vector form of the solution set to the system of equations (Theorem VFSLS[88]). Notice the relationship between the free variables and the set \(F\) above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set \(F\) for the larger examples.
  \[
  \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
  \end{bmatrix}
  =
  \begin{bmatrix}
  1 \\
  2 \\
  -2 \\
  4
  \end{bmatrix}
  \]

- Given a system of equations we can always build a new, related, homogeneous system (Definition HS[52]) by converting the constant terms to zeros and retaining the coefficients of the variables.
Properties of this new system will have precise relationships with various properties of the original system.

\[
\begin{align*}
33x_1 - 16x_2 + 10x_3 - 2x_4 &= 0 \\
99x_1 - 47x_2 + 27x_3 - 7x_4 &= 0 \\
78x_1 - 36x_2 + 17x_3 - 6x_4 &= 0 \\
-9x_1 + 2x_2 + 3x_3 + 4x_4 &= 0
\end{align*}
\]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):
\[
x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0
\]

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system (Notation RREFA [27]). Notice the slight variation for the same analysis of the original system only when the original system was consistent:

\[
r = 4 \quad D = \{1, 2, 3, 4\} \quad F = \{5\}
\]

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations.

\[
\begin{bmatrix}
33 & -16 & 10 & -2 \\
99 & -47 & 27 & -7 \\
78 & -36 & 17 & -6 \\
-9 & 2 & 3 & 4
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Analysis of the row-reduced matrix (Notation RREFA [27]):

\[
r = 4 \quad D = \{1, 2, 3, 4\} \quad F = \{\}
\]
Matrix (coefficient matrix) is nonsingular or singular? \(\text{[Theorem NMRRI]} 62\) at the same time, examine the size of the set \(F\) above. Notice that this property does not apply to matrices that are not square.

Nonsingular.

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix \(\text{[Theorem SSNS]} 107, \text{[Theorem BNS]} 128\). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form \(\text{[Theorem VFSLS]} 88\) to see these vectors arise.

\(
\langle \{ \} \rangle
\)

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \(D\) above. \(\text{[Theorem BCS]} 214\)

\[
\left\{ \begin{bmatrix} 33 \\ 99 \\ 78 \\ -9 \\ -16 \\ -47 \\ -36 \\ 2 \\ 10 \\ 27 \\ 17 \\ 3 \\ -2 \\ -7 \\ -6 \\ 4 \end{bmatrix} \right\}
\]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \(L\) is computed as described in \(\text{Definition EEF} 234\). This is followed by the column space described by a set of linearly independent vectors that span the null space of \(L\), computed as according to \(\text{Theorem FS} 237\) and \(\text{Theorem BNS} 128\). When \(r = m\), the matrix \(L\) has no rows and the column space is all of \(\mathbb{C}^m\).

\[
L = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By \(\text{Theorem CSRST} 221\) and \(\text{Theorem BRS} 220\), and in the style of \(\text{Example CSROI} 221\), this yields a linearly independent set of vectors that span the column space.

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}
\]

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained
from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS 220)
\[
\begin{pmatrix}
[1] & [0] & [0] & [0] \\
[0] & [1] & [0] & [0] \\
[0] & [0] & [1] & [0] \\
[0] & [0] & [0] & [1]
\end{pmatrix}
\]

Inverse matrix, if it exists. The inverse is not defined for matrices that are not square, and if the matrix is square, then the matrix must be nonsingular. (Definition MI 189, Theorem NI 204)
\[
\begin{pmatrix}
\frac{86}{29} & \frac{38}{3} & -\frac{11}{3} & \frac{7}{3} \\
-\frac{129}{2} & \frac{38}{3} & -\frac{11}{3} & \frac{7}{3} \\
-13 & 6 & -2 & 1 \\
-\frac{45}{2} & \frac{29}{3} & -\frac{5}{2} & \frac{13}{6}
\end{pmatrix}
\]

Subspace dimensions associated with the matrix. (Definition NOM 312, Definition ROM 313) Verify Theorem RPNC 314

Matrix columns: 4 Rank: 4 Nullity: 0

Determinant of the matrix, which is only defined for square matrices. The matrix is nonsingular if and only if the determinant is nonzero (Theorem SMZD 351). (Product of all eigenvalues?)

Determinant = \(-18\)

Eigenvalues, and bases for eigenspaces. (Definition EEM 356, Definition EM 364)

\[
\lambda = -1 \quad \mathcal{E}_F(-1) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}
\]

\[
\lambda = 2 \quad \mathcal{E}_F(2) = \left\{ \begin{pmatrix} 2 \\ 5 \\ 2 \\ 1 \end{pmatrix} \right\}
\]

\[
\lambda = 3 \quad \mathcal{E}_F(3) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 7 \\ 17 \\ 45 \\ 21 \\ 0 \end{pmatrix} \right\}
\]

Geometric and algebraic multiplicities. (Definition GME 366, Definition AME 366)

\[
\gamma_F(-1) = 1 \quad \alpha_F(-1) = 1 \\
\gamma_F(2) = 1 \quad \alpha_F(2) = 1 \\
\gamma_F(3) = 2 \quad \alpha_F(3) = 2
\]
Diagonalizable? [Definition DZM (393)]

Yes, full eigenspaces. [Theorem DMFE (396)]

The diagonalization. [Theorem DC (394)]

\[
\begin{bmatrix}
12 & -5 & 1 & -1 \\
-39 & 18 & -7 & 3 \\
27 & -13/7 & 6/7 & -1/7 \\
26/7 & -12/7 & 5/7 & -2/7 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 2 & 1 & 17 \\
99 & -47 & 27 & -7 \\
78 & -36 & 17 & -6 \\
-9 & 2 & 3 & 4 \\
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 3 \\
\end{bmatrix}
\]
Archetype G

Summary System with five equations, two variables. Consistent. Null space of coefficient matrix has dimension 0. Coefficient matrix identical to that of Archetype H, constant vector is different.

A system of linear equations (Definition SLE \[9\]):

\[
\begin{align*}
2x_1 + 3x_2 &= 6 \\
-x_1 + 4x_2 &= -14 \\
3x_1 + 10x_2 &= -2 \\
3x_1 - x_2 &= 20 \\
6x_1 + 9x_2 &= 18
\end{align*}
\]

Some solutions to the system of linear equations (not necessarily exhaustive):

\[x_1 = 6, \quad x_2 = -2\]

Augmented matrix of the linear system of equations (Definition AM \[24\]):

\[
\begin{bmatrix}
2 & 3 & 6 \\
-1 & 4 & -14 \\
3 & 10 & -2 \\
3 & -1 & 20 \\
6 & 9 & 18
\end{bmatrix}
\]

Matrix in reduced row-echelon form, row-equivalent to augmented matrix:

\[
\begin{bmatrix}
1 & 0 & 6 \\
0 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix (Notation RREFA \[27\]):

\[r = 2 \quad D = \{1, 2\} \quad F = \{3\}\]

Vector form of the solution set to the system of equations (Theorem VFSLS \[88\]). Notice the relationship between the free variables and the set \(F\) above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set \(F\) for the larger examples.
\[
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= \begin{bmatrix}
  6 \\
  -2
\end{bmatrix}
\]

Given a system of equations we can always build a new, related, homogeneous system (Definition HS [52]) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system.

\[
\begin{align*}
2x_1 + 3x_2 &= 0 \\
-x_1 + 4x_2 &= 0 \\
3x_1 + 10x_2 &= 0 \\
3x_1 - x_2 &= 0 \\
6x_1 + 9x_2 &= 0
\end{align*}
\]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive): 
\[x_1 = 0, \quad x_2 = 0\]

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system (Notation RREFA [27]). Notice the slight variation for the same analysis of the original system only when the original system was consistent:

\[r = 2 \quad D = \{1, 2\} \quad F = \{3\}\]

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations.

\[
\begin{bmatrix}
2 & 3 \\
-1 & 4 \\
3 & 10 \\
3 & -1 \\
6 & 9
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]
Analysis of the row-reduced matrix (Notation RREF_A): 

\[ r = 2 \quad D = \{1, 2\} \quad F = \{\} \]

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS [107], Theorem BNS [128]). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VFSLS [88]) to see these vectors arise.

\[ \langle{}\rangle \]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \( D \) above. (Theorem BCS [214])

\[ \langle \begin{pmatrix} 2 \\ -1 \\ 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 10 \\ -1 \end{pmatrix} \rangle \]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \( L \) is computed as described in Definition EEF [234]. This is followed by the column space described by a set of linearly independent vectors that span the null space of \( L \), computed as according to Theorem FS [237] and Theorem BNS [128]. When \( r = m \), the matrix \( L \) has no rows and the column space is all of \( \mathbb{C}^m \).

\[ L = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \]

\[ \langle \begin{pmatrix} \frac{1}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \rangle \]

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST [221] and Theorem BRS [220], and in the style of Example CSROI [221], this yields a linearly independent set of vectors that span the column space.
Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS 220)

\[ \langle \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \rangle \]

Subspace dimensions associated with the matrix. (Definition NOM 312, Definition ROM 313) Verify Theorem RPNC 314

Matrix columns: 2 \hspace{1cm} Rank: 2 \hspace{1cm} Nullity: 0
Archetype H

Summary  System with five equations, two variables. Inconsistent, overdetermined. Null space of coefficient matrix has dimension 0. Coefficient matrix identical to that of Archetype G, constant vector is different.

A system of linear equations (Definition SLE):  

\[
egin{align*}
2x_1 + 3x_2 &= 5 \\
-x_1 + 4x_2 &= 6 \\
3x_1 + 10x_2 &= 2 \\
3x_1 - x_2 &= -1 \\
6x_1 + 9x_2 &= 3
\end{align*}
\]

Some solutions to the system of linear equations (not necessarily exhaustive):

None. (Why?)

Augmented matrix of the linear system of equations (Definition AM):

\[
\begin{bmatrix}
2 & 3 & 5 \\
-1 & 4 & 6 \\
3 & 10 & 2 \\
3 & -1 & -1 \\
6 & 9 & 3
\end{bmatrix}
\]

Matrix in reduced row-echelon form, row-equivalent to augmented matrix:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix (Notation RREFA): 

\[ r = 3 \quad D = \{1, 2, 3\} \quad F = \{\} \]

Vector form of the solution set to the system of equations (Theorem VFSLS). Notice the relationship between the free variables and the set \(F\) above. Also, notice the pattern of 0's and 1's in the entries of the vectors corresponding to elements of the set \(F\) for the larger examples.
Inconsistent system, no solutions exist.

Given a system of equations we can always build a new, related, homogeneous system \( \text{Definition HS [52]} \) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system.

\[
\begin{align*}
2x_1 + 3x_2 &= 0 \\
-x_1 + 4x_2 &= 0 \\
3x_1 + 10x_2 &= 0 \\
3x_1 - x_2 &= 0 \\
6x_1 + 9x_2 &= 0
\end{align*}
\]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):

\[x_1 = 0, \quad x_2 = 0\]

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system \( \text{Notation RREFA [27]} \). Notice the slight variation for the same analysis of the original system only when the original system was consistent:

\[r = 2 \quad D = \{1, 2\} \quad F = \{3\}\]

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations.

\[
\begin{bmatrix}
2 & 3 \\
-1 & 4 \\
3 & 10 \\
3 & -1 \\
6 & 9
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]
Analysis of the row-reduced matrix (Notation RREFA 27):

\[ r = 2 \quad D = \{1, 2\} \quad F = \{ \} \]

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS 107, Theorem BNS 128). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VFSLS 88) to see these vectors arise.

\[ \{ \} \]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \( D \) above. (Theorem BCS 214)

\[ \langle \begin{bmatrix} 2 \\ -1 \\ 3 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 10 \\ -1 \\ -9 \end{bmatrix} \rangle \]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \( L \) is computed as described in Definition EEF 234. This is followed by the column space described by a set of linearly independent vectors that span the null space of \( L \), computed as according to Theorem FS 237 and Theorem BNS 128. When \( r = m \), the matrix \( L \) has no rows and the column space is all of \( \mathbb{C}^m \).

\[ L = \left[ \begin{array}{c} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \frac{1}{3} \\ 1 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \end{array} \right] \]

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST 221 and Theorem BRS 220, and in the style of Example CSROI 221, this yields a linearly independent set of vectors that span the column space.
The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix $L$ is computed as described in Definition EEF [234]. This is followed by the column space described by a set of linearly independent vectors that span the null space of $L$, computed as according to Theorem FS [237] and Theorem BNS [128]. When $r = m$, the matrix $L$ has no rows and the column space is all of $\mathbb{C}^m$.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 1 - \frac{1}{3} \\ 0 & 0 & 1 & 1 - 1 \end{bmatrix}$$

$$\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\} \rangle$$

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS [220])

$$\langle \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix} \right\} \rangle$$

Subspace dimensions associated with the matrix. (Definition NOM [312], Definition ROM [313]) Verify Theorem RPNC [314]

Matrix columns: 2  
Rank: 2  
Nullity: 0
Archetype I

Summary System with four equations, seven variables. Consistent. Null space of coefficient matrix has dimension 4.

A system of linear equations (Definition SLE [9]):

\[
\begin{align*}
    x_1 + 4x_2 - x_4 + 7x_6 - 9x_7 &= 3, \\
    2x_1 + 8x_2 - x_3 + 3x_4 + 9x_5 - 13x_6 + 7x_7 &= 9, \\
    2x_3 - 3x_4 - 4x_5 + 12x_6 - 8x_7 &= 1, \\
    -x_1 - 4x_2 + 2x_3 + 4x_4 + 8x_5 - 31x_6 + 37x_7 &= 4.
\end{align*}
\]

Some solutions to the system of linear equations (not necessarily exhaustive):

\[
\begin{align*}
    x_1 &= -25, \quad x_2 = 4, \quad x_3 = 22, \quad x_4 = 29, \quad x_5 = 1, \quad x_6 = 2, \quad x_7 = -3, \\
    x_1 &= -7, \quad x_2 = 5, \quad x_3 = 7, \quad x_4 = 15, \quad x_5 = -4, \quad x_6 = 2, \quad x_7 = 1, \\
    x_1 &= 4, \quad x_2 = 0, \quad x_3 = 2, \quad x_4 = 1, \quad x_5 = 0, \quad x_6 = 0, \quad x_7 = 0.
\end{align*}
\]

Augmented matrix of the linear system of equations (Definition AM [24]):

\[
\begin{pmatrix}
    1 & 4 & 0 & -1 & 0 & 7 & -9 & 3 \\
    2 & 8 & -1 & 3 & 9 & -13 & 7 & 9 \\
    0 & 0 & 2 & -3 & -4 & 12 & -8 & 1 \\
    -1 & -4 & 2 & 4 & 8 & -31 & 37 & 4
\end{pmatrix}
\]

Matrix in reduced row-echelon form, row-equivalent to augmented matrix:

\[
\begin{pmatrix}
    1 & 4 & 0 & 0 & 2 & 1 & -3 & 4 \\
    0 & 0 & 1 & 0 & 1 & -3 & 5 & 2 \\
    0 & 0 & 0 & 1 & 2 & -6 & 6 & 1 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Analysis of the augmented matrix (Notation RREFA [27]):

\[
\begin{align*}
    r &= 3, \\
    D &= \{1, 3, 4\}, \\
    F &= \{2, 5, 6, 7, 8\}.
\end{align*}
\]

Vector form of the solution set to the system of equations (Theorem VFSLS [88]). Notice the relationship between the free variables and the set $F$ above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set $F$ for the larger examples.
Given a system of equations we can always build a new, related, homogeneous system (Definition HS [52]) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system.

\[
\begin{align*}
    x_1 + 4x_2 - x_4 + 7x_6 - 9x_7 &= 0 \\
    2x_1 + 8x_2 - x_3 + 3x_4 + 9x_5 - 13x_6 + 7x_7 &= 0 \\
    2x_3 - 3x_4 - 4x_5 + 12x_6 - 8x_7 &= 0 \\
    -x_1 - 4x_2 + 2x_3 + 4x_4 + 8x_5 - 31x_6 + 37x_7 &= 0
\end{align*}
\]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):
- \(x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0\)
- \(x_1 = 3, x_2 = 0, x_3 = -5, x_4 = -6, x_5 = 0, x_6 = 0, x_7 = 1\)
- \(x_1 = -1, x_2 = 0, x_3 = 3, x_4 = 6, x_5 = 0, x_6 = 1, x_7 = 0\)
- \(x_1 = -2, x_2 = 0, x_3 = -1, x_4 = -2, x_5 = 1, x_6 = 0, x_7 = 0\)
- \(x_1 = -4, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0\)
- \(x_1 = -4, x_2 = 1, x_3 = -3, x_4 = -2, x_5 = 1, x_6 = 1, x_7 = 1\)

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

\[
\begin{bmatrix}
1 & 4 & 0 & 0 & 2 & 1 & -3 & 0 \\
0 & 0 & 1 & 0 & 1 & -3 & 5 & 0 \\
0 & 0 & 0 & 1 & 2 & -6 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system (Notation RREFA [27]). Notice the slight variation for the same analysis of the original system only when the original system was consistent:

\(r = 3\) \quad \(D = \{1, 3, 4\}\) \quad \(F = \{2, 5, 6, 7, 8\}\)

Coefficient matrix of original system of equations, and of associated homogenous system. This
matrix will be the subject of further analysis, rather than the systems of equations.

\[
\begin{bmatrix}
1 & 4 & 0 & -1 & 0 & 7 & -9 \\
2 & 8 & -1 & 3 & 9 & -13 & 7 \\
0 & 0 & 2 & -3 & -4 & 12 & -8 \\
-1 & -4 & 2 & 4 & 8 & -31 & 37
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:

\[
\begin{bmatrix}
1 & 4 & 0 & 0 & 2 & 1 & -3 \\
0 & 0 & 1 & 0 & 1 & -3 & 5 \\
0 & 0 & 0 & 1 & 2 & -6 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis of the row-reduced matrix (Notation RREFA \[27\]):

\[r = 3\]  \[D = \{1, 3, 4\}\]  \[F = \{2, 5, 6, 7\}\]

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS \[107\], Theorem BNS \[128\]). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VFSLS \[88\]) to see these vectors arise.

\[
\begin{bmatrix}
-4 \\
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
-2 \\
0 \\
-1 \\
1
\end{bmatrix},
\begin{bmatrix}
-1 \\
3 \\
6 \\
0
\end{bmatrix},
\begin{bmatrix}
3 \\
0 \\
-5 \\
-6
\end{bmatrix}
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \(D\) above. (Theorem BCS \[214\])

\[
\begin{bmatrix}
1 \\
2 \\
-1
\end{bmatrix},
\begin{bmatrix}
0 \\
-1 \\
2
\end{bmatrix},
\begin{bmatrix}
-1 \\
3 \\
-3
\end{bmatrix},
\begin{bmatrix}
0 \\
-3 \\
4
\end{bmatrix}
\]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \(L\) is computed as described in Definition EEF \[234\]. This is followed by the column space described by a set of linearly independent vectors that span the null space of \(L\), computed as according to Theorem FS \[237\] and Theorem BNS \[128\]. When \(r = m\), the matrix \(L\) has no rows and the column space is all of \(\mathbb{C}^m\).

\[
L = \begin{bmatrix}
1 & -\frac{12}{31} & -\frac{13}{31} & \frac{7}{31}
\end{bmatrix}
\]
\[ \langle \begin{bmatrix} -7 \\ 0 \\ 1 \\ \frac{13}{7} \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ \frac{12}{7} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ \frac{13}{7} \end{bmatrix} \rangle \]

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST \[221\] and Theorem BRS \[220\], and in the style of Example CSROI \[221\], this yields a linearly independent set of vectors that span the column space.

\[ \langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \frac{13}{7} \end{bmatrix} \rangle \]

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS \[220\])

\[ \langle \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 1 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -3 \end{bmatrix} \rangle \]

Subspace dimensions associated with the matrix. (Definition NOM \[312\], Definition ROM \[313\]) Verify Theorem RPNC \[314\]

Matrix columns: 7  
Rank: 3  
Nullity: 4
Archetype J

Summary  System with six equations, nine variables. Consistent. Null space of coefficient matrix has dimension 5.

A system of linear equations (Definition SLE [9]):

\[
\begin{align*}
2x_1 + 2x_2 &- 2x_3 + 9x_4 + 3x_5 - 5x_6 - 2x_7 + x_8 + 27x_9 = -5 \\
2x_1 + 4x_2 + 3x_3 + 4x_4 - x_5 + 4x_6 + 10x_7 + 2x_8 - 23x_9 &= 18 \\
x_1 + 2x_2 + x_3 + 3x_4 + x_5 + x_6 + 5x_7 + 2x_8 - 7x_9 &= 6 \\
2x_1 + 4x_2 + 3x_3 + 4x_4 - 7x_5 + 2x_6 + 4x_7 - 11x_9 &= 20 \\
x_1 + 2x_2 + 5x_4 + 2x_5 - 4x_6 + 3x_7 + 8x_8 + 13x_9 &= -4 \\
-3x_1 - 6x_2 - x_3 - 13x_4 + 2x_5 - 5x_6 - 4x_7 + 13x_8 + 10x_9 &= -29
\end{align*}
\]

Some solutions to the system of linear equations (not necessarily exhaustive):

\[
\begin{align*}
x_1 &= 6, x_2 = 0, x_3 = -1, x_4 = 0, x_5 = -1, x_6 = 2, x_7 = 0, x_8 = 0, x_9 = 0 \\
x_1 &= 4, x_2 = 1, x_3 = -1, x_4 = 0, x_5 = -1, x_6 = 2, x_7 = 0, x_8 = 0, x_9 = 0 \\
x_1 &= -17, x_2 = 7, x_3 = 3, x_4 = 2, x_5 = -1, x_6 = 14, x_7 = -1, x_8 = 3, x_9 = 2 \\
x_1 &= -11, x_2 = -6, x_3 = 1, x_4 = 5, x_5 = -4, x_6 = 7, x_7 = 3, x_8 = 1, x_9 = 1
\end{align*}
\]

Augmented matrix of the linear system of equations (Definition AM [24]):

\[
\begin{bmatrix}
1 & 2 & -2 & 9 & 3 & -5 & -2 & 1 & 27 & -5 \\
2 & 4 & 3 & 4 & -& 1 & 4 & 10 & 2 & -23 & 18 \\
1 & 2 & 1 & 3 & 1 & 1 & 5 & 2 & -7 & 6 \\
2 & 4 & 3 & 4 & -& 7 & 2 & 4 & 0 & -11 & 20 \\
1 & 2 & 0 & 5 & 2 & -4 & 3 & 8 & 13 & -4 \\
-3 & -6 & -1 & -13 & 2 & -5 & -4 & 13 & 10 & -29
\end{bmatrix}
\]

Matrix in reduced row-echelon form, row-equivalent to augmented matrix:

\[
\begin{bmatrix}
1 & 2 & 0 & 5 & 0 & 0 & 0 & 1 & -2 & 3 & 6 \\
0 & 0 & 1 & -2 & 0 & 0 & 3 & 5 & -6 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & -3 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix (Notation RREFA [27]):

\[
r = 4 \quad D = \{1, 3, 5, 6\} \quad F = \{2, 4, 7, 8, 9, 10\}
\]

Version 1.04
Vector form of the solution set to the system of equations (Theorem VFSLS [88]). Notice the relationship between the free variables and the set \( F \) above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set \( F \) for the larger examples.

