A First Course in Linear Algebra

by

Robert A. Beezer
Department of Mathematics and Computer Science
University of Puget Sound

Version 0.92
November 16, 2006
Preface

This textbook is designed to teach the university mathematics student the basics of the subject of linear algebra. There are no prerequisites other than ordinary algebra, but it is probably best used by a student who has the “mathematical maturity” of a sophomore or junior.

The text has two goals: to teach the fundamental concepts and techniques of matrix algebra and abstract vector spaces, and to teach the techniques associated with understanding the definitions and theorems forming a coherent area of mathematics. So there is an emphasis on worked examples of nontrivial size and on proving theorems carefully.

This book is copyrighted. This means that governments have granted the author a monopoly — the exclusive right to control the making of copies and derivative works for many years (too many years in some cases). It also gives others limited rights, generally referred to as “fair use,” such as the right to quote sections in a review without seeking permission. However, the author licenses this book to anyone under the terms of the GNU Free Documentation License (GFDL), which gives you more rights than most copyrights. Loosely speaking, you may make as many copies as you like at no cost, and you may distribute these unmodified copies if you please. You may modify the book for your own use. The catch is that if you make modifications and you distribute the modified version, or make use of portions in excess of fair use in another work, then you must also license the new work with the GFDL. So the book has lots of inherent freedom, and no one is allowed to distribute a derivative work that restricts these freedoms. (See the license itself for all the exact details of the additional rights you have been given.)

Notice that initially most people are struck by the notion that this book is free (the French would say gratuit, at no cost). And it is. However, it is more important that the book has freedom (the French would say liberté, liberty). It will never go “out of print” nor will there ever be trivial updates designed only to frustrate the used book market. Those considering teaching a course with this book can examine it thoroughly in advance. Adding new exercises or new sections has been purposely made very easy, and the hope is that others will contribute these modifications back for incorporation into the book, for the benefit of all.

Depending on how you received your copy, you may want to check for the latest version (and other news) at http://linear.ups.edu/.
Topics  The first half of this text (through Chapter M [197]) is basically a course in matrix algebra, though the foundation of some more advanced ideas is also being formed in these early sections. Vectors are presented exclusively as column vectors (since we also have the typographic freedom to avoid writing a column vector inline as the transpose of a row vector), and linear combinations are presented very early. Spans, null spaces and column spaces are also presented early, simply as sets, saving most of their vector space properties for later, so they are familiar objects before being scrutinized carefully.

You cannot do everything early, so in particular matrix multiplication comes later than usual. However, with a definition built on linear combinations of column vectors, it should seem more natural than the usual definition using dot products of rows with columns. And this delay emphasizes that linear algebra is built upon vector addition and scalar multiplication. Of course, matrix inverses must wait for matrix multiplication, but this does not prevent nonsingular matrices from occurring sooner. Vector space properties are hinted at when vector and matrix operations are first defined, but the notion of a vector space is saved for a more axiomatic treatment later. Once bases and dimension have been explored in the context of vector spaces, linear transformations and their matrix representations follow. The goal of the book is to go as far as canonical forms and matrix decompositions in the Core, with less central topics collected in a section of Topics.

Linear algebra is an ideal subject for the novice mathematics student to learn how to develop a topic precisely, with all the rigor mathematics requires. Unfortunately, much of this rigor seems to have escaped the standard calculus curriculum, so for many university students this is their first exposure to careful definitions and theorems, and the expectation that they fully understand them, to say nothing of the expectation that they become proficient in formulating their own proofs. We have tried to make this text as helpful as possible with this transition. Every definition is stated carefully, set apart from the text. Likewise, every theorem is carefully stated, and almost every one has a complete proof. Theorems usually have just one conclusion, so they can be referenced precisely later. Definitions and theorems are cataloged in order of their appearance in the front of the book, and alphabetical order in the index at the back. Along the way, there are discussions of some more important ideas relating to formulating proofs (Proof Techniques), which is advice mostly.

Origin and History  This book is the result of the confluence of several related events and trends.

- At the University of Puget Sound we teach a one-semester, post-calculus linear algebra course to students majoring in mathematics, computer science, physics, chemistry and economics. Between January 1986 and June 2002, I taught this course seventeen times. For the Spring 2003 semester, I elected to convert my course notes to an electronic form so that it would be easier to incorporate the inevitable and nearly-constant revisions. Central to my new notes was a collection of stock examples that would be used repeatedly to illustrate new concepts. (These would become the Archetypes, Appendix A [717].) It was only a short leap to then decide to distribute copies of these notes and examples to the students in the two sections of this course. As the semester wore on, the notes began to look less like notes and more like a textbook.

- I used the notes again in the Fall 2003 semester for a single section of the course.
Simultaneously, the textbook I was using came out in a fifth edition. A new chapter was added toward the start of the book, and a few additional exercises were added in other chapters. This demanded the annoyance of reworking my notes and list of suggested exercises to conform with the changed numbering of the chapters and exercises. I had an almost identical experience with the third course I was teaching that semester. I also learned that in the next academic year I would be teaching a course where my textbook of choice had gone out of print. I felt there had to be a better alternative to having the organization of my courses buffeted by the economics of traditional textbook publishing.

- I had used \TeX and the Internet for many years, so there was little to stand in the way of typesetting, distributing and “marketing” a free book. With recreational and professional interests in software development, I had long been fascinated by the open-source software movement, as exemplified by the success of GNU and Linux, though public-domain \TeX might also deserve mention. Obviously, this book is an attempt to carry over that model of creative endeavor to textbook publishing.

- As a sabbatical project during the Spring 2004 semester, I embarked on the current project of creating a freely-distributable linear algebra textbook. (Notice the implied financial support of the University of Puget Sound to this project.) Most of the material was written from scratch since changes in notation and approach made much of my notes of little use. By August 2004 I had written half the material necessary for our Math 232 course. The remaining half was written during the Fall 2004 semester as I taught another two sections of Math 232.

- I taught a single section of the course in the Spring 2005 semester, while my colleague, Professor Martin Jackson, graciously taught another section from the constantly shifting sands that was this project (version 0.30). His many suggestions have helped immeasurably. For the Fall 2005 semester, I taught two sections of the course from version 0.50.

However, much of my motivation for writing this book is captured by the sentiments expressed by H.M. Cundy and A.P. Rollet in their Preface to the First Edition of Mathematical Models (1952), especially the final sentence,

This book was born in the classroom, and arose from the spontaneous interest of a Mathematical Sixth in the construction of simple models. A desire to show that even in mathematics one could have fun led to an exhibition of the results and attracted considerable attention throughout the school. Since then the Sherborne collection has grown, ideas have come from many sources, and widespread interest has been shown. It seems therefore desirable to give permanent form to the lessons of experience so that others can benefit by them and be encouraged to undertake similar work.

**How To Use This Book** Chapters, Theorems, etc. are not numbered in this book, but are instead referenced by acronyms. This means that Theorem XYZ will always be Theorem XYZ, no matter if new sections are added, or if an individual decides to remove certain other sections. Within sections, the subsections are acronyms that begin with the acronym of the section. So Subsection XYZ.AB is the subsection AB in Section XYZ. Acronyms are unique within their type, so for example there is just one Definition B, but
there is also a Section B. At first, all the letters flying around may be confusing, but with
time, you will begin to recognize the more important ones on sight. Furthermore, there
are lists of theorems, examples, etc. in the front of the book, and an index that contains
every acronym. If you are reading this in an electronic version (PDF or XML), you will
see that all of the cross-references are hyperlinks, allowing you to click to a definition
or example, and then use the back button to return. In printed versions, you must rely
on the page numbers. However, note that page numbers are not permanent! Different
editions, different margins, or different sized paper will affect what content is on each
page. And in time, the addition of new material will affect the page numbering.

Chapter divisions are not critical to the organization of the book, as Sections are
the main organizational unit. Sections are designed to be the subject of a single lecture
or classroom session, though there is frequently more material than can be discussed
and illustrated in a fifty-minute session. Consequently, the instructor will need to be
selective about which topics to illustrate with other examples and which topics to leave
to the student’s reading. Many of the examples are meant to be large, such as using five
or six variables in a system of equations, so the instructor may just want to “walk” a
class through these examples. The book has been written with the idea that some may
work through it independently, so the hope is that students can learn some of the more
mechanical ideas on their own.

The highest level division of the book is the three Parts: Core, Topics, Applications.
The Core is meant to carefully describe the basic ideas required of a first exposure to
linear algebra. In the final sections of the Core, one should ask the question: which
previous Sections could be removed without destroying the logical development of the
subject? Hopefully, the answer is “none.” The goal of the book is to finish the Core with
the most general representations of linear transformations (Jordan and rational canonical
forms) and perhaps matrix decompositions (LU, QR, singular value). Of course, there
will not be universal agreement on what should, or should not, constitute the Core, but
the main idea will be to limit it to about forty sections. Topics is meant to contain those
subjects that are important in linear algebra, and which would make profitable detours
from the Core for those interested in pursuing them. Applications should illustrate the
power and widespread applicability of linear algebra to as many fields as possible. The
Archetypes [Appendix A 717] cover many of the computational aspects of systems of
linear equations, matrices and linear transformations. The student should consult them
often, and this is encouraged by exercises that simply suggest the right properties to
examine at the right time. But what is more important, they are a repository that con-
tains enough variety to provide abundant examples of key theorems, while also providing
counterexamples to hypotheses or converses of theorems.

I require my students to read each Section prior to the day’s discussion on that section.
For some students this is a novel idea, but at the end of the semester a few always report
on the benefits, both for this course and other courses where they have adopted the
habit. To make good on this requirement, each section contains three Reading Questions.
These sometimes only require parroting back a key definition or theorem, or they require
performing a small example of a key computation, or they ask for musings on key ideas
or new relationships between old ideas. Answers are emailed to me the evening before
the lecture. Given the flavor and purpose of these questions, including solutions seems
foolish.

Formulating interesting and effective exercises is as difficult, or more so, than building
a narrative. But it is the place where a student really learns the material. As such, for
the student’s benefit, complete solutions should be given. As the list of exercises expands, over time solutions will also be provided. Exercises and their solutions are referenced with a section name, followed by a dot, then a letter (C, M, or T) and a number. The letter ‘C’ indicates a problem that is mostly computational in nature, while the letter ‘T’ indicates a problem that is more theoretical in nature. A problem with a letter ‘M’ is somewhere in between (middle, mid-level, median, middling), probably a mix of computation and applications of theorems. So “Solution MO.T34” is a solution to an exercise in Section MO that is theoretical in nature. The number ‘34’ has no intrinsic meaning.

More on Freedom This book is freely-distributable under the terms of the GFDL, along with the underlying \TeX code from which the book is built. This arrangement provides many benefits unavailable with traditional texts.

- No cost, or low cost, to students. With no physical vessel (i.e. paper, binding), no transportation costs (Internet bandwidth being a negligible cost) and no marketing costs (evaluation and desk copies are free to all), anyone with an Internet connection can obtain it, and a teacher could make available paper copies in sufficient quantities for a class. The cost to print a copy is not insignificant, but is just a fraction of the cost of a traditional textbook. Students will not feel the need to sell back their book, and in future years can even pick up a newer edition freely.

- The book will not go out of print. No matter what, a teacher can maintain their own copy and use the book for as many years as they desire. Further, the naming schemes for chapters, sections, theorems, etc. is designed so that the addition of new material will not break any course syllabi or assignment list.

- With many eyes reading the book and with frequent postings of updates, the reliability should become very high. Please report any errors you find that persist into the latest version.

- For those with a working installation of the popular typesetting program \TeX, the book has been designed so that it can be customized. Page layouts, presence of exercises, solutions, sections or chapters can all be easily controlled. Furthermore, many variants of mathematical notation are achieved via \TeX macros. So by changing a single macro, one’s favorite notation can be reflected throughout the text. For example, every transpose of a matrix is coded in the source as $\text{\LaTeX}\{A\}$, which when printed will yield $A^t$. However by changing the definition of $\text{\LaTeX}\{ \}$, any desired alternative notation will then appear throughout the text instead.

- The book has also been designed to make it easy for others to contribute material. Would you like to see a section on symmetric bilinear forms? Consider writing one and contributing it to one of the Topics chapters. Does there need to be more exercises about the null space of a matrix? Send me some. Historical Notes? Contact me, and we will see about adding those in also.

- You have no legal obligation to pay for this book. It has been licensed with no expectation that you pay for it. You do not even have a moral obligation to pay for the book. Thomas Jefferson (1743 – 1826), the author of the United States Declaration of Independence, wrote,
If nature has made any one thing less susceptible than all others of exclusive property, it is the action of the thinking power called an idea, which an individual may exclusively possess as long as he keeps it to himself; but the moment it is divulged, it forces itself into the possession of every one, and the receiver cannot dispossess himself of it. Its peculiar character, too, is that no one possesses the less, because every other possesses the whole of it. He who receives an idea from me, receives instruction himself without lessening mine; as he who lights his taper at mine, receives light without darkening me. That ideas should freely spread from one to another over the globe, for the moral and mutual instruction of man, and improvement of his condition, seems to have been peculiarly and benevolently designed by nature, when she made them, like fire, expansible over all space, without lessening their density in any point, and like the air in which we breathe, move, and have our physical being, incapable of confinement or exclusive appropriation.

Letter to Isaac McPherson
August 13, 1813

However, if you feel a royalty is due the author, or if you would like to encourage the author, or if you wish to show others that this approach to textbook publishing can also bring financial gains, then donations are gratefully received. Moreover, non-financial forms of help can often be even more valuable. A simple note of encouragement, submitting a report of an error, or contributing some exercises or perhaps an entire section for the Topics or Applications chapters are all important ways you can acknowledge the freedoms accorded to this work by the copyright holder and other contributors.

Conclusion Foremost, I hope that students find their time spent with this book profitable. I hope that instructors find it flexible enough to fit the needs of their course. And I hope that everyone will send me their comments and suggestions, and also consider the myriad ways they can help (as listed on the book’s website at linear.ups.edu).

Robert A. Beezer
Tacoma, Washington
January, 2006
Contents

Preface ........................................... i

Contents ................................. vii
  Contributors ........................................... ix
  Definitions ........................................... xi
  Theorems .......................................... xiii
  Notation ......................................... xv
  Examples ....................................... xvii

Part C Core ........................................... 3

Chapter SLE Systems of Linear Equations 3
  WILA What is Linear Algebra? ............... 3
    LA “Linear” + “Algebra” ....................... 3
    A An application: packaging trail mix ... 4
  READ Reading Questions .................... 8
  EXC Exercises ................................. 9
  SOL Solutions ................................ 11
  SSLE Solving Systems of Linear Equations 13
    PSS Possibilities for solution sets ....... 14
    ESEO Equivalent systems and equation operations . 15
    READ Reading Questions ................. 21
    EXC Exercises ............................ 23
    SOL Solutions ................................ 25
  RREF Reduced Row-Echelon Form ........ 29
    MVNSE Matrix and Vector Notation for Systems of Equations . 30
    READ Reading Questions .......... 42
    EXC Exercises ............................. 43
    SOL Solutions ................................ 47
  TSS Types of Solution Sets ............... 51
    READ Reading Questions .......... 58
    EXC Exercises ............................. 61
    SOL Solutions ................................ 63
  HSE Homogeneous Systems of Equations . 65
    SHS Solutions of Homogeneous Systems .... 65
    NSM Null Space of a Matrix ............ 68
    READ Reading Questions .......... 70
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXC Exercises</td>
<td>71</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>73</td>
</tr>
<tr>
<td>NM Nonsingular Matrices</td>
<td>75</td>
</tr>
<tr>
<td>NM Nonsingular Matrices</td>
<td>75</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>82</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>83</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>85</td>
</tr>
<tr>
<td>Chapter V Vectors</td>
<td>87</td>
</tr>
<tr>
<td>VO Vector Operations</td>
<td>87</td>
</tr>
<tr>
<td>VEASM Vector equality, addition, scalar multiplication</td>
<td>88</td>
</tr>
<tr>
<td>VSP Vector Space Properties</td>
<td>90</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>92</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>93</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>95</td>
</tr>
<tr>
<td>LC Linear Combinations</td>
<td>97</td>
</tr>
<tr>
<td>LC Linear Combinations</td>
<td>97</td>
</tr>
<tr>
<td>VFSS Vector Form of Solution Sets</td>
<td>102</td>
</tr>
<tr>
<td>PSHS Particular Solutions, Homogeneous Solutions</td>
<td>113</td>
</tr>
<tr>
<td>URREF Uniqueness of Reduced Row-Echelon Form</td>
<td>116</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>118</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>119</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>121</td>
</tr>
<tr>
<td>SS Spanning Sets</td>
<td>123</td>
</tr>
<tr>
<td>SSV Span of a Set of Vectors</td>
<td>123</td>
</tr>
<tr>
<td>SSNS Spanning Sets of Null Spaces</td>
<td>129</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>134</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>135</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>139</td>
</tr>
<tr>
<td>LI Linear Independence</td>
<td>145</td>
</tr>
<tr>
<td>LIVS Linearly Independent Sets of Vectors</td>
<td>145</td>
</tr>
<tr>
<td>LINM Linear Independence and Nonsingular Matrices</td>
<td>151</td>
</tr>
<tr>
<td>NSSLI Null Spaces, Spans, Linear Independence</td>
<td>152</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>155</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>157</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>161</td>
</tr>
<tr>
<td>LDS Linear Dependence and Spans</td>
<td>167</td>
</tr>
<tr>
<td>LDSS Linearly Dependent Sets and Spans</td>
<td>167</td>
</tr>
<tr>
<td>COV Casting Out Vectors</td>
<td>169</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>176</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>177</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>179</td>
</tr>
<tr>
<td>O Orthogonality</td>
<td>183</td>
</tr>
<tr>
<td>CAV Complex arithmetic and vectors</td>
<td>183</td>
</tr>
<tr>
<td>IP Inner products</td>
<td>184</td>
</tr>
<tr>
<td>N Norm</td>
<td>187</td>
</tr>
<tr>
<td>OV Orthogonal Vectors</td>
<td>188</td>
</tr>
<tr>
<td>GSP Gram-Schmidt Procedure</td>
<td>191</td>
</tr>
<tr>
<td>Chapter M Matrices</td>
<td>197</td>
</tr>
<tr>
<td>-------------------</td>
<td>-----</td>
</tr>
<tr>
<td>MO Matrix Operations</td>
<td>197</td>
</tr>
<tr>
<td>MEASM Matrix equality, addition, scalar multiplication</td>
<td>197</td>
</tr>
<tr>
<td>VSP Vector Space Properties</td>
<td>199</td>
</tr>
<tr>
<td>TSM Transposes and Symmetric Matrices</td>
<td>200</td>
</tr>
<tr>
<td>MCC Matrices and Complex Conjugation</td>
<td>203</td>
</tr>
<tr>
<td>MM Matrix Multiplication</td>
<td>211</td>
</tr>
<tr>
<td>MVP Matrix-Vector Product</td>
<td>211</td>
</tr>
<tr>
<td>MM Matrix Multiplication</td>
<td>215</td>
</tr>
<tr>
<td>MMEE Matrix Multiplication, Entry-by-Entry</td>
<td>216</td>
</tr>
<tr>
<td>PMM Properties of Matrix Multiplication</td>
<td>218</td>
</tr>
<tr>
<td>MISLE Matrix Inverses and Systems of Linear Equations</td>
<td>231</td>
</tr>
<tr>
<td>IM Inverse of a Matrix</td>
<td>232</td>
</tr>
<tr>
<td>CIM Computing the Inverse of a Matrix</td>
<td>234</td>
</tr>
<tr>
<td>PMI Properties of Matrix Inverses</td>
<td>238</td>
</tr>
<tr>
<td>MINM Matrix Inverses and Nonsingular Matrices</td>
<td>249</td>
</tr>
<tr>
<td>NMI Nonsingular Matrices are Invertible</td>
<td>249</td>
</tr>
<tr>
<td>UM Unitary Matrices</td>
<td>252</td>
</tr>
<tr>
<td>CRS Column and Row Spaces</td>
<td>261</td>
</tr>
<tr>
<td>CSSE Column spaces and systems of equations</td>
<td>261</td>
</tr>
<tr>
<td>CSSOC Column space spanned by original columns</td>
<td>263</td>
</tr>
<tr>
<td>CSNM Column Space of a Nonsingular Matrix</td>
<td>266</td>
</tr>
<tr>
<td>RSM Row Space of a Matrix</td>
<td>268</td>
</tr>
<tr>
<td>FS Four Subsets</td>
<td>283</td>
</tr>
<tr>
<td>LNS Left Null Space</td>
<td>283</td>
</tr>
<tr>
<td>CRS Computing Column Spaces</td>
<td>284</td>
</tr>
<tr>
<td>EEF Extended echelon form</td>
<td>287</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>301</td>
</tr>
</tbody>
</table>
## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>SOL Solutions</td>
<td>305</td>
</tr>
<tr>
<td>Chapter VS Vector Spaces</td>
<td>309</td>
</tr>
<tr>
<td>VS Vector Spaces</td>
<td>309</td>
</tr>
<tr>
<td>VS Vector Spaces</td>
<td>309</td>
</tr>
<tr>
<td>EVS Examples of Vector Spaces</td>
<td>311</td>
</tr>
<tr>
<td>VSP Vector Space Properties</td>
<td>316</td>
</tr>
<tr>
<td>RD Recycling Definitions</td>
<td>321</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>321</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>323</td>
</tr>
<tr>
<td>S Subspaces</td>
<td>325</td>
</tr>
<tr>
<td>TS Testing Subspaces</td>
<td>326</td>
</tr>
<tr>
<td>TSS The Span of a Set</td>
<td>331</td>
</tr>
<tr>
<td>SC Subspace Constructions</td>
<td>336</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>337</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>339</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>341</td>
</tr>
<tr>
<td>LISS Linear Independence and Spanning Sets</td>
<td>345</td>
</tr>
<tr>
<td>LI Linear independence</td>
<td>345</td>
</tr>
<tr>
<td>SS Spanning Sets</td>
<td>350</td>
</tr>
<tr>
<td>VR Vector Representation</td>
<td>354</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>356</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>357</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>359</td>
</tr>
<tr>
<td>B Bases</td>
<td>363</td>
</tr>
<tr>
<td>B Bases</td>
<td>363</td>
</tr>
<tr>
<td>BSCV Bases for Spans of Column Vectors</td>
<td>367</td>
</tr>
<tr>
<td>BNM Bases and Nonsingular Matrices</td>
<td>369</td>
</tr>
<tr>
<td>OBC Orthonormal Bases and Coordinates</td>
<td>370</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>373</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>375</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>377</td>
</tr>
<tr>
<td>D Dimension</td>
<td>379</td>
</tr>
<tr>
<td>D Dimension</td>
<td>379</td>
</tr>
<tr>
<td>DVS Dimension of Vector Spaces</td>
<td>383</td>
</tr>
<tr>
<td>RNM Rank and Nullity of a Matrix</td>
<td>385</td>
</tr>
<tr>
<td>RNNM Rank and Nullity of a Nonsingular Matrix</td>
<td>387</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>389</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>391</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>393</td>
</tr>
<tr>
<td>PD Properties of Dimension</td>
<td>397</td>
</tr>
<tr>
<td>GT Goldilocks' Theorem</td>
<td>397</td>
</tr>
<tr>
<td>RT Ranks and Transposes</td>
<td>401</td>
</tr>
<tr>
<td>DFS Dimension of Four Subspaces</td>
<td>402</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>403</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>405</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>407</td>
</tr>
<tr>
<td>CONTENTS</td>
<td></td>
</tr>
<tr>
<td>------------------</td>
<td>------------------</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>.....................</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>.....................</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>.....................</td>
</tr>
<tr>
<td>ILT Injective Linear Transformations</td>
<td>.....................</td>
</tr>
<tr>
<td>EILT Examples of Injective Linear Transformations</td>
<td>.....................</td>
</tr>
<tr>
<td>KLT Kernel of a Linear Transformation</td>
<td>.....................</td>
</tr>
<tr>
<td>ILTI Injective Linear Transformations and Linear Independence</td>
<td>.....................</td>
</tr>
<tr>
<td>ILTD Injective Linear Transformations and Dimension</td>
<td>.....................</td>
</tr>
<tr>
<td>CILT Composition of Injective Linear Transformations</td>
<td>.....................</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>.....................</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>.....................</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>.....................</td>
</tr>
<tr>
<td>SLT Surjective Linear Transformations</td>
<td>.....................</td>
</tr>
<tr>
<td>ESLT Examples of Surjective Linear Transformations</td>
<td>.....................</td>
</tr>
<tr>
<td>RLT Range of a Linear Transformation</td>
<td>.....................</td>
</tr>
<tr>
<td>SSSLT Spanning Sets and Surjective Linear Transformations</td>
<td>.....................</td>
</tr>
<tr>
<td>SLTD Surjective Linear Transformations and Dimension</td>
<td>.....................</td>
</tr>
<tr>
<td>CSLT Composition of Surjective Linear Transformations</td>
<td>.....................</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>.....................</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>.....................</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>.....................</td>
</tr>
<tr>
<td>IVLT Invertible Linear Transformations</td>
<td>.....................</td>
</tr>
<tr>
<td>IVLT Invertible Linear Transformations</td>
<td>.....................</td>
</tr>
<tr>
<td>IV Invertibility</td>
<td>.....................</td>
</tr>
<tr>
<td>SI Structure and Isomorphism</td>
<td>.....................</td>
</tr>
<tr>
<td>RNLT Rank and Nullity of a Linear Transformation</td>
<td>.....................</td>
</tr>
<tr>
<td>SLELT Systems of Linear Equations and Linear Transformations</td>
<td>.....................</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>.....................</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>.....................</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>.....................</td>
</tr>
<tr>
<td>Chapter R Representations</td>
<td>587</td>
</tr>
<tr>
<td>VR Vector Representations</td>
<td>.....................</td>
</tr>
<tr>
<td>CVS Characterization of Vector Spaces</td>
<td>.....................</td>
</tr>
<tr>
<td>CP Coordinatization Principle</td>
<td>.....................</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>.....................</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>.....................</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>.....................</td>
</tr>
<tr>
<td>MR Matrix Representations</td>
<td>.....................</td>
</tr>
<tr>
<td>NRFO New Representations from Old</td>
<td>.....................</td>
</tr>
<tr>
<td>PMR Properties of Matrix Representations</td>
<td>.....................</td>
</tr>
<tr>
<td>IVLT Invertible Linear Transformations</td>
<td>.....................</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>.....................</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>.....................</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>.....................</td>
</tr>
<tr>
<td>CB Change of Basis</td>
<td>.....................</td>
</tr>
<tr>
<td>EELT Eigenvalues and Eigenvectors of Linear Transformations</td>
<td>.....................</td>
</tr>
<tr>
<td>CBM Change-of-Basis Matrix</td>
<td>.....................</td>
</tr>
<tr>
<td>CONTENTS</td>
<td>xiii</td>
</tr>
<tr>
<td>----------</td>
<td>-------------------------</td>
</tr>
<tr>
<td>MRS Matrix Representations and Similarity</td>
<td>644</td>
</tr>
<tr>
<td>CELT Computing Eigenvectors of Linear Transformations</td>
<td>651</td>
</tr>
<tr>
<td>READ Reading Questions</td>
<td>659</td>
</tr>
<tr>
<td>EXC Exercises</td>
<td>661</td>
</tr>
<tr>
<td>SOL Solutions</td>
<td>663</td>
</tr>
<tr>
<td>NLT Nilpotent Linear Transformations</td>
<td>667</td>
</tr>
<tr>
<td>PNLT Properties of Nilpotent Linear Transformations</td>
<td>672</td>
</tr>
<tr>
<td>Appendix CN Computation Notes</td>
<td>679</td>
</tr>
<tr>
<td>MMA Mathematical</td>
<td>679</td>
</tr>
<tr>
<td>ME.MMA Matrix Entry</td>
<td>679</td>
</tr>
<tr>
<td>RR.MMA Row Reduce</td>
<td>679</td>
</tr>
<tr>
<td>LS.MMA Linear Solve</td>
<td>680</td>
</tr>
<tr>
<td>VLC.MMA Vector Linear Combinations</td>
<td>680</td>
</tr>
<tr>
<td>NS.MMA Null Space</td>
<td>681</td>
</tr>
<tr>
<td>VFGS.MMA Vector Form of Solution Set</td>
<td>681</td>
</tr>
<tr>
<td>GSP.MMA Gram-Schmidt Procedure</td>
<td>682</td>
</tr>
<tr>
<td>TM.MMA Transpose of a Matrix</td>
<td>683</td>
</tr>
<tr>
<td>MM.MMA Matrix Multiplication</td>
<td>683</td>
</tr>
<tr>
<td>ML.MMA Matrix Inverse</td>
<td>684</td>
</tr>
<tr>
<td>TI86 Texas Instruments 86</td>
<td>684</td>
</tr>
<tr>
<td>ME.TI86 Matrix Entry</td>
<td>684</td>
</tr>
<tr>
<td>RR.TI86 Row Reduce</td>
<td>685</td>
</tr>
<tr>
<td>VLC.TI86 Vector Linear Combinations</td>
<td>685</td>
</tr>
<tr>
<td>TM.TI86 Transpose of a Matrix</td>
<td>685</td>
</tr>
<tr>
<td>TI83 Texas Instruments 83</td>
<td>686</td>
</tr>
<tr>
<td>ME.TI83 Matrix Entry</td>
<td>686</td>
</tr>
<tr>
<td>RR.TI83 Row Reduce</td>
<td>686</td>
</tr>
<tr>
<td>VLC.TI83 Vector Linear Combinations</td>
<td>686</td>
</tr>
<tr>
<td>Appendix P Preliminaries</td>
<td>687</td>
</tr>
<tr>
<td>CNO Complex Number Operations</td>
<td>687</td>
</tr>
<tr>
<td>CNA Arithmetic with complex numbers</td>
<td>687</td>
</tr>
<tr>
<td>CCN Conjugates of Complex Numbers</td>
<td>689</td>
</tr>
<tr>
<td>MCN Modulus of a Complex Number</td>
<td>690</td>
</tr>
<tr>
<td>SET Sets</td>
<td>693</td>
</tr>
<tr>
<td>SC Set Cardinality</td>
<td>695</td>
</tr>
<tr>
<td>SO Set Operations</td>
<td>695</td>
</tr>
<tr>
<td>PT Proof Techniques</td>
<td>697</td>
</tr>
<tr>
<td>D Definitions</td>
<td>698</td>
</tr>
<tr>
<td>T Theorems</td>
<td>699</td>
</tr>
<tr>
<td>L Language</td>
<td>700</td>
</tr>
<tr>
<td>GS Getting Started</td>
<td>702</td>
</tr>
<tr>
<td>C Constructive Proofs</td>
<td>703</td>
</tr>
<tr>
<td>E Equivalences</td>
<td>704</td>
</tr>
<tr>
<td>N Negation</td>
<td>705</td>
</tr>
<tr>
<td>CP Contrapositives</td>
<td>706</td>
</tr>
</tbody>
</table>

Version 0.92
<table>
<thead>
<tr>
<th>Part T Topics</th>
<th>819</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part A Applications</td>
<td>821</td>
</tr>
</tbody>
</table>
Contributors

Beezer, David. St. Charles Borromeo School
Beezer, Robert. University of Puget Sound [http://buzzard.ups.edu/]
Fellez, Sarah. University of Puget Sound
Jackson, Martin. University of Puget Sound
Linenthal, Jacob. University of Puget Sound
Osborne, Travis. University of Puget Sound
Riegsecker, Joe. Middlebury, Indiana joepye (at) pobox (dot) com
Phelps, Douglas. University of Puget Sound
Shoemaker, Mark. University of Puget Sound
Zimmer, Andy. University of Puget Sound
## Definitions

### Section WILA
- **SLE** - System of Linear Equations ................................................................. 13
- **ESYS** - Equivalent Systems ............................................................................ 15
- **EO** - Equation Operations .............................................................................. 16

### Section SSLE
- **SLE** - System of Linear Equations ................................................................. 13
- **ESYS** - Equivalent Systems ............................................................................ 15
- **EO** - Equation Operations .............................................................................. 16

### Section RREF
- **M** - Matrix ........................................................................................................ 29
- **CV** - Column Vector ......................................................................................... 30
- **ZCV** - Zero Column Vector .............................................................................. 30
- **CM** - Coefficient Matrix .................................................................................. 30
- **VOC** - Vector of Constants .............................................................................. 31
- **SV** - Solution Vector ....................................................................................... 31
- **LSMR** - Matrix Representation of a Linear System ......................................... 32
- **AM** - Augmented Matrix .................................................................................. 32
- **RO** - Row Operations ....................................................................................... 33
- **REM** - Row-Equivalent Matrices .................................................................... 33
- **RREF** - Reduced Row-Echelon Form ............................................................... 35
- **RR** - Row-Reducing ......................................................................................... 42

### Section TSS
- **CS** - Consistent System ................................................................................... 51
- **IDV** - Independent and Dependent Variables ................................................... 53

### Section HSE
- **HS** - Homogeneous System ............................................................................. 65
- **TSHSE** - Trivial Solution to Homogeneous Systems of Equations .................. 66
- **NSM** - Null Space of a Matrix ......................................................................... 68

### Section NM
- **SQM** - Square Matrix ...................................................................................... 75
- **NM** - Nonsingular Matrix ................................................................................ 75
- **IM** - Identity Matrix ........................................................................................ 76

### Section VO
- **VSCV** - Vector Space of Column Vectors ...................................................... 87
- **CVE** - Column Vector Equality ....................................................................... 88
- **CVA** - Column Vector Addition ....................................................................... 89
- **CVSM** - Column Vector Scalar Multiplication ............................................... 89
Section LC

**LCCV** Linear Combination of Column Vectors ................................. 97

Section SS

**SSCV** Span of a Set of Column Vectors ........................................ 123

Section LI

**RLDCV** Relation of Linear Dependence for Column Vectors ..................... 145
**LICV** Linear Independence of Column Vectors .................................. 145

Section LDS

Section O

**CCCV** Complex Conjugate of a Column Vector .................................. 183
**IP** Inner Product ................................................................. 184
**NV** Norm of a Vector ............................................................ 187
**OV** Orthogonal Vectors .......................................................... 189
**OSV** Orthogonal Set of Vectors ................................................. 189
**ONS** OrthoNormal Set ............................................................ 193

Section MO

**VSM** Vector Space of $m \times n$ Matrices .................................... 197
**ME** Matrix Equality ............................................................... 197
**MA** Matrix Addition ............................................................... 198
**MSM** Matrix Scalar Multiplication ............................................. 198
**ZM** Zero Matrix ................................................................. 200
**TM** Transpose of a Matrix ....................................................... 200
**SYM** Symmetric Matrix ........................................................... 201
**CCM** Complex Conjugate of a Matrix .......................................... 203

Section MM

**MVP** Matrix-Vector Product ....................................................... 211
**MM** Matrix Multiplication ........................................................ 215

Section MISLE

**MI** Matrix Inverse ............................................................... 232
**SUV** Standard Unit Vectors ...................................................... 234

Section MINM

**UM** Unitary Matrices ............................................................ 252
**A** Adjoint ............................................................................. 255
**HM** Hermitian Matrix ............................................................. 255

Section CRS

**CSM** Column Space of a Matrix .................................................. 261
**RSM** Row Space of a Matrix ...................................................... 268