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
  x_6 \\
  x_7 \\
  x_8 \\
  x_9
\end{bmatrix} =
\begin{bmatrix}
  6 \\
  0 \\
  -1 \\
  0 \\
  -1 \\
  2 \\
  0 \\
  0 \\
  0
\end{bmatrix}
+ \begin{bmatrix}
  -2 \\
  1 \\
  0 \\
  0 \\
  0 \\
  1 \\
  0 \\
  0 \\
  1
\end{bmatrix}
\begin{bmatrix}
  -5 \\
  2 \\
  0 \\
  1 \\
  0 \\
  0 \\
  1 \\
  0 \\
  0
\end{bmatrix}
+ \begin{bmatrix}
  -1 \\
  0 \\
  0 \\
  0 \\
  0 \\
  -1 \\
  0 \\
  0 \\
  -1
\end{bmatrix}
\begin{bmatrix}
  2 \\
  0 \\
  0 \\
  0 \\
  0 \\
  -1 \\
  2 \\
  2 \\
  1
\end{bmatrix}
+ \begin{bmatrix}
  -3 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  1
\end{bmatrix}
\]

Given a system of equations we can always build a new, related, homogeneous system (Definition HS [52]) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system.

\[
\begin{align*}
x_1 + 2x_2 - 2x_3 + 9x_4 + 3x_5 - 5x_6 - 2x_7 + x_8 + 27x_9 &= 0 \\
2x_1 + 4x_2 + 3x_3 + 4x_4 - x_5 + 4x_6 + 10x_7 + 2x_8 - 23x_9 &= 0 \\
x_1 + 2x_2 + x_3 + 3x_4 + x_5 + x_6 + 5x_7 + 2x_8 - 7x_9 &= 0 \\
2x_1 + 4x_2 + 3x_3 + 4x_4 - 7x_5 + 2x_6 + 4x_7 - 11x_9 &= 0 \\
x_1 + 2x_2 + 5x_4 + 2x_5 - 4x_6 + 3x_7 + 8x_8 + 13x_9 &= 0 \\
-3x_1 - 6x_2 - x_3 - 13x_4 + 2x_5 - 5x_6 - 4x_7 + 13x_8 + 10x_9 &= 0
\end{align*}
\]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):

\[
x_1 = 0, \ x_2 = 0, \ x_3 = 0, \ x_4 = 0, \ x_5 = 0, \ x_6 = 0, \ x_7 = 0, \ x_8 = 0, \ x_9 = 0
\]

\[
x_1 = -2, \ x_2 = 1, \ x_3 = 0, \ x_4 = 0, \ x_5 = 0, \ x_6 = 0, \ x_7 = 0, \ x_8 = 0, \ x_9 = 0
\]

\[
x_1 = -23, \ x_2 = 7, \ x_3 = 4, \ x_4 = 2, \ x_5 = 0, \ x_6 = 12, \ x_7 = -1, \ x_8 = 3, \ x_9 = 2
\]

\[
x_1 = -17, \ x_2 = -6, \ x_3 = 2, \ x_4 = 5, \ x_5 = -3, \ x_6 = 5, \ x_7 = 3, \ x_8 = 1, \ x_9 = 1
\]

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

\[
\begin{bmatrix}
  1 & 2 & 0 & 5 & 0 & 0 & 1 & -2 & 3 & 0 \\
  0 & 0 & 1 & 0 & 0 & 3 & 5 & -6 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & -3 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system (Notation RREFA [27]). Notice
the slight variation for the same analysis of the original system only when the original system was consistent:
\[ r = 4 \quad D = \{1, 3, 5, 6\} \quad F = \{2, 4, 7, 8, 9, 10\} \]

□ Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations.
\[
\begin{bmatrix}
1 & 2 & -2 & 9 & 3 & -5 & -2 & 1 & 27 \\
2 & 4 & 3 & 4 & -1 & 4 & 10 & 2 & -23 \\
1 & 2 & 1 & 3 & 1 & 1 & 5 & 2 & -7 \\
2 & 4 & 3 & 4 & -7 & 2 & 4 & 0 & -11 \\
1 & 2 & 0 & 5 & 2 & -4 & 3 & 8 & 13 \\
-3 & -6 & -1 & -13 & 2 & -5 & -4 & 13 & 10 \\
\end{bmatrix}
\]

□ Matrix brought to reduced row-echelon form:
\[
\begin{bmatrix}
1 & 2 & 0 & 5 & 0 & 0 & 1 & -2 & 3 \\
0 & 0 & 1 & -2 & 0 & 0 & 3 & 5 & -6 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

□ Analysis of the row-reduced matrix [Notation RREFA 27]:
\[ r = 4 \quad D = \{1, 3, 5, 6\} \quad F = \{2, 4, 7, 8, 9\} \]

□ This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix [Theorem SSNS 107, Theorem BNS 128]. Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form [Theorem VFSLS 88] to see these vectors arise.
\[
\begin{bmatrix}
\begin{bmatrix}
-2 \\
1 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix},
\begin{bmatrix}
-5 \\
0 \\
2 \\
1 \\
0 \\
0 \\
\end{bmatrix},
\begin{bmatrix}
-1 \\
0 \\
-3 \\
0 \\
0 \\
0 \\
\end{bmatrix},
\begin{bmatrix}
2 \\
0 \\
-5 \\
1 \\
0 \\
0 \\
\end{bmatrix},
\begin{bmatrix}
-3 \\
0 \\
6 \\
1 \\
0 \\
0 \\
\end{bmatrix}
\end{bmatrix}
\]

□ Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \(D\) above. [Theorem BCS 214]
The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix $L$ is computed as described in Definition EEF [234]. This is followed by the column space described by a set of linearly independent vectors that span the null space of $L$, computed as according to Theorem FS [237] and Theorem BNS [128]. When $r = m$, the matrix $L$ has no rows and the column space is all of $C^m$.

$$L = \begin{bmatrix} 1 & 0 & \frac{186}{131} & \frac{51}{131} & \frac{-188}{131} & \frac{77}{131} \\ 0 & 1 & -\frac{132}{131} & -\frac{45}{131} & -\frac{188}{131} & -\frac{131}{131} \\ -\frac{77}{131} & \frac{188}{131} & \frac{-51}{131} & \frac{-186}{131} & \frac{22}{131} & \frac{22}{131} \\ 0 & 0 & \frac{1}{1} & \frac{1}{1} & \frac{0}{1} & \frac{0}{1} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST [221] and Theorem BRS [220], and in the style of Example CSROI [221], this yields a linearly independent set of vectors that span the column space.

$$\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\frac{29}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{11}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{3}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} \end{bmatrix} \rangle$$

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS [220])

$$\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 5 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 5 & 1 & -2 \\ 3 & -6 & -1 & -3 \end{bmatrix} \rangle$$

Subspace dimensions associated with the matrix. (Definition NOM [312], Definition ROM [312])
Verify Theorem RPNC

Matrix columns: 9  Rank: 4  Nullity: 5
Archetype K

Summary  Square matrix of size 5. Nonsingular. 3 distinct eigenvalues, 2 of multiplicity 2.

A matrix:
\[
\begin{bmatrix}
10 & 18 & 24 & 24 & -12 \\
12 & -2 & -6 & 0 & -18 \\
-30 & -21 & -23 & -30 & 39 \\
27 & 30 & 36 & 37 & -30 \\
18 & 24 & 30 & 30 & -20
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Analysis of the row-reduced matrix (Notation RREFA 27):
\[ r = 5 \quad D = \{1, 2, 3, 4, 5\} \quad F = \{\ \}\]

Matrix (coefficient matrix) is nonsingular or singular? (Theorem NMRRI 62) at the same time, examine the size of the set \(F\) above. Notice that this property does not apply to matrices that are not square.

Nonsingular.

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS 107, Theorem BNS 128). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VFSLS 88) to see these vectors arise.

\[\langle\{\ }\rangle\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \(D\) above. (Theorem BCS 214)
The column space of the matrix, as it arises from the extended echelon form of the matrix.
The matrix $L$ is computed as described in Definition EEF [234]. This is followed by the column
space described by a set of linearly independent vectors that span the null space of
$L$, computed as according to Theorem FS [237] and Theorem BNS [128]. When $r = m$, the matrix $L$ has no rows
and the column space is all of $\mathbb{C}^m$.

$L = \langle \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rangle$

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST [221] and Theorem BRS [220], and in the style of Example CSROI [221], this yields a linearly independent set of vectors that span the column space.

$\langle \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rangle$

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS [220])

$\langle \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rangle$

Inverse matrix, if it exists. The inverse is not defined for matrices that are not square, and if the matrix is square, then the matrix must be nonsingular. (Definition MI [189], Theorem NI [204])
Subspace dimensions associated with the matrix. Verify Theorem RPNC

Matrix columns: 5 Rank: 5 Nullity: 0

Determinant of the matrix, which is only defined for square matrices. The matrix is nonsingular if and only if the determinant is nonzero (Theorem SMZD). (Product of all eigenvalues?)

Determinant = 16

Eigenvalues, and bases for eigenspaces.

\[ \lambda = -2 \]
\[ \mathcal{E}_K (-2) = \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\} \]

\[ \lambda = 1 \]
\[ \mathcal{E}_K (1) = \left\{ \begin{bmatrix} 4 \\ -10 \\ 7 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 18 \\ -17 \\ 5 \\ 0 \end{bmatrix} \right\} \]

\[ \lambda = 4 \]
\[ \mathcal{E}_K (4) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \]

Geometric and algebraic multiplicities.

\[ \gamma_K (-2) = 2 \quad \alpha_K (-2) = 2 \]
\[ \gamma_K (1) = 2 \quad \alpha_K (1) = 2 \]
\[ \gamma_K (4) = 1 \quad \alpha_K (4) = 1 \]

Diagonalizable?
Yes, full eigenspaces. Theorem DMFE \[396\].

The diagonalization. (Theorem DC \[394\])

\[
\begin{bmatrix}
-4 & -3 & -4 & -6 & 7 \\
-7 & -5 & -6 & -8 & 10 \\
1 & -1 & -1 & 1 & -3 \\
1 & 0 & 0 & 1 & -2 \\
2 & 5 & 6 & 4 & 0 \\
\end{bmatrix}
\begin{bmatrix}
10 & 18 & 24 & 24 & -12 \\
12 & -2 & -6 & 0 & -18 \\
-30 & -21 & -23 & -30 & 39 \\
27 & 30 & 36 & 37 & -30 \\
18 & 24 & 30 & 30 & -20 \\
\end{bmatrix}
\begin{bmatrix}
2 & -1 & 4 & -4 & 1 \\
-2 & 2 & -10 & 18 & -1 \\
1 & -2 & 7 & -17 & 0 \\
0 & 1 & 0 & 5 & 1 \\
1 & 0 & 2 & 0 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
-2 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 4 \\
\end{bmatrix}
\]
Archetype L

Summary   Square matrix of size 5. Singular, nullity 2. 2 distinct eigenvalues, each of “high” multiplicity.

A matrix:
\[
\begin{pmatrix}
-2 & -1 & -2 & -4 & 4 \\
-6 & -5 & -4 & -4 & 6 \\
10 & 7 & 7 & 10 & -13 \\
-7 & -5 & -6 & -9 & 10 \\
-4 & -3 & -4 & -6 & 6
\end{pmatrix}
\]

Matrix brought to reduced row-echelon form:
\[
\begin{pmatrix}
1 & 0 & 0 & 1 & -2 \\
0 & 1 & 0 & -2 & 2 \\
0 & 0 & 1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Analysis of the row-reduced matrix (Notation RREFA [27]):
\[ r = 5 \quad D = \{1, 2, 3\} \quad F = \{4, 5\} \]

Matrix (coefficient matrix) is nonsingular or singular? (Theorem NMRRI [62]) at the same time, examine the size of the set \( F \) above. Notice that this property does not apply to matrices that are not square.

Singular.

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS [107], Theorem BNS [128]). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VFSLS [88]) to see these vectors arise.

\[
\left\langle \begin{pmatrix}
-1 \\
2 \\
-2 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
2 \\
-2 \\
1 \\
0 \\
1
\end{pmatrix}\right\rangle
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \( D \) above. (Theorem...
The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix $L$ is computed as described in Definition EEF [234]. This is followed by the column space described by a set of linearly independent vectors that span the null space of $L$, computed as according to Theorem FS [237] and Theorem BNS [128]. When $r = m$, the matrix $L$ has no rows and the column space is all of $\mathbb{C}^m$.

$$L = \begin{bmatrix}
1 & 0 & -2 & -6 & 5 \\
0 & 1 & 4 & 10 & -9
\end{bmatrix}$$

$$\begin{bmatrix}
-5 \\
9 \\
0 \\
0 \\
1
\end{bmatrix} , \begin{bmatrix}
6 \\
-10 \\
0 \\
1 \\
0
\end{bmatrix} , \begin{bmatrix}
2 \\
-4 \\
0 \\
1 \\
0
\end{bmatrix}$$

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST [221] and Theorem BRS [220], and in the style of Example CSROI [221], this yields a linearly independent set of vectors that span the column space.

$$\begin{bmatrix}
1 \\
0 \\
0 \\
2/3 \\
5/2
\end{bmatrix} , \begin{bmatrix}
0 \\
1 \\
0 \\
3/2 \\
3/2
\end{bmatrix} , \begin{bmatrix}
0 \\
0 \\
1 \\
1/2 \\
1
\end{bmatrix}$$

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS [220])

$$\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} , \begin{bmatrix}
0 \\
1 \\
0 \\
-2
\end{bmatrix} , \begin{bmatrix}
0 \\
0 \\
1 \\
2
\end{bmatrix}$$

Inverse matrix, if it exists. The inverse is not defined for matrices that are not square, and if the matrix is square, then the matrix must be nonsingular. (Definition MI [189], Theorem NI [204])

Subspace dimensions associated with the matrix. (Definition NOM [312], Definition ROM [492])
Verify Theorem RPNC

Matrix columns: 5  
Rank: 3  
Nullity: 2

Determinant of the matrix, which is only defined for square matrices. The matrix is nonsingular if and only if the determinant is nonzero (Theorem SMZD). (Product of all eigenvalues?)

Determinant = 0

Eigenvalues, and bases for eigenspaces. (Definition EEM, Definition EM)

\[ \lambda = -1 \]
\[ \mathcal{E}_L(-1) = \left\{ \begin{bmatrix} -5 \\ 9 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -10 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \]

\[ \lambda = 0 \]
\[ \mathcal{E}_L(0) = \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\} \]

Geometric and algebraic multiplicities. (Definition GME, Definition AME)

\[ \gamma_L(-1) = 3 \]
\[ \gamma_L(0) = 2 \]
\[ \alpha_L(-1) = 3 \]
\[ \alpha_L(0) = 2 \]

Diagonalizable? (Definition DZM)

Yes, full eigenspaces, Theorem DMFE.

The diagonalization. (Theorem DC)

\[
\begin{bmatrix}
4 & 3 & 4 & 6 & -6 \\
7 & 5 & 6 & 9 & -10 \\
-10 & -7 & -7 & -10 & 13 \\
-4 & -3 & -4 & -6 & 7 \\
-7 & -5 & -6 & -8 & 10
\end{bmatrix}
= 
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-5 & 6 & 2 & 2 & -1 \\
9 & -10 & -4 & -2 & 2 \\
0 & 0 & 1 & 1 & -2 \\
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
Archetype M

**Summary**  Linear transformation with bigger domain than codomain, so it is guaranteed to not be injective. Happens to not be surjective.

- A linear transformation:  \[[\text{Definition LT}[405]]\]

\[
T: \mathbb{C}^5 \mapsto \mathbb{C}^3, \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 + 4x_5 \\ 3x_1 + x_2 + 4x_3 - 3x_4 + 7x_5 \\ x_1 - x_2 - 5x_4 + x_5 \end{pmatrix}
\]

- A basis for the null space of the linear transformation:  \[[\text{Definition KLT}[429]]\]

\[
\left\{ \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}
\]

- Injective: No.  \[[\text{Definition ILT}[426]]\]

Since the kernel is nontrivial  \[[\text{Theorem KILT}[432]]\] tells us that the linear transformation is not injective. Also, since the rank can not exceed 3, we are guaranteed to have a nullity of at least 2, just from checking dimensions of the domain and the codomain. In particular, verify that

\[
T \begin{pmatrix} 1 \\ 2 \\ -1 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 38 \\ 24 \\ -16 \end{pmatrix} \quad T \begin{pmatrix} 0 \\ -3 \\ 0 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 38 \\ 24 \\ -16 \end{pmatrix}
\]

This demonstration that \(T\) is not injective is constructed with the observation that

\[
\begin{pmatrix} 0 \\ -3 \\ 0 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}
\]

and

\[
z = \begin{pmatrix} -1 \\ -5 \\ 1 \\ 1 \\ 1 \end{pmatrix} \in \mathcal{K}(T)
\]
so the vector $z$ effectively “does nothing” in the evaluation of $T$.

A basis for the range of the linear transformation: \textbf{(Definition RLT [444])}

Evaluate the linear transformation on a standard basis to get a spanning set for the range \textbf{(Theorem SSRLT [448])}:

$$
\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ -5 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix} \right\}
$$

If the linear transformation is injective, then the set above is guaranteed to be linearly independent \textbf{(Theorem ILTLI [433])}. This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows \textbf{(Theorem BRS [220])}, and perhaps un-coordinatizing. A basis for the range is:

$$
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}
$$

Surjective: No. \textbf{(Definition SLT [440])}

Notice that the range is not all of $\mathbb{C}^3$ since its dimension 2, not 3. In particular, verify that

$$
\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \not\in \mathcal{R}(T),
$$

by setting the output equal to this vector and seeing that the resulting system of linear equations has no solution, i.e. is inconsistent. So the preimage, $T^{-1}\left( \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right)$, is empty. This alone is sufficient to see that the linear transformation is not onto.

Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify \textbf{Theorem RPNDD [464]}.

- Domain dimension: 5
- Rank: 2
- Nullity: 3

Invertible: No.

Not injective or surjective.

Matrix representation \textbf{(Theorem MLTCV [411])}:

$$
T: \mathbb{C}^5 \rightarrow \mathbb{C}^3, \quad T(x) = Ax, \quad A = \begin{bmatrix} 1 & 2 & 3 & 4 & 4 \\ 3 & 1 & 4 & -3 & 7 \\ 1 & -1 & 0 & -5 & 1 \end{bmatrix}
$$
**Archetype N**

**Summary**  Linear transformation with domain larger than its codomain, so it is guaranteed to not be injective. Happens to be onto.

- A linear transformation: (Definition LT [405])
  
  \[ T : \mathbb{C}^5 \rightarrow \mathbb{C}^3, \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 + 3x_3 - 4x_4 + 5x_5 \\ x_1 - 2x_2 + 3x_3 - 9x_4 + 3x_5 \\ 3x_1 + 4x_3 - 6x_4 + 5x_5 \end{pmatrix} \]

- A basis for the null space of the linear transformation: (Definition KLT [429])
  
  \[ \left\{ \begin{pmatrix} 1 \\ -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 3 \\ 1 \end{pmatrix} \right\} \]

- Injective: No. (Definition ILT [426])

Since the kernel is nontrivial Theorem KILT [432] tells us that the linear transformation is not injective. Also, since the rank can not exceed 3, we are guaranteed to have a nullity of at least 2, just from checking dimensions of the domain and the codomain. In particular, verify that

\[ T \begin{pmatrix} -3 \\ 1 \\ -2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 19 \\ 6 \end{pmatrix} \]

\[ T \begin{pmatrix} -4 \\ -4 \\ -2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 19 \\ 6 \end{pmatrix} \]

This demonstration that \( T \) is not injective is constructed with the observation that

\[ \begin{pmatrix} -4 \\ -4 \\ -2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -2 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 2 \\ 2 \end{pmatrix} \]

and

\[ z = \begin{pmatrix} -1 \\ -5 \\ 0 \\ 2 \\ 3 \end{pmatrix} \in \mathcal{K}(T) \]
so the vector $z$ effectively “does nothing” in the evaluation of $T$.

A basis for the range of the linear transformation: (Definition RLT [444])

Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT [448]):

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ -6 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$$

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTLI [433]). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS [220]), and perhaps un-coordinatizing. A basis for the range is:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Surjective: Yes. (Definition SLT [440])

Notice that the basis for the range above is the standard basis for $\mathbb{C}^3$. So the range is all of $\mathbb{C}^3$ and thus the linear transformation is surjective.

Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD [464].

Domain dimension: 5  
Rank: 3  
Nullity: 2

Invertible: No.

Not surjective, and the relative sizes of the domain and codomain mean the linear transformation cannot be injective. (Theorem ILTIS [459])

Matrix representation (Theorem MLTCV [411]):

$$T : \mathbb{C}^5 \rightarrow \mathbb{C}^3, \ T (x) = Ax, \ A = \begin{bmatrix} 2 & 1 & 3 & -4 & 5 \\ 1 & -2 & 3 & -9 & 3 \\ 3 & 0 & 4 & -6 & 5 \end{bmatrix}$$
Archetype O

Summary  Linear transformation with a domain smaller than the codomain, so it is guaranteed to not be onto. Happens to not be one-to-one.

- A linear transformation:  (Definition LT [405])
  
  \[ T: \mathbb{C}^3 \rightarrow \mathbb{C}^5, \quad T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 - 3x_3 \\ -x_1 + 2x_2 - 4x_3 \\ x_1 + x_2 + x_3 \\ 2x_1 + 3x_2 + x_3 \\ x_1 + 2x_3 \end{bmatrix} \]

- A basis for the null space of the linear transformation:  (Definition KLT [429])
  
  \[ \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\} \]

- Injective: No.  (Definition ILT [426])
  
  Since the kernel is nontrivial Theorem KILT [432] tells us that the linear transformation is not injective. Also, since the rank can not exceed 3, we are guaranteed to have a nullity of at least 2, just from checking dimensions of the domain and the codomain. In particular, verify that

  \[ T \begin{bmatrix} 1 \\ 5 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -15 \\ -19 \\ 7 \\ 10 \\ 11 \end{bmatrix}, \quad T \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} -15 \\ -19 \\ 7 \\ 10 \\ 11 \end{bmatrix} \]

  This demonstration that \( T \) is not injective is constructed with the observation that

  \[ \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix} \]

  and

  \[ z = \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix} \in \mathcal{K}(T) \]

  so the vector \( z \) effectively “does nothing” in the evaluation of \( T \).

- A basis for the range of the linear transformation:  (Definition RLT [444])
  
  Evaluate the linear transformation on a standard basis to get a spanning set for the range.  (Theorem...
SSRLT [448]:

\[
\begin{bmatrix}
-1 & 1 & -3 \\
-1 & 2 & -4 \\
1 & 1 & 1 \\
2 & 3 & 1 \\
1 & 0 & 2 \\
\end{bmatrix}
\]

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTLI [433]). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS [220]), and perhaps un-coordinatizing. A basis for the range is:

\[
\begin{bmatrix}
1 & 0 & -3 & -7 & -2 \\
0 & 1 & 2 & 5 & 1 \\
\end{bmatrix}
\]

Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD [464].

\[\begin{array}{ccc}
\text{Domain dimension:} & 3 & \text{Rank:} & 2 & \text{Nullity:} & 1 \\
\end{array}\]

Surjective: No. (Definition SLT [440])

The dimension of the range is 2, and the codomain (\(\mathbb{C}^5\)) has dimension 5. So the transformation is not onto. Notice too that since the domain \(\mathbb{C}^3\) has dimension 3, it is impossible for the range to have a dimension greater than 3, and no matter what the actual definition of the function, it cannot possibly be onto.

To be more precise, verify that \(\begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \notin \mathcal{R}(T)\), by setting the output equal to this vector and seeing that the resulting system of linear equations has no solution, i.e. is inconsistent. So the preimage, \(T^{-1}\left(\begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)\), is empty. This alone is sufficient to see that the linear transformation is not onto.

Invertible: No.

Not injective, and the relative dimensions of the domain and codomain prohibit any possibility of being surjective.

Matrix representation (Theorem MLTCV [411]):
Archetype O 689

\[ T : \mathbb{C}^3 \rightarrow \mathbb{C}^5, \quad T(x) = Ax, \quad A = \begin{bmatrix} -1 & 1 & -3 \\ -1 & 2 & -4 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix} \]
**Archetype P**

**Summary**  Linear transformation with a domain smaller that its codomain, so it is guaranteed to not be surjective. Happens to be injective.

- A linear transformation: **(Definition LT 405)**

  \[ T: \mathbb{C}^3 \rightarrow \mathbb{C}^5, \quad T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} -x_1 + x_2 + x_3 \\ -x_1 + 2x_2 + 2x_3 \\ x_1 + x_2 + 3x_3 \\ 2x_1 + 3x_2 + x_3 \\ -2x_1 + x_2 + 3x_3 \end{bmatrix} \]

- A basis for the null space of the linear transformation: **(Definition KLT 429)**

  \{ \}

- Injective: Yes. **(Definition ILT 426)**

  Since \( \mathcal{K}(T) = \{0\} \), **Theorem KILT 432** tells us that \( T \) is injective.

- A basis for the range of the linear transformation: **(Definition RLT 444)**

  Evaluate the linear transformation on a standard basis to get a spanning set for the range **(Theorem SSRLT 448)**:

  \[
  \begin{bmatrix}
  -1 \\
  -1 \\
  1 \\
  2 \\
  -2
  \end{bmatrix},
  \begin{bmatrix}
  1 \\
  2 \\
  3 \\
  1 \\
  3
  \end{bmatrix}
  \]

  If the linear transformation is injective, then the set above is guaranteed to be linearly independent **(Theorem ILTL 433)**. This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows **(Theorem BRS 220)**, and perhaps un-coordinatizing. A basis for the range is:

  \[
  \begin{bmatrix}
  1 \\
  0 \\
  -10 \\
  6
  \end{bmatrix},
  \begin{bmatrix}
  0 \\
  1 \\
  7 \\
  -3
  \end{bmatrix},
  \begin{bmatrix}
  1 \\
  0 \\
  -1 \\
  1
  \end{bmatrix}
  \]

- Surjective: No. **(Definition SLT 440)**
The dimension of the range is 3, and the codomain (\( \mathbb{C}^5 \)) has dimension 5. So the transformation is not surjective. Notice too that since the domain \( \mathbb{C}^3 \) has dimension 3, it is impossible for the range to have a dimension greater than 3, and no matter what the actual definition of the function, it cannot possibly be surjective in this situation.

To be more precise, verify that \[
\begin{bmatrix}
2 \\
1 \\
-3 \\
2 \\
6
\end{bmatrix}
\] \( \not\in \mathcal{R}(T) \), by setting the output equal to this vector and seeing that the resulting system of linear equations has no solution, i.e. is inconsistent. So the preimage, \( T^{-1} \left( \begin{bmatrix}
2 \\
1 \\
-3 \\
2 \\
6
\end{bmatrix} \right) \), is empty. This alone is sufficient to see that the linear transformation is not onto.

Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify [Theorem RPNDD 464].

- Domain dimension: 3
- Rank: 3
- Nullity: 0

Invertible: No.

The relative dimensions of the domain and codomain prohibit any possibility of being surjective, so apply [Theorem ILTIS 459].

Matrix representation (Theorem MLTCV 411):

\[
T: \mathbb{C}^3 \rightarrow \mathbb{C}^5, \quad T(x) = Ax, \quad A = \begin{bmatrix}
-1 & 1 & 1 \\
-1 & 2 & 2 \\
1 & 1 & 3 \\
2 & 3 & 1 \\
-2 & 1 & 3
\end{bmatrix}
\]
Archetype Q

Summary  Linear transformation with equal-sized domain and codomain, so it has the potential to be invertible, but in this case is not. Neither injective nor surjective. Diagonalizable, though.

A linear transformation: \((\text{Definition LT}\ [405])\)

\[
T: \mathbb{C}^5 \mapsto \mathbb{C}^5, \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_1 + 3x_2 + 3x_3 - 6x_4 + 3x_5 \\ -16x_1 + 9x_2 + 12x_3 - 28x_4 + 28x_5 \\ -19x_1 + 7x_2 + 14x_3 - 32x_4 + 37x_5 \\ -21x_1 + 9x_2 + 15x_3 - 35x_4 + 39x_5 \\ -9x_1 + 5x_2 + 7x_3 - 16x_4 + 16x_5 \end{pmatrix}
\]

A basis for the null space of the linear transformation: \((\text{Definition KLT}\ [429])\)

\[
\begin{pmatrix} 3 \\ 4 \\ 1 \\ 3 \\ 3 \end{pmatrix}
\]

Injective: No. \((\text{Definition ILT}\ [426])\)

Since the kernel is nontrivial \(\text{Theorem KILT}\ [432]\) tells us that the linear transformation is not injective. Also, since the rank can not exceed 3, we are guaranteed to have a nullity of at least 2, just from checking dimensions of the domain and the codomain. In particular, verify that

\[
T \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 55 \\ 72 \\ 77 \\ 31 \end{pmatrix}
\]

This demonstration that \(T\) is not injective is constructed with the observation that

\[
\begin{pmatrix} 4 \\ 7 \\ 0 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 1 \\ 3 \\ 3 \end{pmatrix}
\]

and

\[
z = \begin{pmatrix} 3 \\ 4 \\ 1 \\ 3 \\ 3 \end{pmatrix} \in \mathcal{K}(T)
\]
so the vector $z$ effectively “does nothing” in the evaluation of $T$.

A basis for the range of the linear transformation: (Definition RLT [444])

Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT [448]):

$$
\begin{align*}
&\begin{bmatrix}
-2 & 3 & -6 & 3 \\
-16 & 12 & -28 & 28 \\
-19 & 14 & -32 & 37 \\
-21 & 15 & -35 & 39 \\
-9 & 7 & -16 & 16
\end{bmatrix},
&\begin{bmatrix}
3 & 9 & 7 & 9 \\
5 & 7 & -16 & 16
\end{bmatrix},
&\begin{bmatrix}
3 & 12 & -19 & 21 \\
-21 & 15 & -32 & -35 \\
-19 & 14 & -32 & -35 \\
-21 & 15 & -32 & -35 \\
-9 & 7 & -16 & 16
\end{bmatrix}
\end{align*}
$$

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTL [433]). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS [220], and perhaps un-coordinatizing. A basis for the range is:

$$
\begin{align*}
&\begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 2
\end{bmatrix}
\end{align*}
$$

Surjective: No. (Definition SLT [440])

The dimension of the range is 4, and the codomain ($\mathbb{C}^5$) has dimension 5. So $\mathcal{R}(T) \neq \mathbb{C}^5$ and by Theorem RSLT [446] the transformation is not surjective.

To be more precise, verify that $\begin{bmatrix}
-1 \\
2 \\
3 \\
-1 \\
4
\end{bmatrix} \notin \mathcal{R}(T)$, by setting the output equal to this vector and seeing that the resulting system of linear equations has no solution, i.e. is inconsistent. So the preimage, $T^{-1}\begin{bmatrix}
-1 \\
2 \\
3 \\
-1 \\
4
\end{bmatrix}$, is empty. This alone is sufficient to see that the linear transformation is not onto.

Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD [464].

Domain dimension: 5   Rank: 4   Nullity: 1

Invertible: No.

Neither injective nor surjective. Notice that since the domain and codomain have the same dimension, either the transformation is both onto and one-to-one (making it invertible) or else it is both

Version 1.04
not onto and not one-to-one (as in this case) by Theorem RPNDD [464].

Matrix representation (Theorem MLTCV [411]):

\[ T: \mathbb{C}^5 \rightarrow \mathbb{C}^5, \quad T(x) = Ax, \quad A = \begin{bmatrix}
-2 & 3 & 3 & -6 & 3 \\
-16 & 9 & 12 & -28 & 28 \\
-19 & 7 & 14 & -32 & 37 \\
-21 & 9 & 15 & -35 & 39 \\
-9 & 5 & 7 & -16 & 16 \\
\end{bmatrix} \]

Eigenvalues and eigenvectors (Definition EELT [515], Theorem EER [527]):

\[ \lambda = -1 \quad \mathcal{E}_T(-1) = \left\{ \begin{bmatrix} 0 \\ 2 \\ 3 \\ 3 \\ 1 \end{bmatrix} \right\} \]

\[ \lambda = 0 \quad \mathcal{E}_T(0) = \left\{ \begin{bmatrix} 3 \\ 4 \\ 1 \\ 3 \\ 3 \end{bmatrix} \right\} \]

\[ \lambda = 1 \quad \mathcal{E}_T(1) = \left\{ \begin{bmatrix} 5 \\ 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \]

Evaluate the linear transformation with each of these eigenvectors as an interesting check.

A diagonal matrix representation relative to a basis of eigenvectors, \(B\).

\[ B = \begin{bmatrix}
0 & 3 & 5 & -3 & 1 \\
2 & 4 & 3 & 1 & -1 \\
3 & 1 & 0 & 0 & 2 \\
3 & 3 & 0 & 2 & 0 \\
1 & 2 & 0 & 0 & 0 \\
\end{bmatrix} \]

\[ M_{B,B}^T = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \]
Archetype R

Summary  Linear transformation with equal-sized domain and codomain. Injective, surjective, invertible, diagonalizable, the works.

A linear transformation: (Definition LT 405)

\[ T: \mathbb{C}^5 \rightarrow \mathbb{C}^5, \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -65x_1 + 128x_2 + 10x_3 - 262x_4 + 40x_5 \\ 36x_1 - 73x_2 - x_3 + 151x_4 - 16x_5 \\ -44x_1 + 88x_2 + 5x_3 - 180x_4 + 24x_5 \\ 34x_1 - 68x_2 - 3x_3 + 140x_4 - 18x_5 \\ 12x_1 - 24x_2 - x_3 + 49x_4 - 5x_5 \end{pmatrix} \]

A basis for the null space of the linear transformation: (Definition KLT 429)

\[ \{ \} \]

Injective: Yes. (Definition ILT 426)

Since the kernel is trivial Theorem KILT 432 tells us that the linear transformation is injective.

A basis for the range of the linear transformation: (Definition RLT 444)

Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT 448):

\[
\begin{bmatrix} -65 & 128 & 10 & -262 & 40 \\ 36 & -73 & -1 & 151 & -16 \\ -44 & 88 & 5 & -180 & 24 \\ 34 & -68 & -3 & 140 & -18 \\ 12 & -24 & -1 & 49 & -5 \end{bmatrix}
\]

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTL 433). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS 220), and perhaps un-coordinatizing. A basis for the range is:

\[
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\]

Surjective: Yes. (Definition SLT 440)
A basis for the range is the standard basis of $\mathbb{C}^5$, so $\mathcal{R}(T) = \mathbb{C}^5$ and Theorem RSLT [446] tells us $T$ is surjective. Or, the dimension of the range is 5, and the codomain ($\mathbb{C}^5$) has dimension 5. So the transformation is surjective.

Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD [464].

Domain dimension: 5  
Rank: 5  
Nullity: 0

Invertible: Yes.

Both injective and surjective (Theorem ILTIS [459]). Notice that since the domain and codomain have the same dimension, either the transformation is both injective and surjective (making it invertible, as in this case) or else it is both not injective and not surjective.

Matrix representation (Theorem MLTCV [411]):

$$T: \mathbb{C}^5 \rightarrow \mathbb{C}^5, \quad T(x) = Ax, \quad A = \begin{bmatrix} -65 & 128 & 10 & -262 & 40 \\ 36 & -73 & -1 & 151 & -16 \\ -44 & 88 & 5 & -180 & 24 \\ 34 & -68 & -3 & 140 & -18 \\ 12 & -24 & -1 & 49 & -5 \end{bmatrix}$$

The inverse linear transformation (Definition IVLT [456]):

$$T^{-1}: \mathbb{C}^5 \rightarrow \mathbb{C}^5, \quad T^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -47x_1 + 92x_2 + x_3 - 181x_4 - 14x_5 \\ 27x_1 - 55x_2 + \frac{7}{2}x_3 + \frac{221}{4}x_4 + 11x_5 \\ -32x_1 + 64x_2 - x_3 - 126x_4 - 12x_5 \\ 25x_1 - 50x_2 + \frac{3}{2}x_3 + \frac{190}{7}x_4 + 9x_5 \\ 9x_1 - 18x_2 + \frac{1}{2}x_3 + \frac{71}{7}x_4 + 4x_5 \end{bmatrix}$$

Verify that $T(T^{-1}(x)) = x$ and $T^{-1}(T(x)) = x$, and notice that the representations of the transformation and its inverse are matrix inverses (Theorem IMR [499], Definition MI [189]).

Eigenvalues and eigenvectors (Definition EELT [515], Theorem EER [527]):

$$\lambda = -1 \quad \mathcal{E}_T(-1) = \left\{ \begin{bmatrix} -57 \\ 0 \\ -18 \\ 14 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\lambda = 1 \quad \mathcal{E}_T(1) = \left\{ \begin{bmatrix} -10 \\ -5 \\ -6 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}$$
$\lambda = 2 \quad \mathcal{E}_T(2) = \left\{ \begin{pmatrix} -6 \\ 3 \\ -4 \\ 3 \\ 1 \end{pmatrix} \right\}$

Evaluate the linear transformation with each of these eigenvectors as an interesting check.

A diagonal matrix representation relative to a basis of eigenvectors, $B$.

$$B = \begin{cases} \begin{pmatrix} -57 & 2 & -10 & 2 & -6 \\ 0 & 1 & -5 & 3 & 3 \\ -18 & 0 & -6 & 1 & -4 \\ 14 & 0 & 0 & 1 & 3 \\ 5 & 0 & 1 & 0 & 1 \end{pmatrix} \\ \end{cases}$$

$$M_{B,B}^T = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$
Archetype S

Summary  Domain is column vectors, codomain is matrices. Domain is dimension 3 and codomain is dimension 4. Not injective, not surjective.

A linear transformation:  

\[ T : \mathbb{C}^3 \rightarrow M_{22}, \quad T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} a - b & 2a + 2b + c \\ 3a + b + c & -2a - 6b - 2c \end{bmatrix} \]

A basis for the null space of the linear transformation:

\[ \left\{ \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} \right\} \]

Injective: No.  

Since the kernel is nontrivial \[ \text{Theorem KILT} \] tells us that the linear transformation is not injective. Also, since the rank can not exceed 3, we are guaranteed to have a nullity of at least 1, just from checking dimensions of the domain and the codomain. In particular, verify that

\[ T \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{bmatrix} 1 & 9 \\ 10 & -16 \end{bmatrix} \quad T \begin{pmatrix} 0 \\ -1 \\ 11 \end{pmatrix} = \begin{bmatrix} 1 & 9 \\ 10 & -16 \end{bmatrix} \]

This demonstration that \( T \) is not injective is constructed with the observation that

\[ \begin{bmatrix} 0 \\ -1 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \\ 8 \end{bmatrix} \]

and

\[ z = \begin{bmatrix} -2 \\ -2 \\ 8 \end{bmatrix} \in \mathcal{K}(T) \]

so the vector \( z \) effectively “does nothing” in the evaluation of \( T \).

A basis for the range of the linear transformation:

Evaluate the linear transformation on a standard basis to get a spanning set for the range \[ \text{Theorem SSRLT} \] :

\[ \left\{ \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 1 & -6 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \right\} \]
If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTL [433]). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS [220]), and perhaps un-coordinatizing. A basis for the range is:

\[
\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \right\}
\]

\[\square\] Surjective: No. (Definition SLT [440])

The dimension of the range is 2, and the codomain \((M_{22})\) has dimension 4. So the transformation is not surjective. Notice too that since the domain \(\mathbb{C}^3\) has dimension 3, it is impossible for the range to have a dimension greater than 3, and no matter what the actual definition of the function, it cannot possibly be surjective in this situation.

To be more precise, verify that \(\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \notin \mathcal{R}(T)\), by setting the output of \(T\) equal to this matrix and seeing that the resulting system of linear equations has no solution, i.e. is inconsistent. So the preimage, \(T^{-1}\left( \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \right)\), is empty. This alone is sufficient to see that the linear transformation is not onto.

\[\square\] Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD [464].

Domain dimension: 3 
Rank: 2 
Nullity: 1

\[\square\] Invertible: No.

Not injective (Theorem ILTIS [459]), and the relative dimensions of the domain and codomain prohibit any possibility of being surjective.

\[\square\] Matrix representation (Definition MR [485]):

\[
B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

\[
C = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}
\]

\[
M_{B,C}^T = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 1 \\ 3 & 1 & 1 \\ -2 & -6 & -2 \end{bmatrix}
\]
Archetype T

Summary    Domain and codomain are polynomials. Domain has dimension 5, while codomain has dimension 6. Is injective, can’t be surjective.

- A linear transformation: \( T: P_4 \rightarrow P_5, \quad T(p(x)) = (x - 2)p(x) \)

- A basis for the null space of the linear transformation: \( \{ \} \)

  Injective: Yes. \( \text{(Definition ILT) [426]} \)

  Since the kernel is trivial, \( \text{(Theorem KILT)} [432] \) tells us that the linear transformation is injective.

- A basis for the range of the linear transformation: \( \text{(Definition RLT) [444]} \)

  Evaluate the linear transformation on a standard basis to get a spanning set for the range: \( \text{(Theorem SSRLT) [448]} \)

  \[
  \{x - 2, x^2 - 2x, x^3 - 2x^2, x^4 - 2x^3, x^5 - 2x^4, x^6 - 2x^5\}
  \]

  If the linear transformation is injective, then the set above is guaranteed to be linearly independent. \( \text{(Theorem ILTL) [433]} \). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows \( \text{(Theorem BRS) [220]} \), and perhaps un-coordinatizing. A basis for the range is:

  \[
  \left\{ -\frac{1}{32}x^5 + 1, -\frac{1}{16}x^5 + x, -\frac{1}{8}x^5 + x^2, -\frac{1}{4}x^5 + x^3, -\frac{1}{2}x^5 + x^4 \right\}
  \]

- Surjective: No. \( \text{(Definition SLT) [440]} \)

  The dimension of the range is 5, and the codomain \( (P_5) \) has dimension 6. So the transformation is not surjective. Notice too that since the domain \( P_4 \) has dimension 5, it is impossible for the range to have a dimension greater than 5, and no matter what the actual definition of the function, it cannot possibly be surjective in this situation.

  To be more precise, verify that \( 1 + x + x^2 + x^3 + x^4 \not\in R(T) \), by setting the output equal to this vector and seeing that the resulting system of linear equations has no solution, i.e. is inconsistent. So the preimage, \( T^{-1} (1 + x + x^2 + x^3 + x^4) \), is nonempty. This alone is sufficient to see that the linear transformation is not onto.

- Subspace dimensions associated with the linear transformation. Examine parallels with earlier
results for matrices. Verify Theorem RPNDD.

\[
\text{Domain dimension: 5} \quad \text{Rank: 5} \quad \text{Nullity: 0}
\]

- Invertible: No.

The relative dimensions of the domain and codomain prohibit any possibility of being surjective, so apply Theorem ILTIS.

Matrix representation (Definition MR):

\[
\begin{align*}
B &= \{1, x, x^2, x^3, x^4\} \\
C &= \{1, x, x^2, x^3, x^4, x^5\}
\end{align*}
\]

\[
M^{T}_{B,C} = \begin{bmatrix}
-2 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 \\
0 & 1 & -2 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 \\
0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Archetype U

Summary  Domain is matrices, codomain is column vectors. Domain has dimension 6, while
codomain has dimension 4. Can’t be injective, is surjective.

A linear transformation: (Definition LT 405)

\[ T : M_{23} \rightarrow \mathbb{C}^4, \quad T \left( \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \right) = \begin{bmatrix} a + 2b + 12c - 3d + e + 6f \\ 2a - b - c + d - 11f \\ a + b + 7c + 2d + e - 3f \\ a + 2b + 12c + 5e - 5f \end{bmatrix} \]

A basis for the null space of the linear transformation: (Definition KLT 429)

\[
\left\{ \begin{bmatrix} 3 & -4 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \begin{bmatrix} -2 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}
\]

Injective: No. (Definition ILT 426)

Since the kernel is nontrivial (Theorem KILT 432) tells us that the linear transformation is not injective. Also, since the rank can not exceed 4, we are guaranteed to have a nullity of at least 2, just from checking dimensions of the domain and the codomain. In particular, verify that

\[ T \left( \begin{bmatrix} 1 & 10 & -2 \\ 3 & -1 & 1 \end{bmatrix} \right) = \begin{bmatrix} -7 \\ -14 \\ -1 \\ -13 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 & -3 & -1 \\ 5 & 3 & 3 \end{bmatrix} \]

This demonstration that \( T \) is not injective is constructed with the observation that

\[ \begin{bmatrix} 5 & -3 & -1 \\ 5 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 10 & -2 \\ 3 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -13 & 1 \\ 2 & 4 & 2 \end{bmatrix} \]

and

\[ z = \begin{bmatrix} 4 & -13 & 1 \\ 2 & 4 & 2 \end{bmatrix} \in \mathcal{N}(T) \]

so the vector \( z \) effectively “does nothing” in the evaluation of \( T \).

A basis for the range of the linear transformation: (Definition RLT 444)

Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT 448):

\[
\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \\ 2 \\ 12 \end{bmatrix}, \begin{bmatrix} 12 \\ -1 \\ 7 \\ 12 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 2 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 2 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -11 \\ -3 \\ -5 \end{bmatrix}
\]
If the linear transformation is injective, then the set above is guaranteed to be linearly independent \(\text{(Theorem ILTL} [433])\). This spanning set may be converted to a "nice" basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows \(\text{(Theorem BRS} [220])\), and perhaps un-coordinatizing. A basis for the range is:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[\square\] Surjective: Yes. \(\text{(Definition SLT} [440])\)

A basis for the range is the standard basis of \(\mathbb{C}^4\), so \(\mathcal{R}(T) = \mathbb{C}^4\) and \(\text{Theorem RSLT} [446]\) tells us \(T\) is surjective. Or, the dimension of the range is 4, and the codomain \((\mathbb{C}^4)\) has dimension 4. So the transformation is surjective.

\[\square\] Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify \(\text{Theorem RPNDD} [464]\).

- Domain dimension: 6
- Rank: 4
- Nullity: 2

\[\square\] Invertible: No.

The relative sizes of the domain and codomain mean the linear transformation cannot be injective. \(\text{(Theorem ILTIS} [459])\)

\[\square\] Matrix representation \(\text{(Definition MR} [485])\):

\[
B = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
M_{B,C}^T = \begin{bmatrix}
1 & 2 & 12 & -3 & 1 & 6 \\
2 & -1 & -1 & 1 & 0 & -11 \\
1 & 1 & 7 & 2 & 1 & -3 \\
1 & 2 & 12 & 0 & 5 & -5
\end{bmatrix}
\]
Archetype V

Summary  Domain is polynomials, codomain is matrices. Domain and codomain both have dimension 4. Injective, surjective, invertible. Square matrix representation, but domain and codomain are unequal, so no eigenvalue information.

A linear transformation: (Definition LT 405)

\[ T: P_3 \rightarrow M_{2,2}, \quad T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a + b & a - 2c \\ d & b - d \end{bmatrix} \]

A basis for the null space of the linear transformation: (Definition KLT 429)

\{ \}  

Injective: Yes. (Definition ILT 426)

Since the kernel is trivial Theorem KILT 432 tells us that the linear transformation is injective.

A basis for the range of the linear transformation: (Definition RLT 444)

Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT 448):

\[ \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\} \]

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTLI 433). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS 220), and perhaps un-coordinatizing. A basis for the range is:

\[ \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \]

Surjective: Yes. (Definition SLT 440)

A basis for the range is the standard basis of \( M_{2,2} \), so \( R(T) = M_{2,2} \) and Theorem RSLT 440 tells us \( T \) is surjective. Or, the dimension of the range is 4, and the codomain \( M_{2,2} \) has dimension 4. So the transformation is surjective.

Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD 464.

Domain dimension: 4  
Rank: 4  
Nullity: 0
Invertible: Yes.

Both injective and surjective (Theorem ILTIS [459]). Notice that since the domain and codomain have the same dimension, either the transformation is both injective and surjective (making it invertible, as in this case) or else it is both not injective and not surjective.

Matrix representation (Definition MR [485]):

\[ B = \{1, x, x^2, x^3\} \]
\[ C = \begin{\{1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{\}\} \]
\[ M_{B,C}^T = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & -2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1
\end{pmatrix} \]

Since invertible, the inverse linear transformation. (Definition IVLT [456])

\[ T^{-1}: M_{22} \mapsto P_3, \quad T^{-1}\left(\begin{array}{c} a \\ c \\ d \end{array}\right) = (a - c - d) + (c + d)x + \frac{1}{2}(a - b - c - d)x^2 + cx^3 \]
Archetype W

**Summary**  Domain is polynomials, codomain is polynomials. Domain and codomain both have dimension 3. Injective, surjective, invertible, 3 distinct eigenvalues, diagonalizable.

- A linear transformation: (Definition LT 405)
  \[ T: P_2 \rightarrow P_2, \quad T(a + bx + cx^2) = (19a + 6b - 4c) + (-24a - 7b + 4c) + (36a + 12b - 9c) \]

- A basis for the null space of the linear transformation: (Definition KLT 429)
  \[ \{ \} \]

- Injective: Yes. (Definition ILT 426)
  Since the kernel is trivial Theorem KILT 432 tells us that the linear transformation is injective.

- A basis for the range of the linear transformation: (Definition RLT 444)
  Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT 448):
  \[ \{ 19 - 24x + 36x^2, 6 - 7x + 12x^2, -4 + 4x - 9x^2 \} \]
  If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTL 433). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS 220), and perhaps un-coordinatizing. A basis for the range is:
  \[ \{ 1, x, x^2 \} \]

- Surjective: Yes. (Definition SLT 440)
  A basis for the range is the standard basis of \( \mathbb{C}^5 \), so \( R(T) = \mathbb{C}^5 \) and Theorem RSLT 446 tells us \( T \) is surjective. Or, the dimension of the range is 5, and the codomain (\( \mathbb{C}^5 \)) has dimension 5. So the transformation is surjective.

- Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD 464.
  Domain dimension: 3  
  Rank: 3  
  Nullity: 0

- Invertible: Yes.
Both injective and surjective (Theorem ILTIS [459]). Notice that since the domain and codomain have the same dimension, either the transformation is both injective and surjective (making it invertible, as in this case) or else it is both not injective and not surjective.

Matrix representation (Definition MR [485]):

\[
B = \{1, x, x^2\} \\
C = \{1, x, x^2\} \\
M_{B,C}^T = \begin{bmatrix}
19 & 6 & -4 \\
-24 & -7 & 4 \\
36 & 12 & -9
\end{bmatrix}
\]

Since invertible, the inverse linear transformation. (Definition IVLT [456])

\[
T^{-1}: P_2 \mapsto P_2, \quad T^{-1}(a + bx + cx^2) = (-5a - 2b + \frac{4}{3}c) + (24a + 9b - \frac{20}{3}c)x + (12a + 4b - \frac{11}{3}c)x^2
\]

Eigenvalues and eigenvectors (Definition EELT [515], Theorem EER [527]):

\[
\lambda = -1 \quad \mathcal{E}_T(-1) = \langle\{2x + 3x^2\}\rangle \\
\lambda = 1 \quad \mathcal{E}_T(1) = \langle\{-1 + 3x\}\rangle \\
\lambda = 3 \quad \mathcal{E}_T(3) = \langle\{1 - 2x + x^2\}\rangle
\]

Evaluate the linear transformation with each of these eigenvectors as an interesting check.

A diagonal matrix representation relative to a basis of eigenvectors, \(B\).

\[
B = \{2x + 3x^2, -1 + 3x, 1 - 2x + x^2\} \\
M_{B,B}^T = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 3
\end{bmatrix}
\]
Summary  Domain and codomain are square matrices. Domain and codomain both have dimension 4. Not injective, not surjective, not invertible, 3 distinct eigenvalues, diagonalizable.