Section FS

Version 0.92
Section PT
## Theorems

<table>
<thead>
<tr>
<th>Section</th>
<th>Theorem</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>WILA</td>
<td>EOPSS</td>
<td>16</td>
</tr>
<tr>
<td>SSLE</td>
<td>REMES</td>
<td>34</td>
</tr>
<tr>
<td></td>
<td>REMEF</td>
<td>36</td>
</tr>
<tr>
<td>TSS</td>
<td>RCLS</td>
<td>54</td>
</tr>
<tr>
<td></td>
<td>ISRN</td>
<td>55</td>
</tr>
<tr>
<td></td>
<td>CSRN</td>
<td>56</td>
</tr>
<tr>
<td></td>
<td>FVCS</td>
<td>56</td>
</tr>
<tr>
<td></td>
<td>PSSLS</td>
<td>57</td>
</tr>
<tr>
<td></td>
<td>CMVEI</td>
<td>57</td>
</tr>
<tr>
<td>HSE</td>
<td>HSC</td>
<td>65</td>
</tr>
<tr>
<td></td>
<td>HMVEI</td>
<td>67</td>
</tr>
<tr>
<td>NM</td>
<td>NMIRRI</td>
<td>77</td>
</tr>
<tr>
<td></td>
<td>NMTNS</td>
<td>78</td>
</tr>
<tr>
<td></td>
<td>NMUS</td>
<td>79</td>
</tr>
<tr>
<td></td>
<td>NME1</td>
<td>81</td>
</tr>
<tr>
<td>VO</td>
<td>VSPCV</td>
<td>91</td>
</tr>
<tr>
<td>LC</td>
<td>SLSLC</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>VFSLS</td>
<td>107</td>
</tr>
<tr>
<td></td>
<td>PSPHIS</td>
<td>113</td>
</tr>
<tr>
<td></td>
<td>RREFU</td>
<td>116</td>
</tr>
<tr>
<td>SS</td>
<td>SSNS</td>
<td>129</td>
</tr>
<tr>
<td>LI</td>
<td></td>
<td>xxv</td>
</tr>
<tr>
<td>Theorem</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>LIVHS</td>
<td>Linearly Independent Vectors and Homogeneous Systems</td>
<td>147</td>
</tr>
<tr>
<td>LIVRN</td>
<td>Linearly Independent Vectors, r and n</td>
<td>149</td>
</tr>
<tr>
<td>MVSLD</td>
<td>More Vectors than Size implies Linear Dependence</td>
<td>150</td>
</tr>
<tr>
<td>NMLIC</td>
<td>Nonsingular Matrices have Linearly Independent Columns</td>
<td>151</td>
</tr>
<tr>
<td>NME2</td>
<td>Nonsingular Matrix Equivalences, Round 2</td>
<td>152</td>
</tr>
<tr>
<td>BNS</td>
<td>Basis for Null Spaces</td>
<td>154</td>
</tr>
<tr>
<td></td>
<td>Section LDS</td>
<td></td>
</tr>
<tr>
<td>DLDS</td>
<td>Dependency in Linearly Dependent Sets</td>
<td>167</td>
</tr>
<tr>
<td>BS</td>
<td>Basis of a Span</td>
<td>172</td>
</tr>
<tr>
<td></td>
<td>Section O</td>
<td></td>
</tr>
<tr>
<td>CRVA</td>
<td>Conjugation Respects Vector Addition</td>
<td>183</td>
</tr>
<tr>
<td>CRSM</td>
<td>Conjugation Respects Vector Scalar Multiplication</td>
<td>184</td>
</tr>
<tr>
<td>IPVA</td>
<td>Inner Product and Vector Addition</td>
<td>185</td>
</tr>
<tr>
<td>IPSM</td>
<td>Inner Product and Scalar Multiplication</td>
<td>186</td>
</tr>
<tr>
<td>IPAC</td>
<td>Inner Product is Anti-Commutative</td>
<td>186</td>
</tr>
<tr>
<td>IPN</td>
<td>Inner Products and Norms</td>
<td>187</td>
</tr>
<tr>
<td>PIP</td>
<td>Positive Inner Products</td>
<td>188</td>
</tr>
<tr>
<td>OSLI</td>
<td>Orthogonal Sets are Linearly Independent</td>
<td>190</td>
</tr>
<tr>
<td>GSPCV</td>
<td>Gram-Schmidt Procedure, Column Vectors</td>
<td>191</td>
</tr>
<tr>
<td></td>
<td>Section MO</td>
<td></td>
</tr>
<tr>
<td>VSPM</td>
<td>Vector Space Properties of Matrices</td>
<td>199</td>
</tr>
<tr>
<td>SMS</td>
<td>Symmetric Matrices are Square</td>
<td>201</td>
</tr>
<tr>
<td>TMA</td>
<td>Transpose and Matrix Addition</td>
<td>202</td>
</tr>
<tr>
<td>TMSM</td>
<td>Transpose and Matrix Scalar Multiplication</td>
<td>202</td>
</tr>
<tr>
<td>TT</td>
<td>Transpose of a Transpose</td>
<td>202</td>
</tr>
<tr>
<td>CRMA</td>
<td>Conjugation Respects Matrix Addition</td>
<td>203</td>
</tr>
<tr>
<td>CRMSM</td>
<td>Conjugation Respects Matrix Scalar Multiplication</td>
<td>204</td>
</tr>
<tr>
<td>MCT</td>
<td>Matrix Conjugation and Transposes</td>
<td>204</td>
</tr>
<tr>
<td></td>
<td>Section MM</td>
<td></td>
</tr>
<tr>
<td>SLEMM</td>
<td>Systems of Linear Equations as Matrix Multiplication</td>
<td>212</td>
</tr>
<tr>
<td>EMMVP</td>
<td>Equal Matrices and Matrix-Vector Products</td>
<td>214</td>
</tr>
<tr>
<td>EMP</td>
<td>Entries of Matrix Products</td>
<td>216</td>
</tr>
<tr>
<td>MMZM</td>
<td>Matrix Multiplication and the Zero Matrix</td>
<td>218</td>
</tr>
<tr>
<td>MMIM</td>
<td>Matrix Multiplication and Identity Matrix</td>
<td>218</td>
</tr>
<tr>
<td>MMDAA</td>
<td>Matrix Multiplication Distributes Across Addition</td>
<td>219</td>
</tr>
<tr>
<td>MMSMM</td>
<td>Matrix Multiplication and Scalar Matrix Multiplication</td>
<td>219</td>
</tr>
<tr>
<td>MMA</td>
<td>Matrix Multiplication is Associative</td>
<td>220</td>
</tr>
<tr>
<td>MMIP</td>
<td>Matrix Multiplication and Inner Products</td>
<td>220</td>
</tr>
<tr>
<td>MMC0</td>
<td>Matrix Multiplication and Complex Conjugation</td>
<td>221</td>
</tr>
<tr>
<td>MMT</td>
<td>Matrix Multiplication and Transposes</td>
<td>221</td>
</tr>
<tr>
<td></td>
<td>Section MISLE</td>
<td></td>
</tr>
<tr>
<td>TTMi</td>
<td>Two-by-Two Matrix Inverse</td>
<td>234</td>
</tr>
<tr>
<td>CINM</td>
<td>Computing the Inverse of a Nonsingular Matrix</td>
<td>237</td>
</tr>
<tr>
<td>Theorem</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>MIU</td>
<td>Matrix Inverse is Unique</td>
<td>239</td>
</tr>
<tr>
<td>SS</td>
<td>Socks and Shoes</td>
<td>239</td>
</tr>
<tr>
<td>MIMI</td>
<td>Matrix Inverse of a Matrix Inverse</td>
<td>240</td>
</tr>
<tr>
<td>MIT</td>
<td>Matrix Inverse of a Transpose</td>
<td>240</td>
</tr>
<tr>
<td>MISM</td>
<td>Matrix Inverse of a Scalar Multiple</td>
<td>240</td>
</tr>
<tr>
<td>NPNT</td>
<td>Nonsingular Product has Nonsingular Terms</td>
<td>249</td>
</tr>
<tr>
<td>OSIS</td>
<td>One-Sided Inverse is Sufficient</td>
<td>250</td>
</tr>
<tr>
<td>NI</td>
<td>Nonsingularity is Invertibility</td>
<td>251</td>
</tr>
<tr>
<td>NME3</td>
<td>Nonsingular Matrix Equivalences, Round 3</td>
<td>251</td>
</tr>
<tr>
<td>SNCECM</td>
<td>Solution with Nonsingular Coefficient Matrix</td>
<td>252</td>
</tr>
<tr>
<td>UMI</td>
<td>Unitary Matrices are Invertible</td>
<td>253</td>
</tr>
<tr>
<td>CUMOS</td>
<td>Columns of Unitary Matrices are Orthonormal Sets</td>
<td>253</td>
</tr>
<tr>
<td>UMPIP</td>
<td>Unitary Matrices Preserve Inner Products</td>
<td>254</td>
</tr>
<tr>
<td>CSCS</td>
<td>Column Spaces and Consistent Systems</td>
<td>262</td>
</tr>
<tr>
<td>BCS</td>
<td>Basis of the Column Space</td>
<td>264</td>
</tr>
<tr>
<td>CSNM</td>
<td>Column Space of a Nonsingular Matrix</td>
<td>267</td>
</tr>
<tr>
<td>NME4</td>
<td>Nonsingular Matrix Equivalences, Round 4</td>
<td>267</td>
</tr>
<tr>
<td>REMRS</td>
<td>Row-Equivalent Matrices have equal Row Spaces</td>
<td>269</td>
</tr>
<tr>
<td>BRS</td>
<td>Basis for the Row Space</td>
<td>271</td>
</tr>
<tr>
<td>CSRST</td>
<td>Column Space, Row Space, Transpose</td>
<td>273</td>
</tr>
<tr>
<td>PEEF</td>
<td>Properties of Extended Echelon Form</td>
<td>288</td>
</tr>
<tr>
<td>FS</td>
<td>Four Subsets</td>
<td>290</td>
</tr>
<tr>
<td>ZVU</td>
<td>Zero Vector is Unique</td>
<td>317</td>
</tr>
<tr>
<td>AIU</td>
<td>Additive Inverses are Unique</td>
<td>317</td>
</tr>
<tr>
<td>ZSSSM</td>
<td>Zero Scalar in Scalar Multiplication</td>
<td>317</td>
</tr>
<tr>
<td>ZVSM</td>
<td>Zero Vector in Scalar Multiplication</td>
<td>318</td>
</tr>
<tr>
<td>AISM</td>
<td>Additive Inverses from Scalar Multiplication</td>
<td>318</td>
</tr>
<tr>
<td>SMEZV</td>
<td>Scalar Multiplication Equals the Zero Vector</td>
<td>319</td>
</tr>
<tr>
<td>VAC</td>
<td>Vector Addition Cancellation</td>
<td>319</td>
</tr>
<tr>
<td>CSSM</td>
<td>Canceling Scalars in Scalar Multiplication</td>
<td>320</td>
</tr>
<tr>
<td>CVSM</td>
<td>Canceling Vectors in Scalar Multiplication</td>
<td>320</td>
</tr>
<tr>
<td>TSS</td>
<td>Testing Subsets for Subspaces</td>
<td>327</td>
</tr>
<tr>
<td>NSMS</td>
<td>Null Space of a Matrix is a Subspace</td>
<td>330</td>
</tr>
<tr>
<td>SSS</td>
<td>Span of a Set is a Subspace</td>
<td>332</td>
</tr>
<tr>
<td>CSMS</td>
<td>Column Space of a Matrix is a Subspace</td>
<td>336</td>
</tr>
<tr>
<td>RSMS</td>
<td>Row Space of a Matrix is a Subspace</td>
<td>337</td>
</tr>
<tr>
<td>LNSMS</td>
<td>Left Null Space of a Matrix is a Subspace</td>
<td>337</td>
</tr>
</tbody>
</table>
### Theorems

<table>
<thead>
<tr>
<th>Section</th>
<th>Theorem</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>EE</td>
<td>NME7</td>
<td>433</td>
</tr>
<tr>
<td></td>
<td>DRMM</td>
<td>434</td>
</tr>
<tr>
<td></td>
<td>EMHE</td>
<td>445</td>
</tr>
<tr>
<td></td>
<td>EMRCP</td>
<td>449</td>
</tr>
<tr>
<td></td>
<td>EMS</td>
<td>450</td>
</tr>
<tr>
<td></td>
<td>EMNS</td>
<td>451</td>
</tr>
<tr>
<td>PEE</td>
<td>EDELI</td>
<td>467</td>
</tr>
<tr>
<td></td>
<td>SMZE</td>
<td>468</td>
</tr>
<tr>
<td></td>
<td>NME8</td>
<td>468</td>
</tr>
<tr>
<td></td>
<td>ESMM</td>
<td>469</td>
</tr>
<tr>
<td></td>
<td>EOMP</td>
<td>469</td>
</tr>
<tr>
<td></td>
<td>EPM</td>
<td>470</td>
</tr>
<tr>
<td></td>
<td>EIM</td>
<td>471</td>
</tr>
<tr>
<td></td>
<td>ETM</td>
<td>471</td>
</tr>
<tr>
<td></td>
<td>ERMCP</td>
<td>472</td>
</tr>
<tr>
<td></td>
<td>DCP</td>
<td>472</td>
</tr>
<tr>
<td></td>
<td>NEM</td>
<td>473</td>
</tr>
<tr>
<td></td>
<td>ME</td>
<td>474</td>
</tr>
<tr>
<td></td>
<td>MNEM</td>
<td>475</td>
</tr>
<tr>
<td></td>
<td>HMRE</td>
<td>476</td>
</tr>
<tr>
<td></td>
<td>HMOE</td>
<td>477</td>
</tr>
<tr>
<td>SD</td>
<td>SER</td>
<td>485</td>
</tr>
<tr>
<td></td>
<td>SMEE</td>
<td>485</td>
</tr>
<tr>
<td></td>
<td>DC</td>
<td>487</td>
</tr>
<tr>
<td></td>
<td>DMPE</td>
<td>490</td>
</tr>
<tr>
<td></td>
<td>DED</td>
<td>492</td>
</tr>
<tr>
<td>LT</td>
<td>LTTZZ</td>
<td>507</td>
</tr>
<tr>
<td></td>
<td>MBLT</td>
<td>509</td>
</tr>
<tr>
<td></td>
<td>MLTCV</td>
<td>510</td>
</tr>
<tr>
<td></td>
<td>LTLC</td>
<td>512</td>
</tr>
<tr>
<td></td>
<td>LTDB</td>
<td>512</td>
</tr>
<tr>
<td></td>
<td>SLTLT</td>
<td>517</td>
</tr>
<tr>
<td></td>
<td>MLTLT</td>
<td>518</td>
</tr>
<tr>
<td></td>
<td>VSLT</td>
<td>519</td>
</tr>
<tr>
<td></td>
<td>CLTLT</td>
<td>520</td>
</tr>
<tr>
<td>ILT</td>
<td>KLTS</td>
<td>533</td>
</tr>
<tr>
<td></td>
<td>KPI</td>
<td>535</td>
</tr>
<tr>
<td></td>
<td>KILT</td>
<td>535</td>
</tr>
</tbody>
</table>

---

**Version 0.92**
### Section SLT
- **RLTS** Range of a Linear Transformation is a Subspace ........................................ 553
- **RSLT** Range of a Surjective Linear Transformation.................................................. 554
- **SSRLT** Spanning Set for Range of a Linear Transformation........................................ 556
- **RPI** Range and Pre-Image.............................................................................................. 557
- **SLTB** Surjective Linear Transformations and Bases......................................................... 557
- **SLTD** Surjective Linear Transformations and Dimension............................................. 558
- **CSLTS** Composition of Surjective Linear Transformations is Surjective................. 559

### Section IVLT
- **ILTIT** Inverse of a Linear Transformation is a Linear Transformation ......................... 570
- **ILTI** Inverse of an Invertible Linear Transformation .................................................. 570
- **ILTIS** Invertible Linear Transformations are Injective and Surjective......................... 571
- **CIVLT** Composition of Invertible Linear Transformations........................................... 572
- **ICLT** Inverse of a Composition of Linear Transformations........................................... 572
- **IVSED** Isomorphic Vector Spaces have Equal Dimension........................................... 575
- **ROSLT** Rank Of a Surjective Linear Transformation ................................................... 575
- **NOILT** Nullity Of an Injective Linear Transformation .................................................. 576
- **RPNDD** Rank Plus Nullity is Domain Dimension ........................................................... 576

### Section VR
- **VRLT** Vector Representation is a Linear Transformation............................................... 587
- **VRI** Vector Representation is Injective ........................................................................... 592
- **VRS** Vector Representation is Surjective ....................................................................... 592
- **VRILT** Vector Representation is an Invertible Linear Transformation ......................... 593
- **CFDVS** Characterization of Finite Dimensional Vector Spaces .................................... 593
- **IFDVS** Isomorphism of Finite Dimensional Vector Spaces ........................................... 594
- **CLI** Coordinatization and Linear Independence .............................................................. 595
- **CSS** Coordinatization and Spanning Sets ....................................................................... 595

### Section MR
- **FTMR** Fundamental Theorem of Matrix Representation ............................................. 606
- **MRSRLT** Matrix Representation of a Sum of Linear Transformations ......................... 609
- **MRMLT** Matrix Representation of a Multiple of a Linear Transformation .................... 610
- **MRCI** Matrix Representation of a Composition of Linear Transformations .................. 610
- **KNSI** Kernel and Null Space Isomorphism .................................................................... 614
- **RCSI** Range and Column Space Isomorphism ................................................................. 617
- **IMR** Invertible Matrix Representations ......................................................................... 619
- **IMILT** Invertible Matrices, Invertible Linear Transformation ........................................ 621
- **NME9** Nonsingular Matrix Equivalences, Round 9 ....................................................... 622

### Section CB
- **CB** Change-of-Basis ....................................................................................................... 639
<table>
<thead>
<tr>
<th>Section</th>
<th>Theorem Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>NLT</td>
<td>ENLT: Eigenvalues of Nilpotent Linear Transformations</td>
<td>672</td>
</tr>
<tr>
<td></td>
<td>DNLT: Diagonalizable Nilpotent Linear Transformations</td>
<td>673</td>
</tr>
<tr>
<td></td>
<td>KPLT: Kernels of Powers of Linear Transformations</td>
<td>673</td>
</tr>
<tr>
<td></td>
<td>KPNLT: Kernels of Powers of Nilpotent Linear Transformations</td>
<td>675</td>
</tr>
<tr>
<td>CNO</td>
<td>PCNA: Properties of Complex Number Arithmetic</td>
<td>688</td>
</tr>
<tr>
<td></td>
<td>CCRA: Complex Conjugation Respects Addition</td>
<td>690</td>
</tr>
<tr>
<td></td>
<td>CCRM: Complex Conjugation Respects Multiplication</td>
<td>690</td>
</tr>
<tr>
<td></td>
<td>CCT: Complex Conjugation Twice</td>
<td>690</td>
</tr>
</tbody>
</table>
Notation

\begin{tabular}{lll}
\textbf{M} & : Matrix & \textcolor{red}{29} \\
\textbf{MC} & [A]_{ij} : Matrix Components & \textcolor{red}{29} \\
\textbf{CV} & v : Column Vector & \textcolor{red}{30} \\
\textbf{CVC} & [v]_{i} : Column Vector Components & \textcolor{red}{30} \\
\textbf{ZCV} & 0 : Zero Column Vector & \textcolor{red}{30} \\
\textbf{LSMR} & \mathcal{L}(A, b) : Matrix Representation of a Linear System & \textcolor{red}{32} \\
\textbf{AM} & [A | b] : Augmented Matrix & \textcolor{red}{32} \\
\textbf{RO} & R_i \leftrightarrow R_j, \alpha R_i + R_j : Row Operations & \textcolor{red}{33} \\
\textbf{RREFA} & r, D, F : Reduced Row-Echelon Form Analysis & \textcolor{red}{36} \\
\textbf{NSM} & \mathcal{N}(A) : Null Space of a Matrix & \textcolor{red}{68} \\
\textbf{IM} & I_m : Identity Matrix & \textcolor{red}{76} \\
\textbf{VSCV} & \mathbb{C}^m : Vector Space of Column Vectors & \textcolor{red}{87} \\
\textbf{CVE} & u = v : Column Vector Equality & \textcolor{red}{88} \\
\textbf{CVA} & u + v : Column Vector Addition & \textcolor{red}{89} \\
\textbf{CVSM} & \alpha u : Column Vector Scalar Multiplication & \textcolor{red}{89} \\
\textbf{SSV} & (S) : Span of a Set of Vectors & \textcolor{red}{123} \\
\textbf{CCCV} & \overline{u} : Complex Conjugate of a Column Vector & \textcolor{red}{183} \\
\textbf{IP} & (u, v) : Inner Product & \textcolor{red}{184} \\
\textbf{NV} & \|v\| : Norm of a Vector & \textcolor{red}{187} \\
\textbf{VSM} & M_{mn} : Vector Space of Matrices & \textcolor{red}{197} \\
\textbf{ME} & A = B : Matrix Equality & \textcolor{red}{197} \\
\textbf{MA} & A + B : Matrix Addition & \textcolor{red}{198} \\
\textbf{MSM} & \alpha A : Matrix Scalar Multiplication & \textcolor{red}{198} \\
\textbf{ZM} & 0 : Zero Matrix & \textcolor{red}{200} \\
\textbf{TM} & A^t : Transpose of a Matrix & \textcolor{red}{201} \\
\textbf{CCM} & A : Complex Conjugate of a Matrix & \textcolor{red}{203} \\
\textbf{MVP} & Au : Matrix-Vector Product & \textcolor{red}{211} \\
\textbf{MI} & A^{-1} : Matrix Inverse & \textcolor{red}{232} \\
\textbf{CSM} & \mathcal{C}(A) : Column Space of a Matrix & \textcolor{red}{261} \\
\textbf{RSM} & \mathcal{R}(A) : Row Space of a Matrix & \textcolor{red}{268} \\
\textbf{LNS} & \mathcal{L}(A) : Left Null Space & \textcolor{red}{283} \\
\textbf{D} & \text{dim}(V) : Dimension & \textcolor{red}{379} \\
\textbf{NOM} & n(A) : Nullity of a Matrix & \textcolor{red}{385} \\
\textbf{ROM} & r(A) : Rank of a Matrix & \textcolor{red}{385} \\
\textbf{ELEM} & E_{ij}, E_i(\alpha), E_{ij}(\alpha) : Elementary Matrix & \textcolor{red}{410} \\
\textbf{SM} & A(ij) : SubMatrix & \textcolor{red}{414} \\
\textbf{DM} & \text{det}(A), |A| : Determinant of a Matrix & \textcolor{red}{414} \\
\textbf{LT} & T : U \rightarrow V : Linear Transformation & \textcolor{red}{503} \\
\textbf{KLT} & \mathcal{K}(T) : Kernel of a Linear Transformation & \textcolor{red}{532} \\
\textbf{RLT} & \mathcal{R}(T) : Range of a Linear Transformation & \textcolor{red}{551} \\
\end{tabular}
<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ROLT</td>
<td>$r(T)$: Rank of a Linear Transformation</td>
<td>575</td>
</tr>
<tr>
<td>NOLT</td>
<td>$n(T)$: Nullity of a Linear Transformation</td>
<td>575</td>
</tr>
<tr>
<td>JB</td>
<td>$J_n(\lambda)$: Jordan Block</td>
<td>670</td>
</tr>
<tr>
<td>CNE</td>
<td>$\alpha = \beta$: Complex Number Equality</td>
<td>688</td>
</tr>
<tr>
<td>CNA</td>
<td>$\alpha + \beta$: Complex Number Addition</td>
<td>688</td>
</tr>
<tr>
<td>CNM</td>
<td>$\alpha \beta$: Complex Number Multiplication</td>
<td>688</td>
</tr>
<tr>
<td>CCN</td>
<td>$\overline{c}$: Conjugate of a Complex Number</td>
<td>689</td>
</tr>
<tr>
<td>SETM</td>
<td>$x \in S$: Set Membership</td>
<td>693</td>
</tr>
<tr>
<td>SSET</td>
<td>$S \subseteq T$: Subset</td>
<td>693</td>
</tr>
<tr>
<td>ES</td>
<td>$\emptyset$: Empty Set</td>
<td>693</td>
</tr>
<tr>
<td>SE</td>
<td>$S = T$: Set Equality</td>
<td>694</td>
</tr>
<tr>
<td>C</td>
<td>$</td>
<td>S</td>
</tr>
<tr>
<td>SU</td>
<td>$S \cup T$: Set Union</td>
<td>695</td>
</tr>
<tr>
<td>SI</td>
<td>$S \cap T$: Set Intersection</td>
<td>696</td>
</tr>
<tr>
<td>SC</td>
<td>$\overline{S}$: Set Complement</td>
<td>696</td>
</tr>
<tr>
<td>Acronym</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>S</td>
<td>A singular matrix, Archetype A</td>
<td>76</td>
</tr>
<tr>
<td>NM</td>
<td>A nonsingular matrix, Archetype B</td>
<td>76</td>
</tr>
<tr>
<td>IM</td>
<td>An identity matrix</td>
<td>76</td>
</tr>
<tr>
<td>SHR</td>
<td>Singular matrix, row-reduced</td>
<td>77</td>
</tr>
<tr>
<td>NSR</td>
<td>Nonsingular matrix, row-reduced</td>
<td>77</td>
</tr>
<tr>
<td>NSS</td>
<td>Null space of a singular matrix</td>
<td>78</td>
</tr>
<tr>
<td>NSNM</td>
<td>Null space of a nonsingular matrix</td>
<td>78</td>
</tr>
<tr>
<td>VESE</td>
<td>Vector equality for a system of equations</td>
<td>88</td>
</tr>
<tr>
<td>VA</td>
<td>Addition of two vectors in ( \mathbb{C}^4 )</td>
<td>89</td>
</tr>
<tr>
<td>CVSM</td>
<td>Scalar multiplication in ( \mathbb{C}^5 )</td>
<td>90</td>
</tr>
<tr>
<td>TLC</td>
<td>Two linear combinations in ( \mathbb{C}^6 )</td>
<td>97</td>
</tr>
<tr>
<td>ABLC</td>
<td>Archetype B as a linear combination</td>
<td>98</td>
</tr>
<tr>
<td>AALC</td>
<td>Archetype A as a linear combination</td>
<td>99</td>
</tr>
<tr>
<td>VFSAD</td>
<td>Vector form of solutions for Archetype D</td>
<td>102</td>
</tr>
<tr>
<td>VFS</td>
<td>Vector form of solutions</td>
<td>104</td>
</tr>
<tr>
<td>VFSAI</td>
<td>Vector form of solutions for Archetype I</td>
<td>110</td>
</tr>
<tr>
<td>VFSAL</td>
<td>Vector form of solutions for Archetype L</td>
<td>111</td>
</tr>
<tr>
<td>PSHS</td>
<td>Particular solutions, homogeneous solutions, Archetype D</td>
<td>114</td>
</tr>
<tr>
<td>ABS</td>
<td>A basic span</td>
<td>123</td>
</tr>
<tr>
<td>SCAA</td>
<td>Span of the columns of Archetype A</td>
<td>125</td>
</tr>
<tr>
<td>SCAB</td>
<td>Span of the columns of Archetype B</td>
<td>127</td>
</tr>
<tr>
<td>SSNS</td>
<td>Spanning set of a null space</td>
<td>130</td>
</tr>
<tr>
<td>NSDS</td>
<td>Null space directly as a span</td>
<td>131</td>
</tr>
<tr>
<td>SCAD</td>
<td>Span of the columns of Archetype D</td>
<td>132</td>
</tr>
<tr>
<td>LDS</td>
<td>Linearly dependent set in ( \mathbb{C}^5 )</td>
<td>145</td>
</tr>
<tr>
<td>LIS</td>
<td>Linearly independent set in ( \mathbb{C}^5 )</td>
<td>147</td>
</tr>
<tr>
<td>LIHS</td>
<td>Linearly independent, homogeneous system</td>
<td>148</td>
</tr>
<tr>
<td>LDHS</td>
<td>Linearly dependent, homogeneous system</td>
<td>149</td>
</tr>
<tr>
<td>LDRN</td>
<td>Linearly dependent, ( r &lt; n )</td>
<td>150</td>
</tr>
<tr>
<td>LLDS</td>
<td>Large linearly dependent set in ( \mathbb{C}^4 )</td>
<td>150</td>
</tr>
<tr>
<td>LDCAA</td>
<td>Linearly dependent columns in Archetype A</td>
<td>151</td>
</tr>
<tr>
<td>LICAB</td>
<td>Linearly independent columns in Archetype B</td>
<td>151</td>
</tr>
<tr>
<td>LINSB</td>
<td>Linearly independence of null space basis</td>
<td>152</td>
</tr>
<tr>
<td>NSLIL</td>
<td>Null space spanned by linearly independent set, Archetype L</td>
<td>155</td>
</tr>
<tr>
<td>RSC5</td>
<td>Reducing a span in ( \mathbb{C}^5 )</td>
<td>168</td>
</tr>
<tr>
<td>COV</td>
<td>Casting out vectors</td>
<td>170</td>
</tr>
<tr>
<td>RSSC4</td>
<td>Reducing a span in ( \mathbb{C}^4 )</td>
<td>174</td>
</tr>
<tr>
<td>RES</td>
<td>Reworking elements of a span</td>
<td>175</td>
</tr>
<tr>
<td>Section</td>
<td>Example</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>O</td>
<td>CSIP Computing some inner products</td>
<td>184</td>
</tr>
<tr>
<td></td>
<td>CNSV Computing the norm of some vectors</td>
<td>187</td>
</tr>
<tr>
<td></td>
<td>TOV Two orthogonal vectors</td>
<td>189</td>
</tr>
<tr>
<td></td>
<td>SUVOS Standard Unit Vectors are an Orthogonal Set</td>
<td>189</td>
</tr>
<tr>
<td></td>
<td>AOS An orthogonal set</td>
<td>189</td>
</tr>
<tr>
<td></td>
<td>GSTV Gram-Schmidt of three vectors</td>
<td>192</td>
</tr>
<tr>
<td></td>
<td>ONTV Orthonormal set, three vectors</td>
<td>193</td>
</tr>
<tr>
<td></td>
<td>ONFV Orthonormal set, four vectors</td>
<td>194</td>
</tr>
<tr>
<td>MO</td>
<td>MA Addition of two matrices in $M_{23}$</td>
<td>198</td>
</tr>
<tr>
<td></td>
<td>MSM Scalar multiplication in $M_{32}$</td>
<td>198</td>
</tr>
<tr>
<td></td>
<td>TM Transpose of a $3 \times 4$ matrix</td>
<td>201</td>
</tr>
<tr>
<td></td>
<td>SYM A symmetric $5 \times 5$ matrix</td>
<td>201</td>
</tr>
<tr>
<td></td>
<td>CCM Complex conjugate of a matrix</td>
<td>203</td>
</tr>
<tr>
<td>MM</td>
<td>MTV A matrix times a vector</td>
<td>211</td>
</tr>
<tr>
<td></td>
<td>MNSLE Matrix notation for systems of linear equations</td>
<td>212</td>
</tr>
<tr>
<td></td>
<td>MBC Money’s best cities</td>
<td>212</td>
</tr>
<tr>
<td></td>
<td>PTM Product of two matrices</td>
<td>215</td>
</tr>
<tr>
<td></td>
<td>MMNC Matrix Multiplication is not commutative</td>
<td>216</td>
</tr>
<tr>
<td></td>
<td>PTMEE Product of two matrices, entry-by-entry</td>
<td>217</td>
</tr>
<tr>
<td>MISLE</td>
<td>SABMI Solutions to Archetype B with a matrix inverse</td>
<td>231</td>
</tr>
<tr>
<td></td>
<td>MWIAA A matrix without an inverse, Archetype A</td>
<td>232</td>
</tr>
<tr>
<td></td>
<td>MI Matrix Inverse</td>
<td>233</td>
</tr>
<tr>
<td></td>
<td>CMI Computing a Matrix Inverse</td>
<td>235</td>
</tr>
<tr>
<td></td>
<td>CMIAB Computing a Matrix Inverse, Archetype B</td>
<td>238</td>
</tr>
<tr>
<td>MINM</td>
<td>UM3 Unitary matrix of size 3</td>
<td>252</td>
</tr>
<tr>
<td></td>
<td>UPM Unitary permutation matrix</td>
<td>253</td>
</tr>
<tr>
<td></td>
<td>OSMC Orthonormal Set from Matrix Columns</td>
<td>254</td>
</tr>
<tr>
<td>CRS</td>
<td>CSMCS Column space of a matrix and consistent systems</td>
<td>261</td>
</tr>
<tr>
<td></td>
<td>MCSM Membership in the column space of a matrix</td>
<td>263</td>
</tr>
<tr>
<td></td>
<td>CSTW Column space, two ways</td>
<td>264</td>
</tr>
<tr>
<td></td>
<td>CSOCD Column space, original columns, Archetype D</td>
<td>265</td>
</tr>
<tr>
<td></td>
<td>CSAA Column space of Archetype A</td>
<td>266</td>
</tr>
<tr>
<td></td>
<td>CSAB Column space of Archetype B</td>
<td>267</td>
</tr>
<tr>
<td></td>
<td>RSAI Row space of Archetype I</td>
<td>268</td>
</tr>
<tr>
<td></td>
<td>RSREM Row spaces of two row-equivalent matrices</td>
<td>271</td>
</tr>
<tr>
<td></td>
<td>IAS Improving a span</td>
<td>272</td>
</tr>
<tr>
<td>Column</td>
<td>Description</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td>CSROI</td>
<td>Column space from row operations, Archetype I</td>
<td></td>
</tr>
<tr>
<td>LNS</td>
<td>Left null space</td>
<td></td>
</tr>
<tr>
<td>CSANS</td>
<td>Column space as null space</td>
<td></td>
</tr>
<tr>
<td>SEEFS</td>
<td>Submatrices of extended echelon form</td>
<td></td>
</tr>
<tr>
<td>FS1</td>
<td>Four subsets, #1</td>
<td></td>
</tr>
<tr>
<td>FS2</td>
<td>Four subsets, #2</td>
<td></td>
</tr>
<tr>
<td>FSAG</td>
<td>Four subsets, Archetype G</td>
<td></td>
</tr>
<tr>
<td>VSCV</td>
<td>The vector space $\mathbb{C}^m$</td>
<td></td>
</tr>
<tr>
<td>VSM</td>
<td>The vector space of matrices, $\mathbb{M}_{mn}$</td>
<td></td>
</tr>
<tr>
<td>VSP</td>
<td>The vector space of polynomials, $\mathbb{P}_n$</td>
<td></td>
</tr>
<tr>
<td>VSIS</td>
<td>The vector space of infinite sequences</td>
<td></td>
</tr>
<tr>
<td>VSF</td>
<td>The vector space of functions</td>
<td></td>
</tr>
<tr>
<td>VSS</td>
<td>The singleton vector space</td>
<td></td>
</tr>
<tr>
<td>CVS</td>
<td>The crazy vector space</td>
<td></td>
</tr>
<tr>
<td>PCVS</td>
<td>Properties for the Crazy Vector Space</td>
<td></td>
</tr>
<tr>
<td>SC3</td>
<td>A subspace of $\mathbb{C}^3$</td>
<td></td>
</tr>
<tr>
<td>SP4</td>
<td>A subspace of $\mathbb{P}_4$</td>
<td></td>
</tr>
<tr>
<td>NSC2Z</td>
<td>A non-subspace in $\mathbb{C}^2$, zero vector</td>
<td></td>
</tr>
<tr>
<td>NSC2A</td>
<td>A non-subspace in $\mathbb{C}^2$, additive closure</td>
<td></td>
</tr>
<tr>
<td>NSC2S</td>
<td>A non-subspace in $\mathbb{C}^2$, scalar multiplication closure</td>
<td></td>
</tr>
<tr>
<td>RSNS</td>
<td>Recasting a subspace as a null space</td>
<td></td>
</tr>
<tr>
<td>LCM</td>
<td>A linear combination of matrices</td>
<td></td>
</tr>
<tr>
<td>SSP</td>
<td>Span of a set of polynomials</td>
<td></td>
</tr>
<tr>
<td>SM32</td>
<td>A subspace of $\mathbb{M}_{32}$</td>
<td></td>
</tr>
<tr>
<td>LIP4</td>
<td>Linear independence in $\mathbb{P}_4$</td>
<td></td>
</tr>
<tr>
<td>LIM32</td>
<td>Linear Independence in $\mathbb{M}_{32}$</td>
<td></td>
</tr>
<tr>
<td>LIC</td>
<td>Linearly independent set in the crazy vector space</td>
<td></td>
</tr>
<tr>
<td>SSP4</td>
<td>Spanning set in $\mathbb{P}_4$</td>
<td></td>
</tr>
<tr>
<td>SSM22</td>
<td>Spanning set in $\mathbb{M}_{22}$</td>
<td></td>
</tr>
<tr>
<td>SSC</td>
<td>Spanning set in the crazy vector space</td>
<td></td>
</tr>
<tr>
<td>AVR</td>
<td>A vector representation</td>
<td></td>
</tr>
<tr>
<td>BP</td>
<td>Bases for $\mathbb{P}_n$</td>
<td></td>
</tr>
<tr>
<td>BM</td>
<td>A basis for the vector space of matrices</td>
<td></td>
</tr>
<tr>
<td>BSP4</td>
<td>A basis for a subspace of $\mathbb{P}_4$</td>
<td></td>
</tr>
<tr>
<td>BSM22</td>
<td>A basis for a subspace of $\mathbb{M}_{22}$</td>
<td></td>
</tr>
<tr>
<td>BC</td>
<td>Basis for the crazy vector space</td>
<td></td>
</tr>
<tr>
<td>RSB</td>
<td>Row space basis</td>
<td></td>
</tr>
<tr>
<td>RS</td>
<td>Reducing a span</td>
<td></td>
</tr>
</tbody>
</table>
CABAK Columns as Basis, Archetype K .......................... 369
CROB4 Coordinatization relative to an orthonormal basis, \( \mathbb{C}^4 \) .......................... 371
CROB3 Coordinatization relative to an orthonormal basis, \( \mathbb{C}^3 \) .......................... 372