A linear transformation: (Definition LT 405)

\[ T: M_{22} \rightarrow M_{22}, \quad T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} -2a + 15b + 3c + 27d \\ a - 5b - 9d \\ 10b + 6c + 18d \\ -a - 4b - 5c - 8d \end{bmatrix} \]

A basis for the null space of the linear transformation: (Definition KLT 429)

\[ \left\{ \begin{bmatrix} -6 \\ -3 \\ 2 \\ 1 \end{bmatrix} \right\} \]

Injective: No. (Definition ILT 426)

Since the kernel is nontrivial Theorem KILT 432 tells us that the linear transformation is not injective. In particular, verify that

\[ T \left( \begin{bmatrix} -2 \\ 0 \\ 1 \\ -4 \end{bmatrix} \right) = \begin{bmatrix} 115 \\ 78 \\ -38 \\ -35 \end{bmatrix} \quad T \left( \begin{bmatrix} 4 \\ 3 \\ -1 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 115 \\ 78 \\ -38 \\ -35 \end{bmatrix} \]

This demonstration that \( T \) is not injective is constructed with the observation that

\[ \begin{bmatrix} 4 \\ 3 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ -4 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ -2 \\ -1 \end{bmatrix} \]

and

\[ z = \begin{bmatrix} 6 \\ 3 \\ -2 \\ -1 \end{bmatrix} \in \mathcal{K}(T) \]

so the vector \( z \) effectively “does nothing” in the evaluation of \( T \).

A basis for the range of the linear transformation: (Definition RLT 444)

Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT 448):

\[ \left\{ \begin{bmatrix} -2 \\ 15 \\ 3 \\ 27 \\ -9 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ -5 \\ -8 \end{bmatrix} \right\} \]

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTLI 433). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and
retaining the nonzero rows (Theorem BRS \[220\]), and perhaps un-coordinatizing. A basis for the range is:

\[
\begin{bmatrix}
1 & 0 \\
-\frac{1}{2} & 0
\end{bmatrix}, 
\begin{bmatrix}
0 & 1 \\
\frac{1}{4} & 0
\end{bmatrix}, 
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\]

\[\Box \text{ Surjective: No. (Definition SLT \[440\])}\]

The dimension of the range is 3, and the codomain \(M_{22}\) has dimension 5. So \(\mathcal{R}(T) \neq M_{22}\) and by Theorem RSLT \[446\] the transformation is not surjective.

To be more precise, verify that \[
\begin{bmatrix}
2 \\
4 \\
3 \\
1
\end{bmatrix} \not\in \mathcal{R}(T),
\] by setting the output of \(T\) equal to this matrix and seeing that the resulting system of linear equations has no solution, i.e. is inconsistent. So the preimage, \(T^{-1}\left(\begin{bmatrix}
2 \\
4 \\
3 \\
1
\end{bmatrix}\right)\), is empty. This alone is sufficient to see that the linear transformation is not onto.

\[\Box \text{ Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD \[464\].}
\]

\[
\begin{array}{ccc}
\text{Domain dimension:} & 4 & \text{Rank:} & 3 & \text{Nullity:} & 1
\end{array}
\]

\[\Box \text{ Invertible: No.}\]

Neither injective nor surjective (Theorem ILTIS \[459\]). Notice that since the domain and codomain have the same dimension, either the transformation is both injective and surjective or else it is both not injective and not surjective (making it not invertible, as in this case).

\[\Box \text{ Matrix representation (Definition MR \[485\]):}\]

\[
B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},
\]

\[
C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},
\]

\[
M_{B,C}^T =
\begin{bmatrix}
-2 & 15 & 3 & 27 \\
0 & 10 & 6 & 18 \\
1 & -5 & 0 & -9 \\
-1 & -4 & -5 & -8
\end{bmatrix}
\]

\[\Box \text{ Eigenvalues and eigenvectors (Definition EELT \[515\], Theorem EER \[527\]):}\]

\[
\begin{align*}
\lambda = 0 & \quad \mathcal{E}_T(0) = \left\{ \begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix} \right\} \\
\lambda = 1 & \quad \mathcal{E}_T(1) = \left\{ \begin{bmatrix} -7 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \\
\lambda = 3 & \quad \mathcal{E}_T(3) = \left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \right\}
\end{align*}
\]
Evaluate the linear transformation with each of these eigenvectors as an interesting check.

A diagonal matrix representation relative to a basis of eigenvectors, $B$,

\[
B = \left\{ \begin{bmatrix} -6 & -3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -7 & -2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix} \right\}
\]

\[
M_{B,B}^T = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix}
\]
Appendix GFDL
GNU Free Documentation License

Version 1.2, November 2002

59 Temple Place, Suite 330, Boston, MA 02111-1307 USA

Everyone is permitted to copy and distribute verbatim copies of this license document, but changing it is not allowed.

Preamble

The purpose of this License is to make a manual, textbook, or other functional and useful document “free” in the sense of freedom: to assure everyone the effective freedom to copy and redistribute it, with or without modifying it, either commercially or noncommercially. Secondarily, this License preserves for the author and publisher a way to get credit for their work, while not being considered responsible for modifications made by others.

This License is a kind of “copyleft”, which means that derivative works of the document must themselves be free in the same sense. It complements the GNU General Public License, which is a copyleft license designed for free software.

We have designed this License in order to use it for manuals for free software, because free software needs free documentation: a free program should come with manuals providing the same freedoms that the software does. But this License is not limited to software manuals; it can be used for any textual work, regardless of subject matter or whether it is published as a printed book. We recommend this License principally for works whose purpose is instruction or reference.

1. APPLICABILITY AND DEFINITIONS

This License applies to any manual or other work, in any medium, that contains a notice placed by the copyright holder saying it can be distributed under the terms of this License. Such a notice grants a world-wide, royalty-free license, unlimited in duration, to use that work under the conditions stated herein. The “Document”, below, refers to any such manual or work. Any member of the public is a licensee, and is addressed as “you”. You accept the license if you copy, modify or distribute the work in a way requiring permission under copyright law.

A “Modified Version” of the Document means any work containing the Document or a portion of it, either copied verbatim, or with modifications and/or translated into another language.

A “Secondary Section” is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or authors of the Document to the Document’s overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (Thus, if the Document is in part a textbook of mathematics, a Secondary Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.
The "Invariant Sections" are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document is released under this License. If a section does not fit the above definition of Secondary then it is not allowed to be designated as Invariant. The Document may contain zero Invariant Sections. If the Document does not identify any Invariant Sections then there are none.

The "Cover Texts" are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License. A Front-Cover Text may be at most 5 words, and a Back-Cover Text may be at most 25 words.

A "Transparent" copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, that is suitable for revising the document straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings) some widely available drawing editor, and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup, or absence of markup, has been arranged to thwart or discourage subsequent modification by readers is not Transparent. An image format is not Transparent if used for any substantial amount of text. A copy that is not "Transparent" is called "Opaque".

Examples of suitable formats for Transparent copies include plain ASCII without markup, TeXinfo input format, LaTeX input format, SGML or XML using a publicly available DTD, and standard-conforming simple HTML, PostScript or PDF designed for human modification. Examples of transparent image formats include PNG, XCF and JPG. Opaque formats include proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machine-generated HTML, PostScript or PDF produced by some word processors for output purposes only.

The "Title Page" means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires to appear in the title page. For works in formats which do not have any title page as such, "Title Page" means the text near the most prominent appearance of the work’s title, preceding the beginning of the body of the text.

A section "Entitled XYZ" means a named subunit of the Document whose title either is precisely XYZ or contains XYZ in parentheses following text that translates XYZ in another language. (Here XYZ stands for a specific section name mentioned below, such as "Acknowledgements", "Dedications", "Endorsements", or "History"). To "Preserve the Title" of such a section when you modify the Document means that it remains a section "Entitled XYZ" according to this definition.

The Document may include Warranty Disclaimers next to the notice which states that this License applies to the Document. These Warranty Disclaimers are considered to be included by reference in this License, but only as regards disclaiming warranties: any other implication that these Warranty Disclaimers may have is void and has no effect on the meaning of this License.

2. VERBATIM COPYING

You may copy and distribute the Document in any medium, either commercially or noncommercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section 3.

You may also lend copies, under the same conditions stated above, and you may publicly display copies.

3. COPYING IN QUANTITY

If you publish printed copies (or copies in media that commonly have printed covers) of the Document, numbering more than 100, and the Document’s license notice requires Cover Texts, you
must enclose the copies in covers that carry, clearly and legibly, all these Cover Texts: Front-Cover Texts on the front cover, and Back-Cover Texts on the back cover. Both covers must also clearly and legibly identify you as the publisher of these copies. The front cover must present the full title with all words of the title equally prominent and visible. You may add other material on the covers in addition. Copying with changes limited to the covers, as long as they preserve the title of the Document and satisfy these conditions, can be treated as verbatim copying in other respects.

If the required texts for either cover are too voluminous to fit legibly, you should put the first ones listed (as many as fit reasonably) on the actual cover, and continue the rest onto adjacent pages.

If you publish or distribute Opaque copies of the Document numbering more than 100, you must either include a machine-readable Transparent copy along with each Opaque copy, or state in or with each Opaque copy a computer-network location from which the general network-using public has access to download using public-standard network protocols a complete Transparent copy of the Document, free of added material. If you use the latter option, you must take reasonably prudent steps, when you begin distribution of Opaque copies in quantity, to ensure that this Transparent copy will remain thus accessible at the stated location until at least one year after the last time you distribute an Opaque copy (directly or through your agents or retailers) of that edition to the public.

It is requested, but not required, that you contact the authors of the Document well before redistributing any large number of copies, to give them a chance to provide you with an updated version of the Document.

4. MODIFICATIONS

You may copy and distribute a Modified Version of the Document under the conditions of sections 2 and 3 above, provided that you release the Modified Version under precisely this License, with the Modified Version filling the role of the Document, thus licensing distribution and modification of the Modified Version to whoever possesses a copy of it. In addition, you must do these things in the Modified Version:

A. Use in the Title Page (and on the covers, if any) a title distinct from that of the Document, and from those of previous versions (which should, if there were any, be listed in the History section of the Document). You may use the same title as a previous version if the original publisher of that version gives permission.

B. List on the Title Page, as authors, one or more persons or entities responsible for authorship of the modifications in the Modified Version, together with at least five of the principal authors of the Document (all of its principal authors, if it has fewer than five), unless they release you from this requirement.

C. State on the Title page the name of the publisher of the Modified Version, as the publisher.

D. Preserve all the copyright notices of the Document.

E. Add an appropriate copyright notice for your modifications adjacent to the other copyright notices.

F. Include, immediately after the copyright notices, a license notice giving the public permission to use the Modified Version under the terms of this License, in the form shown in the Addendum below.

G. Preserve in that license notice the full lists of Invariant Sections and required Cover Texts given in the Document’s license notice.

H. Include an unaltered copy of this License.
I. Preserve the section Entitled “History”, Preserve its Title, and add to it an item stating at least the title, year, new authors, and publisher of the Modified Version as given on the Title Page. If there is no section Entitled “History” in the Document, create one stating the title, year, authors, and publisher of the Document as given on its Title Page, then add an item describing the Modified Version as stated in the previous sentence.

J. Preserve the network location, if any, given in the Document for public access to a Transparent copy of the Document, and likewise the network locations given in the Document for previous versions it was based on. These may be placed in the “History” section. You may omit a network location for a work that was published at least four years before the Document itself, or if the original publisher of the version it refers to gives permission.

K. For any section Entitled “Acknowledgements” or “Dedications”, Preserve the Title of the section, and preserve in the section all the substance and tone of each of the contributor acknowledgements and/or dedications given therein.

L. Preserve all the Invariant Sections of the Document, unaltered in their text and in their titles. Section numbers or the equivalent are not considered part of the section titles.

M. Delete any section Entitled “Endorsements”. Such a section may not be included in the Modified Version.

N. Do not retitle any existing section to be Entitled “Endorsements” or to conflict in title with any Invariant Section.

O. Preserve any Warranty Disclaimers.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as Invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version’s license notice. These titles must be distinct from any other section titles.

You may add a section Entitled “Endorsements”, provided it contains nothing but endorsements of your Modified Version by various parties—for example, statements of peer review or that the text has been approved by an organization as the authoritative definition of a standard.

You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add another; but you may replace the old one, on explicit permission from the previous publisher that added the old one.

The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version.

5. COMBINING DOCUMENTS

You may combine the Document with other documents released under this License, under the terms defined in section 4 above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified, and list them all as Invariant Sections of your combined work in its license notice, and that you preserve all their Warranty Disclaimers.

The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a
unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work.

In the combination, you must combine any sections Entitled “History” in the various original documents, forming one section Entitled “History”; likewise combine any sections Entitled “Acknowledgements”, and any sections Entitled “Dedications”. You must delete all sections Entitled “Endorsements”.

6. COLLECTIONS OF DOCUMENTS

You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects.

You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document.

7. AGGREGATION WITH INDEPENDENT WORKS

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, is called an “aggregate” if the copyright resulting from the compilation is not used to limit the legal rights of the compilation’s users beyond what the individual works permit. When the Document is included in an aggregate, this License does not apply to the other works in the aggregate which are not themselves derivative works of the Document.

If the Cover Text requirement of section 3 is applicable to these copies of the Document, then if the Document is less than one half of the entire aggregate, the Document’s Cover Texts may be placed on covers that bracket the Document within the aggregate, or the electronic equivalent of covers if the Document is in electronic form. Otherwise they must appear on printed covers that bracket the whole aggregate.

8. TRANSLATION

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section 4. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License, and all the license notices in the Document, and any Warranty Disclaimers, provided that you also include the original English version of this License and the original versions of those notices and disclaimers. In case of a disagreement between the translation and the original version of this License or a notice or disclaimer, the original version will prevail.

If a section in the Document is Entitled “Acknowledgements”, “Dedications”, or “History”, the requirement (section 4) to Preserve its Title (section 1) will typically require changing the actual title.

9. TERMINATION

You may not copy, modify, sublicense, or distribute the Document except as expressly provided for under this License. Any other attempt to copy, modify, sublicense or distribute the Document is void, and will automatically terminate your rights under this License. However, parties who have received copies, or rights, from you under this License will not have their licenses terminated so long as such parties remain in full compliance.

10. FUTURE REVISIONS OF THIS LICENSE
The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See http://www.gnu.org/copyleft/.

Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License “or any later version” applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify a version number of this License, you may choose any version ever published (not as a draft) by the Free Software Foundation.

ADDENDUM: How to use this License for your documents

To use this License in a document you have written, include a copy of the License in the document and put the following copyright and license notices just after the title page:

Copyright ©YEAR YOUR NAME. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled “GNU Free Documentation License”.

If you have Invariant Sections, Front-Cover Texts and Back-Cover Texts, replace the “with...Texts.” line with this:

with the Invariant Sections being LIST THEIR TITLES, with the Front-Cover Texts being LIST, and with the Back-Cover Texts being LIST.

If you have Invariant Sections without Cover Texts, or some other combination of the three, merge those two alternatives to suit the situation.

If your document contains nontrivial examples of program code, we recommend releasing these examples in parallel under your choice of free software license, such as the GNU General Public License, to permit their use in free software.
Part T
Topics
We have chosen to present introductory linear algebra in the Core (Part C [2]) using scalars from the set of complex numbers, $\mathbb{C}$. We could have instead chosen to use scalars from the set of real numbers, $\mathbb{R}$. This would have presented certain difficulties when we encountered characteristic polynomials with complex roots (Definition CP [363]) or when we needed to be sure every matrix had at least one eigenvalue (Theorem EMHE [359]). However, much of the basics would be unchanged. The definition of a vector space would not change, nor would the ideas of linear independence, spanning, or bases. Linear transformations would still behave the same and we would still obtain matrix representations, though our ideas about canonical forms would have to be adjusted slightly.

The real numbers and the complex numbers are both examples of what are called fields, and we can “do” linear algebra in just a bit more generality by letting our scalars take values from some unspecified field. So in this section we will describe exactly what constitutes a field, give some finite examples, and discuss another connection between fields and vector spaces. Vector spaces over finite fields are very important in certain applications, so this is partially background for other topics. As such, we will not prove every claim we make.

**Subsection F**

**Fields**

Like a vector space, a field is a set along with two binary operations. The distinction is that both operations accept two elements of the set, and then produce a new element of the set. In a vector space we have two sets — the vectors and the scalars, and scalar multiplication mixes one of each to produce a vector. Here is the careful definition of a field.

**Definition F**

**Field**

Suppose that $F$ is a set upon which we have defined two operations: (1) **addition**, which combines two elements of $F$ and is denoted by “+”, and (2) **multiplication**, which combines two elements of $F$ and is denoted by juxtaposition. Then $F$, along with the two operations, is a **field** if the following properties hold.

- **ACF**  Additive Closure, Field
  
  If $\alpha, \beta \in F$, then $\alpha + \beta \in F$.

- **MCF**  Multiplicative Closure, Field
  
  If $\alpha, \beta \in F$, then $\alpha \beta \in F$.

- **ACF**  Additive Commutativity, Field
  
  If $\alpha, \beta \in F$, then $\alpha + \beta = \beta + \alpha$.

- **MCF**  Multiplicative Commutativity, Field
  
  If $\alpha, \beta, \gamma \in F$, then $\alpha(\beta \gamma) = (\alpha \beta) \gamma$.

- **AAF**  Additive Associativity, Field
  
  If $\alpha, \beta, \gamma \in F$, then $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

- **MAF**  Multiplicative Associativity, Field
  
  If $\alpha, \beta, \gamma \in F$, then $\alpha (\beta \gamma) = (\alpha \beta) \gamma$. 

Subsection F.FF  Finite Fields

- **DF**  Distributivity, Field
  If $\alpha, \beta, \gamma \in F$, then $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

- **ZF**  Zero, Field
  There is an element, $0 \in F$, called zero, such that $\alpha + 0 = \alpha$ for all $\alpha \in F$.

- **OF**  One, Field
  There is an element, $1 \in F$, called one, such that $\alpha(1) = \alpha$ for all $\alpha \in F$.

- **AIF**  Additive Inverse, Field
  If $\alpha \in F$, then there exists $-\alpha \in V$ so that $\alpha + (-\alpha) = 0$.

- **MIF**  Multiplicative Inverse, Field
  If $\alpha \in F$, $\alpha \neq 0$, then there exists $\frac{1}{\alpha} \in V$ so that $\alpha\left(\frac{1}{\alpha}\right) = 1$.

Mostly this definition says that all the good things you might expect, really do happen in a field. The one technicality is that the special element, 0, the additive identity element, does not have a multiplicative inverse. In other words, no dividing by zero.

This definition should remind you of Theorem PCNA [612], and indeed, Theorem PCNA [612] provides the justification for the statement that the complex numbers form a field. Another example of field is the set of rational numbers $\mathbb{Q} = \{\frac{p}{q} \mid p, q \text{ are integers, } q \neq 0\}$

Of course, the real numbers, $\mathbb{R}$, also form a field. It is this field that you probably studied for many years. You began studying the integers (“counting”), then the rationals (“fractions”), then the reals (“algebra”), along with some excursions in the complex numbers (“imaginary numbers”). So you should have seen three fields already in your previous studies.

Our first observation about fields is that we can go back to our definition of a vector space (Definition VS [251]) and replace every occurrence of $\mathbb{C}$ by some general, unspecified field, $F$, and all our subsequent definitions and theorems are still true, so long as we avoid roots of polynomials (or equivalently, factoring polynomials). So if you consult more advanced texts on linear algebra, you will see this sort of approach. You might study some of the first theorems we proved about vector spaces in Subsection VS.VSP [257] and work through their proofs in the more general setting of an arbitrary field. This exercise should convince you that very little changes when we move from $\mathbb{C}$ to an arbitrary field $F$. (See Exercise F.T10 [723].)

Subsection FF
Finite Fields

It may sound odd at first, but there exist finite fields, and even finite vector spaces. We will find certain of these important in subsequent applications, so we collect some ideas and properties here.

**Definition IMP**

Integers Modulo a Prime

Suppose that $p$ is a prime number. Let $\mathbb{Z}_p = \{0, 1, 2, \ldots, p - 1\}$. Add and multiply elements of $\mathbb{Z}_p$ as integers, but whenever a result lies outside of the set $\mathbb{Z}_p$, find its remainder after division by $p$ and replace the result by this remainder.

We have defined a set, and two binary operations. The result is a field.

**Theorem FIMP**

Field of Integers Modulo a Prime

The set of integers modulo a prime $p$, $\mathbb{Z}_p$, is a field.

**Example IM11**

Integers mod 11

Version 1.04
We can now “do” linear algebra using scalars from a finite field.

**Example VSIM5**

**Vector space over integers mod 5**

Let \((\mathbb{Z}_5)^3\) be the set of all column vectors of length 3 with entries from \(\mathbb{Z}_5\). Use \(\mathbb{Z}_5\) as the set of scalars. Define addition and multiplication the usual way. We exhibit a few sample calculations.

\[
\begin{bmatrix}
2 \\
3 \\
4
\end{bmatrix} +
\begin{bmatrix}
4 \\
1 \\
3
\end{bmatrix} =
\begin{bmatrix}
1 \\
4 \\
2
\end{bmatrix} \quad 3
\begin{bmatrix}
2 \\
0 \\
4
\end{bmatrix} =
\begin{bmatrix}
1 \\
0 \\
2
\end{bmatrix}
\]

We can, of course, build linear combinations, such as

\[
2
\begin{bmatrix}
1 \\
3 \\
0
\end{bmatrix} - 4
\begin{bmatrix}
2 \\
1 \\
4
\end{bmatrix} +
\begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix} =
\begin{bmatrix}
0 \\
4 \\
0
\end{bmatrix}
\]

which almost looks like a relation of linear dependence. The set

\[
\left\{
\begin{bmatrix}
1 \\
3 \\
2
\end{bmatrix},
\begin{bmatrix}
2 \\
1 \\
0
\end{bmatrix}
\right\}
\]

is linearly independent, while the set

\[
\left\{
\begin{bmatrix}
1 \\
3 \\
2
\end{bmatrix},
\begin{bmatrix}
2 \\
2 \\
3
\end{bmatrix},
\begin{bmatrix}
4 \\
0 \\
2
\end{bmatrix}
\right\}
\]

is linearly dependent, as can be seen from the relation of linear dependence formed by the scalars \(a_1 = 2, a_2 = 1\) and \(a_3 = 4\). To find these scalars, one would take the same approach as **Example LDS** [121], but in performing row operations to solve a homogeneous system, you would need to take care that all scalar (field) operations are performed over \(\mathbb{Z}_5\), especially when multiplying a row by a scalar to make a leading entry equal to 1. One more observation about this example — the set

\[
\left\{
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix},
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}
\right\}
\]

is a basis for \((\mathbb{Z}_5)^3\), since it is both linearly independent and spans \((\mathbb{Z}_5)^3\).
Example SM2Z7
Symmetric matrices of size 2 over \( \mathbb{Z}_7 \)
We can employ the field of integers modulo a prime to build other examples of vector spaces with novel fields of scalars. Define
\[
S_{22}(\mathbb{Z}_7) = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \bigg| a, b, c \in \mathbb{Z}_7 \right\}
\]
which is the set of all \( 2 \times 2 \) symmetric matrices with entries from \( \mathbb{Z}_7 \). Use the field \( \mathbb{Z}_7 \) as the set of scalars, and define vector addition and scalar multiplication in the natural way. The result will be a vector space.

Notice that the field of scalars is finite, as is the vector space, since there are \( 7^3 = 343 \) matrices in \( S_{22}(\mathbb{Z}_7) \). The set
\[
\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]
is a basis, so \( \dim (S_{22}(\mathbb{Z}_7)) = 3 \).

In a more advanced algebra course it is possible to prove that the number of elements in a finite field must be of the form \( p^n \), where \( p \) is a prime. We can’t go so far afield as to prove this here, but we can demonstrate an example.

Example FF8
Finite field of size 8
Define the set \( F \) as \( F = \{ a + bt + ct^2 \big| a, b, c \in \mathbb{Z}_2 \} \). Add and multiply these quantities as polynomials in the variable \( t \), but replace any occurrence of \( t^3 \) by \( t + 1 \).

This defines a set, and the two operations on elements of that set. Do not be concerned with what \( t \) “is,” because it isn’t. \( t \) is just a handy device that makes the example a field. We’ll say a bit more about \( t \) when we finish. But first, some examples. Remember that \( 1 + 1 = 0 \) in \( \mathbb{Z}_2 \). Addition is quite simple, for example,
\[
(1 + t + t^2) + (1 + t^2) = (1 + 1) + (1 + 0)t + (1 + 1)t^2 = t
\]

Multiplication gets more involved, for example,
\[
(1 + t + t^2)(1 + t^2) = 1 + t^2 + t + t^3 + t^2 + t^4
\]
\[
= 1 + t + (1 + 1)t^2 + t^3(1 + t)
\]
\[
= 1 + t + (1 + t)(1 + t)
\]
\[
= 1 + t + 1 + t + t^2
\]
\[
= (1 + 1) + (1 + 1 + 1)t + t^2
\]
\[
= t + t^2
\]

Every element has a multiplicative inverse (Property MIF [719]). What is the inverse of \( t + t^2 \)? Check that
\[
(t + t^2)(1 + t) = t + t^2 + t^2 + t^3
\]
\[
= t + (1 + 1)t^2 + (1 + t)
\]
\[
= t + 1 + t
\]
\[
= 1 + (1 + 1)t
\]
\[
= 1
\]

So we can write \( \frac{1}{t + t^2} = 1 + t \). So that you may experiment, we give you the complete addition and multiplication tables for this field. Addition is simple, while multiplication is more interesting,
so verify a few entries of each table. Because of the commutativity of addition and multiplication (Property ACF \[718\], Property MCF \[718\]), we have just listed half of each table.

<math>
\begin{array}{cccccccccc}
+ & 0 & 1 & t & t^2 & t + 1 & t^2 + t & t^2 + t + 1 & t^2 + 1 \\
0 & 0 & 1 & t & t^2 & t + 1 & t^2 + t & t^2 + t + 1 & t^2 + 1 \\
1 & 0 & t + 1 & t^2 + 1 & t & t^2 + t + 1 & t^2 & t^2 & t^2 + 1 \\
t & 0 & t^2 + t & 1 & t^2 & t^2 + 1 & t^2 + t + 1 & t^2 + t & t^2 + 1 \\
t^2 & 0 & t^2 + t + 1 & t & t + 1 & 1 \\
t + 1 & 0 & t^2 + 1 & 1 & t^2 & t^2 + 1 & t^2 + t & t^2 + 1 & 1 \\
t^2 + t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t^2 + t + 1 & t + 1 & t^2 & t^2 + t + 1 & t^2 + 1 & t^2 & t + 1 & t^2 + t & t^2 + t + 1 & t^2 + 1 \\
\end{array}
</math>

Note that every element of \( F \) is a linear combination (with scalars from \( \mathbb{Z}_2 \)) of the polynomials 1, \( t \), \( t^2 \). So \( B = \{ 1, t, t^2 \} \) is a spanning set for \( F \). Further, \( B \) is linearly independent since there is no nontrivial relation of linear dependence, and \( B \) is a basis. So \( \dim (F) = 3 \). Of course, this paragraph presumes that \( F \) is also a vector space over \( \mathbb{Z}_2 \) (which it is).

The defining relation for \( t \) (\( t^3 = t + 1 \)) in Example FF8 \[721\] arises from the polynomial \( t^3 + t + 1 \), which has no factorization with coefficients from \( \mathbb{Z}_2 \). This is an example of an **irreducible polynomial**, which involves considerable theory to fully understand. In the exercises, we provide you with a few more irreducible polynomials to experiment with. See the suggested readings if you would like to learn more.

Trivially, every field (finite or otherwise) is a vector space. Suppose we begin with a field \( F \). From this we know \( F \) has two binary operations defined on it. We need to somehow create a vector space from \( F \), in a general way. First we need a set of vectors. That’ll be \( F \). We also need a set of scalars. That’ll be \( F \) as well. How do we define the addition of two vectors? By the same rule that we use to add them when they are in the field. How do we define scalar multiplication? Since a scalar is an element of \( F \), and a vector is an element of \( F \), we can define scalar multiplication to be the same rule that we use to multiply the two elements as members of the field. With these definitions, \( F \) will be a vector space (Exercise F.T20 \[724\]). This is something of a trivial situation, since the set of vectors and the set of scalars are identical. In particular, do not confuse this with Example FF8 \[721\] where the set of vectors has eight elements, and the set of scalars has just two elements.

**Further Reading**
Subsection EXC
Exercises

C60 Consider the vector space \((\mathbb{Z}_5)^4\) composed of column vectors of size 4 with entries from \(\mathbb{Z}_5\). The matrix \(A\) is a square matrix composed of four such column vectors.

\[
A = \begin{bmatrix}
3 & 3 & 0 & 3 \\
1 & 2 & 3 & 0 \\
1 & 1 & 0 & 2 \\
4 & 2 & 2 & 1 \\
\end{bmatrix}
\]

Find the inverse of \(A\). Use this to find a solution to \(LS(A, b)\) when

\[
b = \begin{bmatrix}
3 \\
3 \\
2 \\
0 \\
\end{bmatrix}
\]

Contributed by Robert Beezer  Solution 725

M10 Suppose we relax the restriction in Definition IMP 719 to allow \(p\) to not be a prime. Will the construction given still be a field? Is \(\mathbb{Z}_6\) a field? Can you generalize?

Contributed by Robert Beezer

M40 Construct a finite field with 9 elements using the set

\[
F = \{a + bt \ | \ a, b \in \mathbb{Z}_3\}
\]

where \(t^2\) is consistently replaced by \(2t + 1\) in any intermediate results obtained with polynomial multiplication. Compute the first nine powers of \(t\) (\(t^0\) through \(t^8\)). Use this information to aid you in the construction of the multiplication table for this field. What is the multiplicative inverse of \(2t\)?

Contributed by Robert Beezer

M45 Construct a finite field with 25 elements using the set

\[
F = \{a + bt \ | \ a, b \in \mathbb{Z}_5\}
\]

where \(t^2\) is consistently replaced by \(t + 3\) in any intermediate results obtained with polynomial multiplication. Compute the first 25 powers of \(t\) (\(t^0\) through \(t^{24}\)). Use this information to aid you in computing in this field. What is the multiplicative inverse of \(2t\)? What is the multiplicative inverse of 4? What is the multiplicative inverse of \(1 + 4t\)?

Find a basis for \(F\) as a vector space with \(\mathbb{Z}_5\) used as the set of scalars.

Contributed by Robert Beezer

M50 Construct a finite field with 16 elements using the set

\[
F = \{a + bt + ct^2 + dt^3 \ | \ a, b, c, d \in \mathbb{Z}_2\}
\]

where \(t^4\) is consistently replaced by \(t + 1\) in any intermediate results obtained with polynomial multiplication. Compute the first 16 powers of \(t\) (\(t^0\) through \(t^{15}\)). Consider the set \(G = \{0, 1, t^5, t^{10}\}\). Then \(G\) will also be a finite field, a subfield of \(F\). Construct the addition and multiplication tables for \(G\). Notice that since both \(G\) and \(F\) are vector spaces over \(\mathbb{Z}_2\), and \(G \subseteq F\), by Definition S 264, \(G\) is a subspace of \(F\).

Contributed by Robert Beezer

T10 Give a new proof of Theorem ZVSM 258 for a vector space whose scalars come from an arbitrary field \(F\).

Contributed by Robert Beezer
T20  By applying Definition VS [251], prove that every field is also a vector space. (See the construction at the end of this section.)
Contributed by Robert Beezer
Remember that every computation must be done with arithmetic in the field, reducing any intermediate number outside of \{0, 1, 2, 3, 4\} to its remainder after division by 5. The matrix inverse can be found with Theorem CINM (and we discover along the way that \(A\) is nonsingular). The inverse is

\[
A^{-1} = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 3 & 4 & 1 & 4 \\ 1 & 4 & 0 & 2 \\ 3 & 0 & 1 & 0 \end{bmatrix}
\]

Then by an application of Theorem SNCM the (unique) solution to the system will be

\[
A^{-1}b = \begin{bmatrix} 1 & 1 & 3 & 1 \\ 3 & 4 & 1 & 4 \\ 1 & 4 & 0 & 2 \\ 3 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}
\]
Chapter MD
Matrix Decompositions

This chapter is about breaking up a matrix $A$ into pieces that somehow combine to recreate $A$. Usually the pieces are again matrices, and usually they are then combined via matrix multiplication (Definition MM [176]). In some cases, the decomposition will be valid for any matrix, but often we might need extra conditions on $A$, such as being square (Definition SQM [61]) or diagonalizable (Definition DZM [393]) before we can guarantee the decomposition. If you are comfortable with topics like decomposing a solution vector into linear combinations (Subsection LC.VFSS [83]) or decomposing vector spaces into direct sums (Subsection PD.DS [325]), then we will be doing similar things in this chapter. If not, review these ideas and take another look at Technique DC [625] on decompositions.

We have studied one matrix decomposition already, so we will review that here in this introduction, both as a way of previewing the topic in a familiar setting, but also since it does not deserve another section all of its own.

A diagonalizable matrix (Definition DZM [393]) is defined to be a square matrix $A$ such that there is an invertible matrix $S$ and a diagonal matrix $D$ where $A = S^{-1}DS$. Here we have a decomposition of $A$ into three matrices, $S^{-1}$, $D$ and $S$, which recombine through matrix multiplication to recreate $A$. We cannot form this decomposition for just any matrix — $A$ must be square and we know from Theorem DC [394] that a matrix of size $n$ is diagonalizable if and only if there is a basis for $\mathbb{C}^n$ composed entirely of eigenvectors of $A$, or by Theorem DMFE [396] we know that $A$ is diagonalizable if and only if each eigenvalue of $A$ has a geometric multiplicity equal to its algebraic multiplicity. Some authors prefer to call this an eigen decomposition of $A$ rather than a matrix diagonalization.

Each section of this chapter is a different matrix decomposition, and they will become progressively more general and more complicated. As such, later decompositions will require greater familiarity with topics from the Core (Part C [2]).

Section ROD
Rank One Decomposition

This Section Under Construction

Our first decomposition applies only to diagonalizable (Definition DZM [393]) matrices, and yields a decomposition into a sum of very simple matrices.

Theorem ROD
Rank One Decomposition
Suppose that $A$ is a diagonalizable matrix of size $n$ and rank $r$. Then there are $r$ square matrices $A_1, A_2, A_3, \ldots, A_r$, each of size $n$ and rank 1 such that

$$A = A_1 + A_2 + A_3 + \cdots + A_r$$
Furthermore, if \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_r \) are the nonzero eigenvalues of \( A \), then there are two sets of \( r \) linearly independent vectors from \( \mathbb{C}^n \),
\[
X = \{ x_1, x_2, x_3, \ldots, x_r \} \quad \quad Y = \{ y_1, y_2, y_3, \ldots, y_r \}
\]
such that \( A_k = \lambda_k x_k y_k^t \), \( 1 \leq k \leq r \).

**Proof**  The proof is constructive. Generally, we will diagonalize \( A \), creating a nonsingular matrix \( S \) and a diagonal matrix \( D \). Then we split up the diagonal matrix into a sum of matrices with a single nonzero entry (on the diagonal). This fundamentally creates the decomposition in the statement of the theorem, the remainder is just bookkeeping. The vectors in \( X \) and \( Y \) will result from the columns of \( S \) and the rows of \( S^{-1} \).

Let \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \) be the eigenvalues of \( A \) (repeated according to their algebraic multiplicity). If \( A \) has rank \( r \), then \( \dim (N(A)) = n - r \) (Theorem [RPNC 314]). The null space of \( A \) is the eigenspace of the eigenvalue \( \lambda = 0 \) (Theorem [EMNS 364]), so it follows that the algebraic multiplicity of \( \lambda = 0 \) is \( n - r \), \( \alpha_A (0) = n - r \). Presume that the complete list of eigenvalues is ordered so that \( \lambda_k = 0 \) for \( r + 1 \leq k \leq n \).

Since \( A \) is hypothesized to be diagonalizable, there exists a diagonal matrix \( D \) and an invertible matrix \( S \), such that \( D = S^{-1} A S \). We can rearrange this equation to read, \( A = S D S^{-1} \). Also, the proof of Theorem [DC 394] says that the diagonal elements of \( D \) are the eigenvalues of \( A \) and we have the flexibility to assume they lie on the diagonal in the same order as we have specified above.

Now, let \( X^* = \{ x_1, x_2, x_3, \ldots, x_n \} \) be the columns of \( S \), and let \( Y^* = \{ y_1, y_2, y_3, \ldots, y_n \} \) be the rows of \( S^{-1} \) converted to column vectors. With little motivation other than the statement of the theorem, define size \( n \) matrices \( A_k \), \( 1 \leq k \leq n \) by \( A_k = \lambda_k x_k y_k^t \). Finally, let \( D_k \) be the size \( n \) matrix that is totally zero, other than having \( \lambda_k \) in row \( k \) and column \( k \).

With everything in place, we compute entry-by-entry,
\[
[A]_{ij} = [SDS^{-1}]_{ij} = \left[ S \left( \sum_{k=1}^{n} D_k \right) S^{-1} \right]_{ij} = \left[ S \left( \sum_{k=1}^{n} D_k S^{-1} \right) \right]_{ij} = \sum_{k=1}^{n} \left[ SD_k S^{-1} \right]_{ij} = \sum_{k=1}^{n} \sum_{\ell=1}^{n} [SDk]_{i\ell} [S^{-1}]_{\ell j} = \sum_{k=1}^{n} \sum_{\ell=1}^{n} \sum_{p=1}^{n} [S]_{ip} [Dk]_{p\ell} [S^{-1}]_{\ell j} = \sum_{k=1}^{n} [S]_{ik} [Dk]_{kk} [S^{-1}]_{kj} = \sum_{k=1}^{n} [S]_{ik} \lambda_k [S^{-1}]_{kj} = \sum_{k=1}^{n} \lambda_k [S]_{ik} [S^{-1}]_{kj} = \sum_{k=1}^{n} \lambda_k [x_k]_{i1} [y_k^t]_{1j} = \sum_{k=1}^{n} \lambda_k x_k y_k^t.
\]

\[\square\]
\[
\sum_{k=1}^{n} \lambda_k \sum_{q=1}^{1} [x_k]_{iq} [y_k]_{qj} = \sum_{k=1}^{n} \lambda_k [x_k y_k^T]_{ij} = \sum_{k=1}^{n} [\lambda_k x_k y_k^T]_{ij} = \sum_{k=1}^{n} [A_k]_{ij} = \sum_{k=1}^{n} A_k
\]

So by Definition ME 163 we have the desired equality of matrices. The careful reader will have noted that \( A_k = 0 \), \( r + 1 \leq k \leq n \), since \( \lambda_k = 0 \) in these instances. To get the sets \( X \) and \( Y \) from \( X^* \) and \( Y^* \), simply discard the last \( n - r \) vectors. We can safely ignore (or remove) \( A_{r+1}, A_{r+2}, \ldots, A_n \) from the summation just derived.

One last assertion to check. What is the rank of \( A_k \), \( 1 \leq k \leq r \)? Every row of \( A_k \) is a scalar multiple of \( y_k^T \), row \( k \) of the nonsingular matrix \( S^{-1} \) (Theorem MIMI 196). As a row of a nonsingular matrix, \( y_k^T \) cannot be all zeros. In particular, row \( i \) of \( A_k \) is obtained as a scalar multiple of \( y_k^T \) by the scalar \( \alpha_k \). We have restricted ourselves to the nonzero eigenvalues of \( A \), and as \( S \) is nonsingular, some entry of \( x_k \) is nonzero. This all implies that some row of \( A_k \) will be nonzero. Now consider row-reducing \( A_k \). Swap the nonzero row up into row 1. Use scalar multiples of this row to zero out every other row. This leaves a single nonzero row in the reduced row-echelon form, so \( A_k \) has rank one.

We record two observations that was not stated in our theorem above. First, the vectors in \( X \), chosen as columns of \( S \), are eigenvectors of \( A \). Second, the product of two vectors from \( X \) and \( Y \) in the opposite order, by which we mean \( y_i^T x_j \), is the entry in row \( i \) and column \( j \) of the matrix product \( S^{-1}S = I_n \) (Theorem EMP 177). In particular,

\[
y_i^T x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]

We give two computational examples. One small, one a bit bigger.

**Example ROD2**

**Rank one decomposition, size 2**

Consider the \( 2 \times 2 \) matrix,

\[
A = \begin{bmatrix} -16 & -6 \\ 45 & 17 \end{bmatrix}
\]

By the techniques of Chapter E 356 we find the eigenvalues and eigenspaces,

\[
\lambda_1 = 2 \quad \mathcal{E}_A(2) = \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\} \quad \lambda_2 = -1 \quad \mathcal{E}_A(-1) = \left\{ \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right\}
\]

With \( n = 2 \) distinct eigenvalues, Theorem DED 398 tells us that \( A \) is diagonalizable, and with no zero eigenvalues we see that \( A \) has full rank. Theorem DC 394 says we can construct the nonsingular matrix \( S \) with eigenvectors of \( A \) as columns, so we have

\[
S = \begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 5 & 2 \\ -3 & -1 \end{bmatrix}
\]
From these matrices we obtain the sets of vectors
\[ X = \left\{ \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -2 \end{bmatrix} \right\} \]
\[ Y = \left\{ \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \end{bmatrix} \right\} \]

And we have the matrices,
\[ A_1 = 2 \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}^t = 2 \begin{bmatrix} 10 \\ -4 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \end{bmatrix} \]
\[ A_2 = (-1) \begin{bmatrix} -2 \\ 5 \\ -3 \end{bmatrix}^t = (-1) \begin{bmatrix} 6 \\ -15 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix} \]

And you can easily verify that \[ A = A_1 + A_2. \]

Here’s a slightly larger example, and the matrix does not have full rank.

**Example ROD4**

**Rank one decomposition, size 4**

Consider the \( 4 \times 4 \) matrix,
\[ B = \begin{bmatrix} 34 & 18 & -1 & -6 \\ -44 & -24 & -1 & 9 \\ 36 & 18 & -3 & -6 \\ 36 & 18 & -6 & -3 \end{bmatrix} \]

By the techniques of [Chapter E 356](#) we find the eigenvalues and eigenvectors,
\[ \lambda_1 = 3 \]
\[ \mathcal{E}_B(3) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} \right\} \]
\[ \lambda_2 = -2 \]
\[ \mathcal{E}_B(-2) = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\} \]
\[ \lambda_3 = 0 \]
\[ \mathcal{E}_A(0) = \left\{ \begin{bmatrix} 2 \\ -3 \\ 2 \\ 2 \end{bmatrix} \right\} \]

The algebraic and geometric multiplicities of each eigenvalue are equal, so [Theorem DMFE 396](#) tells us that \( A \) is diagonalizable. With a single zero eigenvalue we see that \( A \) has rank \( 4 - 1 = 3 \). [Theorem DC 394](#) says we can construct the nonsingular matrix \( S \) with eigenvectors of \( A \) as columns, so we have
\[ S = \begin{bmatrix} 1 & 1 & -1 & 2 \\ -2 & -1 & 2 & -3 \\ 1 & 1 & 0 & 2 \\ -1 & 2 & 0 & 2 \end{bmatrix} \]
\[ S^{-1} = \begin{bmatrix} 4 & 2 & 0 & -1 \\ 8 & 4 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ -6 & -3 & 1 & 1 \end{bmatrix} \]

Since \( r = 3 \), we need only collect three vectors from each of these matrices,
\[ X = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\} \]
\[ Y = \left\{ \begin{bmatrix} 4 \\ 2 \\ 8 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \]
And we obtain the matrices,

\[
B_1 = 3 \begin{bmatrix} 1 & 4 \\ -2 & 2 \\ 1 & 0 \\ -1 & 0 \end{bmatrix}^t = 3 \begin{bmatrix} 4 & 2 & 0 & -1 \\ -8 & -4 & 0 & 2 \\ 4 & 2 & 0 & -1 \\ -4 & -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 6 & 0 & -3 \\ -24 & -12 & 0 & 6 \\ 12 & 6 & 0 & -3 \\ -12 & -6 & 0 & 3 \end{bmatrix}
\]

\[
B_2 = 3 \begin{bmatrix} 1 & 8 \\ -1 & 4 \\ 1 & -1 \\ 2 & -1 \end{bmatrix}^t = 3 \begin{bmatrix} 8 & 4 & -1 & -1 \\ -8 & -4 & 1 & 1 \\ 8 & 4 & -1 & -1 \\ 16 & 8 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 24 & 12 & -3 & -3 \\ -24 & -12 & 3 & 3 \\ 24 & 12 & -3 & -3 \\ 48 & 24 & -6 & -6 \end{bmatrix}
\]

\[
B_3 = (-2) \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^t = (-2) \begin{bmatrix} 1 & 0 & -1 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 2 & 0 \\ 4 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

Then we verify that

\[
B = B_1 + B_2 + B_3
\]

\[
= \begin{bmatrix} 12 & 6 & 0 & -3 \\ -24 & -12 & 0 & 6 \\ 12 & 6 & 0 & -3 \\ -12 & -6 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 24 & 12 & -3 & -3 \\ -24 & -12 & 3 & 3 \\ 24 & 12 & -3 & -3 \\ 48 & 24 & -6 & -6 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 2 & 0 \\ 4 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} 34 & 18 & -1 & -6 \\ -44 & -24 & -1 & 9 \\ 36 & 18 & -3 & -6 \\ 36 & 18 & -6 & -3 \end{bmatrix}
\]

Further Reading
Section TD
Triangular Decomposition

DRAFT: THIS SECTION COMPLETE, BUT SUBJECT TO CHANGE

Our next decomposition will break a square matrix into a product of two matrices, one lower triangular and the other upper triangular. So we will write $A = LU$, and hence many refer to this as LU decomposition. We will see that this decomposition is very easy to compute and that it has a direct application to solving systems of equations. Since this section is about triangular matrices you might want to review the definitions and a couple of basic theorems back in Subsection OD.TM [540].

Subsection TD
Triangular Decomposition

With a slight condition on the nonsingularity of certain submatrices, we can split a matrix into a product of two triangular matrices.

Theorem TD
Triangular Decomposition

Suppose $A$ is a square matrix of size $n$. Let $A_k$ be the $k \times k$ matrix formed from $A$ by taking the first $k$ rows and the first $k$ columns. Suppose that $A_k$ is nonsingular for all $1 \leq k \leq n$. Then there is a lower triangular matrix $L$ with all of its diagonal entries equal to 1 and an upper triangular matrix $U$ such that $A = LU$. Furthermore, this decomposition is unique.

Proof. We will row reduce $A$ to a row-equivalent upper triangular matrix through a series of row operations, forming intermediate matrices $A'_j$, $1 \leq j \leq n$, that denote the state of the conversion after working on column $j$. First, the lone entry of $A_1$ is $[A]_{11}$ and this scalar must be nonzero if $A_1$ is nonsingular (Theorem SMZD [351]). We can use row operations Definition RO [24] of the form $\alpha R_1 + R_k$, $2 \leq k \leq n$, where $\alpha = -[A]_{1k} / [A]_{11}$ to place zeros in the first column below the diagonal. The first two rows and columns of $A'_1$ are a $2 \times 2$ upper triangular matrix whose determinant is equal to the determinant of $A_2$, since the matrices are row-equivalent through a sequence of row operations strictly of the third type [Theorem DRMCA [??]]. As such the diagonal entries of this $2 \times 2$ submatrix of $A'_1$ are nonzero. We can employ this nonzero diagonal element with row operations of the form $\alpha R_2 + R_k$, $3 \leq k \leq n$ to place zeros below the diagonal in the second column. We can continue this process, column by column. The key observations are that our hypothesis on the nonsingularity of the $A_k$ will guarantee a nonzero diagonal entry for each column when we need it, that the row operations employed are always of the third type using a multiple of a row to transform another row with a greater row index, and that the final result will be a nonsingular upper triangular matrix. This is the desired matrix $U$.

Each row operation described in the previous paragraph can be accomplished with matrix multiplication by the appropriate elementary matrix [Theorem EMDRO [334]]. Since every row operation employed is adding a multiple of a row to a subsequent row these elementary matrices are of the form $E_{j,k}(\alpha)$ with $j < k$. By Definition ELEM [333], these matrices are lower triangular with every diagonal entry equal to 1. We know that the product of two such matrices will again be lower triangular [Theorem PTMT [540]], but also, as you can also easily check using a proof with a style similar to one above, that the product maintains all 1’s on the diagonal. Let $E_1, E_2, E_3, \ldots, E_m$ denote the elementary matrices for this sequence of row operations. Then

$$U = E_mE_{m-1} \ldots E_2E_1A = L'A$$

where $L'$ is the product of the elementary matrices, and we know $L'$ is lower triangular with all 1’s on the diagonal. Our desired matrix $L$ is then $L = (L')^{-1}$. By Theorem ITMT [541], $L$ is lower triangular with all 1’s on the diagonal and $A = LU$, as desired.
The process just described is deterministic. That is, the proof is constructive, with no freedom for each of us to walk through it differently. But could there be other matrices with the same properties as $L$ and $U$ that give such a decomposition of $A$. In other words, is the decomposition unique (Technique U [624])? Suppose that we have two triangular decompositions, $A = L_1 U_1$ and $A = L_2 U_2$. Since $A$ is nonsingular, two applications of Theorem NPNT [202] imply that $L_1$, $L_2$, $U_1$, $U_2$ are all nonsingular. We have

\[
L_2^{-1} L_1 = L_2^{-1} I_n L_1 = L_2^{-1} A A^{-1} L_1 = L_2^{-1} L_2 U_2 (L_1 U_1)^{-1} L_1 = L_2^{-1} L_2 U_2 U_1^{-1} L_1^{-1} L_1 = I_n U_2 U_1^{-1} I_n = U_2 U_1^{-1}
\]

which establishes the uniqueness of the decomposition.

Studying the proofs of some previous theorems will perhaps give you an idea for an approach to computing a triangular decomposition. In the proof of Theorem CINM [193] we augmented a nonsingular matrix with an identity matrix of the same size, and row-reduced until the original matrix became the identity matrix (as we knew in advance would happen, since we knew Theorem NMRRI [62]). Theorem PEEF [236] tells us about properties of extended echelon form, and in particular, that $B = JA$, where $A$ is the matrix that begins on the left, and $B$ is the reduced row-echelon form of $A$. The matrix $J$ is the result on the right side of the augmented matrix, which is the result of applying the same row operations to the identity matrix. We should recognize now that $J$ is just the product of the elementary matrices (Subsection DM.EM [333]) that perform these row operations. Theorem ITMT [541] used the extended echelon form to discern properties of the inverse of a triangular matrix. Theorem TD [731] proves the existence of a triangular decomposition by applying specific row operations, and tracking the relevant elementary row operations. It is not a great leap to combine these observations into a computational procedure.