Section D
LDP4 Linearly dependent set in \( P_4 \) .......................... 382
DSM22 Dimension of a subspace of \( M_{22} \) .......................... 383
DSP4 Dimension of a subspace of \( P_4 \) .......................... 384
DC Dimension of the crazy vector space .......................... 385
VSPUD Vector space of polynomials with unbounded degree .......................... 385
RNM Rank and nullity of a matrix .......................... 386
RNSM Rank and nullity of a square matrix .......................... 387

Section PD
BPR Bases for \( P_n \), reprised .......................... 399
BDM22 Basis by dimension in \( M_{22} \) .......................... 399
SVP4 Sets of vectors in \( P_4 \) .......................... 400
RRTI Rank, rank of transpose, Archetype I .......................... 402

Section DM
EMRO Elementary matrices and row operations .......................... 410
SS Some submatrices .......................... 414
D33M Determinant of a \( 3 \times 3 \) matrix .......................... 415
TCSD Two computations, same determinant .......................... 418
DUTM Determinant of an upper-triangular matrix .......................... 419

Section PDM
DRO Determinant by row operations .......................... 428
ZNDAB Zero and nonzero determinant, Archetypes A and B .......................... 433

Section EE
SEE Some eigenvalues and eigenvectors .......................... 442
PM Polynomial of a matrix .......................... 443
CAEHW Computing an eigenvalue the hard way .......................... 446
CPMS3 Characteristic polynomial of a matrix, size 3 .......................... 449
EMS3 Eigenvalues of a matrix, size 3 .......................... 450
ESMS3 Eigenspaces of a matrix, size 3 .......................... 451
EMMS4 Eigenvalue multiplicities, matrix of size 4 .......................... 453
ESMS4 Eigenvalues, symmetric matrix of size 4 .......................... 453
HMEM5 High multiplicity eigenvalues, matrix of size 5 .......................... 454
CEMS6 Complex eigenvalues, matrix of size 6 .......................... 455
DEMS5 Distinct eigenvalues, matrix of size 5 .......................... 457

Section PEE
BDE Building desired eigenvalues .......................... 470

Section SD
SMS5 Similar matrices of size 5 .......................... 483
Examples

SMS3  Similar matrices of size 3 .................................................. 484
EENS  Equal eigenvalues, not similar .............................................. 486
DAB  Diagonalization of Archetype B ................................................ 487
DMS3  Diagonalizing a matrix of size 3 .............................................. 489
NDMS4  A non-diagonalizable matrix of size 4 .................................... 491
DEHD  Distinct eigenvalues, hence diagonalizable .............................. 492
HPDM  High power of a diagonalizable matrix .................................... 493

Section LT
ALT  A linear transformation .......................................................... 504
NLT  Not a linear transformation ..................................................... 505
LTPM  Linear transformation, polynomials to matrices ....................... 506
LTPP  Linear transformation, polynomials to polynomials ................... 507
LTMM  Linear transformation from a matrix ....................................... 508
MFLT  Matrix from a linear transformation ........................................ 509
MOLT  Matrix of a linear transformation ........................................... 511
LTDB1  Linear transformation defined on a basis ............................... 513
LTDB2  Linear transformation defined on a basis ............................... 513
LTDB3  Linear transformation defined on a basis ............................... 514
SPIAS  Sample pre-images, Archetype S ............................................ 515
STLT  Sum of two linear transformations .......................................... 518
SMLT  Scalar multiple of a linear transformation ................................ 519
CTLT  Composition of two linear transformations ................................ 520

Section ILT
NIAQ  Not injective, Archetype Q ................................................... 529
IAR  Injective, Archetype R ............................................................ 530
IAV  Injective, Archetype V ............................................................ 531
NKAO  Nontrivial kernel, Archetype O ............................................... 532
TKAP  Trivial kernel, Archetype P .................................................... 534
NIAQR  Not injective, Archetype Q, revisited .................................... 536
NIAO  Not injective, Archetype O .................................................... 536
IAP  Injective, Archetype P ............................................................. 537
NIDAU  Not injective by dimension, Archetype U ............................... 538

Section SLT
NSAQ  Not surjective, Archetype Q .................................................. 547
SAR  Surjective, Archetype R ........................................................... 548
SAV  Surjective, Archetype V ........................................................... 550
RAO  Range, Archetype O ................................................................. 551
FRAN  Full range, Archetype N .......................................................... 553
NSAQR  Not surjective, Archetype Q, revisited ................................... 555
NSAO  Not surjective, Archetype O .................................................... 555
SAN  Surjective, Archetype N ............................................................ 556
BRLT  A basis for the range of a linear transformation ....................... 557
NSDAT  Not surjective by dimension, Archetype T .............................. 558

Section IVLT

Version 0.92
<table>
<thead>
<tr>
<th>EXAMPLES</th>
<th>xli</th>
</tr>
</thead>
</table>

| AIVLT | An invertible linear transformation | 567 |
| ANILT | A non-invertible linear transformation | 568 |
| IVSAV | Isomorphic vector spaces, Archetype V | 574 |

**Section VR**

| VRC4 | Vector representation in $\mathbb{C}^4$ | 589 |
| VRP2 | Vector representations in $P_2$ | 591 |
| TIVS | Two isomorphic vector spaces | 594 |
| CVSR | Crazy vector space revealed | 594 |
| ASC | A subspace characterized | 594 |
| MIVS | Multiple isomorphic vector spaces | 594 |
| CP2 | Coordinatizing in $P_2$ | 596 |
| CM32 | Coordinatization in $M_{32}$ | 597 |

**Section MR**

| OLTTR | One linear transformation, three representations | 603 |
| ALTMM | A linear transformation as matrix multiplication | 607 |
| MPMR | Matrix product of matrix representations | 611 |
| KVMR | Kernel via matrix representation | 615 |
| RVMR | Range via matrix representation | 618 |
| ILTVR | Inverse of a linear transformation via a representation | 620 |

**Section CB**

| ELTBM | Eigenvectors of linear transformation between matrices | 637 |
| ELTBP | Eigenvectors of linear transformation between polynomials | 638 |
| CBP | Change of basis with polynomials | 640 |
| CBCV | Change of basis with column vectors | 643 |
| MRCM | Matrix representations and change-of-basis matrices | 645 |
| MRBE | Matrix representation with basis with eigenvectors | 648 |
| ELTT | Eigenvectors of a linear transformation, twice | 651 |
| CELT | Complex eigenvectors of a linear transformation | 656 |

**Section NLT**

| NM64 | Nilpotent matrix, size 6, index 4 | 668 |
| NM62 | Nilpotent matrix, size 6, index 2 | 669 |
| JB4 | Jordan block, size 4 | 670 |
| acronym | Nilpotent Jordan block, size 5 | 670 |
| NM83 | Nilpotent matrix, size 8, index 3 | 671 |
| KPNLT | Kernels of Powers of a Nilpotent Linear Transformation | 675 |

**Section CNO**

| ACN | Arithmetic of complex numbers | 688 |
| CSCN | Conjugate of some complex numbers | 689 |
| MSCN | Modulus of some complex numbers | 690 |

**Section SET**

| SETM | Set membership | 693 |
| SSET | Subset | 693 |
Part C
Core
Chapter SLE
Systems of Linear Equations

We will motivate our study of linear algebra by studying solutions to systems of linear equations. While the focus of this chapter is on the practical matter of how to find, and describe, these solutions, we will also be setting ourselves up for more theoretical ideas that will appear later.

Section WILA
What is Linear Algebra?

Subsection LA
"Linear" + "Algebra"

The subject of linear algebra can be partially explained by the meaning of the two terms comprising the title. "Linear" is a term you will appreciate better at the end of this course, and indeed, attaining this appreciation could be taken as one of the primary goals of this course. However for now, you can understand it to mean anything that is "straight" or "flat." For example in the xy-plane you might be accustomed to describing straight lines (is there any other kind?) as the set of solutions to an equation of the form \( y = mx + b \), where the slope \( m \) and the \( y \)-intercept \( b \) are constants that together describe the line. In multivariate calculus, you may have discussed planes. Living in three dimensions, with coordinates described by triples \((x, y, z)\), they can be described as the set of solutions to equations of the form \( ax + by + cz = d \), where \( a, b, c, d \) are constants that together determine the plane. While we might describe planes as "flat," lines in three dimensions might be described as “straight.” From a multivariate calculus course you will recall that lines are sets of points described by equations such as \( x = 3t - 4, \ y = -7t + 2, \ z = 9t \), where \( t \) is a parameter that can take on any value.

Another view of this notion of “flatness” is to recognize that the sets of points just described are solutions to equations of a relatively simple form. These equations involve addition and multiplication only. We will have a need for subtraction, and occasionally we will divide, but mostly you can describe “linear” equations as involving only addition and multiplication. Here are some examples of typical equations we will see in the next
Section WILA What is Linear Algebra?

A few sections:

\[2x + 3y - 4z = 13 \quad 4x_1 + 5x_2 - x_3 + x_4 + x_5 = 0 \quad 9a - 2b + 7c + 2d = -7\]

What we will not see are equations like:

\[xy + 5yz = 13 \quad x_1 + x_2^3/x_4 - x_3x_4x_5^2 = 0 \quad \tan(ab) + \log(c - d) = -7\]

The exception will be that we will on occasion need to take a square root.

You have probably heard the word “algebra” frequently in your mathematical preparation for this course. Most likely, you have spent a good ten to fifteen years learning the algebra of the real numbers, along with some introduction to the very similar algebra of complex numbers (see Section CNO [687]). However, there are many new algebras to learn and use, and likely linear algebra will be your second algebra. Like learning a second language, the necessary adjustments can be challenging at times, but the rewards are many. And it will make learning your third and fourth algebras even easier. Perhaps you have heard of “groups” and “rings” (or maybe you have studied them already), which are excellent examples of other algebras with very interesting properties and applications. In any event, prepare yourself to learn a new algebra and realize that some of the old rules you used for the real numbers may no longer apply to this new algebra you will be learning!

The brief discussion above about lines and planes suggests that linear algebra has an inherently geometric nature, and this is true. Examples in two and three dimensions can be used to provide valuable insight into important concepts of this course. However, much of the power of linear algebra will be the ability to work with “flat” or “straight” objects in higher dimensions, without concerning ourselves with visualizing the situation. While much of our intuition will come from examples in two and three dimensions, we will maintain an algebraic approach to the subject, with the geometry being secondary. Others may wish to switch this emphasis around, and that can lead to a very fruitful and beneficial course, but here and now we are laying our bias bare.

Subsection A
An application: packaging trail mix

We conclude this section with a rather involved example that will highlight some of the power and techniques of linear algebra. Work through all of the details with pencil and paper, until you believe all the assertions made. However, in this introductory example, do not concern yourself with how some of the results are obtained or how you might be expected to solve a similar problem. We will come back to this example later and expose some of the techniques used and properties exploited. For now, use your background in mathematics to convince yourself that everything said here really is correct.

Example TMP
Trail Mix Packaging

Suppose you are the production manager at a food-packaging plant and one of your product lines is trail mix, a healthy snack popular with hikers and backpackers, containing raisins, peanuts and hard-shelled chocolate pieces. By adjusting the mix of these three ingredients, you are able to sell three varieties of this item. The fancy version is sold in
half-kilogram packages at outdoor supply stores and has more chocolate and fewer raisins, thus commanding a higher price. The standard version is sold in one kilogram packages in grocery stores and gas station mini-markets. Since the standard version has roughly equal amounts of each ingredient, it is not as expensive as the fancy version. Finally, a bulk version is sold in bins at grocery stores for consumers to load into plastic bags in amounts of their choosing. To appeal to the shoppers that like bulk items for their economy and healthfulness, this mix has many more raisins (at the expense of chocolate) and therefore sells for less.

Your production facilities have limited storage space and early each morning you are able to receive and store 380 kilograms of raisins, 500 kilograms of peanuts and 620 kilograms of chocolate pieces. As production manager, one of your most important duties is to decide how much of each version of trail mix to make every day. Clearly, you can have up to 1500 kilograms of raw ingredients available each day, so to be the most productive you will likely produce 1500 kilograms of trail mix each day. Also, you would prefer not to have any ingredients leftover each day, so that your final product is as fresh as possible and so that you can receive the maximum delivery the next morning. But how should these ingredients be allocated to the mixing of the bulk, standard and fancy versions?

First, we need a little more information about the mixes. Workers mix the ingredients in 15 kilogram batches, and each row of the table below gives a recipe for a 15 kilogram batch. There is some additional information on the costs of the ingredients and the price the manufacturer can charge for the different versions of the trail mix.

<table>
<thead>
<tr>
<th></th>
<th>Raisins (kg/batch)</th>
<th>Peanuts (kg/batch)</th>
<th>Chocolate (kg/batch)</th>
<th>Cost ($/kg)</th>
<th>Sale Price ($/kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bulk</td>
<td>7</td>
<td>6</td>
<td>2</td>
<td>3.69</td>
<td>4.99</td>
</tr>
<tr>
<td>Standard</td>
<td>6</td>
<td>4</td>
<td>5</td>
<td>3.86</td>
<td>5.50</td>
</tr>
<tr>
<td>Fancy</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>4.45</td>
<td>6.50</td>
</tr>
<tr>
<td>Storage (kg)</td>
<td>380</td>
<td>500</td>
<td>620</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cost ($/kg)</td>
<td>2.55</td>
<td>4.65</td>
<td>4.80</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As production manager, it is important to realize that you only have three decisions to make — the amount of bulk mix to make, the amount of standard mix to make and the amount of fancy mix to make. Everything else is beyond your control or is handled by another department within the company. Principally, you are also limited by the amount of raw ingredients you can store each day. Let us denote the amount of each mix to produce each day, measured in kilograms, by the variable quantities \( b \), \( s \) and \( f \). Your production schedule can be described as values of \( b \), \( s \) and \( f \) that do several things. First, we cannot make negative quantities of each mix, so

\[
b \geq 0 \quad s \geq 0 \quad f \geq 0.
\]

Second, if we want to consume all of our ingredients each day, the storage capacities lead to three (linear) equations, one for each ingredient,

\[
\frac{7}{15}b + \frac{6}{15}s + \frac{2}{15}f = 380 \quad \text{(raisins)}
\]
\[
\frac{6}{15}b + \frac{4}{15}s + \frac{5}{15}f = 500 \quad \text{(peanuts)}
\]
It happens that this system of three equations has just one solution. In other words, as production manager, your job is easy, since there is but one way to use up all of your raw ingredients making trail mix. This single solution is

\[ b = 300 \text{ kg} \quad s = 300 \text{ kg} \quad f = 900 \text{ kg}. \]

We do not yet have the tools to explain why this solution is the only one, but it should be simple for you to verify that this is indeed a solution. (Go ahead, we will wait.) Determining solutions such as this, and establishing that they are unique, will be the main motivation for our initial study of linear algebra.

So we have solved the problem of making sure that we make the best use of our limited storage space, and each day use up all of the raw ingredients that are shipped to us. Additionally, as production manager, you must report weekly to the CEO of the company, and you know he will be more interested in the profit derived from your decisions than in the actual production levels. So you compute,

\[ 300(4.99 - 3.69) + 300(5.50 - 3.86) + 900(6.50 - 4.45) = 2727.00 \]

for a daily profit of $2,727 from this production schedule. The computation of the daily profit is also beyond our control, though it is definitely of interest, and it too looks like a “linear” computation.

As often happens, things do not stay the same for long, and now the marketing department has suggested that your company’s trail mix products standardize on every mix being one-third peanuts. Adjusting the peanut portion of each recipe by also adjusting the chocolate portion, leads to revised recipes, and slightly different costs for the bulk and standard mixes, as given in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Raisins (kg/batch)</th>
<th>Peanuts (kg/batch)</th>
<th>Chocolate (kg/batch)</th>
<th>Cost ($/kg)</th>
<th>Sale Price ($/kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bulk</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>3.70</td>
<td>4.99</td>
</tr>
<tr>
<td>Standard</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3.85</td>
<td>5.50</td>
</tr>
<tr>
<td>Fancy</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>4.45</td>
<td>6.50</td>
</tr>
<tr>
<td>Storage (kg)</td>
<td>380</td>
<td>500</td>
<td>620</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cost ($/kg)</td>
<td>2.55</td>
<td>4.65</td>
<td>4.80</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In a similar fashion as before, we desire values of \( b \), \( s \) and \( f \) so that

\[ b \geq 0, \quad s \geq 0, \quad f \geq 0 \]

and

\[ \frac{7}{15} b + \frac{6}{15} s + \frac{2}{15} f = 380 \quad \text{(raisins)} \]
\[ \frac{5}{15} b + \frac{5}{15} s + \frac{5}{15} f = 500 \quad \text{(peanuts)} \]
\[ \frac{3}{15} b + \frac{4}{15} s + \frac{8}{15} f = 620 \quad \text{(chocolate)} \]
It now happens that this system of equations has infinitely many solutions, as we will now demonstrate. Let $f$ remain a variable quantity. Then if we make $f$ kilograms of the fancy mix, we will make $4f - 3300$ kilograms of the bulk mix and $-5f + 4800$ kilograms of the standard mix. Let us now verify that, for any choice of $f$, the values of $b = 4f - 3300$ and $s = -5f + 4800$ will yield a production schedule that exhausts all of the day’s supply of raw ingredients (right now, do not be concerned about how you might derive expressions like these for $b$ and $s$). Grab your pencil and paper and play along.

$$
\begin{align*}
\frac{7}{15}(4f - 3300) + \frac{6}{15}(-5f + 4800) + \frac{2}{15}f &= 0 \Rightarrow \frac{5700}{15} = 380 \\
\frac{5}{15}(4f - 3300) + \frac{5}{15}(-5f + 4800) + \frac{5}{15}f &= 0 \Rightarrow \frac{7500}{15} = 500 \\
\frac{3}{15}(4f - 3300) + \frac{4}{15}(-5f + 4800) + \frac{8}{15}f &= 0 \Rightarrow \frac{9300}{15} = 620 
\end{align*}
$$

Convince yourself that these expressions for $b$ and $s$ allow us to vary $f$ and obtain an infinite number of possibilities for solutions to the three equations that describe our storage capacities. As a practical matter, there really are not an infinite number of solutions, since we are unlikely to want to end the day with a fractional number of bags of fancy mix, so our allowable values of $f$ should probably be integers. More importantly, we need to remember that we cannot make negative amounts of each mix! Where does this lead us? Positive quantities of the bulk mix requires that

$$b \geq 0 \Rightarrow 4f - 3300 \geq 0 \Rightarrow f \geq 825.$$

Similarly for the standard mix,

$$s \geq 0 \Rightarrow -5f + 4800 \geq 0 \Rightarrow f \leq 960.$$

So, as production manager, you really have to choose a value of $f$ from the finite set $\{825, 826, \ldots , 960\}$ leaving you with 136 choices, each of which will exhaust the day’s supply of raw ingredients. Pause now and think about which you would choose.

Recalling your weekly meeting with the CEO suggests that you might want to choose a production schedule that yields the biggest possible profit for the company. So you compute an expression for the profit based on your as yet undetermined decision for the value of $f$,


Since $f$ has a negative coefficient it would appear that mixing fancy mix is detrimental to your profit and should be avoided. So you will make the decision to set daily fancy mix production at $f = 825$. This has the effect of setting $b = 4(825) - 3300 = 0$ and we stop producing bulk mix entirely. So the remainder of your daily production is standard mix at the level of $s = -5(825) + 4800 = 675$ kilograms and the resulting daily profit is $(-1.04)(825) + 3663 = 2805$. It is a pleasant surprise that daily profit has risen to $2,805, but this is not the most important part of the story. What is important here is that there are a large number of ways to produce trail mix that use all of the day’s worth of raw ingredients and you were able to easily choose the one that netted the largest profit. Notice too how all of the above computations look “linear.”
In the food industry, things do not stay the same for long, and now the sales department says that increased competition has led to the decision to stay competitive and charge just $5.25 for a kilogram of the standard mix, rather than the previous $5.50 per kilogram. This decision has no effect on the possibilities for the production schedule, but will affect the decision based on profit considerations. So you revisit just the profit computation, suitably adjusted for the new selling price of standard mix,

\[(4f - 3300)(4.99 - 3.70) + (-5f + 4800)(5.25 - 3.85) + (f)(6.50 - 4.45) = 0.21f + 2463.\]

Now it would appear that fancy mix is beneficial to the company’s profit since the value of \(f\) has a positive coefficient. So you take the decision to make as much fancy mix as possible, setting \(f = 960\). This leads to \(s = -5(960) + 4800 = 0\) and the increased competition has driven you out of the standard mix market all together. The remainder of production is therefore bulk mix at a daily level of \(b = 4(960) - 3300 = 540\) kilograms and the resulting daily profit is \(0.21(960) + 2463 = 2664.60\). A daily profit of $2,664.60 is less than it used to be, but as production manager, you have made the best of a difficult situation and shown the sales department that the best course is to pull out of the highly competitive standard mix market completely.

This example is taken from a field of mathematics variously known by names such as operations research, system science, or management science. More specifically, this is a perfect example of problems that are solved by the techniques of “linear programming.”

There is a lot going on under the hood in this example. The heart of the matter is the solution to systems of linear equations, which is the topic of the next few sections, and a recurrent theme throughout this course. We will return to this example on several occasions to reveal some of the reasons for its behavior.

**Subsection READ**

**Reading Questions**

1. Is the equation \(x^2 + xy + \tan(y^3) = 0\) linear or not? Why or why not?

2. Find all solutions to the system of two linear equations \(2x + 3y = -8, x - y = 6\).

3. Explain the importance of the procedures described in the trail mix application (Subsection WILA.A [4]) from the point-of-view of the production manager.
Subsection EXC
Exercises

C10 In [Example TMP][4] the first table lists the cost (per kilogram) to manufacture each of the three varieties of trail mix (bulk, standard, fancy). For example, it costs $3.70 to make one kilogram of the bulk variety. Re-compute each of these three costs and notice that the computations are linear in character. Contributed by Robert Beezer

M70 In [Example TMP][4] two different prices were considered for marketing standard mix with the revised recipes (one-third peanuts in each recipe). Selling standard mix at $5.50 resulted in selling the minimum amount of the fancy mix and no bulk mix. At $5.25 it was best for profits to sell the maximum amount of fancy mix and then sell no standard mix. Determine a selling price for standard mix that allows for maximum profits while still selling some of each type of mix. Contributed by Robert Beezer Solution [11]
M70  Contributed by Robert Beezer  Statement 9
If the price of standard mix is set at $5.292, then the profit function has a zero coefficient on the variable quantity \( f \). So, we can set \( f \) to be any integer quantity in \( \{825, 826, \ldots, 960\} \). All but the extreme values \( (f = 825, f = 960) \) will result in production levels where some of every mix is manufactured. No matter what value of \( f \) is chosen, the resulting profit will be the same, at $2,664.60.
We will motivate our study of linear algebra by considering the problem of solving several linear equations simultaneously. The word “solve” tends to get abused somewhat, as in “solve this problem.” When talking about equations we understand a more precise meaning: find all of the values of some variable quantities that make an equation, or several equations, true.

**Example STNE**

Solving two (nonlinear) equations

Suppose we desire the simultaneous solutions of the two equations,

\[
\begin{align*}
x^2 + y^2 &= 1 \\
-x + \sqrt{3}y &= 0
\end{align*}
\]

You can easily check by substitution that \(x = \frac{\sqrt{3}}{2}, y = \frac{1}{2}\) and \(x = -\frac{\sqrt{3}}{2}, y = -\frac{1}{2}\) are both solutions. We need to also convince ourselves that these are the only solutions. To see this, plot each equation on the \(xy\)-plane, which means to plot \((x, y)\) pairs that make an individual equation true. In this case we get a circle centered at the origin with radius 1 and a straight line through the origin with slope \(\frac{1}{\sqrt{3}}\). The intersections of these two curves are our desired simultaneous solutions, and so we believe from our plot that the two solutions we know already are the only ones. We like to write solutions as sets, so in this case we write the set of solutions as

\[
S = \{(\frac{\sqrt{3}}{2}, \frac{1}{2}), (-\frac{\sqrt{3}}{2}, -\frac{1}{2})\}
\]

In order to discuss systems of linear equations carefully, we need a precise definition. And before we do that, we will introduce our periodic discussions about “Proof Techniques.” Linear algebra is an excellent setting for learning how to read, understand and formulate proofs. But this is a difficult step in your development as a mathematician, so we have included a series of vignettes containing advice and explanations to help you along. These can be found back in Section PT [697] of Appendix P [687], and we will reference them as they become appropriate. Be sure to head back to the appendix to read this as they are introduced. With a definition next, now is the time for the first of our proof techniques. Head back to Section PT [697] of Appendix P [687] and study Technique D [698]. We’ll be right here when you get back. See you in a bit.

**Definition SLE**

System of Linear Equations

A system of linear equations is a collection of \(m\) equations in the variable quantities \(x_1, x_2, x_3, \ldots, x_n\) of the form,

\[
\begin{align*}
a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \cdots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \cdots + a_{2n} x_n &= b_2
\end{align*}
\]
Section SSLE  Solving Systems of Linear Equations

\[ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \]

where the values of \( a_{ij}, \ b_i \) and \( x_j \) are from the set of complex numbers, \( \mathbb{C} \).

Don’t let the mention of the complex numbers, \( \mathbb{C} \), rattle you. We will stick with real numbers exclusively for many more sections, and it will sometimes seem like we only work with integers! However, we want to leave the possibility of complex numbers open, and there will be occasions in subsequent sections where they are necessary. You can review the basic properties of complex numbers in Section CNO [687], but these facts will not be critical until we reach Section O [183]. For now, here is an example to illustrate using the notation introduced in Definition SLE [13].

Example NSE
Notation for a system of equations

Given the system of linear equations,

\[ \begin{align*}
x_1 + 2x_2 + x_4 &= 7 \\
x_1 + x_2 + x_3 - x_4 &= 3 \\
3x_1 + x_2 + 5x_3 - 7x_4 &= 1
\end{align*} \]

we have \( n = 4 \) variables and \( m = 3 \) equations. Also,

\[
\begin{align*}
a_{11} &= 1 & a_{12} &= 2 & a_{13} &= 0 & a_{14} &= 1 & b_1 &= 7 \\
a_{21} &= 1 & a_{22} &= 1 & a_{23} &= 1 & a_{24} &= -1 & b_2 &= 3 \\
a_{31} &= 3 & a_{32} &= 1 & a_{33} &= 5 & a_{34} &= -7 & b_3 &= 1
\end{align*}
\]

Additionally, convince yourself that \( x_1 = -2, \ x_2 = 4, \ x_3 = 2, \ x_4 = 1 \) is one solution (but it is not the only one!).

We will often shorten the term “system of linear equations” to “system of equations” leaving the linear aspect implied.

Subsection PSS
Possibilities for solution sets

The next example illustrates the possibilities for the solution set of a system of linear equations. We will not be too formal here, and the necessary theorems to back up our claims will come in subsequent sections. So read for feeling and come back later to revisit this example.

Example TTS
Three typical systems

Consider the system of two equations with two variables,

\[ \begin{align*}
2x_1 + 3x_2 &= 3 \\
x_1 - x_2 &= 4
\end{align*} \]
If we plot the solutions to each of these equations separately on the $x_1x_2$-plane, we get two lines, one with negative slope, the other with positive slope. They have exactly one point in common, $(x_1, x_2) = (3, -1)$, which is the solution $x_1 = 3$, $x_2 = -1$. From the geometry, we believe that this is the only solution to the system of equations, and so we say it is unique.

Now adjust the system with a different second equation,

$$\begin{align*}
2x_1 + 3x_2 &= 3 \\
4x_1 + 6x_2 &= 6
\end{align*}$$

A plot of the solutions to these equations individually results in two lines, one on top of the other! There are infinitely many pairs of points that make both equations true. We will learn shortly how to describe this infinite solution set precisely (see Example SAA 39, Theorem VFSLS 107). Notice now how the second equation is just a multiple of the first.

One more minor adjustment provides a third system of linear equations,

$$\begin{align*}
2x_1 + 3x_2 &= 3 \\
4x_1 + 6x_2 &= 10
\end{align*}$$

A plot now reveals two lines with identical slopes, i.e. parallel lines. They have no points in common, and so the system has a solution set that is empty, $S = \emptyset$.

This example exhibits all of the typical behaviors of a system of equations. A subsequent theorem will tell us that every system of linear equations has a solution set that is empty, contains a single solution or contains infinitely many solutions (Theorem PSSLS 57). Example STNE 13 yielded exactly two solutions, but this does not contradict the forthcoming theorem. The equations in Example STNE 13 are not linear because they do not match the form of Definition SLE 13, and so we cannot apply Theorem PSSLS 57 in this case.
two very different looking systems of equations are equivalent, since they have identical solution sets. It is really like a weaker form of equality, where we allow the systems to be different in some respects, but we use the term equivalent to highlight the situation when their solution sets are equal.

With this definition, we can begin to describe our strategy for solving linear systems. Given a system of linear equations that looks difficult to solve, we would like to have an equivalent system that is easy to solve. Since the systems will have equal solution sets, we can solve the “easy” system and get the solution set to the “difficult” system. Here come the tools for making this strategy viable.

Definition EO 
Equation Operations

Given a system of linear equations, the following three operations will transform the system into a different one, and each is known as an equation operation.

1. Swap the locations of two equations in the list.

2. Multiply each term of an equation by a nonzero quantity.

3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

These descriptions might seem a bit vague, but the proof or the examples that follow should make it clear what is meant by each. We will shortly prove a key theorem about equation operations and solutions to linear systems of equations. We are about to give a rather involved proof, so a discussion about just what a theorem really is would be timely. Head back and read Technique T [699]. In the theorem we are about to prove, the conclusion is that two systems are equivalent. By Definition ESYS [15] this translates to requiring that solution sets be equal for the two systems. So we are being asked to show that two sets are equal. How do we do this? Well, there is a very standard technique, and we will use it repeatedly through the course. If you have not done so already, head to Section SET [693] and familiarize yourself with sets, their operations, and especially the notion of set equality, Definition SE [694] and the nearby discussion about its use.

Theorem EOPSS 
Equation Operations Preserve Solution Sets

If we apply one of the three equation operations of Definition EO [16] to a system of linear equations (Definition SLE [13]), then the original system and the transformed system are equivalent.

Proof We take each equation operation in turn and show that the solution sets of the two systems are equal, using the definition of set equality (Definition SE [694]).

1. It will not be our habit in proofs to resort to saying statements are “obvious,” but in this case, it should be. There is nothing about the order in which we write linear equations that affects their solutions, so the solution set will be equal if the systems only differ by a rearrangement of the order of the equations.
2. Suppose $\alpha \neq 0$ is a number. Let’s choose to multiply the terms of equation $i$ by $\alpha$ to build the new system of equations,

\begin{align*}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &+ \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &+ \cdots + a_{2n}x_n = b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &+ \cdots + a_{3n}x_n = b_3 \\
&\vdots \\
a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 &+ \cdots + a_{in}x_n = b_i \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 &+ \cdots + a_{mn}x_n = b_m.
\end{align*}

Let $S$ denote the solutions to the system in the statement of the theorem, and let $T$ denote the solutions to the transformed system.

(a) Show $S \subseteq T$. Suppose $(x_1, x_2, x_3, \ldots, x_n) = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in S$ is a solution to the original system. Ignoring the $i$-th equation for a moment, we know it makes all the other equations of the transformed system true. We also know that

\begin{align*}
\alpha a_{i1}x_1 &+ \alpha a_{i2}x_2 + \alpha a_{i3}x_3 + \cdots + \alpha a_{in}x_n = \alpha b_i \\
&\vdots \\
a_{m1}x_1 &+ a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m.
\end{align*}

which we can multiply by $\frac{1}{\alpha}$, since $\alpha \neq 0$, to get

\begin{align*}
a_{i1}\beta_1 &+ a_{i2}\beta_2 + a_{i3}\beta_3 + \cdots + a_{in}\beta_n = b_i
\end{align*}

This says that the $i$-th equation of the transformed system is also true, so we have established that $(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in T$, and therefore $S \subseteq T$.

(b) Now show $T \subseteq S$. Suppose $(x_1, x_2, x_3, \ldots, x_n) = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in T$ is a solution to the transformed system. Ignoring the $i$-th equation for a moment, we know it makes all the other equations of the original system true. We also know that

\begin{align*}
\alpha a_{i1}x_1 &+ \alpha a_{i2}x_2 + \alpha a_{i3}x_3 + \cdots + \alpha a_{in}x_n = \alpha b_i \\
&\vdots \\
a_{m1}x_1 &+ a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m.
\end{align*}

which we can multiply by $\frac{1}{\alpha}$, since $\alpha \neq 0$, to get

\begin{align*}
a_{i1}\beta_1 &+ a_{i2}\beta_2 + a_{i3}\beta_3 + \cdots + a_{in}\beta_n = b_i
\end{align*}

This says that the $i$-th equation of the original system is also true, so we have established that $(\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in S$, and therefore $T \subseteq S$. Locate the key point where we required that $\alpha \neq 0$, and consider what would happen if $\alpha = 0$.

3. Suppose $\alpha$ is a number. Let’s choose to multiply the terms of equation $i$ by $\alpha$ and add them to equation $j$ in order to build the new system of equations,

\begin{align*}
a_{11}x_1 + a_{12}x_2 + &\cdots + a_{1n}x_n = b_1 \\
&\vdots \\
\end{align*}
Let \( S \) denote the solutions to the system in the statement of the theorem, and let \( T \) denote the solutions to the transformed system.

(a) Show \( S \subseteq T \). Suppose \((x_1, x_2, x_3, \ldots, x_n) = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in S \) is a solution to the original system. Ignoring the \( j \)-th equation for a moment, we know this solution makes all the other equations of the transformed system true. Using the fact that the solution makes the \( i \)-th and \( j \)-th equations of the original system true, we find

\[
(\alpha a_{i1} + a_{j1})\beta_1 + (\alpha a_{i2} + a_{j2})\beta_2 + \cdots + (\alpha a_{in} + a_{jn})\beta_n = \\
(\alpha a_{i1}\beta_1 + \alpha a_{i2}\beta_2 + \cdots + \alpha a_{in}\beta_n) + (a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n) = \\
\alpha(a_{i1}\beta_1 + a_{i2}\beta_2 + \cdots + a_{in}\beta_n) + (a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n) = \alpha b_i + b_j.
\]

This says that the \( j \)-th equation of the transformed system is also true, so we have established that \((\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in T \), and therefore \( S \subseteq T \).

(b) Now show \( T \subseteq S \). Suppose \((x_1, x_2, x_3, \ldots, x_n) = (\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in T \) is a solution to the transformed system. Ignoring the \( j \)-th equation for a moment, we know it makes all the other equations of the original system true. We then find

\[
a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n = \\
(a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n + \alpha b_i - \alpha b_i = \\
a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n + (\alpha a_{i1}\beta_1 + \alpha a_{i2}\beta_2 + \cdots + \alpha a_{in}\beta_n) - \alpha b_i = \\
a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n + (\alpha a_{i1}\beta_1 + \alpha a_{i2}\beta_2 + \cdots + \alpha a_{in}\beta_n) - \alpha b_i = \\
\alpha b_i + b_j - \alpha b_i = b_j.
\]

This says that the \( j \)-th equation of the original system is also true, so we have established that \((\beta_1, \beta_2, \beta_3, \ldots, \beta_n) \in S \), and therefore \( T \subseteq S \).

Why didn’t we need to require that \( \alpha \neq 0 \) for this row operation? In other words, how does the third statement of the theorem read when \( \alpha = 0 \)? Does our proof require some extra care when \( \alpha = 0 \)? Compare your answers with the similar situation for the second row operation.

[Theorem EOPSS] is the necessary tool to complete our strategy for solving systems of equations. We will use equation operations to move from one system to another, all the while keeping the solution set the same. With the right sequence of operations, we
will arrive at a simpler equation to solve. The next two examples illustrate this idea, while saving some of the details for later.

**Example US**

**Three equations, one solution**

We solve the following system by a sequence of equation operations.

\[
\begin{align*}
  x_1 + 2x_2 + 2x_3 &= 4 \\
  x_1 + 3x_2 + 3x_3 &= 5 \\
  2x_1 + 6x_2 + 5x_3 &= 6
\end{align*}
\]

\[\alpha = -1\] times equation 1, add to equation 2:

\[
\begin{align*}
  x_1 + 2x_2 + 2x_3 &= 4 \\
  0x_1 + 1x_2 + 1x_3 &= 1 \\
  2x_1 + 6x_2 + 5x_3 &= 6
\end{align*}
\]

\[\alpha = -2\] times equation 1, add to equation 3:

\[
\begin{align*}
  x_1 + 2x_2 + 2x_3 &= 4 \\
  0x_1 + 1x_2 + 1x_3 &= 1 \\
  0x_1 + 2x_2 + 1x_3 &= -2
\end{align*}
\]

\[\alpha = -2\] times equation 2, add to equation 3:

\[
\begin{align*}
  x_1 + 2x_2 + 2x_3 &= 4 \\
  0x_1 + 1x_2 + 1x_3 &= 1 \\
  0x_1 + 0x_2 - 1x_3 &= -4
\end{align*}
\]

\[\alpha = -1\] times equation 3:

\[
\begin{align*}
  x_1 + 2x_2 + 2x_3 &= 4 \\
  0x_1 + 1x_2 + 1x_3 &= 1 \\
  0x_1 + 0x_2 + 1x_3 &= 4
\end{align*}
\]

which can be written more clearly as

\[
\begin{align*}
  x_1 + 2x_2 + 2x_3 &= 4 \\
  x_2 + x_3 &= 1 \\
  x_3 &= 4
\end{align*}
\]

This is now a very easy system of equations to solve. The third equation requires that \(x_3 = 4\) to be true. Making this substitution into equation 2 we arrive at \(x_2 = -3\), and finally, substituting these values of \(x_2\) and \(x_3\) into the first equation, we find that \(x_1 = 2\). Note too that this is the only solution to this final system of equations, since we were forced to choose these values to make the equations true. Since we performed equation operations on each system to obtain the next one in the list, all of the systems listed here...
are all equivalent to each other by Theorem EOPSS \[16\]. Thus \((x_1, x_2, x_3) = (2, -3, 4)\) is the unique solution to the original system of equations (and all of the other systems of equations).

\[\alpha = -1 \text{ times equation 1, add to equation 2:}\]
\[
x_1 + 2x_2 + 0x_3 + x_4 = 7 \\
0x_1 - x_2 + x_3 - 2x_4 = -4 \\
3x_1 + x_2 + 5x_3 - 7x_4 = 1
\]

\[\alpha = -3 \text{ times equation 1, add to equation 3:}\]
\[
x_1 + 2x_2 + 0x_3 + x_4 = 7 \\
0x_1 - x_2 + x_3 - 2x_4 = -4 \\
0x_1 - 5x_2 + 5x_3 - 10x_4 = -20
\]

\[\alpha = -5 \text{ times equation 2, add to equation 3:}\]
\[
x_1 + 2x_2 + 0x_3 + x_4 = 7 \\
0x_1 - x_2 + x_3 - 2x_4 = -4 \\
0x_1 + 0x_2 + 0x_3 + 0x_4 = 0
\]

\[\alpha = -1 \text{ times equation 2:}\]
\[
x_1 + 2x_2 + 0x_3 + x_4 = 7 \\
0x_1 + x_2 - x_3 + 2x_4 = 4 \\
0x_1 + 0x_2 + 0x_3 + 0x_4 = 0
\]

\[\alpha = -2 \text{ times equation 2, add to equation 1:}\]
\[
x_1 + 0x_2 + 2x_3 - 3x_4 = -1 \\
0x_1 + x_2 - x_3 + 2x_4 = 4 \\
0x_1 + 0x_2 + 0x_3 + 0x_4 = 0
\]

which can be written more clearly as
\[
x_1 + 2x_3 - 3x_4 = -1
\]
What does the equation 0 = 0 mean? We can choose any values for $x_1, x_2, x_3, x_4$ and this equation will be true, so we only need to consider further the first two equations, since the third is true no matter what. We can analyze the second equation without consideration of the variable $x_1$. It would appear that there is considerable latitude in how we can choose $x_2, x_3, x_4$ and make this equation true. Let’s choose $x_3$ and $x_4$ to be anything we please, say $x_3 = \beta_3$ and $x_4 = \beta_4$.

Now we can take these arbitrary values for $x_3$ and $x_4$, substitute them in equation 1, to obtain

$$x_1 + 2\beta_3 - 3\beta_4 = -1$$
$$x_1 = -1 - 2\beta_3 + 3\beta_4$$

Similarly, equation 2 becomes

$$x_2 - \beta_3 + 2\beta_4 = 4$$
$$x_2 = 4 + \beta_3 - 2\beta_4$$

So our arbitrary choices of values for $x_3$ and $x_4$ ($\beta_3$ and $\beta_4$) translate into specific values of $x_1$ and $x_2$. The lone solution given in Example NSE 14 was obtained by choosing $\beta_3 = 2$ and $\beta_4 = 1$. Now we can easily and quickly find many more (infinitely more). Suppose we choose $\beta_3 = 5$ and $\beta_4 = -2$, then we compute

$$x_1 = -1 - 2(5) + 3(-2) = -17$$
$$x_2 = 4 + 5 - 2(-2) = 13$$

and you can verify that $(x_1, x_2, x_3, x_4) = (-17, 13, 5, -2)$ makes all three equations true. The entire solution set is written as

$$S = \{ (-1 - 2\beta_3 + 3\beta_4, 4 + \beta_3 - 2\beta_4, \beta_3, \beta_4) \mid \beta_3 \in \mathbb{C}, \beta_4 \in \mathbb{C} \}$$

It would be instructive to finish off your study of this example by taking the general form of the solutions given in this set and substituting them into each of the three equations and verify that they are true in each case.

In the next section we will describe how to use equation operations to systematically solve any system of linear equations. But first, read one of our more important pieces of advice about speaking and writing mathematics. See Technique L 700.

Before attacking the exercises in this section, it will be helpful to read some advice on getting started on the construction of a proof. See Technique GS 702.

Subsection READ

Reading Questions

1. How many solutions does the system of equations $3x + 2y = 4$, $6x + 4y = 8$ have? Explain your answer.
2. How many solutions does the system of equations \(3x + 2y = 4, \ 6x + 4y = -2\) have? Explain your answer.

3. What do we mean when we say mathematics is a language?
Subsection EXC  
Exercises

C10  Find a solution to the system in Example IS 20 where $\beta_3 = 6$ and $\beta_4 = 2$. Find two other solutions to the system. Find a solution where $\beta_1 = -17$ and $\beta_2 = 14$. How many possible answers are there to each of these questions?  
Contributed by Robert Beezer

C20  Each archetype (Appendix A 717) that is a system of equations begins by listing some specific solutions. Verify the specific solutions listed in the following archetypes by evaluating the system of equations with the solutions listed.

Archetype A 721
Archetype B 726
Archetype C 731
Archetype D 735
Archetype E 739
Archetype F 743
Archetype G 748
Archetype H 752
Archetype I 757
Archetype J 762

Contributed by Robert Beezer

C50  A three-digit number has two properties. The tens-digit and the ones-digit add up to 5. If the number is written with the digits in the reverse order, and then subtracted from the original number, the result is 792. Use a system of equations to find all of the three-digit numbers with these properties.

Contributed by Robert Beezer  Solution 25

M10  Each sentence below has at least two meanings. Identify the source of the double meaning, and rewrite the sentence (at least twice) to clearly convey each meaning.

1. They are baking potatoes.
2. He bought many ripe pears and apricots.
3. She likes his sculpture.
4. I decided on the bus.

Contributed by Robert Beezer  Solution 25

M11  Discuss the difference in meaning of each of the following three almost identical sentences, which all have the same grammatical structure. (These are due to Keith Devlin.)

1. She saw him in the park with a dog.
2. She saw him in the park with a fountain.
3. She saw him in the park with a telescope.

Contributed by Robert Beezer Solution 26

**M12** The following sentence, due to Noam Chomsky, has a correct grammatical structure, but is meaningless. Critique its faults. “Colorless green ideas sleep furiously.” (Chomsky, Noam. 1957. Syntactic Structures. The Hague/Paris: Mouton. p. 15)

Contributed by Robert Beezer Solution 26

**M13** Read the following sentence and form a mental picture of the situation.

The baby cried and the mother picked it up.

What assumptions did you make about the situation?

Contributed by Robert Beezer Solution 26


Contributed by David Beezer Solution 26

**M40** Solutions to the system in Example IS 20 are given as

\[(x_1, x_2, x_3, x_4) = (-1 - 2\beta_3 + 3\beta_4, 4 + \beta_3 - 2\beta_4, \beta_3, \beta_4)\]

Evaluate the three equations of the original system with these expressions in \(\beta_3\) and \(\beta_4\) and verify that each equation is true, no matter what values are chosen for \(\beta_3\) and \(\beta_4\).

Contributed by Robert Beezer

**M70** We have seen in this section that systems of linear equations have limited possibilities for solution sets, and we will shortly prove Theorem PSSLS 57 that describes these possibilities exactly. This exercise will show that if we relax the requirement that our equations be linear, then the possibilities expand greatly. Consider a system of two equations in the two variables \(x\) and \(y\), where the departure from linearity involves simply squaring the variables.

\[
x^2 - y^2 = 1
\]

\[
x^2 + y^2 = 4
\]

After solving this system of non-linear equations, replace the second equation in turn by

\[
x^2 + 2x + y^2 = 3
\]

\[
x^2 + y^2 = 1
\]

\[
x^2 - x + y^2 = 0
\]

\[
4x^2 + 4y^2 = 1
\]

and solve each resulting system of two equations in two variables.

Contributed by Robert Beezer Solution 26

**T20** Explain why the second equation operation in Definition EO 16 requires that the scalar be nonzero, while in the third equation operation this prohibition on the scalar is not present.

Contributed by Robert Beezer Solution 26

**T10** Technique D 698 asks you to formulate a definition of what it means for a whole number to be odd. What is your definition? (Don’t say “the opposite of even.”) Is 6 odd? Is 11 odd? Justify your answers by using your definition.

Contributed by Robert Beezer Solution 26
Subsection SOL
Solutions

C50 Contributed by Robert Beezer Statement 23
Let \(a\) be the hundreds digit, \(b\) the tens digit, and \(c\) the ones digit. Then the first condition says that \(b + c = 5\). The original number is \(100a + 10b + c\), while the reversed number is \(100c + 10b + a\). So the second condition is

\[
792 = (100a + 10b + c) - (100c + 10b + a) = 99a - 99c
\]

So we arrive at the system of equations

\[
\begin{align*}
b + c &= 5 \\
99a - 99c &= 792
\end{align*}
\]

Using equation operations, we arrive at the equivalent system

\[
\begin{align*}
a - c &= 8 \\
b + c &= 5
\end{align*}
\]

We can vary \(c\) and obtain infinitely many solutions. However, \(c\) must be a digit, restricting us to ten values (0 – 9). Furthermore, if \(c > 2\), then the first equation causes \(a > 9\), an impossibility. Setting \(c = 0\), yields 850 as a solution, and setting \(c = 1\) yields 941 as another solution.

M10 Contributed by Robert Beezer Statement 23

1. Is “baking” a verb or an adjective?
   Potatoes are being baked.
   Those are baking potatoes.

2. Are the apricots ripe, or just the pears? Parentheses could indicate just what the adjective “ripe” is meant to modify. Were there many apricots as well, or just many pears?
   He bought many pears and many ripe apricots.
   He bought apricots and many ripe pears.

3. Is “sculpture” a single physical object, or the sculptor’s style expressed over many pieces and many years?
   She likes his sculpture of the girl.
   She likes his sculptural style.

4. Was a decision made while in the bus, or was the outcome of a decision to choose the bus. Would the sentence “I decided on the car,” have a similar double meaning?
   I made my decision while on the bus.
   I decided to ride the bus.
M11 Contributed by Robert Beezer Statement 23
We know the dog belongs to the man, and the fountain belongs to the park. It is not clear if the telescope belongs to the man, the woman, or the park.

M12 Contributed by Robert Beezer Statement 24
In adjacent pairs the words are contradictory or inappropriate. Something cannot be both green and colorless, ideas do not have color, ideas do not sleep, and it is hard to sleep furiously.

M13 Contributed by Robert Beezer Statement 24
Did you assume that the baby and mother are human?
Did you assume that the baby is the child of the mother?
Did you assume that the mother picked up the baby as an attempt to stop the crying?

M30 Contributed by Robert Beezer Statement 24
If \( x, y \) and \( z \) represent the money held by Dan, Diane and Donna, then \( y = 15 - z \) and \( x = 20 - y = 20 - (15 - z) = 5 + z \). We can let \( z \) take on any value from 0 to 15 without any of the three amounts being negative, since presumably middle-schoolers are too young to assume debt.

Then the total capital held by the three is \( x + y + z = (5 + z) + (15 - z) + z = 20 + z \). So their combined holdings can range anywhere from $20 (Donna is broke) to $35 (Donna is flush).

We will have more to say about this situation in Section TSS 51, and specifically Theorem CMVEI 57.

M70 Contributed by Robert Beezer Statement 24
The equation \( x^2 - y^2 = 1 \) has a solution set by itself that has the shape of a hyperbola when plotted. The five different second equations have solution sets that are circles when plotted individually. Where the hyperbola and circle intersect are the solutions to the system of two equations. As the size and location of the circle varies, the number of intersections varies from four to none (in the order given). Sketching the relevant equations would be instructive, as was discussed in Example STNE 13.

The exact solution sets are (according to the choice of the second equation),

\[
\begin{align*}
x^2 + y^2 &= 4 : \left\{ \left( \frac{\sqrt{5}}{2}, \frac{\sqrt{3}}{2} \right), \left( -\frac{\sqrt{5}}{2}, \frac{\sqrt{3}}{2} \right), \left( \frac{\sqrt{5}}{2}, -\frac{\sqrt{3}}{2} \right), \left( -\frac{\sqrt{5}}{2}, -\frac{\sqrt{3}}{2} \right) \right\} \\
x^2 + 2x + y^2 &= 3 : \left\{ (1,0), (-2,\sqrt{3}), (-2,-\sqrt{3}) \right\} \\
x^2 + y^2 &= 1 : \left\{ (1,0), (-1,0) \right\} \\
x^2 - x + y^2 &= 0 : \left\{ (1,0) \right\} \\
4x^2 + 4y^2 &= 1 : \emptyset
\end{align*}
\]

T10 Contributed by Robert Beezer Statement 24
We can say that an integer is odd if when it is divided by 2 there is a remainder of 1. So 6 is not odd since \( 6 = 3 \times 2 + 0 \), while 11 is odd since \( 11 = 5 \times 2 + 1 \).

T20 Contributed by Robert Beezer Statement 24
Definition EO 16 is engineered to make Theorem EOPSS 16 true. If we were to allow a zero scalar to multiply an equation then that equation would be transformed to the equation \( 0 = 0 \), which is true for any possible values of the variables. Any restrictions on the solution set imposed by the original equation would be lost.
However, in the third operation, it is allowed to choose a zero scalar, multiply an equation by this scalar and add the transformed equation to a second equation (leaving the first unchanged). The result? Nothing. The second equation is the same as it was before. So the theorem is true in this case, the two systems are equivalent. But in practice, this would be a silly thing to actually ever do! We still allow it though, in order to keep our theorem as general as possible.

Notice the location in the proof of Theorem EOPSS [16] where the expression \( \frac{1}{\alpha} \) appears — this explains the prohibition on \( \alpha = 0 \) in the second equation operation.
After solving a few systems of equations, you will recognize that it doesn’t matter so much what we call our variables, as opposed to what numbers act as their coefficients. A system in the variables $x_1, x_2, x_3$ would behave the same if we changed the names of the variables to $a, b, c$ and kept all the constants the same and in the same places. In this section, we will isolate the key bits of information about a system of equations into something called a matrix, and then use this matrix to systematically solve the equations. Along the way we will obtain one of our most important and useful computational tools.

**Definition M**

**Matrix**

An $m \times n$ matrix is a rectangular layout of numbers from $\mathbb{C}$ having $m$ rows and $n$ columns. We will use upper-case Latin letters from the start of the alphabet ($A, B, C, \ldots$) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix $A$, the notation $[A]_{ij}$ will refer to the complex number in row $i$ and column $j$ of $A$.

(This definition contains Notation M.)

Be careful with this notation for individual entries, since it is easy to think that $[A]_{ij}$ refers to the whole matrix. It does not. It is just a number, but is a convenient way to talk about all the entries at once. This notation will get a heavy workout once we get to Chapter M [197].

**Example AM**

A matrix

$$B = \begin{bmatrix} -1 & 2 & 5 & 3 \\ 1 & 0 & -6 & 1 \\ -4 & 2 & 2 & -2 \end{bmatrix}$$

is a matrix with $m = 3$ rows and $n = 4$ columns. We can say that $[B]_{2,3} = -6$ while $[B]_{3,4} = -2$.

A calculator or computer language can be a convenient way to perform calculations with matrices. But first you have to enter the matrix. See: Computation ME.MMA 679 Computation ME.TI86 684 Computation ME.TI83 686.
Subsection MVNSE
Matrix and Vector Notation for Systems of Equations

When we do equation operations on system of equations, the names of the variables really aren’t very important. $x_1$, $x_2$, $x_3$, or $a$, $b$, $c$, or $x$, $y$, $z$, it really doesn’t matter. In this subsection we will describe some notation that will make it easier to describe linear systems, solve the systems and describe the solution sets. Here is a list of definitions, laden with notation.

**Definition CV**
*Column Vector*

A column vector of size $m$ is an ordered list of $m$ numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a vector. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. Some books like to write vectors with arrows, such as $\vec{u}$. Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in $\tilde{u}$. To refer to the entry or component that is number $i$ in the list that is the vector $\mathbf{v}$ we write $[\mathbf{v}]_i$.

Be careful with this notation. While the symbols $[\mathbf{v}]_i$ might look somewhat substantial, as an object this represents just one component of a vector, which is just a single complex number.

**Definition ZCV**
*Zero Column Vector*

The zero vector of size $m$ is the column vector of size $m$ where each entry is the number zero,

$$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or more compactly, $[0]_i = 0$ for $1 \leq i \leq m$.

**Definition CM**
*Coefficient Matrix*

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$
the coefficient matrix is the $m \times n$ matrix

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
  a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
  a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
  \vdots \\
  a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}
\]

Definition VOC

**Vector of Constants**
For a system of linear equations,

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
  a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
  \vdots \\
  a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

the vector of constants is the column vector of size $m$

\[
b = \begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  \vdots \\
  b_m
\end{bmatrix}
\]

Definition SV

**Solution Vector**
For a system of linear equations,

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
  a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
  \vdots \\
  a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

the solution vector is the column vector of size $n$

\[
x = \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  \vdots \\
  x_n
\end{bmatrix}
\]
The solution vector may do double-duty on occasion. It might refer to a list of variable quantities at one point, and subsequently refer to values of those variables that actually form a particular solution to that system.

**Definition LSMR**

**Matrix Representation of a Linear System**

If $A$ is the coefficient matrix of a system of linear equations and $b$ is the vector of constants, then we will write $\mathcal{LS}(A, b)$ as a shorthand expression for the system of linear equations, which we will refer to as the **matrix representation** of the linear system.

(This definition contains Notation LSMR.)

**Example NSLE**

**Notation for systems of linear equations**

The system of linear equations

\[
\begin{align*}
2x_1 + 4x_2 - 3x_3 + 5x_4 + x_5 &= 9 \\
3x_1 + x_2 + x_4 - 3x_5 &= 0 \\
-2x_1 + 7x_2 - 5x_3 + 2x_4 + 2x_5 &= -3
\end{align*}
\]

has coefficient matrix

\[
A = \begin{bmatrix}
2 & 4 & -3 & 5 & 1 \\
3 & 1 & 0 & 1 & -3 \\
-2 & 7 & -5 & 2 & 2
\end{bmatrix}
\]

and vector of constants

\[
b = \begin{bmatrix}
9 \\
0 \\
-3
\end{bmatrix}
\]

and so will be referenced as $\mathcal{LS}(A, b)$.

**Definition AM**

**Augmented Matrix**

Suppose we have a system of $m$ equations in $n$ variables, with coefficient matrix $A$ and vector of constants $b$. Then the **augmented matrix** of the system of equations is the $m \times (n + 1)$ matrix whose first $n$ columns are the columns of $A$ and whose last column (number $n + 1$) is the column vector $b$. This matrix will be written as $[A | b]$.

(This definition contains Notation AM.)

The augmented matrix represents all the important information in the system of equations, since the names of the variables have been ignored, and the only connection with the variables is the location of their coefficients in the matrix. It is important to realize that the augmented matrix is just that, a matrix, and not a system of equations. In particular, the augmented matrix does not have any “solutions,” though it will be useful for finding solutions to the system of equations that it is associated with. (Think about your objects, and review Technique L[700].) However, notice that an augmented matrix always belongs to some system of equations, and vice versa, so it is tempting to try and blur the distinction between the two. Here’s a quick example.
Subsection RREF.MVNSE  Matrix and Vector Notation for Systems of Equations  33

Example AMAA

Augmented matrix for Archetype A

Archetype A is the following system of 3 equations in 3 variables.

\[
\begin{align*}
  x_1 - x_2 + 2x_3 &= 1 \\
  2x_1 + x_2 + x_3 &= 8 \\
  x_1 + x_2 &= 5
\end{align*}
\]

Here is its augmented matrix.

\[
\begin{bmatrix}
  1 & -1 & 2 & 1 \\
  2 & 1 & 1 & 8 \\
  1 & 1 & 0 & 5
\end{bmatrix}
\]

An augmented matrix for a system of equations will save us the tedium of continually writing down the names of the variables as we solve the system. It will also release us from any dependence on the actual names of the variables. We have seen how certain operations we can perform on equations (Definition EO) will preserve their solutions (Theorem EOPSS). The next two definitions and the following theorem carry over these ideas to augmented matrices.

Definition RO

Row Operations

The following three operations will transform an \(m \times n\) matrix into a different matrix of the same size, and each is known as a row operation.

1. Swap the locations of two rows.
2. Multiply each entry of a single row by a nonzero quantity.
3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

1. \(R_i \leftrightarrow R_j\): Swap the location of rows \(i\) and \(j\).
2. \(\alpha R_i\): Multiply row \(i\) by the nonzero scalar \(\alpha\).
3. \(\alpha R_i + R_j\): Multiply row \(i\) by the scalar \(\alpha\) and add to row \(j\).

(This definition contains Notation RO.)

Definition REM

Row-Equivalent Matrices

Two matrices, \(A\) and \(B\), are row-equivalent if one can be obtained from the other by a sequence of row operations.
Example TREM
Two row-equivalent matrices
The matrices

\[
A = \begin{bmatrix}
2 & -1 & 3 & 4 \\
5 & 2 & -2 & 3 \\
1 & 1 & 0 & 6
\end{bmatrix} \quad B = \begin{bmatrix}
1 & 1 & 0 & 6 \\
3 & 0 & -2 & -9 \\
2 & -1 & 3 & 4
\end{bmatrix}
\]

are row-equivalent as can be seen from

\[
\begin{bmatrix}
2 & -1 & 3 & 4 \\
5 & 2 & -2 & 3 \\
1 & 1 & 0 & 6
\end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix}
1 & 1 & 0 & 6 \\
5 & 2 & -2 & 3 \\
2 & -1 & 3 & 4
\end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix}
1 & 1 & 0 & 6 \\
3 & 0 & -2 & -9 \\
2 & -1 & 3 & 4
\end{bmatrix}
\]

We can also say that any pair of these three matrices are row-equivalent.

Notice that each of the three row operations is reversible (Exercise RREF.T10 [45]), so we do not have to be careful about the distinction between “$A$ is row-equivalent to $B$” and “$B$ is row-equivalent to $A$.” (Exercise RREF.T11 [45]) The preceding definitions are designed to make the following theorem possible. It says that row-equivalent matrices represent systems of linear equations that have identical solution sets.

**Theorem REMES**
**Row-Equivalent Matrices represent Equivalent Systems**
Suppose that $A$ and $B$ are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

**Proof** If we perform a single row operation on an augmented matrix, it will have the same effect as if we did the analogous equation operation on the corresponding system of equations. By exactly the same methods as we used in the proof of Theorem EOPSS [16] we can see that each of these row operations will preserve the set of solutions for the corresponding system of equations.

So at this point, our strategy is to begin with a system of equations, represent it by an augmented matrix, perform row operations (which will preserve solutions for the corresponding systems) to get a “simpler” augmented matrix, convert back to a “simpler” system of equations and then solve that system, knowing that its solutions are those of the original system. Here’s a rehash of Example US [19] as an exercise in using our new tools.

**Example USR**
**Three equations, one solution, reprised**
We solve the following system using augmented matrices and row operations. This is the same system of equations solved in Example US [19] using equation operations.

\[
\begin{align*}
x_1 + 2x_2 + 2x_3 &= 4 \\
x_1 + 3x_2 + 3x_3 &= 5 \\
2x_1 + 6x_2 + 5x_3 &= 6
\end{align*}
\]
Form the augmented matrix,

\[
A = \begin{bmatrix}
1 & 2 & 2 & 4 \\
1 & 3 & 3 & 5 \\
2 & 6 & 5 & 6
\end{bmatrix}
\]

and apply row operations,

\[
\begin{align*}
-1R_1 + R_2 & \to \\
-2R_2 + R_3 & \to
\end{align*}
\]

\[
\begin{bmatrix}
1 & 2 & 2 & 4 \\
0 & 1 & 1 & 1 \\
2 & 6 & 5 & 6
\end{bmatrix}
\]

\[
\begin{align*}
-2R_1 + R_3 & \to \\
-2R_2 + R_3 & \to
\end{align*}
\]

\[
\begin{bmatrix}
1 & 2 & 2 & 4 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 4
\end{bmatrix}
\]

So the matrix

\[
B = \begin{bmatrix}
1 & 2 & 2 & 4 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 4
\end{bmatrix}
\]

is row equivalent to \( A \) and by Theorem REMES [34] the system of equations below has the same solution set as the original system of equations.

\[
\begin{align*}
x_1 + 2x_2 + 2x_3 &= 4 \\
x_2 + x_3 &= 1 \\
x_3 &= 4
\end{align*}
\]

Solving this “simpler” system is straightforward and is identical to the process in Example US [19].

The preceding example amply illustrates the definitions and theorems we have seen so far. But it still leaves two questions unanswered. Exactly what is this “simpler” form for a matrix, and just how do we get it? Here’s the answer to the first question, a definition of reduced row-echelon form.

**Definition RREF**

**Reduced Row-Echelon Form**

A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

1. A row where every entry is zero lies below any row that contains a nonzero entry.
2. The leftmost nonzero entry of a row is equal to 1.
3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
4. Consider any two different leftmost nonzero entries, one located in row \( i \), column \( j \) and the other located in row \( s \), column \( t \). If \( s > i \), then \( t > j \).

A row of only zero entries will be called a **zero row** and the leftmost nonzero entry of a nonzero row will be called a **leading 1**. The number of nonzero rows will be denoted by \( r \).

A column containing a leading 1 will be called a **pivot column**. The set of column indices for all of the pivot columns will be denoted by \( D = \{d_1, d_2, d_3, \ldots, d_r\} \) where
$d_1 < d_2 < d_3 < \cdots < d_r$, while the columns that are not pivot columns will be denoted as $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

(This definition contains Notation RREFA.)

The principal feature of reduced row-echelon form is the pattern of leading 1’s guaranteed by conditions (2) and (4), reminiscent of a flight of geese, or steps in a staircase, or water cascading down a mountain stream.

There are a number of new terms and notation introduced in this definition, which should make you suspect that this is an important definition. Given all there is to digest here, we will save the use of $D$ and $F$ until Section TSS \[51\].

**Example RREF**

**A matrix in reduced row-echelon form**

The matrix $C$ is in reduced row-echelon form.

$$
\begin{bmatrix}
1 & -3 & 0 & 6 & 0 & 0 & -5 & 9 \\
0 & 0 & 0 & 0 & 1 & 0 & 3 & -7 \\
0 & 0 & 0 & 0 & 0 & 1 & 7 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

This matrix has two zero rows and three leading 1’s. $r = 3$. Columns 1, 5, and 6 are pivot columns.

**Example NRREF**

**A matrix not in reduced row-echelon form**

The matrix $D$ is not in reduced row-echelon form, as it fails each of the four requirements once.

$$
\begin{bmatrix}
1 & 0 & -3 & 0 & 6 & 0 & 7 & -5 & 9 \\
0 & 0 & 0 & 5 & 0 & 1 & 0 & 3 & -7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -4 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 7 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Our next theorem has a “constructive” proof. Learn about the meaning of this term in Technique C \[703\].

**Theorem REMEF**

**Row-Equivalent Matrix in Echelon Form**

Suppose $A$ is a matrix. Then there is a matrix $B$ so that

1. $A$ and $B$ are row-equivalent.
2. $B$ is in reduced row-echelon form.

**Proof** Suppose that $A$ has $m$ rows and $n$ columns. We will describe a process for converting $A$ into $B$ via row operations. This procedure is known as Gauss–Jordan elimination. Tracing through this procedure will be easier if you recognize that $i$ refers to a row that is being converted, $j$ refers to a column that is being converted, and $r$ keeps track of the number of nonzero rows. Here we go.
1. Set $j = 0$ and $r = 0$.

2. Increase $j$ by 1. If $j$ now equals $n + 1$, then stop.

3. Examine the entries of $A$ in column $j$ located in rows $r + 1$ through $m$.
   If all of these entries are zero, then go to Step 2.

4. Choose a row from rows $r + 1$ through $m$ with a nonzero entry in column $j$.
   Let $i$ denote the index for this row.

5. Increase $r$ by 1.

6. Use the first row operation to swap rows $i$ and $r$.

7. Use the second row operation to convert the entry in row $r$ and column $j$ to a 1.

8. Use the third row operation with row $r$ to convert every other entry of column $j$ to zero.

9. Go to Step 2.

The result of this procedure is that the matrix $A$ is converted to a matrix in reduced row-echelon form, which we will refer to as $B$. We need to now prove this claim by showing that the converted matrix has the requisite properties of Definition RREF [35].

First, the matrix is only converted through row operations (Step 6, Step 7, Step 8), so $A$ and $B$ are row-equivalent (Definition REM [33]).

It is a bit more work to be certain that $B$ is in reduced row-echelon form. We claim that as we begin Step 2, the first $j$ columns of the matrix are in reduced row-echelon form with $r$ nonzero rows. Certainly this is true at the start when $j = 0$, since the matrix has no columns and so vacuously meets the conditions of Definition RREF [35] with $r = 0$ nonzero rows.

In Step 2 we increase $j$ by 1 and begin to work with the next column. There are two possible outcomes for Step 3. Suppose that every entry of column $j$ in rows $r + 1$ through $m$ is zero. Then with no changes we recognize that the first $j$ columns of the matrix has its first $r$ rows still in reduced row-echelon form, with the final $m - r$ rows still all zero.

Suppose instead that the entry in row $i$ of column $j$ is nonzero. Notice that since $r + 1 \leq i \leq m$, we know the first $j - 1$ entries of this row are all zero. Now, in Step 5 we increase $r$ by 1, and then embark on building a new nonzero row. In Step 6 we swap row $r$ and row $i$. In the first $j$ columns, the first $r - 1$ rows remain in reduced row-echelon form after the swap. In Step 7 we multiply row $r$ by a nonzero scalar, creating a 1 in the entry in column $j$ of row $i$, and not changing any other rows. This new leading 1 is the first nonzero entry in its row, and is located to the right of all the leading 1’s in the preceding $r - 1$ rows. With Step 8 we insure that every entry in the column with this new leading 1 is now zero, as required for reduced row-echelon form. Also, rows $r + 1$ through $m$ are now all zeros in the first $j$ columns, so we now only have one new nonzero row, consistent with our increase of $r$ by one. Furthermore, since the first $j - 1$ entries of row $r$ are zero, the employment of the third row operation does not destroy any of the necessary features of rows 1 through $r - 1$ and rows $r + 1$ through $m$, in columns 1 through $j - 1$.

So at this stage, the first $j$ columns of the matrix are in reduced row-echelon form. When Step 2 finally increases $j$ to $n + 1$, then the procedure is completed and the full
The procedure given in the proof of Theorem REMEF \[36\] can be more precisely described using a pseudo-code version of a computer program, as follows:

\[
\text{input } m, n \text{ and } A \\
r \leftarrow 0 \\
\text{for } j \leftarrow 1 \text{ to } n \\
\quad i \leftarrow r + 1 \\
\quad \text{while } i \leq m \text{ and } [A]_{ij} = 0 \\
\quad\quad i \leftarrow i + 1 \\
\quad\quad \text{if } i \neq m + 1 \\
\quad\quad\quad r \leftarrow r + 1 \\
\quad\quad\quad \text{swap rows } i \text{ and } r \text{ of } A \text{ (row op 1)} \\
\quad\quad\quad \text{scale entry in row } r, \text{ column } j \text{ of } A \text{ to a leading 1 (row op 2)} \\
\quad\quad\quad \text{for } k \leftarrow 1 \text{ to } m, k \neq r \\
\quad\quad\quad\quad \text{zero out entry in row } k, \text{ column } j \text{ of } A \text{ (row op 3 using row } r) \\
\text{output } r \text{ and } A
\]

Notice that as a practical matter the “and” used in the conditional statement of the while statement should be of the “short-circuit” variety so that the array access that follows is not out-of-bounds.

So now we can put it all together. Begin with a system of linear equations (Definition SLE \[13\]), and represent the system by its augmented matrix (Definition AM \[32\]). Use row operations (Definition RO \[33\]) to convert this matrix into reduced row-echelon form (Definition RREF \[35\]), using the procedure outlined in the proof of Theorem REMEF \[36\]. Theorem REMEF \[36\] also tells us we can always accomplish this, and that the result is row-equivalent (Definition REM \[33\]) to the original augmented matrix. Since the matrix in reduced-row echelon form has the same solution set, we can analyze the row-reduced version instead of the original matrix, viewing it as the augmented matrix of a different system of equations. The beauty of augmented matrices in reduced row-echelon form is that the solution sets to their corresponding systems can be easily determined, as we will see in the next few examples and in the next section.

We will see through the course that almost every interesting property of a matrix can be discerned by looking at a row-equivalent matrix in reduced row-echelon form. For this reason it is important to know that the matrix \(B\) guaranteed to exist by Theorem REMEF \[36\] is also unique. We could prove this result right now, but the proof will be much easier to state and understand a few sections from now when we have a few more definitions. However, the proof we will provide does not explicitly require any more theorems than we have right now, so we can, and will, make use of the uniqueness of \(B\) between now and then by citing Theorem RREFU \[116\]. You might want to jump forward now to read the statement of this important theorem and save studying its proof for later, once the rest of us get there.