To find the triangular decomposition of $A$, augment $A$ with the identity matrix of the same size and call this new $2n \times n$ matrix, $M$. Perform row operations on $M$ that convert the first $n$ columns to an upper triangular matrix. Do this using only row operations that add a scalar multiple of one row to another row with higher index (i.e. lower down). In this way, the last $n$ columns of $M$ will be converted into a lower triangular matrix with 1’s on the diagonal (since $M$ has 1’s in these locations initially). We could think of this process as doing about half of the work required to compute the inverse of $A$. Take the first $n$ columns of the row-equivalent version of $M$ and call this matrix $U$. Take the final $n$ columns of the row-equivalent version of $M$ and call this matrix $L’$. Then by a proof employing elementary matrices, or a proof similar in spirit to the one used to prove Theorem PEEF [236], we arrive at a result similar to the second assertion of Theorem PEEF [236]. Namely, $U = L’ A$. Multiplication on the left, by the inverse of $L’$, will give us a decomposition of $A$ (which we know to be unique). Ready? Let’s try it.

Example TD4

Triangular decomposition, size 4

In this example, we will illustrate the process for computing a triangular decomposition, as described
In the previous paragraphs. Consider the nonsingular square matrix $A$ of size 4,

$$
A = \begin{bmatrix}
-2 & 6 & -8 & 7 \\
-4 & 16 & -14 & 15 \\
-6 & 22 & -23 & 26 \\
-6 & 26 & -18 & 17
\end{bmatrix}
$$

We form $M$ by augmenting $A$ with the size 4 identity matrix $I_4$. We will perform the allowed operations, column by column, only reporting intermediate results as we finish converting each column. It is easy to determine exactly which row operations we perform, since the final four columns contain a record of each such operation. We will not verify our hypotheses about the nonsingularity of the $A_k$, since if we do not have these conditions, we will reach a stage where a diagonal entry is zero and we cannot create the row operations we need to zero out the bottom portion of the associated column. In other words, we can boldly proceed and the necessity of our hypotheses will become apparent.

$$
M = \begin{bmatrix}
-2 & 6 & -8 & 7 & 1 & 0 & 0 & 0 \\
-4 & 16 & -14 & 15 & 0 & 1 & 0 & 0 \\
-6 & 22 & -23 & 26 & 0 & 0 & 1 & 0 \\
-6 & 26 & -18 & 17 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\rightarrow \begin{bmatrix}
-2 & 6 & -8 & 7 & 1 & 0 & 0 & 0 \\
0 & 4 & 2 & 1 & -2 & 1 & 0 & 0 \\
0 & 4 & 1 & 5 & -3 & 0 & 1 & 0 \\
0 & 8 & 6 & -4 & -3 & 0 & 0 & 1
\end{bmatrix}
$$

$$
\rightarrow \begin{bmatrix}
-2 & 6 & -8 & 7 & 1 & 0 & 0 & 0 \\
0 & 4 & 2 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & -1 & 4 & -1 & -1 & 0 & 1 \\
0 & 0 & 2 & -6 & 1 & -2 & 0 & 1
\end{bmatrix}
$$

$$
\rightarrow \begin{bmatrix}
-2 & 6 & -8 & 7 & 1 & 0 & 0 & 0 \\
0 & 4 & 2 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & -1 & 4 & -1 & -1 & 1 & 0 \\
0 & 0 & 0 & 2 & -1 & -4 & 2 & 1
\end{bmatrix}
$$

So at this point, we have $U$ and $L'$,

$$
U = \begin{bmatrix}
-2 & 6 & -8 & 7 \\
0 & 4 & 2 & 1 \\
0 & 0 & -1 & 4 \\
0 & 0 & 0 & 2
\end{bmatrix}
$$

$$
L' = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 \\
-1 & -1 & 1 & 0 \\
-1 & -4 & 2 & 1
\end{bmatrix}
$$

Then by whatever procedure we like (such as Theorem CINM [193]), we find

$$
L = (L')^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
3 & 1 & 1 & 0 \\
3 & 2 & -2 & 1
\end{bmatrix}
$$

It is instructive to verify that indeed $LU = A$. □

Subsection TDSSE
Triangular Decomposition and Solving Systems of Equations

In this section we give an explanation of why you might be interested in a triangular decomposition for a matrix. Many of the computational problems in linear algebra revolve around solving large
systems of equations, or nearly equivalently, finding inverses of large matrices. Suppose we have a system of equations with coefficient matrix $A$ and vector of constants $b$, and suppose further that $A$ has the triangular decomposition $A = LU$.

Let $y$ be the solution to the linear system $LS(L, b)$, so that by Theorem SLEMM \[173\], we have $Ly = b$. Notice that since $L$ is nonsingular, this solution is unique, and the form of $L$ makes it trivial to solve the system. The first component of $y$ is determined easily, and we can continue on through determining the components of $y$, without even ever dividing. Now, with $y$ in hand, consider the linear system, $LS(U, y)$. Let $x$ be the unique solution to this system, so by Theorem SLEMM \[173\] we have $Ux = y$. Notice that a system of equations with $U$ as a coefficient matrix is also straightforward to solve, though we will compute the bottom entries of $x$ first, and we will need to divide. The upshot of all this is that $x$ is a solution to $LS(A, b)$, as we now show,

$$Ax = LUx = L(Ux) = Ly = b$$

An application of Theorem SLEMM \[173\] demonstrates that $x$ is a solution to $LS(A, b)$.

**Example TDSSE**

**Triangular decomposition solves a system of equations**

Here we illustrate the previous discussion, recycling the decomposition found previously in Example TD4 \[732\]. Consider the linear system $LS(A, b)$ with

$$A = \begin{bmatrix}
-2 & 6 & -8 & 7 \\
-4 & 16 & -14 & 15 \\
-6 & 22 & -23 & 26 \\
-6 & 26 & -18 & 17
\end{bmatrix}, \quad b = \begin{bmatrix}
-10 \\
-2 \\
-1 \\
-8
\end{bmatrix}$$

First we solve the system $LS(L, b)$ (see Example TD4 \[732\] for $L$),

$$y_1 = -10$$
$$2y_1 + y_2 = -2$$
$$3y_1 + y_2 + y_3 = -1$$
$$3y_1 + 2y_2 - 2y_3 + y_4 = -8$$

Then

$$y_1 = -10$$
$$y_2 = -2 - 2y_1 = -2 - 2(-10) = 18$$
$$y_3 = -1 - 3y_1 - y_2 = -1 - 3(-10) - 18 = 11$$
$$y_4 = -8 - 3y_1 - 2y_2 + 2y_3 = -8 - 3(-10) - 2(18) + 2(11) = 8$$

so

$$y = \begin{bmatrix}
-10 \\
18 \\
11 \\
8
\end{bmatrix}$$

Then we solve the system $LS(U, y)$ (see Example TD4 \[732\] for $U$),

$$-2x_1 + 6x_2 - 8x_3 + 7x_4 = -10$$
$$4x_2 + 2x_3 + x_4 = 18$$
$$-x_3 + 4x_4 = 11$$
$$2x_4 = 8$$

Then

$$x_4 = 8/2 = 4$$
\[ x_3 = (11 - 4x_4) / (-1) = (11 - 4(4)) / (-1) = 5 \]
\[ x_2 = (18 - 2x_3 - x_4) / 4 = (18 - 2(5) - 4) / 4 = 1 \]
\[ x_1 = (-10 - 6x_2 + 8x_3 - 7x_4) / (-2) = (-10 - 6(1) + 8(5) - 7(4)) / (-2) = 2 \]

And so

\[
x = \begin{bmatrix} 4 \\ 5 \\ 1 \\ 2 \end{bmatrix}
\]

is the solution to \( LS(U, y) \) and consequently is the unique solution to \( LS(A, b) \), as you can easily verify.

\[ \Box \]

Subsection CTD

Computing Triangular Decompositions

It would be a simple matter to adjust the algorithm for converting a matrix to reduced row-echelon form and obtain an algorithm to compute the triangular decomposition of the matrix, along the lines of Example TD4 \[732\] and the discussion preceding this example. However, it is possible to obtain relatively simple formulas for the entries of the decomposition, and if computed in the proper order, an implementation will be straightforward. We will state the result as a theorem and then give an example of its use.

**Theorem TDEE**

**Triangular Decomposition, Entry by Entry**

Suppose that \( A \) is a square matrix of size \( n \) with a triangular decomposition \( A = LU \), where \( L \) is lower triangular with diagonal entries all equal to 1, and \( U \) is upper triangular. Then

\[
[U]_{ij} = [A]_{ij} - \sum_{k=1}^{i-1} [L]_{ik} [U]_{kj} \hspace{1cm} 1 \leq i \leq j \leq n
\]

\[
[L]_{ij} = \frac{1}{[U]_{jj}} \left( [A]_{ij} - \sum_{k=1}^{j-1} [L]_{ik} [U]_{kj} \right) \hspace{1cm} 1 \leq j < i \leq n
\]

\[ \Box \]

**Proof** Consider a single scalar product of an entry of \( L \) with an entry of \( U \) of the form \([L]_{ik} [U]_{kj}\). By Definition LTM \[540\], if \( k > i \) then \([L]_{ik} = 0\), while Definition UTM \[540\], says that if \( k > j \) then \([U]_{kj} = 0\). So we can combine these two facts to assert that if \( k > \min(i, j) \), \([L]_{ik} [U]_{kj} = 0\) since at least one term of the product will be zero. Employing this observation,

\[
[A]_{ij} = \sum_{k=1}^{n} [L]_{ik} [U]_{kj} \hspace{1cm} \text{Theorem EMP} \[177\]
\]

Now, assume that \( 1 \leq i \leq j \leq n \),

\[
[U]_{ij} = [A]_{ij} - \sum_{k=1}^{\min(i, j)} [L]_{ik} [U]_{kj} + [U]_{ij}
\]

\[ \Box \]
\begin{align*}
[A]_{ij} &= \sum_{k=1}^{i} [L]_{ik} [U]_{kj} + [U]_{ij} \\
&= \sum_{k=1}^{i-1} [L]_{ik} [U]_{kj} - [L]_{ii} [U]_{ij} + [U]_{ij} \\
&= \sum_{k=1}^{i-1} [L]_{ik} [U]_{kj} - [U]_{ij} + [U]_{ij} \\
&= [A]_{ij} - \sum_{k=1}^{i-1} [L]_{ik} [U]_{kj}
\end{align*}

And for $1 \leq j < i \leq n$,

\begin{align*}
[L]_{ij} &= \frac{1}{[U]_{jj}} ([L]_{ij} [U]_{jj}) \\
&= \frac{1}{[U]_{jj}} ([A]_{ij} - [A]_{ij} + [L]_{ij} [U]_{jj}) \\
&= \frac{1}{[U]_{jj}} ([A]_{ij} - \sum_{k=1}^{\min(i, j)} [L]_{ik} [U]_{kj} + [L]_{ij} [U]_{jj}) \\
&= \frac{1}{[U]_{jj}} ([A]_{ij} - \sum_{k=1}^{j} [L]_{ik} [U]_{kj} + [L]_{ij} [U]_{jj}) \\
&= \frac{1}{[U]_{jj}} ([A]_{ij} - \sum_{k=1}^{j-1} [L]_{ik} [U]_{kj} - [L]_{ij} [U]_{jj} + [L]_{ij} [U]_{jj}) \\
&= \frac{1}{[U]_{jj}} ([A]_{ij} - \sum_{k=1}^{j-1} [L]_{ik} [U]_{kj})
\end{align*}

At first glance, these formulas may look exceedingly complex. Upon closer examination, it looks even worse. We have expressions for entries of $U$ that depend on other entries of $U$ and also on entries of $L$. But then the formula for entries of $L$ depend on entries from $L$ and entries from $U$. Do these formula have circular dependencies? Or perhaps equivalently, how do we get started? The key is to be organized about the computations and employ these two (similar) formulas in a specific order. First compute the first row of $L$, followed by the first column of $U$. Then the second row of $L$, followed by the second column of $U$. And so on. In this way, all of the values required for each new entry will have already been computed previously.

Of course, the formula for entries of $L$ require division by diagonal entries of $U$. These entries might be zero, but in this case $A$ is nonsingular and does not have a triangular decomposition. So we need not check the hypothesis carefully and can launch into the arithmetic dictated by the formulas, confident that we will be reminded when a decomposition is not possible. Note that these formula give us all of the values that we need for the decomposition, since we require that $L$ has 1’s on the diagonal. If we replace the 1’s on the diagonal of $L$ by zeros, and add the matrix $U$, we get an $n \times n$ matrix containing all the information we need to resurrect the triangular decomposition. This is mostly a notational convenience, but it is a frequent way of presenting the information. We’ll employ it in the next example.

Example TDEE6
Triangular decomposition, entry by entry, size 6
We illustrate the application of the formulas in Theorem TDEE [735] for the $6 \times 6$ matrix $A$.

\[ A = \begin{bmatrix}
 3 & 3 & -3 & -2 & -1 & 0 \\
 -6 & -4 & 5 & 2 & 4 & 2 \\
 9 & 9 & -7 & -7 & 0 & 1 \\
 -6 & -10 & 8 & 10 & -1 & -7 \\
 6 & 4 & -9 & -2 & -10 & 1 \\
 9 & 3 & -12 & -3 & -21 & -2 \\
\end{bmatrix} \]

Using the notational convenience of packaging the two triangular matrices into one matrix, and using the ordering of the computations mentioned above, we display the results after computing a single row and column of each of the two triangular matrices.

\[ L = \begin{bmatrix}
 1 & 0 & 0 & 0 & 0 & 0 \\
 -2 & 1 & 0 & 0 & 0 & 0 \\
 3 & 0 & 1 & 0 & 0 & 0 \\
 -2 & -2 & 0 & 1 & 0 & 0 \\
 2 & -1 & -2 & -1 & 1 & 0 \\
 3 & -3 & -3 & -3 & 0 & 1 \\
\end{bmatrix} \quad U = \begin{bmatrix}
 3 & 3 & -3 & -2 & -1 & 0 \\
 -2 & 2 & -1 & -2 & 2 & 2 \\
 3 & 0 & -2 & -1 & -2 & 2 \\
 -2 & 2 & 2 & 2 & 2 & 1 \\
 2 & -1 & -2 & -1 & 1 & 2 \\
 3 & -3 & -3 & -3 & 0 & -2 \\
\end{bmatrix} \]

Further Reading
Part A
Applications
Index

A (appendix), 630
A (archetype), 634
A (definition), 207
A (notation), 207
AA (Property), 251
AA (subsection, section WILA), 3
AAC (Property), 75
AACN (Property), 612
AAF (Property), 718
AALC (example), 81
AAM (Property), 165
ABLC (example), 80
ABS (example), 102
AC (Property), 251
ACC (Property), 75
ACCN (Property), 612
ACF (Property), 718
ACM (Property), 165
ACN (example), 611
additive associativity
column vectors
Property AAC, 75
column vectors
Property AACN, 612
complex numbers
Property AAM, 165
matrices
Property AIC, 75
matrices
Property AIM, 165
vectors
Property AIM, 165
vectors
Property AICN, 612
from scalar multiplication
theorem AISM, 259
additive inverses
column vectors
Property AIC, 75
matrices
Property AIM, 165
unique
theorem AIU, 258
vectors
Property AI, 252
additive closure
column vectors
Property ACC, 75
field
Property ACF, 718
matrices
Property ACM, 165
vectors
Property AC, 251
adjoint
definition A, 207
notation, 207
AHSAC (example), 52
AI (Property), 252
AIC (Property), 75
AICN (Property), 612
AIF (Property), 719
AIM (Property), 165
AISM (theorem), 259
AIU (theorem), 258
AIVLT (example), 456
ALT (example), 406
ALTMM (example), 488
AM (definition), 24
AM (example), 21
AM (notation), 24
AMAA (example), 24
AME (definition), 306
ANILT (example), 457
ANM (example), 545
AOS (example), 557
Archetype A
column space, 215
linearly dependent columns, 126
singular matrix, 61
solving homogeneous system, 53
system as linear combination, 81
archetype A
augmented matrix
definition A, 207
example AMAA, 24
Archetype B
column space, 216
inverse
definition A, 207
definition A, 207
example CMIAB, 194
linearly independent columns, 126
nonsingular matrix, 62
not invertible

739
example MWIAA, 189
solutions via inverse
example SABMI, 188
solving homogeneous system, 62
system as linear combination, 80
vector equality, 73
archetype B
solutions
example SAB, 29
Archetype C
homogeneous system, 52
Archetype D
column space, original columns, 214
solving homogeneous system, 63
vector form of solutions, 83
Archetype I
column space from row operations, 221
null space, 54
row space, 217
vector form of solutions, 90
Archetype I: casting out vectors, 141
Archetype L
null space span, linearly independent, 129
vector form of solutions, 92
ASC (example), 479
augmented matrix
notation, 24
AVR (example), 288

B (archetype), 638
B (definition), 294
B (section), 294
B (subsection, section B), 294
basis
columns nonsingular matrix
example CABAK, 299
common size
theorem BIS, 310
crazy vector space
example BC, 297
definition B, 294
matrices
example BM, 295
example BSM22, 296
polynomials
example BP, 295
example BPR, 321
example BSP4, 295
example SVP4, 322
subspace of matrices
example BDM22, 322

BDM22 (example), 322
best cities
money magazine
example MBC, 174
BIS (theorem), 128
BM (example), 295
BNM (subsection, section B), 299
BNS (theorem), 128
BP (example), 295
BPR (example), 321
BRLT (example), 448
BRS (theorem), 220
BS (theorem), 143
BSCV (subsection, section B), 297
BSM22 (example), 296
BSP4 (example), 295

C (archetype), 643
C (definition), 616
C (notation), 616
C (part), 2
C (Property), 251
C (technique, section PT), 621
CABAK (example), 299
CAEHW (example), 361
cancellation
vector addition
theorem VAC, 260
canonical form
nilpotent linear transformation
example CFNLT, 561
theorem CFNLT, 557
CAV (subsection, section O), 151
Cayley-Hamilton
theorem CHT, 603
CB (section), 515
CB (theorem), 516
CBM (example), 517
CBM (definition), 616
CBM (subsection, section CB), 516
CBP (example), 517
CC (Property), 75
CCCV (definition), 151
CCCV (notation), 151
CCM (definition), 108
CCM (example), 168
CCM (notation), 168
CCN (definition), 613
CCN (notation), 613
CCN (subsection, section CNO), 613
CCRA (theorem), 613
CCRM (theorem), 613
CCT (theorem), 613
CD (subsection, section DM), 338
Property C, 251
complex m-space
example VSCV, 253
complex arithmetic
example ACN, 611
complex number
conjugate
example CSCN, 613
modulus
example MSCN, 614
complex number
definition CCN, 613
modulus
definition MCN, 614
complex numbers
addition
definition CNA, 612
notation, 612
arithmetic properties
theorem PCNA, 612
equality
definition CNE, 612
notation, 612
multiplication
definition CNM, 612
notation, 612
complex vector space
dimension
theorem DCM, 310
composition
injective linear transformations
theorem CILTI, 435
surjective linear transformations
theorem CSLTS, 450
conjugate
addition
theorem CCRA, 613
column vector
definition CCCV, 151
matrix
definition CCM, 168
notation, 168
multiplication
theorem CCRM, 613
notation, 613
scalar multiplication
theorem CRSM, 151
twice
theorem CCT, 613
vector addition
theorem CRVA, 151
conjugate of a vector
notation, 151
conjugation
matrix addition
theorem CRMA, 168
matrix scalar multiplication
theorem CRMSM, 169
matrix transpose
theorem MCT, 169
consistent linear system
45
consistent linear systems
theorem CSRN, 46
consistent system
definition CS, 42
constructive proofs
technique C, 621
contradiction
technique CD, 623
contrapositive
technique CP, 623
converse
technique CV, 623
coordinates
orthonormal basis
theorem COB, 300
coordinatization
linear combination of matrices
example CM32, 481
linear independence
theorem CLI, 479
orthonormal basis
example CRONB3, 302
example CRONB4, 301
spanning sets
theorem CSS, 480
coordinatization principle, 481
coonordinatizing
polynomials
example CP2, 480
COV (example), 141
COV (subsection, section LDS), 141
CP (definition), 363
CP (subsection, section VR), 479
CP (technique, section PT), 623
CP2 (example), 480
CPMS3 (example), 363
crazy vector space
example CVSR, 479
properties
example PCVS, 259
CRMA (theorem), 168
CRMSM (theorem), 169
CRN (theorem), 313
CROB3 (example), 302
CROB4 (example), 301
CRS (section), 211
INDEX 743

CRS (subsection, section FS), 232
CRSM (theorem), 151
CRVA (theorem), 151
CS (definition), 42
CS (example), 616
CS (subsection, section TSS), 42
CSAA (example), 215
CSAB (example), 216
CSANS (example), 232
CSCN (example), 613
CSCS (theorem), 212
C SIP (example), 152
CSLT (subsection, section SLT), 450
CS LTS (theorem), 450
CSM (definition), 211
CSM (notation), 211
CSMCS (example), 211
CSMS (theorem), 274
CS NM (subsection, section CRS), 215
CSNM (theorem), 216
CSOC (example), 214
CSR (theorem), 46
CSROI (example), 221
CSRO (definition), 221
CSS (theorem), 480
CSS (subsection, section CRS), 211
CSSM (theorem), 260
CSSOC (subsection, section CRS), 213
CSTW (example), 213
CTD (subsection, section TD), 735
CTLT (example), 420
CUMOS (theorem), 206
CV (definition), 21
CV (notation), 22
CV (technique, section PT), 623
CVA (definition), 73
CVA (notation), 73
CVC (notation), 22
CVE (definition), 73
CVE (notation), 73
CVS (example), 255
CVS (subsection, section VR), 178
CVSM (definition), 74
CVSM (example), 74
CVSM (notation), 74
CVSM (theorem), 261
CVSR (example), 479

D (archetype), 647
D (chapter), 333
D (definition), 307
D (notation), 307
D (section), 307
D (subsection, section D), 307

D (subsection, section SD), 393
D (technique, section PT), 619
D33M (example), 338
DAB (example), 393
DC (example), 312
DC (technique, section PT), 625
DC (theorem), 394
DCM (theorem), 310
DCN (Property), 612
DCP (theorem), 383
DD (subsection, section DM), 337
DEC (theorem), 340
decomposition

technique DC, 625

DED (theorem), 398

definition
A, 207
AM, 23
AME, 366
B, 291
C, 616
CBM, 516
CCCV, 151
CCM, 168
CCN, 613
CM, 22
CNA, 612
CNE, 612
CNM, 612
CP, 363
CS, 42
CSM, 211
CV, 21
CVA, 73
CVE, 73
CVSM, 74
D, 307
DIM, 393
DM, 337
DS, 326
DZM, 393
EEF, 234
EELT, 515
EEM, 356
ELEM, 333
EM, 364
EO, 11
ES, 613
ESYS, 11
F, 718
GES, 568
GEV, 568
GME, 366
HM, 208
INDEX 744

HS, 52
IDLT, 456
IDV, 44
IE, 578
ILT, 426
IM, 62
IMP, 719
IP, 152
IS, 563
IVLT, 456
IVS, 461
JB, 550
JCF, 590
KLT, 429
LC, 269
LCCV, 79
Li, 280
LICV, 121
LNS, 231
LSMR, 23
LT, 405
LTA, 417
LTC, 419
LTM, 540
LTR, 572
LTSM, 418
M, 21
MA, 163
MCN, 614
ME, 163
MI, 189
MM, 176
MR, 485
MSM, 164
MVP, 173
NLT, 548
NM, 61
NOLT, 463
NOM, 312
NRML, 545
NSM, 54
NV, 155
ONS, 160
OSV, 157
OV, 156
pl, 415
REM, 25
RLD, 280
RLDCV, 121
RLT, 444
RO, 24
ROLT, 463
ROM, 313
RR, 32
RREF, 26
RSM, 217
S, 264
SC, 617
SE, 616
SET, 615
SL, 617
SIM, 390
SLE, 9
SLT, 440
SM, 337
SQM, 61
SS, 270
SSCV, 102
SSSET, 616
SU, 617
SUV, 190
SV, 23
SYM, 166
technique D, 619
TM, 166
TS, 268
TSHSE, 52
TTS, 284
UM, 205
UTM, 540
VOC, 22
VR, 173
VS, 251
VSCV, 72
VSM, 163
ZCV, 22
ZM, 166