We will now run through some examples of using these definitions and theorems to solve some systems of equations. From now on, when we have a matrix in reduced row-echelon form, we will mark the leading 1’s with a small box. In your work, you can box ‘em, circle ‘em or write ‘em in a different color — just identify ‘em somehow. This device will prove very useful later and is a very good habit to start developing right now.
Example SAB

Solutions for Archetype B

Let’s find the solutions to the following system of equations,

\[-7x_1 - 6x_2 - 12x_3 = -33\]
\[5x_1 + 5x_2 + 7x_3 = 24\]
\[x_1 + 4x_3 = 5\]

First, form the augmented matrix,

\[
\begin{bmatrix}
-7 & -6 & -12 & -33 \\
5 & 5 & 7 & 24 \\
1 & 0 & 4 & 5
\end{bmatrix}
\]

and work to reduced row-echelon form, first with \(i = 1\),

\[
R_1 \rightarrow R_3 \\
\begin{bmatrix}
1 & 0 & 4 & 5 \\
5 & 5 & 7 & 24 \\
-7 & -6 & -12 & -33
\end{bmatrix}
\]

\[
-5R_1 + R_2 \\
\begin{bmatrix}
1 & 0 & 4 & 5 \\
0 & 5 & -13 & -1 \\
-7 & -6 & -12 & -33
\end{bmatrix}
\]

\[
7R_1 + R_3 \\
\begin{bmatrix}
1 & 0 & 4 & 5 \\
0 & 5 & -13 & -1 \\
0 & -6 & 16 & 2
\end{bmatrix}
\]

Now, with \(i = 2\),

\[
\frac{1}{5}R_2 \\
\begin{bmatrix}
1 & 0 & 4 & 5 \\
0 & 1 & -\frac{13}{5} & \frac{1}{5} \\
0 & -6 & 16 & 2
\end{bmatrix}
\]

\[
6R_2 + R_3 \\
\begin{bmatrix}
1 & 0 & 4 & 5 \\
0 & 1 & -\frac{13}{5} & \frac{1}{5} \\
0 & 0 & \frac{2}{5} & \frac{4}{5}
\end{bmatrix}
\]

And finally, with \(i = 3\),

\[
\frac{2}{5}R_3 \\
\begin{bmatrix}
1 & 0 & 4 & 5 \\
0 & 1 & \frac{-13}{5} & \frac{-1}{5} \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

\[
\frac{12}{5}R_3 + R_2 \\
\begin{bmatrix}
1 & 0 & 4 & 5 \\
0 & 1 & 0 & 5 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]

This is now the augmented matrix of a very simple system of equations, namely \(x_1 = -3\), \(x_2 = 5\), \(x_3 = 2\), which has an obvious solution. Furthermore, we can see that this is the only solution to this system, so we have determined the entire solution set. You might compare this example with the procedure we used in Example US [19].

Archetypes A and B are meant to contrast each other in many respects. So let’s solve Archetype A now.

Example SAA

Solutions for Archetype A

Let’s find the solutions to the following system of equations,

\[x_1 - x_2 + 2x_3 = 1\]
\( \begin{align*}
2x_1 + x_2 + x_3 &= 8 \\
x_1 + x_2 &= 5
\end{align*} \)

First, form the augmented matrix,
\[
\begin{bmatrix}
1 & -1 & 2 & 1 \\
2 & 1 & 1 & 8 \\
1 & 1 & 0 & 5
\end{bmatrix}
\]
and work to reduced row-echelon form, first with \( i = 1 \),
\[
\begin{align*}
-2R_1 + R_2 &\rightarrow \\
\begin{bmatrix}
1 & -1 & 2 & 1 \\
0 & 3 & -3 & 6 \\
1 & 1 & 0 & 5
\end{bmatrix} &\rightarrow \\
-1R_1 + R_3 &\rightarrow \\
\begin{bmatrix}
1 & -1 & 2 & 1 \\
0 & 3 & -3 & 6 \\
0 & 2 & -2 & 4
\end{bmatrix}
\end{align*}
\]
Now, with \( i = 2 \),
\[
\begin{align*}
\frac{1}{3}R_2 &\rightarrow \\
\begin{bmatrix}
1 & -1 & 2 & 1 \\
0 & 1 & -1 & 2 \\
0 & 2 & -2 & 4
\end{bmatrix} &\rightarrow \\
1R_2 + R_1 &\rightarrow \\
\begin{bmatrix}
1 & 0 & 1 & 3 \\
0 & 1 & -1 & 2 \\
0 & 2 & -2 & 4
\end{bmatrix}
\end{align*}
\]
The system of equations represented by this augmented matrix needs to be considered a bit differently than that for Archetype B. First, the last row of the matrix is the equation 0 = 0, which is always true, so it imposes no restrictions on our possible solutions and therefore we can safely ignore it as we analyze the other two equations. These equations are,
\[
\begin{align*}
x_1 + x_3 &= 3 \\
x_2 - x_3 &= 2.
\end{align*}
\]
While this system is fairly easy to solve, it also appears to have a multitude of solutions. For example, choose \( x_3 = 1 \) and see that then \( x_1 = 2 \) and \( x_2 = 3 \) will together form a solution. Or choose \( x_3 = 0 \), and then discover that \( x_1 = 3 \) and \( x_2 = 2 \) lead to a solution. Try it yourself: pick any value of \( x_3 \) you please, and figure out what \( x_1 \) and \( x_2 \) should be to make the first and second equations (respectively) true. We’ll wait while you do that. Because of this behavior, we say that \( x_3 \) is a “free” or “independent” variable. But why do we vary \( x_3 \) and not some other variable? For now, notice that the third column of the augmented matrix does not have any leading 1’s in its column. With this idea, we can rearrange the two equations, solving each for the variable that corresponds to the leading 1 in that row.
\[
\begin{align*}
x_1 &= 3 - x_3 \\
x_2 &= 2 + x_3
\end{align*}
\]
To write the set of solution vectors in set notation, we have
\[
S = \left\{ \begin{bmatrix} 3 - x_3 \\ 2 + x_3 \\ x_3 \end{bmatrix} \mid x_3 \in \mathbb{C} \right\}
\]
We’ll learn more in the next section about systems with infinitely many solutions and how to express their solution sets. Right now, you might look back at Example IS 20.

Example SAE

Solutions for Archetype E

Let’s find the solutions to the following system of equations,

\[
\begin{align*}
2x_1 + x_2 + 7x_3 - 7x_4 &= 2 \\
-3x_1 + 4x_2 - 5x_3 - 6x_4 &= 3 \\
x_1 + x_2 + 4x_3 - 5x_4 &= 2
\end{align*}
\]

First, form the augmented matrix,

\[
\begin{bmatrix}
2 & 1 & 7 & -7 & 2 \\
-3 & 4 & -5 & -6 & 3 \\
1 & 1 & 4 & -5 & 2
\end{bmatrix}
\]

and work to reduced row-echelon form, first with \(i = 1\),

\[
R_1 \leftrightarrow R_3 \rightarrow \begin{bmatrix}
1 & 1 & 4 & -5 & 2 \\
-3 & 4 & -5 & -6 & 3 \\
2 & 1 & 7 & -7 & 2
\end{bmatrix}, \quad 3R_1 + R_2 \rightarrow \begin{bmatrix}
1 & 1 & 4 & -5 & 2 \\
0 & 7 & -21 & 9 & 13 \\
2 & 1 & 7 & -7 & 2
\end{bmatrix}
\]

\[-2R_1 + R_3 \rightarrow \begin{bmatrix}
1 & 1 & 4 & -5 & 2 \\
0 & 7 & 7 & -21 & 9 \\
0 & -1 & -1 & 3 & -2
\end{bmatrix}
\]

Now, with \(i = 2\),

\[
R_2 \rightarrow R_3 \rightarrow \begin{bmatrix}
1 & 1 & 4 & -5 & 2 \\
0 & -1 & -1 & 3 & -2 \\
0 & 7 & 7 & -21 & 9
\end{bmatrix}, \quad -1R_2 \rightarrow \begin{bmatrix}
1 & 1 & 4 & -5 & 2 \\
0 & 1 & 1 & -3 & 2 \\
0 & 7 & 7 & -21 & 9
\end{bmatrix}
\]

\[-1R_2 + R_1 \rightarrow \begin{bmatrix}
1 & 0 & 3 & -2 & 0 \\
0 & 1 & 1 & -3 & 2 \\
0 & 7 & 7 & -21 & 9
\end{bmatrix}, \quad -7R_2 + R_3 \rightarrow \begin{bmatrix}
1 & 0 & 3 & -2 & 0 \\
0 & 1 & 1 & -3 & 2 \\
0 & 0 & 0 & 0 & -5
\end{bmatrix}
\]

And finally, with \(i = 3\),

\[
-\frac{1}{5}R_3 \rightarrow \begin{bmatrix}
1 & 0 & 3 & -2 & 0 \\
0 & 1 & 1 & -3 & 2 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad -2R_3 + R_2 \rightarrow \begin{bmatrix}
1 & 0 & 3 & -2 & 0 \\
0 & 1 & 1 & -3 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Let’s analyze the equations in the system represented by this augmented matrix. The third equation will read \(0 = 1\). This is patently false, all the time. No choice of values for our variables will ever make it true. We’re done. Since we cannot even make the last equation true, we have no hope of making all of the equations simultaneously true. So this system has no solutions, and its solution set is the empty set, \(\emptyset = \{\}\) (Definition ES 693).

Notice that we could have reached this conclusion sooner. After performing the row operation \(-7R_2 + R_3\), we can see that the third equation reads \(0 = -5\), a false
statement. Since the system represented by this matrix has no solutions, none of the systems represented has any solutions. However, for this example, we have chosen to bring the matrix fully to reduced row-echelon form for the practice.

These three examples (Example SAB [39], Example SAA [39], Example SAE [41]) illustrate the full range of possibilities for a system of linear equations — no solutions, one solution, or infinitely many solutions. In the next section we'll examine these three scenarios more closely.

**Definition RR**

**Row-Reducing**

To row-reduce the matrix $A$ means to apply row operations to $A$ and arrive at a row-equivalent matrix $B$ in reduced row-echelon form.

So the term row-reduce is used as a verb. Theorem REMEF [36] tells us that this process will always be successful and Theorem RREFU [116] tells us that the result will be unambiguous. Typically, the analysis of $A$ will proceed by analyzing $B$ and applying theorems whose hypotheses include the row-equivalence of $A$ and $B$.

After some practice by hand, you will want to use your favorite computing device to do the computations required to bring a matrix to reduced row-echelon form (Exercise RREF.C30 [41]). See: Computation RR.MMA 679 Computation RR.TI86 685 Computation RR.TI83 686.

**Subsection READ**

**Reading Questions**

1. Is the matrix below in reduced row-echelon form? Why or why not?

$$
\begin{bmatrix}
1 & 5 & 0 & 6 & 8 \\
0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

2. Use row operations to convert the matrix below to reduced row-echelon form and report the final matrix.

$$
\begin{bmatrix}
2 & 1 & 8 \\
-1 & 1 & -1 \\
-2 & 5 & 4 \\
\end{bmatrix}
$$

3. Find all the solutions to the system below by using an augmented matrix and row operations. Report your final matrix in reduced row-echelon form and the set of solutions.

$$
\begin{align*}
2x_1 + 3x_2 - x_3 &= 0 \\
x_1 + 2x_2 + x_3 &= 3 \\
x_1 + 3x_2 + 3x_3 &= 7
\end{align*}
$$
Subsection EXC Exercises

C05 Each archetype below is a system of equations. Form the augmented matrix of the system of equations, convert the matrix to reduced row-echelon form by using equation operations and then describe the solution set of the original system of equations.

Archetype A [721]
Archetype B [726]
Archetype C [731]
Archetype D [735]
Archetype E [739]
Archetype F [743]
Archetype G [748]
Archetype H [752]
Archetype I [757]
Archetype J [762]

Contributed by Robert Beezer

For problems C10–C16, find all solutions to the system of linear equations. Write the solutions as a set, using correct set notation.

C10

\[
\begin{align*}
2x_1 - 3x_2 + x_3 + 7x_4 &= 14 \\
2x_1 + 8x_2 - 4x_3 + 5x_4 &= -1 \\
x_1 + 3x_2 - 3x_3 &= 4 \\
-5x_1 + 2x_2 + 3x_3 + 4x_4 &= -19
\end{align*}
\]

Contributed by Robert Beezer Solution [47]

C11

\[
\begin{align*}
3x_1 + 4x_2 - x_3 + 2x_4 &= 6 \\
x_1 - 2x_2 + 3x_3 + x_4 &= 2 \\
10x_2 - 10x_3 - x_4 &= 1
\end{align*}
\]

Contributed by Robert Beezer Solution [47]

C12

\[
\begin{align*}
2x_1 + 4x_2 + 5x_3 + 7x_4 &= -26 \\
x_1 + 2x_2 + x_3 - x_4 &= -4 \\
-2x_1 - 4x_2 + x_3 + 11x_4 &= -10
\end{align*}
\]

Contributed by Robert Beezer Solution [47]
C13

\[ x_1 + 2x_2 + 8x_3 - 7x_4 = -2 \]
\[ 3x_1 + 2x_2 + 12x_3 - 5x_4 = 6 \]
\[ -x_1 + x_2 + x_3 - 5x_4 = -10 \]

Contributed by Robert Beezer   Solution 47

C14

\[ 2x_1 + x_2 + 7x_3 - 2x_4 = 4 \]
\[ 3x_1 - 2x_2 + 11x_4 = 13 \]
\[ x_1 + x_2 + 5x_3 - 3x_4 = 1 \]

Contributed by Robert Beezer   Solution 48

C15

\[ 2x_1 + 3x_2 - x_3 - 9x_4 = -16 \]
\[ x_1 + 2x_2 + x_3 = 0 \]
\[ -x_1 + 2x_2 + 3x_3 + 4x_4 = 8 \]

Contributed by Robert Beezer   Solution 48

C16

\[ 2x_1 + 3x_2 + 19x_3 - 4x_4 = 2 \]
\[ x_1 + 2x_2 + 12x_3 - 3x_4 = 1 \]
\[ -x_1 + 2x_2 + 8x_3 - 5x_4 = 1 \]

Contributed by Robert Beezer   Solution 48

C30  Row-reduce the matrix below without the aid of a calculator, indicating the row operations you are using at each step.

\[
\begin{bmatrix}
2 & 1 & 5 & 10 \\
1 & -3 & -1 & -2 \\
4 & -2 & 6 & 12 \\
\end{bmatrix}
\]

Contributed by Robert Beezer   Solution 48

C31  Convert the matrix \( D \) to reduced row-echelon form by performing row operations without the aid of a calculator. Indicate clearly which row operations you are doing at each step.

\[
D = \begin{bmatrix}
1 & 2 & -4 \\
-3 & -1 & -3 \\
-2 & 1 & -7 \\
\end{bmatrix}
\]
M50  A parking lot has 66 vehicles (cars, trucks, motorcycles and bicycles) in it. There are four times as many cars as trucks. The total number of tires (4 per car or truck, 2 per motorcycle or bicycle) is 252. How many cars are there? How many bicycles?

T10  Prove that each of the three row operations (Definition RO) is reversible. More precisely, if the matrix $B$ is obtained from $A$ by application of a single row operation, show that there is a single row operation that will transform $B$ back into $A$.

T11  Suppose that $A$, $B$ and $C$ are $m \times n$ matrices. Use the definition of row-equivalence (Definition REM) to prove the following three facts.

1. $A$ is row-equivalent to $A$.
2. If $A$ is row-equivalent to $B$, then $B$ is row-equivalent to $A$.
3. If $A$ is row-equivalent to $B$, and $B$ is row-equivalent to $C$, then $A$ is row-equivalent to $C$.

A relationship that satisfies these three properties is known as an equivalence relation, an important idea in the study of various algebras. This is a formal way of saying that a relationship behaves like equality, without requiring the relationship to be as strict as equality itself. We’ll see it again in Theorem SER.

T12  Suppose that $B$ is an $m \times n$ matrix in reduced row-echelon form. Build a new, likely smaller, $k \times \ell$ matrix $C$ as follows. Keep any collection of $k$ adjacent rows, $k \leq m$. From these rows, keep columns 1 through $\ell$, $\ell \leq n$. Prove that $C$ is in reduced row-echelon form.
Subsection SOL
Solutions

C10 Contributed by Robert Beezer Statement 43
The augmented matrix row-reduces to
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & -3 \\
0 & 0 & 1 & 0 & -4 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix}
\]
and we see from the locations of the leading 1’s that the system is consistent (Theorem RCLS 54) and that \(n - r = 4 - 4 = 0\) and so the system has no free variables (Theorem CSRN 56) and hence has a unique solution. This solution is \(\{(1, -3, -4, 1)\}\).

C11 Contributed by Robert Beezer Statement 43
The augmented matrix row-reduces to
\[
\begin{bmatrix}
1 & 0 & 1 & 4/5 & 0 \\
0 & 1 & -1 & -1/10 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
and a leading 1 in the last column tells us that the system is inconsistent (Theorem RCLS 54). So the solution set is \(\emptyset = \{\}\).

C12 Contributed by Robert Beezer Statement 43
The augmented matrix row-reduces to
\[
\begin{bmatrix}
1 & 2 & 0 & -4 & 2 \\
0 & 0 & 1 & 3 & -6 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
(Theorem RCLS 54) and (Theorem CSRN 56) tells us the system is consistent and the solution set can be described with \(n - r = 4 - 2 = 2\) free variables, namely \(x_2\) and \(x_4\). Solving for the dependent variables \((D = \{x_1, x_3\})\) the first and second equations represented in the row-reduced matrix yields,
\[
x_1 = 2 - 2x_2 + 4x_4 \\
x_3 = -6 - 3x_4
\]
As a set, we write this as
\[
\begin{Bmatrix}
2 - 2x_2 + 4x_4 \\
x_2 \\
-6 - 3x_4 \\
x_4
\end{Bmatrix}
|\begin{Bmatrix} x_2, x_4 \in \mathbb{C} \end{Bmatrix}
\]

C13 Contributed by Robert Beezer Statement 44
The augmented matrix of the system of equations is
\[
\begin{bmatrix}
1 & 2 & 8 & -7 & -2 \\
3 & 2 & 12 & -5 & 6 \\
-1 & 1 & 1 & -5 & -10
\end{bmatrix}
\]
which row-reduces to
\[
\begin{bmatrix}
1 & 0 & 2 & 1 & 0 \\
0 & 1 & 3 & -4 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

With a leading one in the last column Theorem RCLS tells us the system of equations is inconsistent, so the solution set is the empty set, \( \emptyset \).

C14 Contributed by Robert Beezer Statement
The augmented matrix of the system of equations is
\[
\begin{bmatrix}
2 & 1 & 7 & -2 & 4 \\
3 & -2 & 0 & 11 & 13 \\
1 & 1 & 5 & -3 & 1
\end{bmatrix}
\]

which row-reduces to
\[
\begin{bmatrix}
1 & 0 & 2 & 1 & 3 \\
0 & 1 & 3 & -4 & -2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Then \( D = \{1, 2\} \) and \( F = \{3, 4, 5\} \), so the system is consistent (5 \( \not\in \) D) and can be described by the two free variables \( x_3 \) and \( x_4 \). Rearranging the equations represented by the two nonzero rows to gain expressions for the dependent variables \( x_1 \) and \( x_2 \), yields the solution set,
\[
S = \left\{ \begin{bmatrix} 3 - 2x_3 - x_4 \\ -2 - 3x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} \middle| x_3, x_4 \in \mathbb{C} \right\}
\]

C15 Contributed by Robert Beezer Statement
The augmented matrix of the system of equations is
\[
\begin{bmatrix}
2 & 3 & -1 & -9 & -16 \\
1 & 2 & 1 & 0 & 0 \\
-1 & 2 & 3 & 4 & 8
\end{bmatrix}
\]

which row-reduces to
\[
\begin{bmatrix}
1 & 0 & 0 & 2 & 3 \\
0 & 1 & 0 & -3 & -5 \\
0 & 0 & 1 & 4 & 7
\end{bmatrix}
\]

Then \( D = \{1, 2, 3\} \) and \( F = \{4, 5\} \), so the system is consistent (5 \( \not\in \) D) and can be described by the one free variable \( x_4 \). Rearranging the equations represented by the three nonzero rows to gain expressions for the dependent variables \( x_1 \), \( x_2 \) and \( x_3 \), yields the solution set,
\[
S = \left\{ \begin{bmatrix} 3 - 2x_4 \\ -5 + 3x_4 \\ 7 - 4x_4 \\ x_4 \end{bmatrix} \middle| x_4 \in \mathbb{C} \right\}
\]

C16 Contributed by Robert Beezer Statement
The augmented matrix of the system of equations is
\[
\begin{bmatrix}
2 & 3 & 19 & -4 & 2 \\
1 & 2 & 12 & -3 & 1 \\
-1 & 2 & 8 & -5 & 1
\end{bmatrix}
\]
which row-reduces to

\[
\begin{bmatrix}
1 & 0 & 2 & 1 & 0 \\
0 & 1 & 5 & -2 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

With a leading one in the last column Theorem RCLS \[54\] tells us the system of equations is inconsistent, so the solution set is the empty set, \( \emptyset = \{\} \).

C30 Contributed by Robert Beezer Statement [44]

\[
\begin{bmatrix}
2 & 1 & 5 & 10 \\
1 & -3 & -1 & -2 \\
4 & -2 & 6 & 12
\end{bmatrix} \rightarrow R_1 \leftrightarrow R_2 \\
\begin{bmatrix}
1 & -3 & -1 & -2 \\
0 & 7 & 7 & 14 \\
4 & -2 & 6 & 12
\end{bmatrix} \rightarrow -2R_1 + R_2 \\
\begin{bmatrix}
1 & -3 & -1 & -2 \\
0 & 1 & 1 & 2 \\
0 & 10 & 10 & 20
\end{bmatrix} \rightarrow \frac{1}{2} R_2 \\
\begin{bmatrix}
1 & 0 & 2 & 4 \\
0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix} \rightarrow -10R_2 + R_3
\]

C31 Contributed by Robert Beezer Statement [44]

\[
\begin{bmatrix}
1 & 2 & -4 \\
-3 & -1 & -3 \\
-2 & 1 & -7
\end{bmatrix} \rightarrow 3R_1 + R_2 \\
\begin{bmatrix}
1 & 2 & -4 \\
0 & 5 & -15 \\
0 & 5 & -15
\end{bmatrix} \rightarrow 2R_1 + R_3 \\
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & -3 \\
0 & 5 & -15
\end{bmatrix} \rightarrow -2R_2 + R_1 \\
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & -3 \\
0 & 0 & 0
\end{bmatrix} \rightarrow -5R_2 + R_3
\]

M50 Contributed by Robert Beezer Statement [45]

Let \( c, t, m, b \) denote the number of cars, trucks, motorcycles, and bicycles. Then the statements from the problem yield the equations:

\[
c + t + m + b = 66 \\
c - 4t = 0 \\
4c + 4t + 2m + 2b = 252
\]

The augmented matrix for this system is

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 66 \\
1 & -4 & 0 & 0 & 0 \\
4 & 4 & 2 & 2 & 252
\end{bmatrix}
\]
which row-reduces to
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 48 \\
0 & 1 & 0 & 0 & 12 \\
0 & 0 & 1 & 1 & 6
\end{bmatrix}
\]

c = 48 is the first equation represented in the row-reduced matrix so there are 48 cars. 
m + b = 6 is the third equation represented in the row-reduced matrix so there are 
anywhere from 0 to 6 bicycles. We can also say that b is a free variable, but the context 
of the problem limits it to 7 integer values since cannot have a negative number of 
motorcycles.

**T10** Contributed by Robert Beezer. Statement 45.

If we can reverse each row operation individually, then we can reverse a sequence of 
row operations. The operations that reverse each operation are listed below, using our 
shorthand notation,

\[
\begin{align*}
R_i \leftrightarrow R_j & \quad R_i \leftrightarrow R_j \\
\alpha R_i, \alpha \neq 0 & \quad \frac{1}{\alpha} R_i \\
\alpha R_i + R_j & \quad -\alpha R_i + R_j
\end{align*}
\]
Section TSS
Types of Solution Sets

We will now be more careful about analyzing the reduced row-echelon form derived from the augmented matrix of a system of linear equations. In particular, we will see how to systematically handle the situation when we have infinitely many solutions to a system, and we will prove that every system of linear equations has either zero, one or infinitely many solutions. With these tools, we will be able to solve any system by a well-described method.

The computer scientist Donald Knuth said, “Science is what we understand well enough to explain to a computer. Art is everything else.” In this section we’ll remove solving systems of equations from the realm of art, and into the realm of science. We begin with a definition.

Definition CS
Consistent System
A system of linear equations is consistent if it has at least one solution. Otherwise, the system is called inconsistent.

We will want to first recognize when a system is inconsistent or consistent, and in the case of consistent systems we will be able to further refine the types of solutions possible. We will do this by analyzing the reduced row-echelon form of a matrix, using the value of \( r \), and the sets of column indices, \( D \) and \( F \), first defined back in Definition RREF [35].

Use of the notation for the elements of \( D \) and \( F \) can be a bit confusing, since we have subscripted variables that are in turn equal to integers used to index the matrix. However, many questions about matrices and systems of equations can be answered once we know \( r \), \( D \) and \( F \). The choice of the letters \( D \) and \( F \) refer to our upcoming definition of dependent and free variables (Definition IDV [53]). An example will help us begin to get comfortable with this aspect of reduced row-echelon form.

Example RREFN
Reduced row-echelon form notation
For the \( 5 \times 9 \) matrix

\[
B = \begin{bmatrix}
1 & 5 & 0 & 0 & 2 & 8 & 0 & 5 & -1 \\
0 & 0 & 1 & 0 & 4 & 7 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & 3 & 9 & 0 & 3 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

in reduced row-echelon form we have

\[
\begin{align*}
r &= 4 \\
d_1 &= 1 \\
d_2 &= 3 \\
d_3 &= 4 \\
d_4 &= 7 \\
f_1 &= 2 \\
f_2 &= 5 \\
f_3 &= 6 \\
f_4 &= 8 \\
f_5 &= 9.
\end{align*}
\]

Notice that the sets \( D = \{d_1, d_2, d_3, d_4\} = \{1, 3, 4, 7\} \) and \( F = \{f_1, f_2, f_3, f_4, f_5\} = \{2, 5, 6, 8, 9\} \) have nothing in common and together account for all of the columns of \( B \) (we say it is a partition of the set of column indices).
The number $r$ is the single most important piece of information we can get from the reduced row-echelon form of a matrix. It is defined as the number of non-zero rows, but since each non-zero row has a leading 1, it is also the number of leading 1’s present. For each leading 1, we have a pivot column, so $r$ is also the number of pivot columns. Repeating ourselves, $r$ is the number of leading 1’s, the number of non-zero rows and the number of pivot columns. Across different situations, each of these interpretations of the meaning of $r$ will be useful.

Before proving some theorems about the possibilities for solution sets to systems of equations, let’s analyze one particular system with an infinite solution set very carefully as an example. We’ll use this technique frequently, and shortly we’ll refine it slightly.

Archetypes I and J are both fairly large for doing computations by hand (though not impossibly large). Their properties are very similar, so we will frequently analyze the situation in Archetype I, and leave you the joy of analyzing Archetype J yourself. So work through Archetype I with the text, by hand and/or with a computer, and then tackle Archetype J yourself (and check your results with those listed). Notice too that the archetypes describing systems of equations each lists the values of $r$, $D$ and $F$. Here we go. . .

**Example ISSI**

**Describing infinite solution sets, Archetype I**

Archetype I is the system of $m = 4$ equations in $n = 7$ variables.

\[
\begin{align*}
1x_1 + 4x_2 - x_4 + 7x_6 - 9x_7 &= 3 \\
2x_1 + 8x_2 - x_3 + 3x_4 + 9x_5 - 13x_6 + 7x_7 &= 9 \\
2x_3 - 3x_4 - 4x_5 + 12x_6 - 8x_7 &= 1 \\
-x_1 - 4x_2 + 2x_3 + 4x_4 + 8x_5 - 31x_6 + 37x_7 &= 4
\end{align*}
\]

This system has a $4 \times 8$ augmented matrix that is row-equivalent to the following matrix (check this!), and which is in reduced row-echelon form (the existence of this matrix is guaranteed by Theorem REMEF),

\[
\begin{bmatrix}
1 & 4 & 0 & 0 & 2 & 1 & -3 & 4 \\
0 & 0 & 1 & 0 & 1 & -3 & 5 & 2 \\
0 & 0 & 0 & 1 & 2 & -6 & 6 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

So we find that $r = 3$ and

\[D = \{d_1, d_2, d_3\} = \{1, 3, 4\} \quad F = \{f_1, f_2, f_3, f_4, f_5\} = \{2, 5, 6, 7, 8\}.\]

Let $i$ denote one of the $r = 3$ non-zero rows, and then we see that we can solve the corresponding equation represented by this row for the variable $x_{d_i}$ and write it as a linear function of the variables $x_{f_1}, x_{f_2}, x_{f_3}, x_{f_4}$ (notice that $f_5 = 8$ does not reference a variable). We’ll do this now, but you can already see how the subscripts upon subscripts takes some getting used to.

\[
\begin{align*}
(i = 1) \quad x_{d_1} &= x_1 = 4 - 4x_2 - 2x_5 - x_6 + 3x_7 \\
(i = 2) \quad x_{d_2} &= x_3 = 2 - x_5 + 3x_6 - 5x_7 \\
(i = 3) \quad x_{d_3} &= x_4 = 1 - 2x_5 + 6x_6 - 6x_7
\end{align*}
\]
Each element of the set $F = \{f_1, f_2, f_3, f_4, f_5\} = \{2, 5, 6, 7, 8\}$ is the index of a variable, except for $f_5 = 8$. We refer to $x_{f_1} = x_2$, $x_{f_2} = x_5$, $x_{f_3} = x_6$ and $x_{f_4} = x_7$ as “free” (or “independent”) variables since they are allowed to assume any possible combination of values that we can imagine and we can continue on to build a solution to the system by solving individual equations for the values of the other (“dependent”) variables.

Each element of the set $D = \{d_1, d_2, d_3\} = \{1, 3, 4\}$ is the index of a variable. We refer to the variables $x_{d_1} = x_1$, $x_{d_2} = x_3$ and $x_{d_3} = x_4$ as “dependent” variables since they depend on the independent variables. More precisely, for each possible choice of values for the independent variables we get exactly one set of values for the dependent variables that combine to form a solution of the system.

To express the solutions as a set, we write

$$\begin{cases}
4 - 4x_2 - 2x_5 - x_6 + 3x_7 & = 0 \\
2 - x_5 + 3x_6 - 5x_7 & = 0 \\
1 - 2x_5 + 6x_6 - 6x_7 & = 0 \\
x_5 & = 0 \\
x_6 & = 0 \\
x_7 & = 0 \\
x_2, x_5, x_6, x_7 & \in \mathbb{C}
\end{cases}$$

The condition that $x_2, x_5, x_6, x_7 \in \mathbb{C}$ is how we specify that the variables $x_2, x_5, x_6, x_7$ are “free” to assume any possible values.

This systematic approach to solving a system of equations will allow us to create a precise description of the solution set for any consistent system once we have found the reduced row-echelon form of the augmented matrix. It will work just as well when the set of free variables is empty and we get just a single solution. And we could program a computer to do it! Now have a whack at Archetype J (Exercise TSS.T10 [61]), mimicking the discussion in this example. We’ll still be here when you get back.

Using the reduced row-echelon form of the augmented matrix of a system of equations to determine the nature of the solution set of the system is a very key idea. So let’s look at one more example like the last one. But first a definition, and then the example. We mix our metaphors a bit when we call variables free versus dependent. Maybe we should call dependent variables “enslaved”?

**Definition IDV**

**Independent and Dependent Variables**

Suppose $A$ is the augmented matrix of a consistent system of linear equations and $B$ is a row-equivalent matrix in reduced row-echelon form. Suppose $j$ is the index of a column of $B$ that contains the leading 1 for some row (i.e. column $j$ is a pivot column), and this column is not the last column. Then the variable $x_j$ is dependent. A variable that is not dependent is called independent or free.

**Example FDV**

**Free and dependent variables**

Consider the system of five equations in five variables,

\[
\begin{align*}
 x_1 - x_2 - 2x_3 + x_4 + 11x_5 & = 13 \\
 x_1 - x_2 + x_3 + x_4 + 5x_5 & = 16 \\
 2x_1 - 2x_2 + x_4 + 10x_5 & = 21
\end{align*}
\]
\[2x_1 - 2x_2 - x_3 + 3x_4 + 20x_5 = 38\]
\[2x_1 - 2x_2 + x_3 + x_4 + 8x_5 = 22\]

whose augmented matrix row-reduces to
\[
\begin{bmatrix}
1 & -1 & 0 & 0 & 3 & 6 \\
0 & 0 & 1 & 0 & -2 & 1 \\
0 & 0 & 0 & 1 & 4 & 9 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

There are leading 1’s in columns 1, 3 and 4, so \(D = \{1, 3, 4\}\). From this we know that the variables \(x_1, x_3\) and \(x_4\) will be dependent variables, and each of the \(r = 3\) nonzero rows of the row-reduced matrix will yield an expression for one of these three variables. The set \(F\) is all the remaining column indices, \(F = \{2, 5, 6\}\). Since \(6 \in F\) we know there is no leading 1 in the final column, so the system is consistent by Theorem RCLS \[54\]. The remaining indices in \(F\) will correspond to free variables, so \(x_2\) and \(x_5\) are our free variables. The resulting three equations that describe our solution set are then,

\[
(x_{d_1} = x_1) \quad \quad x_1 = 6 + x_2 - 3x_5
\]
\[
(x_{d_2} = x_3) \quad \quad x_3 = 1 + 2x_5
\]
\[
(x_{d_3} = x_4) \quad \quad x_4 = 9 - 4x_5
\]

Make sure you understand where these three equations came from, and notice how the location of the leading 1’s determined the variables on the left-hand side of each equation. We can compactly describe the solution set as,

\[
S = \left\{ \begin{bmatrix} 6 + x_2 - 3x_5 \\ x_2 \\ 1 + 2x_5 \\ 9 - 4x_5 \\ x_5 \end{bmatrix} \mid x_2, x_5 \in \mathbb{C} \right\}
\]

Notice how we express the freedom for \(x_2\) and \(x_5\): \(x_2, x_5 \in \mathbb{C}\). 

 Sets are an important part of algebra, and we’ve seen a few already. Being comfortable with sets is important for understanding and writing proofs. If you haven’t already, pay a visit now to Section SET \[693\].

We can now use the values of \(m, n, r\), and the independent and dependent variables to categorize the solution sets for linear systems through a sequence of theorems. Through the following sequence of proofs, you will want to consult three proof techniques. See Technique E \[704\]. See Technique N \[705\]. See Technique CP \[706\].

First we have a theorem that explores the distinction between consistent and inconsistent linear systems.

**Theorem RCLS**

**Recognizing Consistency of a Linear System**

Suppose \(A\) is the augmented matrix of a system of linear equations with \(m\) equations in \(n\) variables. Suppose also that \(B\) is a row-equivalent matrix in reduced row-echelon form with \(r\) rows that are not zero rows. Then the system of equations is inconsistent if and only if the leading 1 of row \(r\) is located in column \(n + 1\) of \(B\). 

\(\square\)
**Proof**  
(⇐) The first half of the proof begins with the assumption that the leading 1 of row $r$ is located in column $n+1$ of $B$. Then row $r$ of $B$ begins with $n$ consecutive zeros, finishing with the leading 1. This is a representation of the equation $0 = 1$, which is false. Since this equation is false for any collection of values we might choose for the variables, there are no solutions for the system of equations, and it is inconsistent.

(⇒) For the second half of the proof, we wish to show that if we assume the system is inconsistent, then the final leading 1 is located in the last column. But instead of proving this directly, we’ll form the logically equivalent statement that is the contrapositive, and prove that instead (see [Technique CP][706]). Turning the implication around, and negating each portion, we arrive at the logically equivalent statement: If the leading 1 of row $r$ is not in column $n+1$, then the system of equations is consistent.