DEHD (example), 398
DEM (theorem), 350
DEMM (theorem), 350
DEMS5 (example), 370
DER (theorem), 339
DERC (theorem), 347
determinant
computed two ways
example TCSD, 340
definition DM, 337
equal rows or columns
theorem DERC, 347
expansion, columns
theorem DEC, 340
expansion, rows
theorem DER, 339
identity matrix
theorem DIM, 349
matrix multiplication
theorem DRMM, 353
nonsingular matrix, 351
notation, 338
row or column multiple
  theorem DRCM, 346
row or column swap
  theorem DRCS, 345
size 2 matrix
  theorem DMST, 338
size 3 matrix
  example D33M, 338
transpose
  theorem DT, 340
via row operations
  example DRO, 348
zero
  theorem SMZD, 351
zero row or column
  theorem DZRC, 345
zero versus nonzero
  example ZNDAB, 352
determinant, upper triangular matrix
  example DUTM, 341
determinants
  elementary matrices
    theorem DEMMM, 350
DF (Property), 718
DFS (subsection, section PD), 324
DFS (theorem), 325
DGES (theorem), 590
diagonal matrix
  definition DIM, 393
diagonalizable
  definition DZM, 393
distinct eigenvalues
  example DEHD, 398
  theorem DED, 398
diagonalization
  Archetype B
    example DAB, 393
criteria
    theorem DC, 394
    example DMS3, 395
DIM (definition), 393
DIM (theorem), 349
dimension
  crazy vector space
    example DC, 312
  definition D, 307
notation, 307
polynomial subspace
  example DSP4, 312
proper subspaces
  theorem PSSD, 323
subspace
  example DSM22, 311
direct sum
  decomposing zero vector
    theorem DSZV, 327
definition DS, 326
dimension
  theorem DSD, 396
  example SDS, 326
  from a basis
    theorem DSFB, 326
  from one subspace
    theorem DSFOS, 327
  notation, 326
  zero intersection
    theorem DSZI, 328
direct sums
  linear independence
    theorem DSLI, 328
  repeated
    theorem RDS, 330
distributivity
  complex numbers
    Property DCN, 612
  field
    Property DF, 718
distributivity, matrix addition
  matrices
    Property DMAM, 165
distributivity, scalar addition
column vectors
    Property DSAC, 75
matrices
    Property DSAM, 165
  vectors
    Property DSA, 252
distributivity, vector addition
column vectors
    Property DVAC, 75
  vectors
    Property DVA, 252
DLDS (theorem), 139
DM (definition), 337
DM (notation), 338
DM (section), 333
DM (theorem), 311
DMAM (Property), 165
DMFE (theorem), 396
DMS3 (example), 395
INDEX 746

DMST (theorem), 338
DNLT (theorem), 553
DNMMM (subsection, section PDM), 351
DP (theorem), 311
DRCM (theorem), 346
DRCMA (theorem), 347
DRCS (theorem), 345
DRMM (theorem), 353
DRO (example), 348
DRO (subsection, section PDM), 345
DROEM (subsection, section PDM), 349
DS (definition), 326
DS (notation), 326
DS (subsection, section PD), 325
DSA (Property), 252
DSAC (Property), 75
DSAM (Property), 165
DSD (theorem), 329
DSFB (theorem), 326
DSFOS (theorem), 327
DSLI (theorem), 328
DSM22 (example), 311
DSP4 (example), 312
DSZI (theorem), 328
DSZV (theorem), 327
DT (theorem), 340
DUTM (example), 341
DVA (Property), 252
DVAC (Property), 75
DVS (subsection, section D), 310
DZM (definition), 393
DZRC (theorem), 345

E (archetype), 651
E (chapter), 356
E (technique, section PT), 622
ECEE (subsection, section EE), 365
EDELI (theorem), 378
EDYES (theorem), 323
EE (section), 356
EEE (subsection, section EE), 359
EEF (definition), 234
EEF (subsection, section FS), 234
EELT (definition), 515
EELT (subsection, section CB), 515
EEM (definition), 356
EEM (subsection, section EE), 356
EENS (example), 392
EER (theorem), 327
EHM (subsection, section PEE), 386

eigenspace
  as null space
    theorem EMNS, 364
  definition EM, 364

invariant subspace
  theorem EIS, 566
subspace
  theorem EMS, 364
eigenvalue
  algebraic multiplicity
    definition AME, 366
  complex
    example CEMS6, 368
  definition EEM, 356
  existence
    example CAEHW, 361
    theorem EMHE, 359
  geometric multiplicity
    definition GME, 366
index, 578
linear transformation
  definition EELT, 515
  multiplicities
    example EMMS4, 366
power
  theorem EOMP, 380
  root of characteristic polynomial
    theorem EMRCP, 363
scalar multiple
  theorem ESMM, 379
symmetric matrix
  example ESMS4, 367
  zero
    theorem SMZE, 379
eigenvalues
  building desired
    example BDE, 381
  complex, of a linear transformation
    example CELT, 532
  conjugate pairs
    theorem ERMCP, 382
  distinct
    example DEMS5, 370
    example SEE, 356
  Hermitian matrices
    theorem HMRE, 386
  inverse
    theorem EIM, 381
  maximum number
    theorem MNEM, 385
  multiplicities
    example HMEMS5, 367
    theorem ME, 384
  number
    theorem NEM, 383
  of a polynomial
    theorem EPM, 380
  size 3 matrix
example EMS3, 363
example ESMS3, 365
transpose
theorem ETM, 382
eigenvalues, eigenvectors
theorem EER, 527
eigenvector, 356
linear transformation, 515
eigenvectors, 357
conjugate pairs, 382
Hermitian matrices
theorem HMOE, 386
linear transformation
example ELTBM, 515
example ELTBP, 516
linearly independent
theorem EDELI, 378
of a linear transformation
example ELTT, 528
EILT (subsection, section ILT), 426
EIM (theorem), 381
EIS (example), 567
EIS (theorem), 566
ELEM (definition), 333
ELEM (notation), 334
elementary matrices
definition ELEM, 333
determinants
theorem DEM, 350
nonsingular
theorem EMN, 336
notation, 334
row operations
example EMRO, 334
theorem EMDRO, 334
ELIS (theorem), 320
ELTBM (example), 515
ELTBP (example), 516
ELTT (example), 528
EM (definition), 364
EM (subsection, section DM), 333
EMDRO (theorem), 334
EMHE (theorem), 358
EMMS4 (example), 366
EMMVP (theorem), 175
EMN (theorem), 336
EMNS (theorem), 364
EMP (theorem), 177
empty set, 615
notation, 615
EMRCACP (theorem), 363
EMRO (example), 334
EMS (theorem), 364
EMS3 (example), 363
ENLT (theorem), 553
EO (definition), 11
EOMP (theorem), 380
EOPSS (theorem), 12
EPM (theorem), 380
equal matrices
via equal matrix-vector products
theorem EMMVP, 175
equation operations
definition EO, 11
theorem EOPSS, 12
equivalence statements
technique E, 622
equivalences
technique ME, 624
equivalent systems
definition ESYS, 11
ERMCP (theorem), 382
ES (definition), 615
ES (notation), 615
ESEO (subsection, section SSLE), 11
ESLT (subsection, section SLT), 440
ESMM (theorem), 379
ESMS3 (example), 365
ESMS4 (example), 367
ESYS (definition), 11
ETM (theorem), 382
EVS (subsection, section VS), 252
example
AALC, 81
ABLC, 80
ABS, 102
ACN, 611
AHSAC, 52
AIVLT, 456
ALT, 406
ALTMM, 458
AM, 21
AMAA, 24
ANILT, 457
ANM, 545
AOS, 157
ASC, 479
AVR, 288
BC, 297
BDE, 381
BDM22, 322
BM, 295
BP, 295
BPR, 321
BRLT, 448
BSM22, 296
BSPT, 295
Version 1.04
INDEX 748

CABAK, 299
CAEHW, 361
CBCV, 520
CBP, 517
CCM, 168
CELT, 532
CEMS6, 368
CFNLT, 561
CFV, 47
CM32, 481
CMI, 192
CMIAB, 194
CNS1, 55
CNS2, 56
CNSV, 155
COV, 141
CP2, 480
CPMS3, 363
CROB3, 302
CROB4, 301
CS, 616
CSAA, 215
CSAB, 216
CSANS, 232
CSCN, 613
CSIP, 152
CSMCS, 211
CSOCD, 214
CSROI, 221
CSTW, 213
CTLT, 420
CVS, 255
CVSM, 74
CVSR, 479
D33M, 338
DAB, 393
DC, 312
DEHD, 398
DEMS5, 370
DMS3, 395
DRO, 348
DSM22, 311
DSP4, 312
DUTM, 341
EENS, 392
EIS, 567
ELTBM, 515
ELTBP, 516
ELTT, 528
EMMS4, 366
EMRO, 334
EMS3, 363
ESMS3, 365
ESMS4, 367

FDV, 44
FF8, 721
FRAN, 445
FS1, 241
FS2, 241
FSAG, 242
GE4, 570
GE6, 571
GENR6, 578
GSTV, 160
HISAA, 53
HISAD, 53
HMEM5, 367
HPDM, 399
HISAB, 52
IAP, 433
IAR, 427
IAS, 220
IAV, 428
ILTVR, 500
IM, 62
IM11, 719
IS, 15
ISJB, 568
ISMR4, 575
ISMR6, 576
ISSI, 43
IVSAV, 462
JB4, 550
JCF10, 591
KPNLT, 555
KVMR, 395
LCM, 269
LDCA, 126
LDHS, 124
LDP4, 310
LDRN, 125
LDS, 121
LIC, 283
LICAB, 126
LIHS, 123
LIM32, 252
LNSB, 127
LIP4, 280
LIS, 122
LLDS, 125
LNS, 231
LTDB1, 413
LTDB2, 414
LTDB3, 415
LTM, 409
LTPM, 407
LTPP, 408
LTRGE, 573
INDEX 751

FS (theorem), 237
FS1 (example), 241
FS2 (example), 241
FSAG (example), 242
FTMR (theorem), 487
FV (subsection, section TSS), 46
FVCS (theorem), 46

G (archetype), 659
G (theorem), 320
GE4 (example), 570
GE6 (example), 571
GEE (subsection, section IS), 568
GEK (theorem), 569
generalized eigenspace
as kernel
  theorem GEK, 569
  definition GES, 568
dimension
  theorem DGES, 590
dimension 4 domain
  example GE4, 570
dimension 6 domain
  example GE6, 571
invariant subspace
  theorem GESIS, 569
nilpotent restriction
  theorem RGEN, 578
nilpotent restrictions, dimension 6 domain
  example GENR6, 578
notation, 569
generalized eigenspace decomposition
  theorem GESD, 584
generalized eigenvector
  definition GEV, 568
GENR6 (example), 578
GES (definition), 568
GES (notation), 569
GESD (subsection, section JCF), 584
GESD (theorem), 584
GESIS (theorem), 569
GEV (definition), 568
GFDL (appendix), 711
GME (definition), 566
goldilocks
  theorem G, 320
Gram-Schmidt
  column vectors
    theorem GSPPCV, 158
  three vectors
    example GSTV, 160
gram-schmidt
  mathematica, 607
GS (technique, section PT), 621

GSP (subsection, section O), 158
GSP.MMA (computation, section MMA), 607
GSPCV (theorem), 158
GSTV (example), 160
GT (subsection, section PD), 320

H (archetype), 663
  hermitian
    definition HM, 208
    HISAA (example), 53
    HISAD (example), 53
    HM (definition), 208
    HMEM5 (example), 367
    HMOE (theorem), 386
    HMRE (theorem), 386
    HMVEI (theorem), 54
homogeneous system
  consistent
    theorem HSC, 52
    definition HS, 52
  infinitely many solutions
    theorem HMVEI, 54
homogeneous systems
  linear independence, 123
homogenous system
  Archetype C
    example AHSAC, 52
HPDM (example), 399
HS (definition), 52
HSC (theorem), 52
HSE (section), 52
HUSAB (example), 52

I (archetype), 667
I (technique, section PT), 626
IAP (example), 433
IAR (example), 427
IAS (example), 220
IAV (example), 428
ICBM (theorem), 517
ICLT (theorem), 460
identities
  technique PI, 625
identity matrix
  determinant, 349
  example IM, 62
  notation, 62
IDLT (definition), 456
IDV (definition), 44
IE (definition), 578
IE (notation), 578
IFDVS (theorem), 479
IILT (theorem), 459
ILT (definition), 426
ILT (section), 426

Version 1.04
ILTB (theorem), 433
ILTD (subsection, section ILT), 434
ILTD (theorem), 434
ILTL (theorem), 433
ILTL (subsection, section ILT), 433
ILTL (theorem), 433
ILTVR (example), 500
IM (definition), 62
IM (example), 62
IM (notation), 62
IM (subsection, section MISLE), 189
IM11 (example), 719
IMILT (theorem), 501
IMP (definition), 719
IMR (theorem), 499
inconsistent linear systems
  theorem ISRN, 46
independent, dependent variables
  definition IDV, 44
index
  example SM2Z7, 720
  example SSET, 615
index
  example CSIP, 154
  example ISJJB, 568
  example TIS, 565
  example ISJB, 568
  example EIS, 567
  example TIS, 565
  theorem KPIS, 567
induction
  technique I, 626
infinite solution set
  example ISSI, 43
infinite solutions, $3 \times 4$
  example IS, 15
injective
  example IAP, 433
  example IAR, 427
not
  example NIAO, 432
  example NIAQ, 426
  example NIAQR, 432
not, by dimension
  example NIDAU, 434
polynomials to matrices
  example IAV, 428
injective linear transformation
  bases
    theorem ILTB, 433
injective linear transformations
  dimension
    theorem ILTD, 434
inner product
  anti-commutative
    theorem IPAC, 154
  example CSIP, 154
  norm
    theorem IPN, 152
  notation, 152
  positive
    theorem PIP, 156
  scalar multiplication
    theorem IPSM, 153
  vector addition
    theorem IPVA, 153
integers
  mod $p$
    definition IMP, 719
    mod $p$, field
      theorem FIMP, 719
    mod 11
      example IM11, 719
invariant subspace
  definition IS, 565
  eigenspace, 566
  eigenspaces
    example EIS, 567
    example TIS, 565
  Jordan block
    example ISJB, 568
  kernels of powers
    theorem KPIS, 567
inverse
  composition of linear transformations
    theorem ICLT, 460
    example CMI, 192
  example MI, 190
  notation, 189
  of a matrix, 189
invertible linear transformation
  defined by invertible matrix
    theorem IMILT, 501
invertible linear transformations
  composition
    theorem CIVLT, 460
IP (definition), 152
IP (notation), 152
IP (subsection, section O), 152
IPAC (theorem), 154
IPN (theorem), 155
IPSM (theorem), 153
IPVA (theorem), 153
IS (definition), 565
IS (example), 15
IS (section), 565
IS (subsection, section IS), 565
ISJJB (example), 568
ISMR4 (example), 575
ISMR6 (example), 576
isomorphic
  multiple vector spaces
     example MIVS, 479
  vector spaces
     example IVSAV, 462
isomorphic vector spaces
dimension
  theorem IVSED, 462
  example TIVS, 478
ISRN (theorem), 46
ISSI (example), 43
ITMT (theorem), 541
IV (subsection, section IVLT), 459
IVLT (definition), 456
IVLT (section), 456
IVLT (subsection, section IVLT), 456
IVLT (subsection, section MR), 499
IVS (definition), 461
IVSAV (example), 462
IVSED (theorem), 462
J (archetype), 671
JB (definition), 550
JB (notation), 550
JB4 (example), 550
JCF (definition), 590
JCF (section), 582
JCF (subsection, section JCF), 590
JCF10 (example), 591
JCFLT (theorem), 590
Jordan block
  definition JB, 550
  nilpotent
     theorem NJB, 552
     notation, 550
  size 4
     example JB4, 550
Jordan canonical form
  definition JCF, 590
  size 10
     example JCF10, 591
K (archetype), 676
kernel
  injective linear transformation
     theorem KILT, 432
  isomorphic to null space
     theorem KNSI, 495
linear transformation
  example NKAO, 429
  notation, 429
  of a linear transformation
     definition KLT, 429
pre-image, 131
subspace
  theorem KLTS, 430
  trivial
     example TKAP, 430
  via matrix representation
     example KVMR, 495
KILT (theorem), 432
KLT (definition), 429
KLT (notation), 429
KLT (subsection, section ILT), 429
KLTS (theorem), 430
KNSI (theorem), 495
KPI (theorem), 431
KPIS (theorem), 567
KPLT (theorem), 554
KPNLT (example), 555
KPNLT (theorem), 555
KVMR (example), 495
L (archetype), 680
L (technique, section PT), 620
LA (subsection, section WILA), 2
LC (definition), 269
LC (section), 79
LC (subsection, section LC), 79
LC (technique, section PT), 627
LCCV (definition), 79
LCM (example), 269
LDCAA (example), 126
LDHS (example), 124
LDP4 (example), 310
LDRN (example), 125
LDS (example), 121
LDS (section), 139
LDSS (subsection, section LDS), 139
left null space
  as row space, 237
  definition LNS, 231
  example LNS, 231
  notation, 231
  subspace
     theorem LNSMS, 274
lemma
  technique LC, 627
LI (definition), 280
LI (section), 121
LI (subsection, section LISS), 280
LIC (example), 283
LICAB (example), 126
LICV (definition), 121
LIHS (example), 123
LIM32 (example), 282
linear combination
system of equations
  example ABLC, 80
  definition LC, 269
  definition LCCV, 79
  example TLC, 79
linear transformation, 413
  matrices
  example LCM, 269
  system of equations
  example AALC, 81
linear combinations
  solutions to linear systems
  theorem SLSLC, 82
linear dependence
  more vectors than size
  theorem MVSLD, 126
linear independence
  definition LI, 280
  definition LICV, 121
homogeneous systems
  theorem LIVHS, 123
injective linear transformation
  theorem ILTLI, 433
  matrices
  example LIM32, 282
orthogonal, 157
  r and n
  theorem LIVRN, 125
linear solve
  mathematica, 604
linear system
  consistent
  theorem RCLS, 45
matrix representation
  definition LSMR, 23
notation, 23
linear systems
  notation
  example MNSLE, 174
  example NSLE, 23
linear transformation
  polynomials to polynomials
  example LTPP, 408
  addition
  definition LTA, 417
  theorem MLTLT, 418
  theorem SLTLT, 417
as matrix multiplication
  example ALTMM, 488
basis of range
  example BRLT, 448
checking
  example ALT, 406
composition
  definition LTC, 419
  theorem CLTLT, 419
defined by a matrix
  example LTM, 409
defined on a basis
  example LTDB1, 413
  example LTDB2, 414
  example LTDB3, 415
  theorem LTDB, 413
definition LT, 405
  identity
  definition IDLT, 456
  injection
  definition ILT, 426
inverse
  theorem ILTLT, 458
  inverse of inverse
  theorem IILT, 459
invertible
  definition IVLT, 456
  example AIVLT, 456
invertible, injective and surjective
  theorem ILTIS, 459
Jordan canonical form
  theorem JCFLT, 590
kernels of powers
  theorem KPLT, 554
linear combination
  theorem LTLC, 413
matrix of
  example MFLT, 410
  example MOLT, 412
not
  example NLT, 407
not invertible
  example ANILT, 457
notation, 405
polynomials to matrices
  example LTPM, 407
rank plus nullity
  theorem RPND, 164
restriction
  definition LTR, 572
notation, 573
scalar multiple
  example SMLT, 419
scalar multiplication
  definition LTSM, 418
spanning range
  theorem SSRLT, 448
sum
  example STLT, 418
surjection
  definition SLT, 440
vector space of, 419
zero vector
thm. LTTZZ, 408
linear transformation inverse
via matrix representation
exmpl. ILTVR, 500
linear transformation restriction
on generalization eigenspace
exmpl. LTRGE, 573
linear transformations
compositions
exmpl. CTLT, 420
from matrices
thm. MBLT, 410
linearly dependent
$r < n$
exmpl. LDRN, 125
via homogeneous system
exmpl. LDHS, 124
linearly dependent columns
Archetype A
exmpl. LDCAA, 126
linearly dependent set
exmpl. LDS, 121
linear combinations within
theor. DLDS, 139
polynomials
exmpl. LDP4, 310
linearly independent
crazy vector space
exmpl. LIC, 283
extending sets
thm. ELIS, 320
polynomials
exmpl. LIP4, 280
via homogeneous system
exmpl. LIHS, 123
linearly independent columns
Archetype B
exmpl. LICAB, 126
linearly independent set
exmpl. LIS, 122
exmpl. LLDS, 125
LINM (subsect. sect. LI), 126
LNSB (exmpl.), 127
LIP4 (exmpl.), 280
LIS (exmpl.), 122
LISS (sect.), 280
LISV (subsect. sect. LI), 121
LIVHS (thm.), 123
LIVRN (thm.), 125
LLDS (exmpl.), 125
LNS (def.), 231
LNS (exmpl.), 231
LNS (subsect. sect. FS), 231
LNSMS (thm.), 274
lt. triangular matrix
def. LTM, 540
LS.MMA (computation sect. MMA), 604
LSMR (def.), 23
LSMR (not.), 23
LT (chap.), 405
LT (def.), 405
LT (not.), 405
LT (sect.), 405
LT (subsect. sect. LT), 405
LTR (def.), 417
LTRC (def.), 419
LTDB (thm.), 413
LTDB1 (exmpl.), 413
LTDB2 (exmpl.), 414
LTDB3 (exmpl.), 415
LTLT (subsect. sect. LT), 413
LTLT (def.), 413
LTM (def.), 540
LTM (exmpl.), 409
LTPM (exmpl.), 407
LTPP (exmpl.), 408
LTR (def.), 572
LTR (not.), 573
LTRGE (exmpl.), 573
LTSM (exmpl.), 418
LTTZZ (thm.), 408
M (arch.), 683
M (chap.), 163
M (def.), 21
M (not.), 21
MA (def.), 163
MA (exmpl.), 164
MA (not.), 164
MACN (Prop.), 612
MAF (Prop.), 718
mathematica
gs. (computation), 607
ls. (computation), 604
mat. entry (computation), 604
mat. inverse (computation), 608
mat. multiplication (computation), 608
null sp. (computation), 605
row reduce (computation), 604
t. of a mat. (computation), 607
vec. form of solns. (computation), 606
vec. linear combinations (computation), 605
math. language
technique L, 620
mat.
addition  
definition MA, 163  
notation, 164  
augmented  
definition AM, 24  
column space  
definition CSM, 211  
complex conjugate  
definition CCM, 168  
equality  
definition ME, 163  
notation, 163  
example AM, 21  
identity  
definition IM, 62  
inverse  
definition MI, 189  
nonsingular  
definition NM, 61  
notation, 21  
of a linear transformation  
theorem MLTCV, 411  
product  
example PTM, 176  
example PTMEE, 178  
product with vector  
definition MVP, 173  
rectangular, 61  
row space  
definition RSM, 217  
scalar multiplication  
definition MSM, 164  
notation, 164  
singular, 61  
square  
definition SQM, 61  
submatrices  
example SS, 337  
submatrix  
definition SM, 337  
symmetric  
definition SYM, 166  
transpose  
definition TM, 166  
unitary  
definition UM, 205  
unitary is invertible  
theorem UMI, 205  
zero  
definition ZM, 166  
matrix addition  
example MA, 164  
matrix components  
notation, 21  
matrix entry  
mathematica, 604  
ti83, 609  
ti86, 608  
matrix inverse  
Archetype B, [194]  
computation  
thm CINM, 193  
mathematica, 608  
nonsingular matrix  
thm NI, 204  
of a matrix inverse  
thm MIMI, 196  
one-sided  
thm OSIS, 203  
product  
thm SS, 195  
scalar multiple  
thm MISM, 196  
size 2 matrices  
thm TTMI, 191  
transpose  
thm MIT, 196  
uniqueness  
thm MIU, 195  
matrix multiplication  
assciativity  
thm MMA, 180  
complex conjugation  
thm MMCC, 181  
definition MM, 176  
distributivity  
thm MMDAA, 179  
entry-by-entry  
thm EMP, 177  
identity matrix  
thm MIMI, 179  
inner product  
thm MMIP, 181  
mathematica, 608  
noncommutative  
example MMNC, 177  
scalar matrix multiplication  
thm MMSMM, 180  
systems of linear equations  
thm SLEMM, 173  
transposes  
thm MMT, 182  
zero matrix  
thm MMZM, 178  
matrix product  
as composition of linear transformations  
example MPMR, 492
matrix representation
  basis of eigenvectors
    example MRBE, 524
composition of linear transformations
  theorem MRCLT, 491
  definition MR, 485
invertible
  theorem IMR, 499
multiple of a linear transformation
  theorem MRMLT, 491
restriction to generalized eigenspace
  theorem MRRGE, 580
sum of linear transformations
  theorem MRSLT, 490
theorem FTMR, 487
upper triangular
  theorem UTMR, 541, 582
matrix representations
  converting with change-of-basis
    example MRCM, 522
    example OLTTR, 485
matrix scalar multiplication
  example MSM, 164
matrix vector space
  dimension
    theorem DM, 311
matrix-vector product
  example MTV, 173
  notation, 173
MBC (example), 174
MBLT (theorem), 410
MC (notation), 21
MCC (subsection, section MO), 168
MCCN (Property), 612
MCF (Property), 718
MCN (definition), 614
MCN (subsection, section CNO), 614
MCNM (example), 212
MCT (theorem), 169
MD (chapter), 726
ME (definition), 163
ME (notation), 163
ME (subsection, section PEE), 383
ME (technique, section PT), 624
ME (theorem), 384
ME.MMA (computation, section MMA), 604
ME.TI83 (computation, section TI83), 609
ME.TI86 (computation, section TI86), 608
MEASM (subsection, section MO), 163
MFLT (example), 410
MI (definition), 189
MI (example), 190
MI (notation), 189
MI.MMA (computation, section MMA), 608
MICN (Property), 613
MIF (Property), 719
MIMI (theorem), 196
MINM (section), 202
MISLE (section), 188
MISM (theorem), 196
MIT (theorem), 196
MIU (theorem), 195
MIVS (example), 479
MLT (subsection, section LT), 409
MLTCTV (theorem), 411
MLTCLT (theorem), 418
MM (definition), 176
MM (section), 173
MM (subsection, section MM), 176
MM.MMA (computation, section MMA), 608
MMA (section), 604
MMA (theorem), 180
MMCC (theorem), 181
MMDAA (theorem), 179
MMEE (subsection, section MM), 177
MMIM (theorem), 179
MMIP (theorem), 181
MMNC (example), 177
MMSMM (theorem), 180
MMT (theorem), 182
MMZM (theorem), 178
MNEM (theorem), 385
MNSLE (example), 174
MO (section), 163
MOLT (example), 412
more variables than equations
  example OSGMD, 48
  theorem CMVEI, 47
MPMR (example), 492
MR (definition), 485
MR (section), 485
MRBE (example), 524
MRCB (theorem), 521
MRCLT (theorem), 491
MRCM (example), 522
MRMLT (theorem), 491
MRRGE (theorem), 580
MRS (subsection, section CB), 521
MRSLT (theorem), 490
MSCN (example), 614
MSM (definition), 164
MSM (example), 164
MSM (notation), 164
MVT (example), 173
multiplicative associativity
  complex numbers
    Property MACN, 612
multiplicative closure

Version 1.04
field
  Property MCF, 718
multiplicative commutativity
  complex numbers
  Property MCCN, 612
multiplicative inverse
  complex numbers
  Property MICN, 613
MVNSE (subsection, section RREF), 21
MVP (definition), 173
MVP (notation), 173
MVP (subsection, section MM), 173
MVSLD (theorem), 126
MWIAIA (example), 189
N (archetype), 685
N (subsection, section O), 154
N (technique, section PT), 622
NDMS4 (example), 397
negation of statements
  technique N, 622
NEM (theorem), 383
NI (theorem), 204
NIAO (example), 432
NIAQ (example), 426
NIAQR (example), 432
NIDAU (example), 434
nilpotent
  linear transformation
    definition NLT, 548
NJB (theorem), 552
NJB5 (example), 550
NKAO (example), 429
NLT (definition), 548
NLT (example), 407
NLT (section), 548
NLT (subsection, section NLT), 548
NLTO (subsection, section LT), 417
NM (definition), 61
NM (example), 62
NM (section), 61
NM (subsection, section NM), 61
NM (subsection, section OD), 545
NM62 (example), 549
NM64 (example), 548
NM83 (example), 551
NME1 (theorem), 66
NME2 (theorem), 127
NME3 (theorem), 204
NME4 (theorem), 216
NME5 (theorem), 299
NME6 (theorem), 315
NME7 (theorem), 352
NME8 (theorem), 379
NME9 (theorem), 502
NMI (subsection, section MINM), 202
NMLIC (theorem), 126
NMPEM (theorem), 337
NMRRI (theorem), 62
NMTNS (theorem), 64
NMUS (theorem), 64
NOILT (theorem), 463
NOLT (definition), 463
NOLT (notation), 463
NOM (definition), 312
NOM (notation), 312
nonsingular
  columns as basis
    theorem CNMB, 299
nonsingular matrices
  linearly independent columns
    theorem NMLIC, 126
nonsingular matrix
  Archetype B
    example NM, 62
  column space, 216
elementary matrices
  theorem NMPEM, 337
equivalences
  theorem NME1, 66
  theorem NME2, 127
  theorem NME3, 204
  theorem NME4, 216
  theorem NME5, 299
  theorem NME6, 315
  theorem NME7, 352
  theorem NME8, 379
  theorem NME9, 502
matrix inverse, 204
null space
  example NSNM, 63
  nullity, 314
product of nonsingular matrices
  theorem NPNT, 202
rank
  theorem RNNM, 314
row-reduced
  theorem NMRRI, 62
trivial null space
  theorem NMTNS, 64
unique solutions
  theorem NMUS, 64
nonsingular matrix, row-reduced
  example NSR, 63
norm
  example CNSV, 155
  inner product, 155
  notation, 155

Version 1.04
normal matrix
  definition NRML, 545
  example ANM, 545

notation
  A, 207
  AM, 24
  C, 616
  CCCV, 151
  CCM, 168
  CCN, 613
  CNA, 612
  CNE, 612
  CNM, 612
  CSM, 211
  CV, 22
  CVA, 73
  CVC, 22
  CVE, 73
  CVSM, 74
  D, 307
  DM, 338
  DS, 326
  ELEM, 334
  ES, 615
  GES, 569
  IE, 578
  IM, 62
  IP, 152
  JB, 550
  KLT, 429
  LNS, 231
  LSMR, 23
  LT, 405
  LTR, 573
  M, 21
  MA, 164
  MC, 21
  ME, 163
  MI, 189
  MSM, 164
  MVP, 173
  NOLT, 463
  NOM, 312
  NSM, 54
  NV, 155
  RLT, 444
  RO, 25
  ROLT, 463
  ROM, 313
  RRFA, 27
  RSM, 217
  SC, 617
  SE, 616
  SETM, 615

null space
  as a span
    example NSDS, 109

null space
  Archetype I
    example NSEAI, 54
  basis
    theorem BNS, 128
  computation
    example CNS1, 55
    example CNS2, 96
  isomorphic to kernel, 495
  linearly independent basis
    example LINSB, 127
  matrix
theorem DP, 311
practice
technique P, 627
pre-image
definition PI, 415
kernel
theorem KPI, 431
pre-images
element SPIAS, 415
product of triangular matrices
theorem PTMT, 540
Property
AA, 251
AAC, 75
AACN, 612
AAM, 165
AC, 251
ACC, 75
ACCN, 612
ACF, 718
ACM, 165
AI, 252
AIC, 75
AICN, 612
AIF, 719
AIM, 165
C, 251
CC, 75
CM, 165
DCN, 612
DF, 718
DMAM, 165
DSA, 252
DSAC, 75
DSAM, 165
DVA, 252
DVAC, 75
MACN, 612
MAF, 718
MCCN, 612
MCF, 718
MICN, 613
MIF, 719
O, 252
OC, 75
OCN, 612
OF, 719
OM, 165
SC, 251
SCC, 75
SCM, 165
SMA, 252
SMAC, 75
SMAM, 165
Z, 251
ZC, 75
ZCN, 612
ZF, 719
ZM, 165
PSHS (example), 94
PSHS (subsection, section LC), 93
PSM (subsection, section SD), 391
PSPHS (theorem), 93
PSS (subsection, section SSLE), 10
PSSD (theorem), 323
PSSLS (theorem), 47
PT (section), 619
PTM (example), 176
PTMEE (example), 178
PTMT (example), 540
Q (archetype), 692
R (archetype), 695
R (chapter), 473
range
full
element FRAN, 445
isomorphic to column space
theorem RCSI, 497
linear transformation
element RAO, 444
notation, 444
of a linear transformation
definition RLT, 444
pre-image
theorem RPI, 449
subspace
theorem RLTS, 445
surjective linear transformation
theorem RSLT, 446
via matrix representation
element RVMR, 498
rank
computing
theorem CRN, 313
linear transformation
definition ROLT, 463
matrix
definition ROM, 313
element RNM, 313
notation, 313 463
of transpose
element RRTI, 324
square matrix
element RNSM, 314
surjective linear transformation
theorem ROSLT, 463

Version 1.04
row space
Archetype I
  example RSAI, 217
as column space, 221
  basis
    example RSB, 297
    theorem BRS, 220
  matrix, 217
  notation, 217
row-equivalent matrices
  theorem REMRS, 218
subspace
  theorem RSMS, 274
row-equivalent matrices
  definition REM, 25
  example TREM, 25
row space, 218
  row spaces
    example RSREM, 219
  theorem REMES, 25
row-reduce
  the verb
    definition RR, 32
row-reduced matrices
  theorem REMEF, 27
  RPI (theorem), 449
  RPNC (theorem), 314
  RPND (theorem), 464
RR (definition), 32
  RR.MMA (computation, section MMA), 604
  RR.TI83 (computation, section TI83), 610
  RR.TI86 (computation, section TI86), 609
RREF (definition), 26
  RREF (example), 27
  RREF (section), 21
  RREF (subsection, section RREF), 26
RREFA (notation), 27
  RREFN (example), 42
  RREFU (theorem), 96
RRTI (example), 324
RS (example), 298
RSAI (example), 217
RSB (example), 297
RSC5 (example), 140
RSLT (theorem), 446
RSM (definition), 217
  (notation), 217
  (subsection, section CRS), 217
RSMS (theorem), 274
RSNS (example), 268
RSREM (example), 219
RSSC4 (example), 145
RT (subsection, section PD), 323
RVMR (example), 498
S (archetype), 698
  (definition), 264
S (example), 61
S (section), 264
SAA (example), 30
SAB (example), 29
SABMI (example), 188
SAE (example), 31
SAN (example), 447
SAR (example), 441
SAV (example), 442
SC (definition), 617
SC (example), 618
  (notation), 617
SC (Property), 251
SC (subsection, section S), 274
SC (subsection, section SET), 616
SC3 (example), 264
SCAA (example), 104
SCAB (example), 106
SCAD (example), 110
scalar closure
  column vectors
    Property SCC, 75
  matrices
    Property SCM, 165
  vectors
    Property SC, 251
scalar multiple
  matrix inverse, 196
scalar multiplication
  canceling scalars
    theorem CSSM, 260
  canceling vectors
    theorem CVSM, 261
  zero scalar
    theorem ZSSM, 258
  zero vector
    theorem ZVSM, 258
  zero vector result
    theorem SMEZV, 259
scalar multiplication associativity
  column vectors
    Property SMAC, 75
  matrices
    Property SMAM, 165
  vectors
    Property SMA, 252
SCB (theorem), 524
SCC (Property), 75
SCM (Property), 165
SD (section), 390
SDS (example), 326
SE (definition), 616
set
  cardinality
    definition C, 616
    example CS, 616
    notation, 616
  complement
    definition SC, 617
    example SC, 618
    notation, 617
  definition SET, 615
  empty
    definition ES, 615
  equality
    definition SE, 616
    notation, 616
  intersection
    definition SI, 617
    example SI, 617
    notation, 617
  membership
    example SETM, 615
    notation, 615
  size, 616
  subset, 615
  union
    definition SU, 617
    example SU, 617
    notation, 617
  SET (definition), 615
  SET (section), 615
  SETM (example), 615
  SETM (notation), 615
  shoes, 195
  SHS (subsection, section HSE), 52
  SI (definition), 617
  SI (example), 617
  SI (notation), 617
  SI (subsection, section IVLT), 461
  SIM (definition), 390

similar matrices
  equal eigenvalues
    example EENS, 392
  equal eigenvalues
    theorem SMEE, 392
  example SMS3, 391
  example SMS5, 390

similarity
  definition SIM, 390
  equivalence relation
    theorem SER, 391

singular matrix
  Archetype A
    example S, 61
  null space
    example NSS, 63
  singular matrix, row-reduced
    example SRR, 62

SLE (chapter), 2
SLE (definition), 9
SLE (subsection, section SSLE), 9
SLELT (subsection, section IVLT), 466
SLEMM (theorem), 173
SLSLC (theorem), 82
SLT (definition), 440
SLT (section), 440
SLTB (theorem), 449
SLTD (subsection, section SLT), 450
SLTD (theorem), 450
SLTLLT (theorem), 417
SM (definition), 337
SM (notation), 337
SM (subsection, section SD), 390
SM2Z7 (example), 720
SM32 (example), 272
SMA (Property), 252
SMAC (Property), 75
SMAM (Property), 165
SME (theorem), 392
SMEZV (theorem), 259
SMLT (example), 419
SMS (theorem), 167
SMS3 (example), 391
SMS5 (example), 390
SMZD (theorem), 351
SMZE (theorem), 379
SNCM (theorem), 204
SO (subsection, section SET), 617
socks, 195
SOL (subsection, section B), 305
SOL (subsection, section CB), 537
SOL (subsection, section CRS), 227
SOL (subsection, section D), 318
SOL (subsection, section DM), 344
SOL (subsection, section EE), 374
SOL (subsection, section F), 725
SOL (subsection, section FS), 247
SOL (subsection, section HSE), 59
SOL (subsection, section ILT), 438
SOL (subsection, section IVLT), 470
SOL (subsection, section LC), 100
SOL (subsection, section LDS), 149
SOL (subsection, section LI), 134
SOL (subsection, section LISS), 291
SOL (subsection, section LT), 424
SOL (subsection, section MINM), 210
SOL (subsection, section MISLE), 200
SOL (subsection, section MM), 186
SOL (subsection, section MO), 172
SOL (subsection, section MR), 307
SOL (subsection, section NM), 70
SOL (subsection, section PD), 332
SOL (subsection, section PDM), 355
SOL (subsection, section PEE), 389
SOL (subsection, section RREF), 37
SOL (subsection, section S), 277
SOL (subsection, section SD), 402
SOL (subsection, section SLT), 454
SOL (subsection, section SS), 115
SOL (subsection, section SSLE), 19
SOL (subsection, section TSS), 51
SOL (subsection, section VO), 78
SOL (subsection, section VR), 484
SOL (subsection, section WILA), 8
solution set
Archetype A
example SAA, 30
archetype E
example SAE, 31
theorem PSPHS, 93
solution sets
possibilities
theorem PSSLS, 47
solution vector
definition SV, 23
solving homogeneous system
Archetype A
example HISAA, 53
Archetype B
example HUSAB, 52
Archetype D
example HISAD, 53
solving nonlinear equations
example STNE, 9
SP4 (example), 266
span
basic
eexample ABS, 102
basis
theorem BS, 143
definition SS, 270
definition SSCV, 102
improved
eexample IAS, 220
notation, 102
reducing
eexample RSSC4, 145
reduction
eexample RS, 298
removing vectors
eexample COV, 141
reworking elements
eexample RES, 146
set of polynomials
eexample SSP, 271
subspace
theorem SSS, 271
span of columns
Archetype A
example SCAA, 104
Archetype B
example SCAB, 106
Archetype D
example SCAD, 110
spanning set
crazy vector space
example SSC, 287
definition TSVS, 284
matrices
example SSM22, 286
more vectors
theorem SSLD, 307
polynomials
example SSP4, 284
SPIAS (example), 415
SQM (definition), 61
SRR (example), 62
SS (definition), 270
SS (example), 337
SS (section), 102
SS (subsection, section LISS), 284
SS (theorem), 195
SSC (example), 287
SSCV (definition), 102
SSET (definition), 615
SSET (example), 615
SSET (notation), 615
SSLD (theorem), 307
SSLE (section), 9
SSM22 (example), 286
SSNS (example), 108
SSNS (subsection, section SS), 107
SSNS (theorem), 107
SSP (example), 271
SSP4 (example), 284
SSRLT (theorem), 448
SSS (theorem), 270
SSSSLT (subsection, section SLT), 448
SSV (notation), 102
SSV (subsection, section SS), 102
starting proofs
technique GS, 621
STLT (example), 418
<table>
<thead>
<tr>
<th>ISRN</th>
<th>46</th>
<th>NMPEM</th>
<th>337</th>
</tr>
</thead>
<tbody>
<tr>
<td>ITMT</td>
<td>541</td>
<td>NMRI</td>
<td>62</td>
</tr>
<tr>
<td>IVSED</td>
<td>462</td>
<td>NMTNS</td>
<td>64</td>
</tr>
<tr>
<td>JCFIT</td>
<td>590</td>
<td>NMTS</td>
<td>64</td>
</tr>
<tr>
<td>KILT</td>
<td>432</td>
<td>NOILT</td>
<td>463</td>
</tr>
<tr>
<td>KLTS</td>
<td>430</td>
<td>NPNT</td>
<td>202</td>
</tr>
<tr>
<td>KNSI</td>
<td>495</td>
<td>NSMS</td>
<td>268</td>
</tr>
<tr>
<td>KPL</td>
<td>431</td>
<td>OBLT</td>
<td>544</td>
</tr>
<tr>
<td>KPS</td>
<td>567</td>
<td>OBLT</td>
<td>544</td>
</tr>
<tr>
<td>KPLT</td>
<td>554</td>
<td>OSIS</td>
<td>205</td>
</tr>
<tr>
<td>KPNLT</td>
<td>555</td>
<td>OSLI</td>
<td>157</td>
</tr>
<tr>
<td>LIVHS</td>
<td>123</td>
<td>PCNA</td>
<td>612</td>
</tr>
<tr>
<td>LIVRN</td>
<td>125</td>
<td>PEEF</td>
<td>236</td>
</tr>
<tr>
<td>LNSMS</td>
<td>274</td>
<td>PIP</td>
<td>156</td>
</tr>
<tr>
<td>LTDB</td>
<td>413</td>
<td>PSPHS</td>
<td>93</td>
</tr>
<tr>
<td>LTLC</td>
<td>413</td>
<td>PSSD</td>
<td>323</td>
</tr>
<tr>
<td>LTZZ</td>
<td>408</td>
<td>PSSL</td>
<td>47</td>
</tr>
<tr>
<td>MBLT</td>
<td>410</td>
<td>PTTMT</td>
<td>540</td>
</tr>
<tr>
<td>MCT</td>
<td>169</td>
<td>RCLS</td>
<td>45</td>
</tr>
<tr>
<td>ME</td>
<td>384</td>
<td>RCSI</td>
<td>497</td>
</tr>
<tr>
<td>MIMI</td>
<td>196</td>
<td>RDS</td>
<td>330</td>
</tr>
<tr>
<td>MISM</td>
<td>196</td>
<td>REMEF</td>
<td>27</td>
</tr>
<tr>
<td>MIT</td>
<td>196</td>
<td>REMES</td>
<td>25</td>
</tr>
<tr>
<td>MIU</td>
<td>195</td>
<td>REMRS</td>
<td>218</td>
</tr>
<tr>
<td>MLTCV</td>
<td>411</td>
<td>RGEN</td>
<td>578</td>
</tr>
<tr>
<td>MLTLT</td>
<td>418</td>
<td>RLTS</td>
<td>445</td>
</tr>
<tr>
<td>MMA</td>
<td>180</td>
<td>RMRT</td>
<td>324</td>
</tr>
<tr>
<td>MMCC</td>
<td>181</td>
<td>RNMM</td>
<td>314</td>
</tr>
<tr>
<td>MMDAA</td>
<td>179</td>
<td>ROD</td>
<td>726</td>
</tr>
<tr>
<td>MMIM</td>
<td>179</td>
<td>ROSLT</td>
<td>463</td>
</tr>
<tr>
<td>MMIP</td>
<td>181</td>
<td>RPI</td>
<td>449</td>
</tr>
<tr>
<td>MMSMM</td>
<td>180</td>
<td>RPNC</td>
<td>314</td>
</tr>
<tr>
<td>MMT</td>
<td>182</td>
<td>RPNDD</td>
<td>464</td>
</tr>
<tr>
<td>MMZM</td>
<td>178</td>
<td>RREFU</td>
<td>96</td>
</tr>
<tr>
<td>MNEM</td>
<td>385</td>
<td>RSLT</td>
<td>446</td>
</tr>
<tr>
<td>MRCLT</td>
<td>521</td>
<td>RSMS</td>
<td>274</td>
</tr>
<tr>
<td>MRLT</td>
<td>491</td>
<td>SCB</td>
<td>324</td>
</tr>
<tr>
<td>MRRGE</td>
<td>580</td>
<td>SER</td>
<td>391</td>
</tr>
<tr>
<td>MRSLT</td>
<td>490</td>
<td>SLEMM</td>
<td>173</td>
</tr>
<tr>
<td>MVSLD</td>
<td>126</td>
<td>SLESLC</td>
<td>82</td>
</tr>
<tr>
<td>NEM</td>
<td>383</td>
<td>SLTB</td>
<td>449</td>
</tr>
<tr>
<td>NI</td>
<td>204</td>
<td>SLTD</td>
<td>450</td>
</tr>
<tr>
<td>NJB</td>
<td>552</td>
<td>SLTT</td>
<td>417</td>
</tr>
<tr>
<td>NME1</td>
<td>66</td>
<td>SMS</td>
<td>167</td>
</tr>
<tr>
<td>NME2</td>
<td>127</td>
<td>SMZD</td>
<td>351</td>
</tr>
<tr>
<td>NME3</td>
<td>204</td>
<td>SMZE</td>
<td>379</td>
</tr>
<tr>
<td>NME4</td>
<td>216</td>
<td>SNCM</td>
<td>204</td>
</tr>
<tr>
<td>NME5</td>
<td>299</td>
<td>SS</td>
<td>195</td>
</tr>
<tr>
<td>NME6</td>
<td>315</td>
<td>SSLD</td>
<td>307</td>
</tr>
<tr>
<td>NME7</td>
<td>352</td>
<td>SSNS</td>
<td>107</td>
</tr>
<tr>
<td>NME8</td>
<td>379</td>
<td>SSRLT</td>
<td>348</td>
</tr>
<tr>
<td>NME9</td>
<td>392</td>
<td>SSS</td>
<td>270</td>
</tr>
<tr>
<td>NMLIC</td>
<td>126</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
INDEX 769

SUVB, 294
TD, 731
TDEE, 735
technique T, 620
TMA, 167
TMSM, 167
TSS, 265
TT, 167
TTMI, 191
UMI, 205
UMPIP, 207
UTMR, 541, 582
VAC, 260
VFSLS, 88
VRI, 477
VRILT, 478
VRLT, 473
VRRB, 288
VRS, 478
VSLT, 419
VSPCV, 75
VSPM, 164
ZSSM, 258
ZVSM, 258
ZVU, 257

ti83
matrix entry (computation), 609
row reduce (computation), 610
vector linear combinations (computation), 610

TI83 (section), 609

iti86
matrix entry (computation), 608
row reduce (computation), 609
transpose of a matrix (computation), 609
vector linear combinations (computation), 609

TI86 (section), 608
TIS (example), 505
TIVS (example), 478
TKAP (example), 430
TLC (example), 79
TM (definition), 166
TM (example), 166
TM (notation), 166
TM (subsection, section OD), 540
TM.TI86 (computation, section TI86), 609
TMA (theorem), 167
TMP (example), 3
TMSM (theorem), 167
TOV (example), 156
trail mix
eample TMP, 3
transpose
matrix scalar multiplication

theorem TMSM, 167
eample TM, 166
matrix addition
theorem TMA, 167
matrix inverse, 196
notation, 166
scalar multiplication, 168
transpose of a matrix
mathematica, 607
ti86, 609
transpose of a transpose
theorem TT, 167
TREM (example), 25
triamgular decomposition
entry by entry, size 6
eample TDEE6, 736
entry by entry
theorem TDEE, 735
size 4
eample TD4, 732
olving systems of equations
eample TDSSE, 734
theorem TD, 731
triamgular matrix
verse
theorem ITMT, 541
trivial solution
ystem of equations
definition TSHSE, 52
TS (definition), 268
TS (subsection, section S), 265
TSHSE (definition), 52
TSM (subsection, section MO), 166
TSS (section), 12
TSS (subsection, section S), 269
TSS (theorem), 265
TSVS (definition), 284
TT (theorem), 167
TTMI (theorem), 191
TTS (example), 10
typical systems, 2 × 2
eample TTS, 10
U (archetype), 702
U (technique, section PT), 624
UM (definition), 205
UM (subsection, section MINM), 205
UM3 (example), 205
UMI (theorem), 205
UMPIP (theorem), 207
unique solution, 3 × 3
eample US, 14
eample USR, 25
uniqueness

Version 1.04
technique U, 624
unit vectors
  basis
    theorem SUVB, 294
    definition SUV, 190
  orthogonal
    example SUVOS, 157
unitary
  permutation matrix
    example UPM, 205
  size 3
    example UM3, 205
unitary matrices
  columns
    theorem CUMOS, 200
unitary matrix
  inner product
    theorem UMPIP, 207
    UPM (example), 205
upper triangular matrix
  definition UTM, 540
URREF (subsection, section LC), 95
US (example), 14
USR (example), 25
UTM (definition), 540
UTMR (subsection, section JCF), 582
UTMR (subsection, section OD), 541
UTMR (theorem), 541, 582
V (archetype), 704
V (chapter), 72
VA (example), 74
VAC (theorem), 260
VEASM (subsection, section VO), 72
vector
  addition
    definition CVA, 73
  column
    definition CV, 21
  equality
    definition CVE, 73
    notation, 73
  inner product
    definition IP, 152
    norm
    definition NV, 155
    notation, 22
  of constants
    definition VOC, 22
  product with matrix, 173, 176
  scalar multiplication
    definition CVSM, 74
  vector addition
    example VA, 74
vector component
  notation, 22
vector form of solutions
  Archetype D
    example VFSAD, 83
  Archetype I
    example VFSAI, 90
  Archetype L
    example VFSAL, 91
    example VFS, 89
    mathematica, 606
    theorem VFSLS, 88
vector linear combinations
  mathematica, 605
  ti83, 610
  ti86, 609
vector representation
  example AVR, 288
  example VRC4, 474
  injective
    theorem VRI, 477
  invertible
    theorem VRILT, 478
  linear transformation
    definition VR, 473
    theorem VRLT, 473
  surjective
    theorem VRS, 478
    theorem VRRB, 288
vector representations
  polynomials
    example VRP2, 476
vector scalar multiplication
  example CVSM, 74
vector space
  characterization
    theorem CFDVS, 478
  column vectors
    definition VSCV, 72
    definition VS, 251
  infinite dimension
    example VSPUD, 312
  linear transformations
    theorem VSLT, 419
  over integers mod 5
    example VSIM5, 720
vector space of column vectors
  notation, 72
vector space of functions
  example VSF, 254
vector space of infinite sequences
  example VSIS, 254
vector space of matrices
  definition VSM, 163