If the leading 1 for row $r$ is located somewhere in columns 1 through $n$, then every preceding row’s leading 1 is also located in columns 1 through $n$. In other words, since the last leading 1 is not in the last column, no leading 1 for any row is in the last column, due to the echelon layout of the leading 1’s. Let $b_{i,n+1}$, $1 \leq i \leq r$, denote the entries of the last column of $B$ for the first $r$ rows. Employ our notation for columns of the reduced row-echelon form of a matrix (see [Notation RREFA][36]) to $B$ and set $x_{fi} = 0$, $1 \leq i \leq n-r$ and then set $x_{di} = b_{i,n+1}$, $1 \leq i \leq r$. In other words, set the dependent variables equal to the corresponding values in the final column and set all the free variables to zero. These values for the variables make the equations represented by the first $r$ rows all true (convince yourself of this). Rows $r+1$ through $m$ (if any) are all zero rows, hence represent the equation $0 = 0$ and are also all true. We have now identified one solution to the system, so we can say the system is consistent.

The beauty of this theorem being an equivalence is that we can unequivocally test to see if a system is consistent or inconsistent by looking at just a single entry of the reduced row-echelon form matrix. We could program a computer to do it!

Notice that for a consistent system the row-reduced augmented matrix has $n+1 \in F$, so the largest element of $F$ does not refer to a variable. Also, for an inconsistent system, $n+1 \in D$, and it then does not make much sense to discuss whether or not variables are free or dependent since there is no solution. With the characterization of [Theorem RCLS][54], we can explore the relationships between $r$ and $n$ in light of the consistency of a system of equations. First, a situation where we can quickly conclude the inconsistency of a system.

**Theorem ISRN**  
**Inconsistent Systems, $r$ and $n$**

Suppose $A$ is the augmented matrix of a system of linear equations with $m$ equations in $n$ variables. Suppose also that $B$ is a row-equivalent matrix in reduced row-echelon form with $r$ rows that are not completely zeros. If $r = n+1$, then the system of equations is inconsistent. □

**Proof**  
If $r = n+1$, then $D = \{1, 2, 3, \ldots, n, n+1\}$ and every column of $B$ contains a leading 1 and is a pivot column. In particular, the entry of column $n+1$ for row $r = n+1$ is a leading 1. [Theorem RCLS][54] then says that the system is inconsistent. □

Do not confuse [Theorem ISRN][55] with its converse! Go check out [Technique CV][707] right now.

Next, if a system is consistent, we can distinguish between a unique solution and infinitely many solutions, and furthermore, we recognize that these are the only two possibilities.
Theorem CSRN
Consistent Systems, $r$ and $n$

Suppose $A$ is the augmented matrix of a consistent system of linear equations with $m$ equations in $n$ variables. Suppose also that $B$ is a row-equivalent matrix in reduced row-echelon form with $r$ rows that are not zero rows. Then $r \leq n$. If $r = n$, then the system has a unique solution, and if $r < n$, then the system has infinitely many solutions.

\[\square\]

Proof This theorem contains three implications that we must establish. Notice first that $B$ has $n + 1$ columns, so there can be at most $n + 1$ pivot columns, i.e. $r \leq n + 1$. If $r = n + 1$, then Theorem ISRN \[55\] tells us that the system is inconsistent, contrary to our hypothesis. We are left with $r \leq n$.

When $r = n$, we find $n - r = 0$ free variables (i.e. $F = \{n + 1\}$) and any solution must equal the unique solution given by the first $n$ entries of column $n + 1$ of $B$.

When $r < n$, we have $n - r > 0$ free variables, corresponding to columns of $B$ without a leading 1, excepting the final column, which also does not contain a leading 1 by Theorem RCLS \[54\]. By varying the values of the free variables suitably, we can demonstrate infinitely many solutions.

\[\blacksquare\]

The next theorem simply states a conclusion from the final paragraph of the previous proof, allowing us to state explicitly the number of free variables for a consistent system.

Theorem FVCS
Free Variables for Consistent Systems

Suppose $A$ is the augmented matrix of a consistent system of linear equations with $m$ equations in $n$ variables. Suppose also that $B$ is a row-equivalent matrix in reduced row-echelon form with $r$ rows that are not completely zeros. Then the solution set can be described with $n - r$ free variables.

\[\square\]

Proof See the proof of Theorem CSRN \[56\].

\[\blacksquare\]

Example CFV
Counting free variables

For each archetype that is a system of equations, the values of $n$ and $r$ are listed. Many also contain a few sample solutions. We can use this information profitably, as illustrated by four examples.

1. Archetype A \[721\] has $n = 3$ and $r = 2$. It can be seen to be consistent by the sample solutions given. Its solution set then has $n - r = 1$ free variables, and therefore will be infinite.

2. Archetype B \[726\] has $n = 3$ and $r = 3$. It can be seen to be consistent by the single sample solution given. Its solution set can then be described with $n - r = 0$ free variables, and therefore will have just the single solution.

3. Archetype H \[752\] has $n = 2$ and $r = 3$. In this case, $r = n + 1$, so Theorem ISRN \[55\] says the system is inconsistent. We should not try to apply Theorem FVCS \[56\] to count free variables, since the theorem only applies to consistent systems. (What would happen if you did?)

4. Archetype E \[739\] has $n = 4$ and $r = 3$. However, by looking at the reduced row-echelon form of the augmented matrix, we find a leading 1 in row 3, column 4. By
Theorem RCLS \[54\] we recognize the system is then inconsistent. (Why doesn’t this example contradict Theorem ISRN \[55\]?)

We have accomplished a lot so far, but our main goal has been the following theorem, which is now very simple to prove. The proof is so simple that we ought to call it a corollary, but the result is important enough that it deserves to be called a theorem. (See Technique LC \[716\] .) Notice that this theorem was presaged first by Example TTS \[14\] and further foreshadowed by other examples.

**Theorem PSSLS**

**Possible Solution Sets for Linear Systems**

A system of linear equations has no solutions, a unique solution or infinitely many solutions.

**Proof** By definition, a system is either inconsistent or consistent. The first case describes systems with no solutions. For consistent systems, we have the remaining two possibilities as guaranteed by, and described in, Theorem CSRN \[56\].

We have one more theorem to round out our set of tools for determining solution sets to systems of linear equations.

**Theorem CMVEI**

**Consistent, More Variables than Equations, Infinite solutions**

Suppose a consistent system of linear equations has \( m \) equations in \( n \) variables. If \( n > m \), then the system has infinitely many solutions.

**Proof** Suppose that the augmented matrix of the system of equations is row-equivalent to \( B \), a matrix in reduced row-echelon form with \( r \) nonzero rows. Because \( B \) has \( m \) rows in total, the number that are nonzero rows is less. In other words, \( r \leq m \). Follow this with the hypothesis that \( n > m \) and we find that the system has a solution set described by at least one free variable because

\[
n - r \geq n - m > 0.
\]

A consistent system with free variables will have an infinite number of solutions, as given by Theorem CSRN \[56\].

Notice that to use this theorem we need only know that the system is consistent, together with the values of \( m \) and \( n \). We do not necessarily have to compute a row-equivalent reduced row-echelon form matrix, even though we discussed such a matrix in the proof. This is the substance of the following example.

**Example OSGMD**

**One solution gives many, Archetype D**

Archetype D is the system of \( m = 3 \) equations in \( n = 4 \) variables,

\[
\begin{align*}
2x_1 + x_2 + 7x_3 - 7x_4 &= 8 \\
-3x_1 + 4x_2 - 5x_3 - 6x_4 &= -12 \\
x_1 + x_2 + 4x_3 - 5x_4 &= 4
\end{align*}
\]

and the solution \( x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 1 \) can be checked easily by substitution. Having been *handed* this solution, we know the system is consistent. This, together with
$n > m$, allows us to apply Theorem CMVEI \[57\] and conclude that the system has infinitely many solutions.

These theorems give us the procedures and implications that allow us to completely solve any system of linear equations. The main computational tool is using row operations to convert an augmented matrix into reduced row-echelon form. Here’s a broad outline of how we would instruct a computer to solve a system of linear equations.

1. Represent a system of linear equations by an augmented matrix (an array is the appropriate data structure in most computer languages).

2. Convert the matrix to a row-equivalent matrix in reduced row-echelon form using the procedure from the proof of Theorem REMEF \[36\].

3. Determine $r$ and locate the leading 1 of row $r$. If it is in column $n+1$, output the statement that the system is inconsistent and halt.

4. With the leading 1 of row $r$ not in column $n+1$, there are two possibilities:

   (a) $r = n$ and the solution is unique. It can be read off directly from the entries in rows 1 through $n$ of column $n+1$.

   (b) $r < n$ and there are infinitely many solutions. If only a single solution is needed, set all the free variables to zero and read off the dependent variable values from column $n+1$, as in the second half of the proof of Theorem RCLS \[54\]. If the entire solution set is required, figure out some nice compact way to describe it, since your finite computer is not big enough to hold all the solutions (we’ll have such a way soon).

The above makes it all sound a bit simpler than it really is. In practice, row operations employ division (usually to get a leading entry of a row to convert to a leading 1) and that will introduce round-off errors. Entries that should be zero sometimes end up being very, very small nonzero entries, or small entries lead to overflow errors when used as divisors. A variety of strategies can be employed to minimize these sorts of errors, and this is one of the main topics in the important subject known as numerical linear algebra.

Solving a linear system is such a fundamental problem in so many areas of mathematics, and its applications, that any computational device worth using for linear algebra will have a built-in routine to do just that. See: Computation LS,MMA \[680\]. In this section we’ve gained a foolproof procedure for solving any system of linear equations, no matter how many equations or variables. We also have a handful of theorems that allow us to determine partial information about a solution set without actually constructing the whole set itself. Donald Knuth would be proud.

**Subsection READ**

**Reading Questions**

1. How do we recognize when a system of linear equations is inconsistent?

2. Suppose we have converted the augmented matrix of a system of equations into reduced row-echelon form. How do we then identify the dependent and independent (free) variables?
3. What are the possible solution sets for a system of linear equations?
Subsection EXC
Exercises

C10  In the spirit of Example ISSI [52], describe the infinite solution set for Archetype J [762].
Contributed by Robert Beezer

M45  Prove that Archetype J [762] has infinitely many solutions without row-reducing the augmented matrix.
Contributed by Robert Beezer  Solution [63]

For Exercises M51–M54 say as much as possible about each system's solution set. Be sure to make it clear which theorems you are using to reach your conclusions.

M51  A consistent system of 8 equations in 6 variables.
Contributed by Robert Beezer  Solution [63]

M52  A consistent system of 6 equations in 8 variables.
Contributed by Robert Beezer  Solution [63]

M53  A system of 5 equations in 9 variables.
Contributed by Robert Beezer  Solution [63]

M54  A system with 12 equations in 35 variables.
Contributed by Robert Beezer  Solution [63]

M60  Without doing any computations, and without examining any solutions, say as much as possible about the form of the solution set for each archetype that is a system of equations.
Arche type A [721]
Arche type B [726]
Arche type C [731]
Arche type D [735]
Arche type E [739]
Arche type F [743]
Arche type G [748]
Arche type H [752]
Arche type I [757]
Arche type J [762]
Contributed by Robert Beezer

T10  An inconsistent system may have \( r > n \). If we try (incorrectly!) to apply Theorem FVCS [56] to such a system, how many free variables would we discover?
Contributed by Robert Beezer  Solution [63]
Subsection SOL
Solutions

M45  Contributed by Robert Beezer  Statement 61
Demonstrate that the system is consistent by verifying any one of the four sample solutions provided. Then because \( n = 9 > 6 = m \), Theorem CMVEI 57 gives us the conclusion that the system has infinitely many solutions.

Notice that we only know the system will have at least \( 9 - 6 = 3 \) free variables, but very well could have more. We do not know know that \( r = 6 \), only that \( r \leq 6 \).

M51  Contributed by Robert Beezer  Statement 61
Consistent means there is at least one solution (Definition CS 51). It will have either a unique solution or infinitely many solutions (Theorem PSSLS 57).

M52  Contributed by Robert Beezer  Statement 61
With 6 rows in the augmented matrix, the row-reduced version will have \( r \leq 6 \). Since the system is consistent, apply Theorem CSRN 56 to see that \( n - r \geq 2 \) implies infinitely many solutions.

M53  Contributed by Robert Beezer  Statement 61
The system could be inconsistent. If it is consistent, then because it has more variables than equations, Theorem CMVEI 57 implies that there would be infinitely many solutions. So, of all the possibilities in Theorem PSSLS 57, only the case of a unique solution can be ruled out.

M54  Contributed by Robert Beezer  Statement 61
The system could be inconsistent. If it is consistent, then Theorem CMVEI 57 tells us the solution set will be infinite. So we can be certain that there is not a unique solution.

T10  Contributed by Robert Beezer  Statement 61
Theorem FVCS 56 will indicate a negative number of free variables, but we can say even more. If \( r > n \), then the only possibility is that \( r = n + 1 \), and then we compute \( n - r = n - (n + 1) = -1 \) free variables.
In this section we specialize to systems of linear equations where every equation has a zero as its constant term. Along the way, we will begin to express more and more ideas in the language of matrices and begin a move away from writing out whole systems of equations. The ideas initiated in this section will carry through the remainder of the course.

Subsection SHS
Solutions of Homogeneous Systems

As usual, we begin with a definition.

Definition HS
Homogeneous System

A system of linear equations, \( \mathcal{L}(A, b) \) is homogeneous if the vector of constants is the zero vector, in other words, \( b = 0 \).

Example AHSAC
Archetype C as a homogeneous system

For each archetype that is a system of equations, we have formulated a similar, yet different, homogeneous system of equations by replacing each equation’s constant term with a zero. To wit, for Archetype C \([731]\), we can convert the original system of equations into the homogeneous system,

\[
\begin{align*}
2x_1 - 3x_2 + x_3 - 6x_4 &= 0 \\
4x_1 + x_2 + 2x_3 + 9x_4 &= 0 \\
3x_1 + x_2 + x_3 + 8x_4 &= 0
\end{align*}
\]

Can you quickly find a solution to this system without row-reducing the augmented matrix?

As you might have discovered by studying Example AHSAC \([65]\), setting each variable to zero will always be a solution of a homogeneous system. This is the substance of the following theorem.

Theorem HSC
Homogeneous Systems are Consistent

Suppose that a system of linear equations is homogeneous. Then the system is consistent.

Proof Set each variable of the system to zero. When substituting these values into each equation, the left-hand side evaluates to zero, no matter what the coefficients are. Since a homogeneous system has zero on the right-hand side of each equation as the constant term, each equation is true. With one demonstrated solution, we can call the system consistent.
Since this solution is so obvious, we now define it as the trivial solution.

**Definition TSHSE**

**Trivial Solution to Homogeneous Systems of Equations**

Suppose a homogeneous system of linear equations has $n$ variables. The solution $x_1 = 0$, $x_2 = 0, \ldots, x_n = 0$ (i.e. $\mathbf{x} = \mathbf{0}$) is called the **trivial solution**.

Here are three typical examples, which we will reference throughout this section. Work through the row operations as we bring each to reduced row-echelon form. Also notice what is similar in each example, and what differs.

**Example HUSAB**

**Homogeneous, unique solution, Archetype B**

Archetype B can be converted to the homogeneous system,

\[-11x_1 + 2x_2 - 14x_3 = 0
23x_1 - 6x_2 + 33x_3 = 0
14x_1 - 2x_2 + 17x_3 = 0\]

whose augmented matrix row-reduces to

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

By **Theorem HSC** [65], the system is consistent, and so the computation $n-r = 3-3 = 0$ means the solution set contains just a single solution. Then, this lone solution must be the trivial solution.

**Example HISAA**

**Homogeneous, infinite solutions, Archetype A**

Archetype A [721] can be converted to the homogeneous system,

\[x_1 - x_2 + 2x_3 = 0
2x_1 + x_2 + x_3 = 0
x_1 + x_2 = 0\]

whose augmented matrix row-reduces to

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

By **Theorem HSC** [65], the system is consistent, and so the computation $n-r = 3-2 = 1$ means the solution set contains one free variable by **Theorem FVCS** [56], and hence has infinitely many solutions. We can describe this solution set using the free variable $x_3$,

\[S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1 = -x_3, x_2 = x_3 \right\} = \left\{ \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} \middle| x_3 \in \mathbb{C} \right\}\]

Geometrically, these are points in three dimensions that lie on a line through the origin.
Example HISAD
Homogeneous, infinite solutions, Archetype D
Archetype D \[735\] (and identically, Archetype E \[739\]) can be converted to the homogeneous system,
\[
\begin{align*}
2x_1 + x_2 + 7x_3 - 7x_4 &= 0 \\
-3x_1 + 4x_2 - 5x_3 - 6x_4 &= 0 \\
x_1 + x_2 + 4x_3 - 5x_4 &= 0
\end{align*}
\]
whose augmented matrix row-reduces to
\[
\begin{bmatrix}
1 & 0 & 3 & -2 & 0 \\
0 & 1 & 1 & -3 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
By Theorem HSC \[65\], the system is consistent, and so the computation \(n - r = 4 - 2 = 2\) means the solution set contains two free variables by Theorem FVCS \[56\], and hence has infinitely many solutions. We can describe this solution set using the free variables \(x_3\) and \(x_4\),
\[
S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \bigg| x_1 = -3x_3 + 2x_4, \ x_2 = -x_3 + 3x_4 \right\}
\]
\[
= \left\{ \begin{bmatrix} -3x_3 + 2x_4 \\ -x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix} \bigg| x_3, \ x_4 \in \mathbb{C} \right\}
\]

After working through these examples, you might perform the same computations for the slightly larger example, Archetype J \[762\].

Notice that when we do row operations on the augmented matrix of a homogeneous system of linear equations the last column of the matrix is all zeros. Any one of the three allowable row operations will convert zeros to zeros and thus, the final column of the matrix in reduced row-echelon form will also be all zeros. So in this case, we may be as likely to reference only the coefficient matrix and presume that we remember that the final column begins with zeros, and after any number of row operations is still zero.

Example HISAD \[67\] suggests the following theorem.

Theorem HMVEI
Homogeneous, More Variables than Equations, Infinite solutions
Suppose that a homogeneous system of linear equations has \(m\) equations and \(n\) variables with \(n > m\). Then the system has infinitely many solutions. □

Proof We are assuming the system is homogeneous, so Theorem HSC \[65\] says it is consistent. Then the hypothesis that \(n > m\), together with Theorem CMVEI \[57\], gives infinitely many solutions. □
Example HUSAB 66 and Example HISAA 66 are concerned with homogeneous systems where \( n = m \) and expose a fundamental distinction between the two examples. One has a unique solution, while the other has infinitely many. These are exactly the only two possibilities for a homogeneous system and illustrate that each is possible (unlike the case when \( n > m \) where Theorem HMVEI 67 tells us that there is only one possibility for a homogeneous system).

Subsection NSM
Null Space of a Matrix

The set of solutions to a homogeneous system (which by Theorem HSC 65 is never empty) is of enough interest to warrant its own name. However, we define it as a property of the coefficient matrix, not as a property of some system of equations.

**Definition NSM**
**Null Space of a Matrix**

The null space of a matrix \( A \), denoted \( \mathcal{N}(A) \), is the set of all the vectors that are solutions to the homogeneous system \( \mathcal{L}S(A, 0) \).

(This definition contains Notation NSM.)

In the Archetypes (Appendix A 717) each example that is a system of equations also has a corresponding homogeneous system of equations listed, and several sample solutions are given. These solutions will be elements of the null space of the coefficient matrix. We'll look at one example.

**Example NSEAI**
**Null space elements of Archetype I**

The write-up for Archetype I 757 lists several solutions of the corresponding homogeneous system. Here are two, written as solution vectors. We can say that they are in the null space of the coefficient matrix for the system of equations in Archetype I 757.

\[
\begin{bmatrix}
 3 \\
 0 \\
-5 \\
-6 \\
 0 \\
 0 \\
 1
\end{bmatrix}
\quad
\begin{bmatrix}
-4 \\
 1 \\
-3 \\
-2 \\
 1 \\
 1 \\
 1
\end{bmatrix}
\]

However, the vector

\[
\begin{bmatrix}
1 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 2
\end{bmatrix}
\]

is not in the null space, since it is not a solution to the homogeneous system. For example, it fails to even make the first equation true.

\[\boxtimes\]
Here are two (prototypical) examples of the computation of the null space of a matrix. Notice that we will now begin writing solutions as vectors.

**Example CNS1**

**Computing a null space, #1**

Let's compute the null space of

\[
A = \begin{bmatrix}
2 & -1 & 7 & -3 & -8 \\
1 & 0 & 2 & 4 & 9 \\
2 & 2 & -2 & -1 & 8
\end{bmatrix}
\]

which we write as \( \mathcal{N}(A) \). Translating [Definition NSM][68], we simply desire to solve the homogeneous system \( \mathcal{L}S(A, 0) \). So we row-reduce the augmented matrix to obtain

\[
\begin{bmatrix}
1 & 0 & 2 & 0 & 1 & 0 \\
0 & 1 & -3 & 0 & 4 & 0 \\
0 & 0 & 0 & 1 & 2 & 0
\end{bmatrix}
\]

The variables (of the homogeneous system) \( x_3 \) and \( x_5 \) are free (since columns 1, 2 and 4 are pivot columns), so we arrange the equations represented by the matrix in reduced row-echelon form to

\[
\begin{align*}
\begin{aligned}
x_1 &= -2x_3 - x_5 \\
x_2 &= 3x_3 - 4x_5 \\
x_4 &= -2x_5
\end{aligned}
\end{align*}
\]

So we can write the infinite solution set as sets using column vectors,

\[
\mathcal{N}(A) = \left\{ \begin{bmatrix}
-2x_3 - x_5 \\
3x_3 - 4x_5 \\
x_3 \\
-2x_5 \\
x_5
\end{bmatrix} \bigg| x_3, x_5 \in \mathbb{C} \right\}
\]

**Example CNS2**

**Computing a null space, #2**

Let's compute the null space of

\[
C = \begin{bmatrix}
-4 & 6 & 1 \\
-1 & 4 & 1 \\
5 & 6 & 7 \\
4 & 7 & 1
\end{bmatrix}
\]

which we write as \( \mathcal{N}(C) \). Translating [Definition NSM][68], we simply desire to solve the homogeneous system \( \mathcal{L}S(C, 0) \). So we row-reduce the augmented matrix to obtain

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
There are no free variables in the homogenous system represented by the row-reduced matrix, so there is only the trivial solution, the zero vector, \( \mathbf{0} \). So we can write the (trivial) solution set as

\[ \mathcal{N}(C) = \{ \mathbf{0} \} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

Subsection READ

Reading Questions

1. What is *always* true of the solution set for a homogenous system of equations?

2. Suppose a homogenous sytem of equations has 13 variables and 8 equations. How many solutions will it have? Why?

3. Describe in words (not symbols) the null space of a matrix.
Subsection EXC
Exercises

C10  Each archetype (Appendix A) that is a system of equations has a corresponding homogeneous system with the same coefficient matrix. Compute the set of solutions for each. Notice that these solution sets are the null spaces of the coefficient matrices.

Archetype A  721
Archetype B  726
Archetype C  731
Archetype D  735
Archetype E  739
Archetype F  743
Archetype G  748
Archetype H  752
Archetype I  757
and Archetype J  762
Contributed by Robert Beezer

C20  Archetype K  767 and Archetype L  771 are simply $5 \times 5$ matrices (i.e. they are not systems of equations). Compute the null space of each matrix.
Contributed by Robert Beezer

C30  Compute the null space of the matrix $A$, $\mathcal{N}(A)$.

$$A = \begin{bmatrix} 2 & 4 & 1 & 3 & 8 \\ -1 & -2 & -1 & -1 & 1 \\ 2 & 4 & 0 & -3 & 4 \\ 2 & 4 & -1 & -7 & 4 \end{bmatrix}$$

Contributed by Robert Beezer  Solution  73

M45  Without doing any computations, and without examining any solutions, say as much as possible about the form of the solution set for corresponding homogeneous system of equations of each archetype that is a system of equations.

Archetype A  721
Archetype B  726
Archetype C  731
Archetype D  735
Archetype E  739
Archetype F  743
Archetype G  748
Archetype H  752
Archetype I  757
Archetype J  762
Contributed by Robert Beezer

For Exercises M50–M52 say as much as possible about each system’s solution set. Be sure to make it clear which theorems you are using to reach your conclusions.

M50  A homogeneous system of 8 equations in 8 variables.
Contributed by Robert Beezer  Solution  73

M51  A homogeneous system of 8 equations in 9 variables.
Contributed by Robert Beezer  Solution  73
M52 A homogeneous system of 8 equations in 7 variables.
Contributed by Robert Beezer Solution 73

T10 Prove or disprove: A system of linear equations is homogeneous if and only if the system has the zero vector as a solution.
Contributed by Martin Jackson Solution 73

T20 Consider the homogeneous system of linear equations $\mathcal{LS}(A, 0)$, and suppose that
\[
\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix}
\]
is one solution to the system of equations. Prove that
\[
\mathbf{v} = \begin{bmatrix} 4u_1 \\ 4u_2 \\ 4u_3 \\ \vdots \\ 4u_n \end{bmatrix}
\]
is also a solution to $\mathcal{LS}(A, 0)$.
Contributed by Robert Beezer Solution 74
Subsection SOL
Solutions

**C30** Contributed by Robert Beezer Statement [71]

Definition NSM [68] tells us that the null space of $A$ is the solution set to the homogeneous system $LS(A, 0)$. The augmented matrix of this system is

$$
\begin{bmatrix}
2 & 4 & 1 & 3 & 8 & 0 \\
-1 & -2 & -1 & 1 & 1 & 0 \\
2 & 4 & 0 & -3 & 4 & 0 \\
2 & 4 & -1 & -7 & 4 & 0
\end{bmatrix}
$$

To solve the system, we row-reduce the augmented matrix and obtain,

$$
\begin{bmatrix}
1 & 2 & 0 & 0 & 5 & 0 \\
0 & 0 & 1 & 0 & -8 & 0 \\
0 & 0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

This matrix represents a system with equations having three dependent variables ($x_1$, $x_3$, and $x_4$) and two independent variables ($x_2$ and $x_5$). These equations rearrange to

$$
x_1 = -2x_2 - 5x_5 \\
x_3 = 8x_5 \\
x_4 = -2x_5
$$

So we can write the solution set (which is the requested null space) as

$$
N(A) = \left\{ \begin{bmatrix} -2x_2 - 5x_5 \\ x_2 \\ 8x_5 \\ -2x_5 \\ x_5 \end{bmatrix} \mid x_2, x_5 \in \mathbb{C} \right\}
$$

**M50** Contributed by Robert Beezer Statement [71]

Since the system is homogeneous, we know it has the trivial solution (Theorem HSC [65]). We cannot say anymore based on the information provided, except to say that there is either a unique solution or infinitely many solutions (Theorem PSSLS [57]). See Archetype A [721] and Archetype B [726] to understand the possibilities.

**M51** Contributed by Robert Beezer Statement [71]

Since there are more variables than equations, Theorem HMVEI [67] applies and tells us that the solution set is infinite. From the proof of Theorem HSC [65] we know that the zero vector is one solution.

**M52** Contributed by Robert Beezer Statement [72]

By Theorem HSC [65], we know the system is consistent because the zero vector is always a solution of a homogeneous system. There is no more that we can say, since both a unique solution and infinitely many solutions are possibilities.

**T10** Contributed by Robert Beezer Statement [72]

This is a true statement. A proof is:
(⇐) Suppose we have a homogeneous system $\mathcal{L}S(A, 0)$. Then by substituting the scalar zero for each variable, we arrive at true statements for each equation. So the zero vector is a solution. This is the content of Theorem HSC [65].

(⇒) Suppose now that we have a generic (i.e. not necessarily homogeneous) system of equations, $\mathcal{L}S(A, b)$ that has the zero vector as a solution. Upon substituting this solution into the system, we discover that each component of $b$ must also be zero. So $b = 0$.

T20 Contributed by Robert Beezer Statement [72]
Suppose that a single equation from this system (the $i$-th one) has the form,

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \cdots + a_{in}x_n = 0$$

Evaluate the left-hand side of this equation with the components of the proposed solution vector $v$,

$$a_{i1} (4u_1) + a_{i2} (4u_2) + a_{i3} (4u_3) + \cdots + a_{in} (4u_n)$$

$$= 4a_{i1}u_1 + 4a_{i2}u_2 + 4a_{i3}u_3 + \cdots + 4a_{in}u_n \quad \text{Commutativity}$$

$$= 4 \left( a_{i1}u_1 + a_{i2}u_2 + a_{i3}u_3 + \cdots + a_{in}u_n \right) \quad \text{Distributivity}$$

$$= 4(0) \quad u \text{ solution to } \mathcal{L}S(A, 0)$$

$$= 0$$

So $v$ makes each equation true, and so is a solution to the system.

Notice that this result is not true if we change $\mathcal{L}S(A, 0)$ from a homogeneous system to a non-homogeneous system. Can you create an example of a (non-homogeneous) system with a solution $u$ such that $v$ is not a solution?
Section NM
Nonsingular Matrices

In this section we specialize and consider matrices with equal numbers of rows and columns, which when considered as coefficient matrices lead to systems with equal numbers of equations and variables. We will see in the second half of the course (Chapter D [409], Chapter E [441], Chapter LT [503], Chapter R [587]) that these matrices are especially important.

Subsection NM
Nonsingular Matrices

Our theorems will now establish connections between systems of equations (homogeneous or otherwise), augmented matrices representing those systems, coefficient matrices, constant vectors, the reduced row-echelon form of matrices (augmented and coefficient) and solution sets. Be very careful in your reading, writing and speaking about systems of equations, matrices and sets of vectors. A system of equations is not a matrix, a matrix is not a solution set, and a solution set is not a system of equations. Now would be a great time to review the discussion about speaking and writing mathematics in Technique L [700].

Definition SQM
Square Matrix
A matrix with $m$ rows and $n$ columns is square if $m = n$. In this case, we say the matrix has size $n$. To emphasize the situation when a matrix is not square, we will call it rectangular. $\triangle$

We can now present one of the central definitions of linear algebra.

Definition NM
Nonsingular Matrix
Suppose $A$ is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $LS(A, \mathbf{0})$ is $\{\mathbf{0}\}$, i.e. the system has only the trivial solution. Then we say that $A$ is a nonsingular matrix. Otherwise we say $A$ is a singular matrix. $\triangle$

We can investigate whether any square matrix is nonsingular or not, no matter if the matrix is derived somehow from a system of equations or if it is simply a matrix. The definition says that to perform this investigation we must construct a very specific system of equations (homogeneous, with the matrix as the coefficient matrix) and look at its solution set. We will have theorems in this section that connect nonsingular matrices with systems of equations, creating more opportunities for confusion. Convince yourself now of two observations, (1) we can decide nonsingularity for any square matrix, and (2) the determination of nonsingularity involves the solution set for a certain homogenous system of equations.
Notice that it makes no sense to call a system of equations nonsingular (the term does not apply to a system of equations), nor does it make any sense to call a $5 \times 7$ matrix singular (the matrix is not square).

**Example S**

**A singular matrix, Archetype A**

Example HISAA [66] shows that the coefficient matrix derived from Archetype A [721], specifically the $3 \times 3$ matrix,

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

is a singular matrix since there are nontrivial solutions to the homogeneous system $LS(A, 0)$.

**Example NM**

**A nonsingular matrix, Archetype B**

Example HUSAB [66] shows that the coefficient matrix derived from Archetype B [726], specifically the $3 \times 3$ matrix,

$$B = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$$

is a nonsingular matrix since the homogeneous system, $LS(B, 0)$, has only the trivial solution.

Notice that we will not discuss Example HISAD [67] as being a singular or nonsingular coefficient matrix since the matrix is not square.

The next theorem combines with our main computational technique (row-reducing a matrix) to make it easy to recognize a nonsingular matrix. But first a definition.

**Definition IM**

**Identity Matrix**

The $m \times m$ identity matrix, $I_m$, is defined by

$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

(This definition contains Notation IM.)

**Example IM**

**An identity matrix**

The $4 \times 4$ identity matrix is

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
Notice that an identity matrix is square, and in reduced row-echelon form. So in particular, if we were to arrive at the identity matrix while bringing a matrix to reduced row-echelon form, then it would have all of the diagonal entries circled as leading 1’s.

**Theorem NMRRI**

**Nonsingular Matrices Row Reduce to the Identity matrix**

Suppose that $A$ is a square matrix and $B$ is a row-equivalent matrix in reduced row-echelon form. Then $A$ is nonsingular if and only if $B$ is the identity matrix. □

**Proof** (⇐) Suppose $B$ is the identity matrix. When the augmented matrix $[A | 0]$ is row-reduced, the result is $[B | 0] = [I_n | 0]$. The number of nonzero rows is equal to the number of variables in the linear system of equations $LS(A, 0)$, so $n = r$ and Theorem FVCS gives $n - r = 0$ free variables. Thus, the homogeneous system $LS(A, 0)$ has just one solution, which must be the trivial solution. This is exactly the definition of a nonsingular matrix.

(⇒) If $A$ is nonsingular, then the homogeneous system $LS(A, 0)$ has a unique solution, and has no free variables in the description of the solution set. The homogeneous system is consistent (Theorem HSC) so Theorem FVCS applies and tells us there are $n - r$ free variables. Thus, $n - r = 0$, and so $n = r$. So $B$ has $n$ pivot columns among its total of $n$ columns. This is enough to force $B$ to be the $n \times n$ identity matrix $I_n$. ■

Notice that since this theorem is an equivalence it will always allow us to determine if a matrix is either nonsingular or singular. Here are two examples of this, continuing our study of Archetype A and Archetype B.

**Example SRR**

**Singular matrix, row-reduced**

The coefficient matrix for Archetype A is

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

which when row-reduced becomes the row-equivalent matrix

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$  

Since this matrix is not the $3 \times 3$ identity matrix, Theorem NMRRI tells us that $A$ is a singular matrix.

**Example NSR**

**Nonsingular matrix, row-reduced**

The coefficient matrix for Archetype B is

$$A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$$

which when row-reduced becomes the row-equivalent matrix

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
Since this matrix is the $3 \times 3$ identity matrix, Theorem NMRRI tells us that $A$ is a nonsingular matrix.

**Example NSS**

**Null space of a singular matrix**

Given the coefficient matrix from Archetype A,

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

the null space is the set of solutions to the homogeneous system of equations $LS(A, 0)$ has a solution set and null space constructed in Example HISAA as

$$N(A) = \left\{ \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} \bigg| x_3 \in \mathbb{C} \right\}$$

**Example NSNM**

**Null space of a nonsingular matrix**

Given the coefficient matrix from Archetype B,

$$A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$$

the homogeneous system $LS(A, 0)$ has a solution set constructed in Example HUSAB that contains only the trivial solution, so the null space has only a single element,

$$N(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

These two examples illustrate the next theorem, which is another equivalence.

**Theorem NMTNS**

**Nonsingular Matrices have Trivial Null Spaces**

Suppose that $A$ is a square matrix. Then $A$ is nonsingular if and only if the null space of $A$, $N(A)$, contains only the zero vector, i.e. $N(A) = \{0\}$.

**Proof**  The null space of a square matrix, $A$, is equal to the set of solutions to the homogeneous system, $LS(A, 0)$. A matrix is nonsingular if and only if the set of solutions to the homogeneous system, $LS(A, 0)$, has only a trivial solution. These two observations may be chained together to construct the two proofs necessary for each half of this theorem.

The next theorem pulls a lot of ideas together. Two proof techniques are applicable to the proof. So first, head out and read two more proof techniques: Technique CD and Technique U. Theorem NMUS tells us that we can learn a
lot about solutions to a system of linear equations with a square coefficient matrix by examining a similar homogeneous system.

**Theorem NMUS**  
Nonsingular Matrices and Unique Solutions

Suppose that $A$ is a square matrix. $A$ is a nonsingular matrix if and only if the system $Ls(A, b)$ has a unique solution for every choice of the constant vector $b$.

**Proof**  
($\Leftarrow$) The hypothesis for this half of the proof is that the system $Ls(A, b)$ has a unique solution for every choice of the constant vector $b$. We will make a very specific choice for $b$: $b = 0$. Then we know that the system $Ls(A, 0)$ has a unique solution. But this is precisely the definition of what it means for $A$ to be nonsingular (Definition NM [75]). That almost seems too easy! Notice that we have not used the full power of our hypothesis, but there is nothing that says we must use a hypothesis to its fullest.

If the first half of the proof seemed easy, perhaps we’ll have to work a bit harder to get the implication in the opposite direction. We provide two different proofs for the second half. The first is suggested by Asa Scherer and relies on the uniqueness of the reduced row-echelon form of a matrix (Theorem RREFU [116]), a result that we could have proven earlier, but we have decided to delay until later. The second proof is lengthier and more involved, but does not rely on the uniqueness of the reduced row-echelon form of a matrix, a result we have not proven yet. It is also a good example of the types of proofs we will encounter throughout the course.

($\Rightarrow$, Round 1) We assume that $A$ is nonsingular, so we know there is a sequence of row operations that will convert $A$ into the identity matrix $I_n$ (Theorem NMRR [77]). Form the augmented matrix $A' = [A | b]$ and apply this same sequence of row operations to $A'$. The result will be the matrix $B' = [I_n | c]$, which is in reduced row-echelon form. It should be clear that $c$ is a solution to $Ls(A, b)$. Furthermore, since $B'$ is unique (Theorem RREFU [116]), the vector $c$ must be unique, and therefore is a unique solution of $Ls(A, b)$.

($\Rightarrow$, Round 2) We will assume $A$ is nonsingular, and try to solve the system $Ls(A, b)$ without making any assumptions about $b$. To do this we will begin by constructing a new homogeneous linear system of equations that looks very much like the original. Suppose $A$ has size $n$ (why must it be square?) and write the original system as,

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
    a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
    \vdots \\
    a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= b_n
\end{align*}
\]

form the new, homogeneous system in $n$ equations with $n + 1$ variables, by adding a new variable $y$, whose coefficients are the negatives of the constant terms,

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n - b_1y &= 0 \\
    a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n - b_2y &= 0 \\
    a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n - b_3y &= 0
\end{align*}
\]
\[ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n - b_ny = 0 \]  

Since this is a homogeneous system with more variables than equations \((m = n + 1 > n)\), Theorem HMVEI \([67]\) says that the system has infinitely many solutions. We will choose one of these solutions, \textit{any} one of these solutions, so long as it is \textit{not} the trivial solution. Write this solution as

\[ x_1 = c_1, \quad x_2 = c_2, \quad x_3 = c_3, \quad \ldots, \quad x_n = c_n, \quad y = c_{n+1} \]

We know that at least one value of the \(c_i\) is nonzero, but we will now show that in particular \(c_{n+1} \neq 0\). We do this using a proof by contradiction (Technique CD \([708]\)). So suppose the \(c_i\) form a solution as described, and in addition that \(c_{n+1} = 0\). Then we can write the \(i\)-th equation of system (***) as,

\[ a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3 + \cdots + a_{in}c_n - b_i(0) = 0 \]

which becomes

\[ a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3 + \cdots + a_{in}c_n = 0 \]

Since this is true for each \(i\), we have that \(x_1 = c_1, x_2 = c_2, x_3 = c_3, \ldots, x_n = c_n\) is a solution to the homogeneous system \(L S(A, 0)\) formed with a nonsingular coefficient matrix. This means that the only possible solution is the trivial solution, so \(c_1 = 0, c_2 = 0, c_3 = 0, \ldots, c_n = 0\). So, assuming simply that \(c_{n+1} = 0\), we conclude that \textit{all} of the \(c_i\) are zero. But this contradicts our choice of the \(c_i\) as not being the trivial solution to the system (**). So \(c_{n+1} \neq 0\).

We now propose and verify a solution to the original system (*). Set

\[ x_1 = \frac{c_1}{c_{n+1}}, \quad x_2 = \frac{c_2}{c_{n+1}}, \quad x_3 = \frac{c_3}{c_{n+1}}, \quad \ldots, \quad x_n = \frac{c_n}{c_{n+1}} \]

Notice how it was necessary that we know that \(c_{n+1} \neq 0\) for this step to succeed. Now, evaluate the \(i\)-th equation of system (*) with this proposed solution, and recognize in the third line that \(c_1, c_2, \ldots, c_{n+1}\) appear as if they were substituted into the left-hand side of the \(i\)-th equation of system (**),

\[
\begin{align*}
&\left( a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3 + \cdots + a_{in}c_n \right) \frac{c_n}{c_{n+1}} = b_i(0) + b_i \\
&= \frac{1}{c_{n+1}} \left( a_{i1}c_1 + a_{i2}c_2 + a_{i3}c_3 + \cdots + a_{in}c_n - b_i c_{n+1} \right) \\
&= \frac{1}{c_{n+1}} (0) + b_i \\
&= b_i
\end{align*}
\]

Since this equation is true for every \(i\), we have found a solution to system (*). To finish, we still need to establish that this solution is \textit{unique}.

Version 0.92
With one solution in hand, we will entertain the possibility of a second solution. So assume system (*) has two solutions,

\[ \begin{align*}
  x_1 &= d_1 \\
  x_2 &= d_2 \\
  x_3 &= d_3 \\
  \vdots \\
  x_n &= d_n \\
  x_1 &= e_1 \\
  x_2 &= e_2 \\
  x_3 &= e_3 \\
  \vdots \\
  x_n &= e_n
\end{align*} \]

Then,

\[\begin{align*}
(a_1(d_1 - e_1) + a_2(d_2 - e_2) + a_3(d_3 - e_3) + \cdots + a_n(d_n - e_n)) \\
= (a_1d_1 + a_2d_2 + a_3d_3 + \cdots + a_n d_n) - (a_1e_1 + a_2e_2 + a_3e_3 + \cdots + a_n e_n) \\
= b_i - b_i \\
= 0
\end{align*}\]

This is the \(i\)-th equation of the homogeneous system \(LS(A, 0)\) evaluated with \(x_j = d_j - e_j\), \(1 \leq j \leq n\). Since \(A\) is nonsingular, we must conclude that this solution is the trivial solution, and so \(0 = d_j - e_j\), \(1 \leq j \leq n\). That is, \(d_j = e_j\) for all \(j\) and the two solutions are identical, meaning any solution to (*) is unique. \(\blacksquare\)

This important theorem deserves several comments. First, notice that the proposed solution \((x_i = \frac{c_i}{c_{i+1}})\) appeared in the Round 2 proof with no motivation whatsoever. This is just fine in a proof. A proof should convince you that a theorem is true. It is your job to read the proof and be convinced of every assertion. Questions like “Where did that come from?” or “How would I think of that?” have no bearing on the validity of the proof.

Second, this theorem helps to explain part of our interest in nonsingular matrices. If a matrix is nonsingular, then no matter what vector of constants we pair it with, using the matrix as the coefficient matrix will always yield a linear system of equations with a solution, and the solution is unique. To determine if a matrix has this property (nonsingularity) it is enough to just solve one linear system, the homogeneous system with the matrix as coefficient matrix and the zero vector as the vector of constants (or any other vector of constants, see Exercise MM.T10 [225]).

Finally, formulating the negation of the second part of this theorem is a good exercise. A singular matrix has the property that for some value of the vector \(b\), the system \(LS(A, b)\) does not have a unique solution (which means that it has no solution or infinitely many solutions). We will be able to say more about this case later (see the discussion following Theorem PSPHS [113]). Square matrices that are nonsingular have a list of interesting properties, which we will start to catalog in the following, recurring, theorem. Of course, singular matrices will then have all of the opposite properties. The following theorem is a list of equivalences. We want to understand just what is involved with understanding and proving a theorem that says several conditions are equivalent. So have a look at Technique ME [710] before studying the first in this series of theorems.

**Theorem NME1**

**Nonsingular Matrix Equivalences, Round 1**

Suppose that \(A\) is a square matrix. The following are equivalent.

1. \(A\) is nonsingular.
2. \(A\) row-reduces to the identity matrix.
3. The null space of $A$ contains only the zero vector, $\mathcal{N}(A) = \{0\}$.

4. The linear system $\mathcal{L}S(A, b)$ has a unique solution for every possible choice of $b$.

\[ \square \]

**Proof** That $A$ is nonsingular is equivalent to each of the subsequent statements by, in turn, Theorem NMRRI, Theorem NMTNS, and Theorem NMUS. So the statement of this theorem is just a convenient way to organize all these results.

---

**Subsection READ**

**Reading Questions**

1. What is the definition of a nonsingular matrix?

2. What is the easiest way to recognize a nonsingular matrix?

3. Suppose we have a system of equations and its coefficient matrix is nonsingular. What can you say about the solution set for this system?
Subsection EXC
Exercises

In Exercises C30–C33 determine if the matrix is nonsingular or singular. Give reasons for your answer.

C30
\[
\begin{bmatrix}
-3 & 1 & 2 & 8 \\
2 & 0 & 3 & 4 \\
1 & 2 & 7 & -4 \\
5 & -1 & 2 & 0
\end{bmatrix}
\]
Contributed by Robert Beezer Solution 85

C31
\[
\begin{bmatrix}
2 & 3 & 1 & 4 \\
1 & 1 & 1 & 0 \\
-1 & 2 & 3 & 5 \\
1 & 2 & 1 & 3
\end{bmatrix}
\]
Contributed by Robert Beezer Solution 85

C32
\[
\begin{bmatrix}
9 & 3 & 2 & 4 \\
5 & -6 & 1 & 3 \\
4 & 1 & 3 & -5
\end{bmatrix}
\]
Contributed by Robert Beezer Solution 85

C33
\[
\begin{bmatrix}
-1 & 2 & 0 & 3 \\
1 & -3 & -2 & 4 \\
-2 & 0 & 4 & 3 \\
-3 & 1 & -2 & 3
\end{bmatrix}
\]
Contributed by Robert Beezer Solution 85

C40 Each of the archetypes below is a system of equations with a square coefficient matrix, or is itself a square matrix. Determine if these matrices are nonsingular, or singular. Comment on the null space of each matrix.

Archetype A 721
Archetype B 726
Archetype F 743
Archetype K 767
Archetype L 771
Contributed by Robert Beezer

For Exercises M51–M52 say as much as possible about each system’s solution set. Be sure to make it clear which theorems you are using to reach your conclusions.

M51 6 equations in 6 variables, singular coefficient matrix.
Contributed by Robert Beezer Solution 85
M52  A system with a nonsingular coefficient matrix, not homogeneous.
Contributed by Robert Beezer  Solution

T10  Suppose that $A$ is a singular matrix, and $B$ is a matrix in reduced row-echelon form that is row-equivalent to $A$. Prove that the last row of $B$ is a zero row.
Contributed by Robert Beezer  Solution
Subsection SOL
Solutions

C30 Contributed by Robert Beezer Statement S3
The matrix row-reduces to
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
which is the $4 \times 4$ identity matrix. By Theorem NMRRI \([77]\) the original matrix must be nonsingular.

C31 Contributed by Robert Beezer Statement S3
Row-reducing the matrix yields,
\[
\begin{bmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Since this is not the $4 \times 4$ identity matrix, Theorem NMRRI \([77]\) tells us the matrix is singular.

C32 Contributed by Robert Beezer Statement S3
The matrix is not square, so neither term is applicable. See Definition NM \([75]\), which is stated for just square matrices.

C33 Contributed by Robert Beezer Statement S3
Theorem NMRRI \([77]\) tells us we can answer this question by simply row-reducing the matrix. Doing this we obtain,
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Since the reduced row-echelon form of the matrix is the $4 \times 4$ identity matrix $I_4$, we know that $B$ is nonsingular.

M51 Contributed by Robert Beezer Statement S3
Theorem NMRRI \([77]\) tells us that the coefficient matrix will not row-reduce to the identity matrix. So if were to row-reduce the augmented matrix of this system of equations, we would not get a unique solution. So by Theorem PSSLS \([57]\) there remaining possibilities are no solutions, or infinitely many.

M52 Contributed by Robert Beezer Statement S4
Any system with a nonsingular coefficient matrix will have a unique solution by Theorem NMUS \([79]\). If the system is not homogeneous, the solution cannot be the zero vector (Exercise HSE.T10 \([72]\)).

T10 Contributed by Robert Beezer Statement S4
Let $n$ denote the size of the square matrix $A$. By Theorem NMRRI \([77]\) the hypothesis
that $A$ is singular implies that $B$ is not the identity matrix $I_n$. If $B$ has $n$ pivot columns, then it would have to be $I_n$, so $B$ must have fewer than $n$ pivot columns. But the number of nonzero rows in $B$ ($r$) is equal to the number of pivot columns as well. So the $n$ rows of $B$ have fewer than $n$ nonzero rows, and $B$ must contain at least one zero row. By Definition RREF\,[35], this row must be at the bottom of $B$. 
Chapter V
Vectors

We have worked extensively in the last chapter with matrices, and some with vectors. In this chapter we will develop the properties of vectors, while preparing to study vector spaces. Initially we will depart from our study of systems of linear equations, but in Section LC we will forge a connection between linear combinations and systems of linear equations in Theorem SLSLC. This connection will allow us to understand systems of linear equations at a higher level, while consequently discussing them less frequently.

Section VO
Vector Operations

In this section we define some new operations involving vectors, and collect some basic properties of these operations. Begin by recalling our definition of a column vector as an ordered list of complex numbers, written vertically (Definition CV). The collection of all possible vectors of a fixed size is a commonly used set, so we start with its definition.

Definition VSCV
Vector Space of Column Vectors

The vector space $\mathbb{C}^m$ is the set of all column vectors of size $m$ with entries from the set of complex numbers, $\mathbb{C}$.

When a set similar to this is defined using only column vectors where all the entries are from the real numbers, it is written as $\mathbb{R}^m$ and is known as Euclidean $m$-space.

The term “vector” is used in a variety of different ways. We have defined it as an ordered list written vertically. It could simply be an ordered list of numbers, and written as $(2, 3, -1, 6)$. Or it could be interpreted as a point in $m$ dimensions, such as $(3, 4, -2)$ representing a point in three dimensions relative to $x$, $y$ and $z$ axes. With an interpretation as a point, we can construct an arrow from the origin to the point which is consistent with the notion that a vector has direction and magnitude.

All of these ideas can be shown to be related and equivalent, so keep that in mind as you connect the ideas of this course with ideas from other disciplines. For now, we’ll stick with the idea that a vector is a just a list of numbers, in some particular order.
Subsection VEASM
Vector equality, addition, scalar multiplication

We start our study of this set by first defining what it means for two vectors to be the same.

Definition CVE
Column Vector Equality

The vectors \( u \) and \( v \) are equal, written \( u = v \) provided that

\[
[u]_i = [v]_i, \quad 1 \leq i \leq m
\]

(This definition contains Notation CVE.)

Now this may seem like a silly (or even stupid) thing to say so carefully. Of course two vectors are equal if they are equal for each corresponding entry! Well, this is not as silly as it appears. We will see a few occasions later where the obvious definition is not the right one. And besides, in doing mathematics we need to be very careful about making all the necessary definitions and making them unambiguous. And we’ve done that here.

Notice now that the symbol ‘=’ is now doing triple-duty. We know from our earlier education what it means for two numbers (real or complex) to be equal, and we take this for granted. In Definition SE [694] we defined what it meant for two sets to be equal. Now we have defined what it means for two vectors to be equal, and that definition builds on our definition for when two numbers are equal when we use the condition \( u_i = v_i \) for all \( 1 \leq i \leq m \). So think carefully about your objects when you see an equal sign and think about just which notion of equality you have encountered. This will be especially important when you are asked to construct proofs whose conclusion states that two objects are equal.

OK, let’s do an example of vector equality that begins to hint at the utility of this definition.

Example VESE
Vector equality for a system of equations

Consider the system of linear equations in Archetype B [726],

\[
\begin{align*}
-7x_1 - 6x_2 - 12x_3 &= -33 \\
5x_1 + 5x_2 + 7x_3 &= 24 \\
x_1 + 4x_3 &= 5
\end{align*}
\]

Note the use of three equals signs — each indicates an equality of numbers (the linear expressions are numbers when we evaluate them with fixed values of the variable quantities). Now write the vector equality,

\[
\begin{bmatrix}
-7x_1 - 6x_2 - 12x_3 \\
5x_1 + 5x_2 + 7x_3 \\
x_1 + 4x_3
\end{bmatrix}
= \begin{bmatrix}
-33 \\
24 \\
5
\end{bmatrix}.
\]

By Definition CVE [88], this single equality (of two column vectors) translates into three simultaneous equalities of numbers that form the system of equations. So with this new
notion of vector equality we can become less reliant on referring to systems of simultaneous equations. There’s more to vector equality than just this, but this is a good example for starters and we will develop it further.

We will now define two operations on the set $\mathbb{C}^m$. By this we mean well-defined procedures that somehow convert vectors into other vectors. Here are two of the most basic definitions of the entire course.

**Definition CVA**

**Column Vector Addition**

Given the vectors $\mathbf{u}$ and $\mathbf{v}$ the *sum* of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} + \mathbf{v}$ defined by

$$[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i, \quad 1 \leq i \leq m$$

(This definition contains Notation CVA.)

So vector addition takes two vectors of the same size and combines them (in a natural way!) to create a new vector of the same size. Notice that this definition is required, even if we agree that this is the obvious, right, natural or correct way to do it. Notice too that the symbol ‘+’ is being recycled. We all know how to add numbers, but now we have the same symbol extended to double-duty and we use it to indicate how to add two new objects, vectors. And this definition of our new meaning is built on our previous meaning of addition via the expressions $u_i + v_i$. Think about your objects, especially when doing proofs. Vector addition is easy, here’s an example from $\mathbb{C}^4$.

**Example VA**

**Addition of two vectors in $\mathbb{C}^4$**

If

$$\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 5 \\ 2 \\ -7 \end{bmatrix}$$

then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \\ 2 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 + (-1) \\ -3 + 5 \\ 4 + 2 \\ 2 + (-7) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \\ -5 \end{bmatrix}.$$
Notice that we are doing a kind of multiplication here, but we are defining a new type, perhaps in what appears to be a natural way. We use juxtaposition (smashing two symbols together side-by-side) to denote this operation rather than using a symbol like we did with vector addition. So this can be another source of confusion. When two symbols are next to each other, are we doing regular old multiplication, the kind we’ve done for years, or are we doing scalar vector multiplication, the operation we just defined? Think about your objects — if the first object is a scalar, and the second is a vector, then it must be that we are doing our new operation, and the result of this operation will be another vector.

Notice how consistency in notation can be an aid here. If we write scalars as lower case Greek letters from the start of the alphabet (such as \( \alpha, \beta, \ldots \)) and write vectors in bold Latin letters from the end of the alphabet (\( \mathbf{u}, \mathbf{v}, \ldots \)), then we have some hints about what type of objects we are working with. This can be a blessing and a curse, since when we go read another book about linear algebra, or read an application in another discipline (physics, economics, \ldots) the types of notation employed may be very different and hence unfamiliar.

Again, computationally, vector scalar multiplication is very easy.

**Example CVSM**

**Scalar multiplication in \( \mathbb{C}^5 \)**

If

\[
\mathbf{u} = \begin{bmatrix}
3 \\
1 \\
-2 \\
4 \\
-1
\end{bmatrix}
\]

and \( \alpha = 6 \), then

\[
\alpha \mathbf{u} = 6 \begin{bmatrix}
3 \\
1 \\
-2 \\
4 \\
-1
\end{bmatrix} = \begin{bmatrix}
6(3) \\
6(1) \\
6(-2) \\
6(4) \\
6(-1)
\end{bmatrix} = \begin{bmatrix}
18 \\
6 \\
-12 \\
24 \\
-6
\end{bmatrix}.
\]

Vector addition and scalar multiplication are the most natural and basic operations to perform on vectors, so it should be easy to have your computational device form a linear combination. See: [Computation VLC.MMA 680](#) [Computation VLC.TI86 685](#) [Computation VLC.TI83 686](#).

**Subsection VSP**

**Vector Space Properties**

With definitions of vector addition and scalar multiplication we can state, and prove, several properties of each operation, and some properties that involve their interplay. We now collect ten of them here for later reference.
Theorem VSPCV
Vector Space Properties of Column Vectors

Suppose that \( \mathbb{C}^m \) is the set of column vectors of size \( m \) (Definition VSCV \[87\]) with addition and scalar multiplication as defined in Definition CVA \[89\] and Definition CVSM \[89\]. Then

- **ACC** Additive Closure, Column Vectors
  If \( u, v \in \mathbb{C}^m \), then \( u + v \in \mathbb{C}^m \).

- **SCC** Scalar Closure, Column Vectors
  If \( \alpha \in \mathbb{C} \) and \( u \in \mathbb{C}^m \), then \( \alpha u \in \mathbb{C}^m \).

- **CC** Commutativity, Column Vectors
  If \( u, v \in \mathbb{C}^m \), then \( u + v = v + u \).

- **AAC** Additive Associativity, Column Vectors
  If \( u, v, w \in \mathbb{C}^m \), then \( u + (v + w) = (u + v) + w \).

- **ZC** Zero Vector, Column Vectors
  There is a vector, \( 0 \), called the **zero vector**, such that \( u + 0 = u \) for all \( u \in \mathbb{C}^m \).

- **AIC** Additive Inverses, Column Vectors
  If \( u \in \mathbb{C}^m \), then there exists a vector \( -u \in \mathbb{C}^m \) so that \( u + (-u) = 0 \).

- **SMAC** Scalar Multiplication Associativity, Column Vectors
  If \( \alpha, \beta \in \mathbb{C} \) and \( u \in \mathbb{C}^m \), then \( \alpha(\beta u) = (\alpha\beta)u \).

- **DVAC** Distributivity across Vector Addition, Column Vectors
  If \( \alpha, \beta \in \mathbb{C} \) and \( u, v \in \mathbb{C}^m \), then \( \alpha(u + v) = \alpha u + \alpha v \).

- **DSAC** Distributivity across Scalar Addition, Column Vectors
  If \( \alpha, \beta \in \mathbb{C} \) and \( u \in \mathbb{C}^m \), then \( (\alpha + \beta)u = \alpha u + \beta u \).

- **OC** One, Column Vectors
  If \( u \in \mathbb{C}^m \), then \( 1u = u \).

**Proof** While some of these properties seem very obvious, they all require proof. However, the proofs are not very interesting, and border on tedious. We’ll prove one version of distributivity very carefully, and you can test your proof-building skills on some of the others. We need to establish an equality, so we will do so by beginning with one side of the equality, apply various definitions and theorems (listed to the right of each step) to massage the expression from the left into the expression on the right. Now would be a good time to read **Technique PI** \[71\], just below. Here we go with a proof of Property DSAC \[91\]. For \( 1 \leq i \leq m \),

\[
[(\alpha + \beta)u]_i = (\alpha + \beta)[u]_i \\
= \alpha[u]_i + \beta[u]_i \\
= [\alpha u]_i + [\beta u]_i \\
= [\alpha u + \beta u]_i
\]

Definition CVSM \[89\]
Distributivity in \( \mathbb{C} \)
Definition CVSM \[89\]
Definition CVA \[89\]
Since the individual components of the vectors \((\alpha + \beta)\mathbf{u}\) and \(\alpha \mathbf{u} + \beta \mathbf{u}\) are equal for all \(i\), \(1 \leq i \leq m\), Definition CVE \[88\] tells us the vectors are equal. \[\square\]

Many of the conclusions of our theorems can be characterized as “identities,” especially when we are establishing basic properties of operations such as those in this section. So some advice about the style we use for proving identities is appropriate right now. Have a look at Technique PI \[711\].

Be careful with the notion of the vector \(-\mathbf{u}\). This is a vector that we add to \(\mathbf{u}\) so that the result is the particular vector \(\mathbf{0}\). This is basically a property of vector addition. It happens that we can compute \(-\mathbf{u}\) using the other operation, scalar multiplication. We can prove this directly by writing that

\[
[-\mathbf{u}]_i = -[\mathbf{u}]_i = (-1)[\mathbf{u}]_i = [(-1)\mathbf{u}]_i
\]

We will see later how to derive this property as a consequence of several of the ten properties listed in Theorem VSPCV \[91\].

**Subsection READ**

**Reading Questions**

1. Where have you seen vectors used before in other courses? How were they different?
2. In words, when are two vectors equal?
3. Perform the following computation with vector operations

   \[
   2 \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}
   \]
Subsection EXC
Exercises

C10 Compute

\[
\begin{bmatrix}
2 \\
-3 \\
4 \\
1 \\
0
\end{bmatrix}
+ (-2)
\begin{bmatrix}
1 \\
-5 \\
2 \\
2 \\
4
\end{bmatrix}
+ 
\begin{bmatrix}
-1 \\
3 \\
0 \\
1 \\
2
\end{bmatrix}
\]

Contributed by Robert Beezer Solution 95

T13 Prove Property CC of Theorem VSPCV. Write your proof in the style of the proof of Property DSAC given in this section.
Contributed by Robert Beezer Solution 95

T17 Prove Property SMAC of Theorem VSPCV. Write your proof in the style of the proof of Property DSAC given in this section.
Contributed by Robert Beezer

T18 Prove Property DVAC of Theorem VSPCV. Write your proof in the style of the proof of Property DSAC given in this section.
Contributed by Robert Beezer
Subsection SOL
Solutions

C10 Contributed by Robert Beezer Statement 93
\[
\begin{bmatrix}
5 \\
-13 \\
26 \\
1 \\
-6
\end{bmatrix}
\]

T13 Contributed by Robert Beezer Statement 93
For all \(1 \leq i \leq m\),
\[
[u + v]_i = [u]_i + [v]_i = [v]_i + [u]_i = [v + u]_i
\]
Definition CVA 89
Commutativity in \(\mathbb{C}\)
Definition CVA 89

With equality of each component of the vectors \(u + v\) and \(v + u\) being equal Definition CVE 88 tells us the two vectors are equal.
Section LC  
Linear Combinations  

Subsection LC  
Linear Combinations  

In Section VO we defined vector addition and scalar multiplication. These two operations combine nicely to give us a construction known as a linear combination, a construct that we will work with throughout this course.

Definition LCCV  
Linear Combination of Column Vectors

Given \( n \) vectors \( u_1, u_2, u_3, \ldots, u_n \) from \( \mathbb{C}^m \) and \( n \) scalars \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \), their linear combination is the vector

\[
\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_n u_n.
\]

So this definition takes an equal number of scalars and vectors, combines them using our two new operations (scalar multiplication and vector addition) and creates a single brand-new vector, of the same size as the original vectors. When a definition or theorem employs a linear combination, think about the nature of the objects that go into its creation (lists of scalars and vectors), and the type of object that results (a single vector). Computationally, a linear combination is pretty easy.

Example TLC  
Two linear combinations in \( \mathbb{C}^6 \)

Suppose that

\[
\alpha_1 = 1 \quad \alpha_2 = -4 \quad \alpha_3 = 2 \quad \alpha_4 = -1
\]

and

\[
\begin{align*}
\mathbf{u}_1 &= \begin{bmatrix} 2 \\ 4 \\ -3 \\ 1 \\ 2 \\ 9 \end{bmatrix} \\
\mathbf{u}_2 &= \begin{bmatrix} 6 \\ 3 \\ 0 \\ -2 \\ 1 \\ 4 \end{bmatrix} \\
\mathbf{u}_3 &= \begin{bmatrix} -5 \\ 2 \\ 1 \\ -3 \\ 0 \end{bmatrix} \\
\mathbf{u}_4 &= \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}
\end{align*}
\]

then their linear combination is

\[
\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \alpha_4 \mathbf{u}_4 = (1) \begin{bmatrix} 2 \\ 4 \\ -3 \\ 1 \\ 2 \\ 9 \end{bmatrix} + (2) \begin{bmatrix} 6 \\ 3 \\ 0 \\ -2 \\ 1 \\ 4 \end{bmatrix} + (3) \begin{bmatrix} -5 \\ 2 \\ 1 \\ -3 \\ 0 \end{bmatrix} + (4) \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}
\]
\[
\begin{bmatrix}
2 \\
4 \\
-3 \\
1 \\
2 \\
9
\end{bmatrix}
+ \begin{bmatrix}
-24 \\
-12 \\
0 \\
8 \\
-4 \\
-16
\end{bmatrix}
+ \begin{bmatrix}
-10 \\
4 \\
2 \\
2 \\
-6 \\
0
\end{bmatrix}
+ \begin{bmatrix}
-3 \\
-2 \\
5 \\
-7 \\
-1 \\
-3
\end{bmatrix}
= \begin{bmatrix}
-35 \\
-6 \\
4 \\
4 \\
-9 \\
-10
\end{bmatrix}.
\]

A different linear combination, of the same set of vectors, can be formed with different scalars. Take
\[
\beta_1 = 3 \quad \beta_2 = 0 \quad \beta_3 = 5 \quad \beta_4 = -1
\]
and form the linear combination
\[
\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 = (3) \begin{bmatrix}
2 \\
4 \\
-3 \\
1 \\
2 \\
9
\end{bmatrix} + (0) \begin{bmatrix}
-24 \\
-12 \\
0 \\
8 \\
-4 \\
-16
\end{bmatrix} + (5) \begin{bmatrix}
-10 \\
4 \\
2 \\
2 \\
-6 \\
0
\end{bmatrix} + (-1) \begin{bmatrix}
-3 \\
-2 \\
5 \\
-7 \\
-1 \\
-3
\end{bmatrix} = \begin{bmatrix}
-35 \\
-6 \\
4 \\
4 \\
-9 \\
-10
\end{bmatrix}.
\]

Notice how we could keep our set of vectors fixed, and use different sets of scalars to construct different vectors. You might build a few new linear combinations of \(u_1, u_2, u_3, u_4\) right now. We'll be right here when you get back. What vectors were you able to create? Do you think you could create the vector
\[
w = \begin{bmatrix}
13 \\
15 \\
5 \\
-17 \\
2 \\
25
\end{bmatrix}
\]
with a “suitable” choice of four scalars? Do you think you could create any possible vector from \(\mathbb{C}^6\) by choosing the proper scalars? These last two questions are very fundamental, and time spent considering them now will prove beneficial later.

Our next two examples are key ones, and a discussion about decompositions is timely. Have a look at Technique DC 712 before studying the next two examples.

**Example ABLC**

**Archetype B as a linear combination**

In this example we will rewrite Archetype B 726 in the language of vectors, vector equality and linear combinations. In Example VESE 88 we wrote the system of \(m = 3\) equations as the vector equality
\[
\begin{bmatrix}
-7x_1 - 6x_2 - 12x_3 \\
5x_1 + 5x_2 + 7x_3 \\
x_1 + 4x_3
\end{bmatrix}
= \begin{bmatrix}
-33 \\
24 \\
5
\end{bmatrix}.
\]
Now we will bust up the linear expressions on the left, first using vector addition,

\[
\begin{bmatrix}
-7x_1 \\
5x_1 \\
x_1
\end{bmatrix}
+ \begin{bmatrix}
-6x_2 \\
5x_2 \\
0x_2
\end{bmatrix}
+ \begin{bmatrix}
-12x_3 \\
7x_3 \\
4x_3
\end{bmatrix}
= \begin{bmatrix}
-33 \\
24 \\
5
\end{bmatrix}.
\]

Now we can rewrite each of these \( n = 3 \) vectors as a scalar multiple of a fixed vector, where the scalar is one of the unknown variables, converting the left-hand side into a linear combination

\[
x_1 \begin{bmatrix}
-7 \\
5 \\
1
\end{bmatrix}
+ x_2 \begin{bmatrix}
-6 \\
5 \\
0
\end{bmatrix}
+ x_3 \begin{bmatrix}
-12 \\
7 \\
4
\end{bmatrix}
= \begin{bmatrix}
-33 \\
24 \\
5
\end{bmatrix}.
\]

We can now interpret the problem of solving the system of equations as determining values for the scalar multiples that make the vector equation true. In the analysis of Archetype B \[726\], we were able to determine that it had only one solution. A quick way to see this is to row-reduce the coefficient matrix to the \( 3 \times 3 \) identity matrix and apply Theorem NMRRI \[77\] to determine that the coefficient matrix is nonsingular. Then Theorem NMUS \[79\] tells us that the system of equations has a unique solution. This solution is

\[
x_1 = -3 \quad x_2 = 5 \quad x_3 = 2.
\]

So, in the context of this example, we can express the fact that these values of the variables are a solution by writing the linear combination,

\[
(-3) \begin{bmatrix}
-7 \\
5 \\
1
\end{bmatrix}
+ (5) \begin{bmatrix}
-6 \\
5 \\
0
\end{bmatrix}
+ (2) \begin{bmatrix}
-12 \\
7 \\
4
\end{bmatrix}
= \begin{bmatrix}
-33 \\
24 \\
5
\end{bmatrix}.
\]

Furthermore, these are the only three scalars that will accomplish this equality, since they come from a unique solution.

Notice how the three vectors in this example are the columns of the coefficient matrix of the system of equations. This is our first hint of the important interplay between the vectors that form the columns of a matrix, and the matrix itself.

With any discussion of Archetype A \[721\] or Archetype B \[726\] we should be sure to contrast with the other.

**Example AALC**

**Archetype A as a linear combination**

As a vector equality, Archetype A \[721\] can be written as

\[
\begin{bmatrix}
x_1 - x_2 + 2x_3 \\
2x_1 + x_2 + x_3 \\
x_1 + x_2
\end{bmatrix}
= \begin{bmatrix}
1 \\
8 \\
5
\end{bmatrix}.
\]

Now bust up the linear expressions on the left, first using vector addition,

\[
\begin{bmatrix}
x_1 \\
2x_1 \\
x_1
\end{bmatrix}
+ \begin{bmatrix}
-x_2 \\
x_2 \\
0x_3
\end{bmatrix}
+ \begin{bmatrix}
2x_3 \\
x_3 \\
0x_3
\end{bmatrix}
= \begin{bmatrix}
1 \\
8 \\
5
\end{bmatrix}.
\]
Rewrite each of these \( n = 3 \) vectors as a scalar multiple of a fixed vector, where the scalar is one of the unknown variables, converting the left-hand side into a linear combination

\[
x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix}.
\]

Row-reducing the augmented matrix for [Archetype A](721) leads to the conclusion that the system is consistent and has free variables, hence infinitely many solutions. So for example, the two solutions

\[
x_1 = 2 \quad x_2 = 3 \quad x_3 = 1 \\
x_1 = 3 \quad x_2 = 2 \quad x_3 = 0
\]

can be used together to say that,

\[
(2) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (3) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = (3) \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix} + (2) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + (0) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.
\]

Ignore the middle of this equation, and move all the terms to the left-hand side,

\[
(2) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (3) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + (-0) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

Regrouping gives

\[
(-1) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

Notice that these three vectors are the columns of the coefficient matrix for the system of equations in [Archetype A](721). This equality says there is a linear combination of those columns that equals the vector of all zeros. Give it some thought, but this says that

\[
x_1 = -1 \quad x_2 = 1 \quad x_3 = 1
\]

is a nontrivial solution to the homogeneous system of equations with the coefficient matrix for the original system in [Archetype A](721). In particular, this demonstrates that this coefficient matrix is singular.

There’s a lot going on in the last two examples. Come back to them in a while and make some connections with the intervening material. For now, we will summarize and explain some of this behavior with a theorem.

**Theorem SLSLC**

**Solutions to Linear Systems are Linear Combinations**

Denote the columns of the \( m \times n \) matrix \( A \) as the vectors \( A_1, A_2, A_3, \ldots, A_n \). Then \( x \) is a solution to the linear system of equations \( L S(A, b) \) if and only if

\[
[x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n = b
\]

\[\square\]
Proof  The proof of this theorem is as much about a change in notation as it is about making logical deductions. Write the system of equations $\mathcal{L}S(A, \mathbf{b})$ as

$$
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
    a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
    \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m.
\end{align*}
$$

Notice then that the entry of the coefficient matrix $A$ in row $i$ and column $j$ has two names: $a_{ij}$ as the coefficient of $x_j$ in equation $i$ of the system and $[A_i]_j$, as the $i$-th entry of the column vector $\mathbf{b}$ has two names: $b_i$ from the linear system and $[\mathbf{b}]_i$ as an entry of a vector. Our theorem is an equivalence (Technique E [704]) so we need to prove both “directions.”

($\Rightarrow$) Suppose we have the vector equality between $\mathbf{b}$ and the linear combination of the columns of $A$. Then for $1 \leq i \leq n,$

$$
\begin{align*}
    b_i &= [\mathbf{b}]_i \\
    &= [[x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n]_i \\
    &= [x]_1 [A_1]_i + [x]_2 [A_2]_i + [x]_3 [A_3]_i + \cdots + [x]_n [A_n]_i \quad \text{Notation} \\
    &= [x]_1 a_{i1} + [x]_2 a_{i2} + [x]_3 a_{i3} + \cdots + [x]_n a_{in} \\
    &= a_{i1} [x]_1 + a_{i2} [x]_2 + a_{i3} [x]_3 + \cdots + a_{in} [x]_n \quad \text{Commutativity in } \mathbb{C}
\end{align*}
$$

This says that the entries of $\mathbf{x}$ form a solution to equation $i$ of $\mathcal{L}S(A, \mathbf{b})$ for all $1 \leq i \leq n$, i.e. $\mathbf{x}$ is a solution to $\mathcal{L}S(A, \mathbf{b})$.

($\Leftarrow$) Suppose now that $\mathbf{x}$ is a solution to the linear system $\mathcal{L}S(A, \mathbf{b})$. Then for all $1 \leq i \leq n,$

$$
\begin{align*}
    [\mathbf{b}]_i &= b_i \\
    &= a_{i1} [x]_1 + a_{i2} [x]_2 + a_{i3} [x]_3 + \cdots + a_{in} [x]_n \\
    &= [x]_1 a_{i1} + [x]_2 a_{i2} + [x]_3 a_{i3} + \cdots + [x]_n a_{in} \quad \text{Commutativity in } \mathbb{C} \\
    &= [x]_1 [A_1]_i + [x]_2 [A_2]_i + [x]_3 [A_3]_i + \cdots + [x]_n [A_n]_i \quad \text{Notation} \\
    &= [[x]_1 A_1]_i + [[x]_2 A_2]_i + [[x]_3 A_3]_i + \cdots + [[x]_n A_n]_i \quad \text{Definition CVSM [89]} \\
    &= ([x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n]_i \quad \text{Definition CVE [88]}
\end{align*}
$$

Since the components of $\mathbf{b}$ and the linear combination of the columns of $A$ agree for all $1 \leq i \leq n$, Definition CVE [88] tells us that the vectors are equal. $
$}

In other words, this theorem tells us that solutions to systems of equations are linear combinations of the column vectors of the coefficient matrix $(A_i)$ which yield the constant vector $\mathbf{b}$. Or said another way, a solution to a system of equations $\mathcal{L}S(A, \mathbf{b})$ is an answer to the question “How can I form the vector $\mathbf{b}$ as a linear combination of the columns of $A$?” Look through the archetypes that are systems of equations and examine a few of the advertised solutions. In each case use the solution to form a linear combination of the columns of the coefficient matrix and verify that the result equals the constant vector (see Exercise LC.C21 [119]).
Subsection VFSS
Vector Form of Solution Sets

We have written solutions to systems of equations as column vectors. For example \textit{Archetype B} \footnote[726]{Archetype B} has the solution $x_1 = -3$, $x_2 = 5$, $x_3 = 2$ which we now write as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}.$$

Now, we will use column vectors and linear combinations to express all of the solutions to a linear system of equations in a compact and understandable way. First, here’s two examples that will motivate our next theorem. This is a valuable technique, almost the equal of row-reducing a matrix, so be sure you get comfortable with it over the course of this section.

\textbf{Example VFSAD}

\textit{Vector form of solutions for Archetype D} \footnote[735]{Archetype D} is a linear system of 3 equations in 4 variables. Row-reducing the augmented matrix yields

$$\begin{bmatrix} 1 & 0 & 3 & -2 & 4 \\ 0 & 1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and we see $r = 2$ nonzero rows. Also, $D = \{1, 2\}$ so the dependent variables are then $x_1$ and $x_2$. $F = \{3, 4, 5\}$ so the two free variables are $x_3$ and $x_4$. We will express a generic solution for the system by two slightly different methods, though both arrive at the same conclusion.

First, we will decompose \footnote[712]{Technique DC} a solution vector. Rearranging each equation represented in the row-reduced form of the augmented matrix by solving for the dependent variable in each row yields the vector equality,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 - 3x_3 + 2x_4 \\ -x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix}.$$

Now we will use the definitions of column vector addition and scalar multiplication to express this vector as a linear combination,

$$= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_4 \\ 3x_4 \\ 0 \\ x_4 \end{bmatrix} \quad \text{Definition CVA} \footnote[89]{Definition CVA}$$

$$= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \quad \text{Definition CVSM} \footnote[89]{Definition CVSM}$$
We will develop the same linear combination a bit quicker, using three steps. While the method above is instructive, the method below will be our preferred approach.

Step 1. Write the vector of variables as a fixed vector, plus a linear combination of \( n-r \) vectors, using the free variables as the scalars.

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \end{bmatrix} + x_3 \begin{bmatrix} \end{bmatrix} + x_4 \begin{bmatrix} \end{bmatrix}
\]

Step 2. Use 0’s and 1’s to ensure equality for the entries of the vectors with indices in \( F \) (corresponding to the free variables).

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

Step 3. For each dependent variable, use the augmented matrix to formulate an equation expressing the dependent variable as a constant plus multiples of the free variables. Convert this equation into entries of the vectors that ensure equality for each dependent variable, one at a time.

\[
x_1 = 4 - 3x_3 + 2x_4 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

\[
x_2 = 0 - 1x_3 + 3x_4 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}
\]

This final form of a typical solution is especially pleasing and useful. For example, we can build solutions quickly by choosing values for our free variables, and then compute a linear combination. Such as

\[
x_3 = 2, \ x_4 = -5 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 2 \end{bmatrix} + (2) \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (-5) \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ -17 \\ 2 \\ -5 \end{bmatrix}
\]

or,

\[
x_3 = 1, \ x_4 = 3 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 2 \end{bmatrix} + (1) \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (3) \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 1 \\ 3 \end{bmatrix}
\]

Version 0.92
You’ll find the second solution listed in the write-up for Archetype D 735, and you might check the first solution by substituting it back into the original equations.

While this form is useful for quickly creating solutions, its even better because it tells us exactly what every solution looks like. We know the solution set is infinite, which is pretty big, but now we can say that a solution is some multiple of

\[
\begin{bmatrix}
-3 \\
-1 \\
1 \\
0
\end{bmatrix}
\]  
plus a multiple of

\[
\begin{bmatrix}
2 \\
3 \\
0 \\
1
\end{bmatrix}
\]

plus the fixed vector

\[
\begin{bmatrix}
4 \\
0 \\
0 \\
0
\end{bmatrix}
\] . Period. So it only takes us three vectors to describe the entire infinite solution set, provided we also agree on how to combine the three vectors into a linear combination.

This is such an important and fundamental technique, we’ll do another example.

**Example VFS**

**Vector form of solutions**

Consider a linear system of \( m = 5 \) equations in \( n = 7 \) variables, having the augmented matrix \( A \).

\[
A = \begin{bmatrix}
2 & 1 & -1 & -2 & 2 & 1 & 5 \\
1 & 1 & -3 & 1 & 1 & 1 & 2 & -5 \\
1 & 2 & -8 & 5 & 1 & 1 & -6 & -15 \\
3 & 3 & -9 & 3 & 6 & 5 & 2 & -24 \\
-2 & -1 & 1 & 2 & 1 & 1 & -9 & -30
\end{bmatrix}
\]

Row-reducing we obtain the matrix

\[
B = \begin{bmatrix}
1 & 0 & 2 & -3 & 0 & 0 & 9 & 15 \\
0 & 1 & -5 & 4 & 0 & 0 & -8 & -10 \\
0 & 0 & 0 & 0 & 1 & 0 & -6 & 11 \\
0 & 0 & 0 & 0 & 0 & 1 & 7 & -21 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and we see \( r = 4 \) nonzero rows. Also, \( D = \{1, 2, 5, 6\} \) so the dependent variables are then \( x_1, x_2, x_5, \) and \( x_6 \). \( F = \{3, 4, 7, 8\} \) so the \( n - r = 3 \) free variables are \( x_3, x_4 \) and \( x_7 \). We will express a generic solution for the system by two different methods: both a decomposition and a construction.

First, we will decompose \([\text{Technique DC 712}]\) a solution vector. Rearranging each equation represented in the row-reduced form of the augmented matrix by solving for the dependent variable in each row yields the vector equality,

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}
= \begin{bmatrix}
15 - 2x_3 + 3x_4 - 9x_7 \\
-10 + 5x_3 - 4x_4 + 8x_7 \\
x_3 \\
x_4 \\
11 + 6x_7 \\
-21 + 7x_7 \\
x_7
\end{bmatrix}
\]
Now we will use the definitions of column vector addition and scalar multiplication to decompose this generic solution vector as a linear combination,

\[
\begin{bmatrix}
15 \\
-10 \\
0 \\
0 \\
11 \\
-21 \\
0
\end{bmatrix}
+ \begin{bmatrix}
-2x_3 \\
5x_3 \\
x_3 \\
x_4 \\
0 \\
0 \\
0
\end{bmatrix}
+ \begin{bmatrix}
3x_4 \\
-4x_4 \\
0 \\
0 \\
6x_7 \\
-7x_7 \\
0
\end{bmatrix}
+ \begin{bmatrix}
-9x_7 \\
8x_7 \\
0 \\
0 \\
x_7 \\
x_7 \\
x_7
\end{bmatrix}
\]

Definition CVA \[89\]

We will now develop the same linear combination a bit quicker, using three steps. While the method above is instructive, the method below will be our preferred approach.

Step 1. Write the vector of variables as a fixed vector, plus a linear combination of \(n - r\) vectors, using the free variables as the scalars.

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}
= \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}
+ x_3 \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}
+ x_4 \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}
+ x_7 \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}
\]

Step 2. Use 0’s and 1’s to ensure equality for the entries of the the vectors with indices in \(F\) (corresponding to the free variables).

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}
= \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}
+ x_3 \begin{bmatrix}
0 \\
0 \\
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
+ x_4 \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
0 \\
0 \\
0
\end{bmatrix}
+ x_7 \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

Step 3. For each dependent variable, use the augmented matrix to formulate an equation expressing the dependent variable as a constant plus multiples of the free variables. Convert this equation into entries of the vectors that ensure equality for each dependent
variable, one at a time.

\[
x_1 = 15 - 2x_3 + 3x_4 - 9x_7 \quad \Rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

\[
x_2 = -10 + 5x_3 - 4x_4 + 8x_7 \quad \Rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

\[
x_5 = 11 + 6x_7 \quad \Rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

\[
x_6 = -21 - 7x_7 \quad \Rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

This final form of a typical solution is especially pleasing and useful. For example, we can build solutions quickly by choosing values for our free variables, and then compute a linear combination. For example

\[
x_3 = 2, \ x_4 = -4, \ x_7 = 3 \quad \Rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} -2 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (-4) \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (3) \begin{bmatrix} -9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}
\]

or perhaps,

\[
x_3 = 5, \ x_4 = 2, \ x_7 = 1 \quad \Rightarrow \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} -2 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (-4) \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (3) \begin{bmatrix} -9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \end{bmatrix}
\]
Subsection LC.VFSS  Vector Form of Solution Sets 107

\[
\mathbf{x} = \begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7
\end{bmatrix} = \begin{bmatrix}
15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0
\end{bmatrix} + (5) \begin{bmatrix}
-2 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix} + (2) \begin{bmatrix}
3 \\ -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix} + (1) \begin{bmatrix}
-9 \\ 8 \\ 0 \\ 0 \\ 0 \\ -7 \\ 1
\end{bmatrix} = \begin{bmatrix}
2 \\ 15 \\ 5 \\ 1 \\ 6 \\ -21 \\ 0
\end{bmatrix} + \mathbf{c}
\]

or even,

\[
x_3 = 0, \ x_4 = 0, \ x_7 = 0 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7
\end{bmatrix} = \begin{bmatrix}
15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0
\end{bmatrix} + (0) \begin{bmatrix}
-2 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix} + (0) \begin{bmatrix}
3 \\ -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix} + (0) \begin{bmatrix}
-9 \\ 8 \\ 0 \\ 0 \\ 0 \\ -7 \\ 1
\end{bmatrix} = \begin{bmatrix}
2 \\ 15 \\ 5 \\ 1 \\ 6 \\ -21 \\ 0
\end{bmatrix} = \mathbf{w}
\]

So we can compactly express all of the solutions to this linear system with just 4 fixed vectors, provided we agree how to combine them in a linear combinations to create solution vectors.

Suppose you were told that the vector \( \mathbf{w} \) below was a solution to this system of equations. Could you turn the problem around and write \( \mathbf{w} \) as a linear combination of the four vectors \( \mathbf{c}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \)? (See Exercise LC.M11 [120].)

\[
\mathbf{w} = \begin{bmatrix}
100 \\ -75 \\ 7 \\ 9 \\ -37 \\ 35 \\ -8
\end{bmatrix} \quad \mathbf{c} = \begin{bmatrix}
15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0
\end{bmatrix} \quad \mathbf{u}_1 = \begin{bmatrix}
-2 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix}
3 \\ -4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1
\end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix}
-9 \\ 8 \\ 0 \\ 0 \\ 0 \\ -7 \\ 1
\end{bmatrix}
\]

\( \blacklozenge \)

Did you think a few weeks ago that you could so quickly and easily list all the solutions to a linear system of 5 equations in 7 variables?

We’ll now formalize the last two (important) examples as a theorem.

Theorem VFSLS

Vector Form of Solutions to Linear Systems

Suppose that \([A \mid \mathbf{b}]\) is the augmented matrix for a consistent linear system \(\mathcal{L}(A, \mathbf{b})\) of \(m\) equations in \(n\) variables. Let \(B\) be a row-equivalent \(m \times (n + 1)\) matrix in reduced row-echelon form. Suppose that \(B\) has \(r\) nonzero rows, columns without leading 1’s with indices \(F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}\), and columns with leading 1’s (pivot columns) having indices \(D = \{d_1, d_2, d_3, \ldots, d_r\}\). Define vectors \(\mathbf{c}, \mathbf{u}_j, 1 \leq j \leq n - r\) of size \(n\) by

\[
[c]_i = \begin{cases}
0 & \text{if } i \in F \\
[B]_{k,n+1} & \text{if } i \in D, \ i = d_k
\end{cases}
\]
Then the set of solutions to the system of equations $\mathcal{L}S(A, b)$ is

$$S = \{ c + x_{f_1}u_1 + x_{f_2}u_2 + x_{f_3}u_3 + \cdots + x_{f_{n-r}}u_{n-r} \mid x_{f_1}, x_{f_2}, x_{f_3}, \ldots, x_{f_{n-r}} \in \mathbb{C} \}$$

\[ \begin{align*}
[u_j]_i &= \begin{cases} 
1 & \text{if } i \in F, \ i = f_j \\
0 & \text{if } i \in F, \ i \neq f_j \\
-B_{k,f_j} & \text{if } i \in D, \ i = d_k
\end{cases}
\end{align*} \]

Proof  We are being asked to prove that the solution set has a particular form. First, $\mathcal{L}S(A, b)$ is equivalent to the linear system of equations that has the matrix $B$ as its augmented matrix \( \text{Theorem REMES} \ [34] \), so we need only show that $S$ is the solution set for the system with $B$ as its augmented matrix.

We begin by showing that every element of $S$ is a solution to the system. Let $x_{f_1} = \alpha_1$, $x_{f_2} = \alpha_2$, $x_{f_3} = \alpha_3$, $\ldots$, $x_{f_{n-r}} = \alpha_{n-r}$ be one choice of the values of $x_{f_1}, x_{f_2}, x_{f_3}, \ldots, x_{f_{n-r}}$. So a proposed solution is

$$x = c + \alpha_{f_1}u_1 + \alpha_{f_2}u_2 + \alpha_{f_3}u_3 + \cdots + \alpha_{f_{n-r}}u_{n-r}$$

So we evaluate equation $\ell$ of the system represented by $B$ with the solution vector $x$,

$$\beta = [B]_{\ell f_1} [x]_1 + [B]_{\ell f_2} [x]_2 + [B]_{\ell f_3} [x]_3 + \cdots + [B]_{\ell f_n} [x]_n$$

When $r + 1 \leq \ell \leq m$, row $\ell$ of the matrix $B$ is a zero row, so the equation represented by that row is always true, no matter which solution vector we propose. So assume $1 \leq \ell \leq r$. Then $[B]_{\ell d_i} = 0$ for all $1 \leq i \leq r$, except that $[B]_{\ell d_\ell} = 1$, so $\beta$ simplifies to

$$\beta = [x]_{d_\ell} + [B]_{\ell f_1} [x]_{f_1} + [B]_{\ell f_2} [x]_{f_2} + [B]_{\ell f_3} [x]_{f_3} + \cdots + [B]_{\ell f_{n-r}} [x]_{f_{n-r}}$$

Notice that for $1 \leq i \leq n - r$

$$[x]_{f_i} = [c]_{f_i} + \alpha_{f_1} [u_1]_{f_i} + \alpha_{f_2} [u_2]_{f_i} + \alpha_{f_3} [u_3]_{f_i} + \cdots + \alpha_{f_{n-r}} [u_{n-r}]_{f_i}$$

$$= 0 + \alpha_{f_1}(0) + \alpha_{f_2}(0) + \alpha_{f_3}(0) + \cdots + \alpha_{f_{n-r}}(0)$$

$$= \alpha_{f_i}$$

So $\beta$ simplifies further to

$$\beta = [x]_{d_\ell} + [B]_{\ell f_1} \alpha_{f_1} + [B]_{\ell f_2} \alpha_{f_2} + [B]_{\ell f_3} \alpha_{f_3} + \cdots + [B]_{\ell f_{n-r}} \alpha_{f_{n-r}}$$

Now examine the $[x]_{d_\ell}$ term of $\beta$,

$$[x]_{d_\ell} = [c]_{d_\ell} + \alpha_{f_1} [u_1]_{d_\ell} + \alpha_{f_2} [u_2]_{d_\ell} + \alpha_{f_3} [u_3]_{d_\ell} + \cdots + \alpha_{f_{n-r}} [u_{n-r}]_{d_\ell}$$

$$= [B]_{\ell c,n+1} + \alpha_{f_1}(-[B]_{\ell f_1}) + \alpha_{f_2}(-[B]_{\ell f_2}) + \alpha_{f_3}(-[B]_{\ell f_3}) + \cdots + \alpha_{f_{n-r}}(-[B]_{\ell f_{n-r}})$$

Replacing this term into the expression for $\beta$, we obtain

$$\beta = [x]_{d_\ell} + [B]_{\ell f_1} \alpha_{f_1} + [B]_{\ell f_2} \alpha_{f_2} + [B]_{\ell f_3} \alpha_{f_3} + \cdots + [B]_{\ell f_{n-r}} \alpha_{f_{n-r}}$$

$$= [B]_{\ell c,n+1} + \alpha_{f_1}(-[B]_{\ell f_1}) + \alpha_{f_2}(-[B]_{\ell f_2}) + \alpha_{f_3}(-[B]_{\ell f_3}) + \cdots + \alpha_{f_{n-r}}(-[B]_{\ell f_{n-r}}) +$$

$$[B]_{\ell f_1} \alpha_{f_1} + [B]_{\ell f_2} \alpha_{f_2} + [B]_{\ell f_3} \alpha_{f_3} + \cdots + [B]_{\ell f_{n-r}} \alpha_{f_{n-r}}$$

Version 0.92
\[ |B|_{\ell,n+1} \]

So \( \beta \) began as the left-hand side of equation \( \ell \) from the system represented by \( B \) and we now know it equals \([B]_{\ell,n+1}\), the constant term for equation \( \ell \). So this arbitrarily chosen vector from \( S \) makes every equation true, and therefore is a solution to the system.

For the second half of the proof, assume that \( x_1 = \alpha_1, x_2 = \alpha_2, x_3 = \alpha_3, \ldots, x_n = \alpha_n \) are the components of a solution vector for the system having \( B \) as its augmented matrix, and show that this solution vector is an element of the set \( S \). Begin with the observation that this solution makes equation \( \ell \) of the system true for \( 1 \leq \ell \leq m \),

\[ [B]_{\ell,1} \alpha_1 + [B]_{\ell,2} \alpha_2 + [B]_{\ell,3} \alpha_3 + \cdots + [B]_{\ell,n} \alpha_n = [B]_{\ell,n+1} \]

Since \( B \) is in reduced row-echelon form, when \( \ell > r \) we know that all the entries of \( B \) in row \( \ell \) are all zero and this equation is true. For \( \ell \leq r \), we can further exploit the knowledge of the structure of \( B \), specifically recalling that \( B \) has no leading 1’s in the final column since the system is consistent (Theorem RCLS\textsuperscript{[54]}). Equation \( \ell \) then reduces to

\[ (1)\alpha_{d_1} + [B]_{\ell,f_1} \alpha_{f_1} + [B]_{\ell,f_2} \alpha_{f_2} + [B]_{\ell,f_3} \alpha_{f_3} + \cdots + [B]_{\ell,f_n} \alpha_{f_n} = [B]_{\ell,n+1} \]

Rearranging, this becomes,

\[
\alpha_{d_\ell} = [B]_{\ell,n+1} - [B]_{\ell,f_1} \alpha_{f_1} - [B]_{\ell,f_2} \alpha_{f_2} - [B]_{\ell,f_3} \alpha_{f_3} - \cdots - [B]_{\ell,f_n} \alpha_{f_n}
\]

\[ = [c]_\ell + \alpha_{f_1} [u_1]_\ell + \alpha_{f_2} [u_2]_\ell + \alpha_{f_3} [u_3]_\ell + \cdots + \alpha_{f_n} [u_n-r]_\ell \]

This tells us that the components of the solution vector corresponding to dependent variables (indices in \( D \)), are of the same form as stated for membership in the set \( S \). We still need to check the components that correspond to the free variables (indices in \( F \)). To this end, suppose \( i \in F \) and \( i = f_j \). Then

\[
\alpha_i = 1 \alpha_{f_j} = 0 + 0 \alpha_{f_1} + 0 \alpha_{f_2} + 0 \alpha_{f_3} + \cdots + 0 \alpha_{f_{j-1}} + 1 \alpha_{f_j} + 0 \alpha_{f_{j+1}} + \cdots + 0 \alpha_{f_{n-r}}
\]

\[
= [c]_i + \alpha_{f_1} [u_1]_i + \alpha_{f_2} [u_2]_i + \alpha_{f_3} [u_3]_i + \cdots + \alpha_{f_{n-r}} [u_{n-r}]_i
\]

So our solution vector is also of the right form in the remaining slots, and hence qualifies for membership in the set \( S \).

\textbf{Theorem VFSLS}\textsuperscript{[107]} formalizes what happened in the three steps of \textbf{Example VFSAD}\textsuperscript{[102]}. The theorem will be useful in proving other theorems, and it is useful since it tells us an exact procedure for simply describing an infinite solution set. We could program a computer to implement it, once we have the augmented matrix row-reduced and have checked that the system is consistent. By Knuth’s definition, this completes our conversion of linear equation solving from art into science. Notice that it even applies (but is overkill) in the case of a unique solution. However, as a practical matter, I prefer the three-step process of \textbf{Example VFSAD}\textsuperscript{[102]} when I need to describe an infinite solution set. So let’s practice some more, but with a bigger example.
Example VFSAI

Vector form of solutions for Archetype I

Archetype I [757] is a linear system of \( m = 4 \) equations in \( n = 7 \) variables. Row-reducing the augmented matrix yields

\[
\begin{bmatrix}
1 & 4 & 0 & 0 & 2 & 1 & -3 & 4 \\
0 & 0 & 1 & 0 & 1 & -3 & 5 & 2 \\
0 & 0 & 0 & 1 & 2 & -6 & 6 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and we see \( r = 3 \) nonzero rows. The columns with leading 1’s are \( D = \{1, 3, 4\} \) so the \( r \) dependent variables are \( x_1, x_3, x_4 \). The columns without leading 1’s are \( F = \{2, 5, 6, 7, 8\} \), so the \( n - r = 4 \) free variables are \( x_2, x_5, x_6, x_7 \).

Step 1. Write the vector of variables (\( \mathbf{x} \)) as a fixed vector (\( \mathbf{c} \)), plus a linear combination of \( n - r = 4 \) vectors (\( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4 \)), using the free variables as the scalars.

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_5 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_6 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} x_7 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

Step 2. For each free variable, use 0’s and 1’s to ensure equality for the corresponding entry of the the vectors. Take note of the pattern of 0’s and 1’s at this stage, because this is the best look you’ll have at it. We’ll state an important theorem in the next section and the proof will essentially rely on this observation.

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_5 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_6 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_7 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\]

Step 3. For each dependent variable, use the augmented matrix to formulate an equation expressing the dependent variable as a constant plus multiples of the free variables. Convert this equation into entries of the vectors that ensure equality for each dependent variable, one at a time.

\[
x_1 = 4 - 4x_2 - 2x_5 - x_6 + 3x_7 \quad \Rightarrow
\]

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ -4 \\ 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

Version 0.92
We can now use this final expression to quickly build solutions to the system. You might try to recreate each of the solutions listed in the write-up for Archetype I \[757\]. (Hint: look at the values of the free variables in each solution, and notice that the vector $c$ has 0’s in these locations.)

Even better, we have a description of the infinite solution set, based on just 5 vectors, which we combine in linear combinations to produce solutions.

Whenever we discuss Archetype I \[757\] you know that’s your cue to go work through Archetype J \[762\] by yourself. Remember to take note of the 0/1 pattern at the conclusion of Step 2. Have fun — we won’t go anywhere while you’re away.

This technique is so important, that we’ll do one more example. However, an important distinction will be that this system is homogeneous.

Example VFSAL

Vector form of solutions for Archetype L

Archetype L \[771\] is presented simply as the $5 \times 5$ matrix

\[
L = \begin{bmatrix}
-2 & -1 & -2 & -4 & 4 \\
-6 & -5 & -4 & -4 & 6 \\
10 & 7 & 7 & 10 & -13 \\
-7 & -5 & -6 & -9 & 10 \\
-4 & -3 & -4 & -6 & 6 \\
\end{bmatrix}
\]

We’ll interpret it here as the coefficient matrix of a homogeneous system and reference this matrix as $L$. So we are solving the homogeneous system $LS(L, 0)$ having $m = 5$ equations in $n = 5$ variables. If we built the augmented matrix, we would add a sixth column to $L$ containing all zeros. As we did row operations, this sixth column would remain all zeros. So instead we will row-reduce the coefficient matrix, and mentally
remember the missing sixth column of zeros. This row-reduced matrix is
\[
\begin{bmatrix}
1 & 0 & 0 & 1 & -2 \\
0 & 1 & 0 & -2 & 2 \\
0 & 0 & 1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
and we see \( r = 3 \) nonzero rows. The columns with leading 1’s are \( D = \{1, 2, 3\} \) so the \( r \) dependent variables are \( x_1, x_2, x_3 \). The columns without leading 1’s are \( F = \{4, 5\} \), so the \( n - r = 2 \) free variables are \( x_4, x_5 \). Notice that if we had included the all-zero vector of constants to form the augmented matrix for the system, then the index 6 would have appeared in the set \( F \), and subsequently would have been ignored when listing the free variables.

Step 1. Write the vector of variables \( \mathbf{x} \) as a fixed vector \( \mathbf{c} \), plus a linear combination of \( n - r = 2 \) vectors \( \mathbf{u}_1, \mathbf{u}_2 \), using the free variables as the scalars.

\[
\mathbf{x} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
-1 \\
2 \\
0 \\
0 \\
0
\end{bmatrix} + x_5 \begin{bmatrix}
2 \\
0 \\
1 \\
0 \\
1
\end{bmatrix}
\]

Step 2. For each free variable, use 0’s and 1’s to ensure equality for the corresponding entry of the the vectors. Take note of the pattern of 0’s and 1’s at this stage, even if it is not as illuminating as in other examples.

\[
\mathbf{x} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
-1 \\
2 \\
0 \\
0 \\
0
\end{bmatrix} + x_5 \begin{bmatrix}
2 \\
0 \\
1 \\
0 \\
1
\end{bmatrix}
\]

Step 3. For each dependent variable, use the augmented matrix to formulate an equation expressing the dependent variable as a constant plus multiples of the free variables. Don’t forget about the “missing” sixth column being full of zeros. Convert this equation into entries of the vectors that ensure equality for each dependent variable, one at a time.

\[
x_1 = 0 - 1x_4 + 2x_5 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
-1 \\
1 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
-1 \\
2 \\
0 \\
0 \\
1
\end{bmatrix} + x_5 \begin{bmatrix}
2 \\
0 \\
2 \\
0 \\
1
\end{bmatrix}
\]
\[
x_2 = 0 + 2x_4 - 2x_5 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
-1 \\
2 \\
0 \\
0 \\
1
\end{bmatrix} + x_5 \begin{bmatrix}
2 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]
The vector \( c \) will always have 0’s in the entries corresponding to free variables. However, since we are solving a homogeneous system, the row-reduced augmented matrix has zeros in column \( n + 1 = 6 \), and hence all the entries of \( c \) are zero. So we can write

\[
\mathbf{x} = \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix} + x_4 \begin{bmatrix}
  -1 \\
  2 \\
  -2 \\
  1 \\
  0
\end{bmatrix} + x_5 \begin{bmatrix}
  2 \\
  -2 \\
  1 \\
  0 \\
  1
\end{bmatrix}
\]

It will always happen that the solutions to a homogeneous system has \( c = 0 \) (even in the case of a unique solution?). So our expression for the solutions is a bit more pleasing. In this example it says that the solutions are all possible linear combinations of the two vectors \( \mathbf{u}_1 = \begin{bmatrix}
  -1 \\
  2 \\
  -2 \\
  1 \\
  0
\end{bmatrix} \) and \( \mathbf{u}_2 = \begin{bmatrix}
  0 \\
  -2 \\
  1 \\
  0 \\
  1
\end{bmatrix} \), with no mention of any fixed vector entering into the linear combination.

This observation will motivate our next section and the main definition of that section, and after that we will conclude the section by formalizing this situation.

Subsection PSHS

Particular Solutions, Homogeneous Solutions

The next theorem tells us that in order to find all of the solutions to a linear system of equations, it is sufficient to find just one solution, and then find all of the solutions to the corresponding homogeneous system. This explains part of our interest in the null space, the set of all solutions to a homogeneous system.

**Theorem PSPHS**

**Particular Solution Plus Homogeneous Solutions**

Suppose that \( \mathbf{w} \) is one solution to the linear system of equations \( \mathbf{L} \mathbf{s}(\mathbf{A}, \mathbf{b}) \). Then \( \mathbf{y} \) is a solution to \( \mathbf{L} \mathbf{s}(\mathbf{A}, \mathbf{b}) \) if and only if \( \mathbf{y} = \mathbf{w} + \mathbf{z} \) for some vector \( \mathbf{z} \in \mathcal{N}(\mathbf{A}) \).

**Proof**

Let \( \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n \) be the columns of the coefficient matrix \( \mathbf{A} \).

(\( \Rightarrow \)) Suppose \( \mathbf{y} = \mathbf{w} + \mathbf{z} \) and \( \mathbf{z} \in \mathcal{N}(\mathbf{A}) \). Then

\[
\mathbf{b} = [\mathbf{w}]_1 \mathbf{A}_1 + [\mathbf{w}]_2 \mathbf{A}_2 + [\mathbf{w}]_3 \mathbf{A}_3 + \cdots + [\mathbf{w}]_n \mathbf{A}_n
\]

By **Theorem SLSLC**

\[
= [\mathbf{w}]_1 \mathbf{A}_1 + [\mathbf{w}]_2 \mathbf{A}_2 + [\mathbf{w}]_3 \mathbf{A}_3 + \cdots + [\mathbf{w}]_n \mathbf{A}_n + 0
\]

By **Theorem SLSLC**

\[
= [\mathbf{w}]_1 \mathbf{A}_1 + [\mathbf{w}]_2 \mathbf{A}_2 + [\mathbf{w}]_3 \mathbf{A}_3 + \cdots + [\mathbf{w}]_n \mathbf{A}_n + [\mathbf{z}]_1 \mathbf{A}_1 + [\mathbf{z}]_2 \mathbf{A}_2 + [\mathbf{z}]_3 \mathbf{A}_3 + \cdots + [\mathbf{z}]_n \mathbf{A}_n
\]

By **Theorem SLSLC**

\[
\begin{aligned}
\mathbf{b} &= [\mathbf{w}]_1 \mathbf{A}_1 + [\mathbf{w}]_2 \mathbf{A}_2 + [\mathbf{w}]_3 \mathbf{A}_3 + \cdots + [\mathbf{w}]_n \mathbf{A}_n + [\mathbf{z}]_1 \mathbf{A}_1 + [\mathbf{z}]_2 \mathbf{A}_2 + [\mathbf{z}]_3 \mathbf{A}_3 + \cdots + [\mathbf{z}]_n \mathbf{A}_n \\
&= \mathbf{L} \mathbf{s}(\mathbf{A}, \mathbf{b})
\end{aligned}
\]
= ([w]_1 + [z]_1) A_1 + ([w]_2 + [z]_2) A_2 + \cdots + ([w]_n + [z]_n) A_n \quad \text{Theorem VSPCV [91]}
= [w + z]_1 A_1 + [w + z]_2 A_2 + [w + z]_3 A_3 + \cdots + [w + z]_n A_n \quad \text{Definition CVA [89]}
= [y]_1 A_1 + [y]_2 A_2 + [y]_3 A_3 + \cdots + [y]_n A_n \quad \text{Definition of y}

Applying Theorem SLSLC [100] we see that y is a solution to \( LS(A, b) \).

(\Rightarrow) Suppose y is a solution to \( LS(A, b) \). Then

\[
0 = b - b = [y]_1 A_1 + [y]_2 A_2 + [y]_3 A_3 + \cdots + [y]_n A_n
- ([w]_1 A_1 + [w]_2 A_2 + [w]_3 A_3 + \cdots + [w]_n A_n)
= ([y]_1 - [w]_1) A_1 + ([y]_2 - [w]_2) A_2 + \cdots + ([y]_n - [w]_n) A_n
= [y - w]_1 A_1 + [y - w]_2 A_2 + [y - w]_3 A_3 + \cdots + [y - w]_n A_n 
\] 

By Theorem SLSLC [100] we see that \( y - w \) is a solution to the homogeneous system \( LS(A, 0) \) and by Definition NSM [68], \( y - w \in \mathcal{N}(A) \). In other words, \( y - w = z \) for some vector \( z \in \mathcal{N}(A) \). Rewritten, this is \( y = w + z \), as desired. ■

After proving Theorem NMUS [79] we commented (insufficiently) on the negation of one half of the theorem. Nonsingular coefficient matrices lead to unique solutions for every choice of the vector of constants. What does this say about singular matrices? A singular matrix \( A \) has a nontrivial null space (Theorem NMTNS [78]). For a given vector of constants, \( b \), the system \( LS(A, b) \) could be inconsistent, meaning there are no solutions. But if there is at least one solution \( (w) \), then Theorem PSPHS [113] tells us there will be infinitely many solutions because of the role of the infinite null space for a singular matrix. So a system of equations with a singular coefficient matrix never has a unique solution. Either there are no solutions, or infinitely many solutions, depending on the choice of the vector of constants \( (b) \).

Example PSHS

Particular solutions, homogeneous solutions, Archetype D

Archetype D [735] is a consistent system of equations with a nontrivial null space. Let \( A \) denote the coefficient matrix of this system. The write-up for this system begins with three solutions,

\[
y_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \quad y_2 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad y_3 = \begin{bmatrix} 7 \\ 8 \\ 1 \\ 3 \end{bmatrix}
\]

We will choose to have \( y_1 \) play the role of \( w \) in the statement of Theorem PSPHS [113], any one of the three vectors listed here (or others) could have been chosen. To illustrate the theorem, we should be able to write each of these three solutions as the vector \( w \) plus a solution to the corresponding homogeneous system of equations. Since \( 0 \) is always a solution to a homogeneous system we can easily write

\[
y_1 = w = w + 0.
\]

The vectors \( y_2 \) and \( y_3 \) will require a bit more effort. Solutions to the homogeneous system \( LS(A, 0) \) are exactly the elements of the null space of the coefficient matrix, which by

Version 0.92
an application of Theorem VFSLS is

\[ \mathcal{N}(A) = \begin{cases} x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \mid x_3, x_4 \in \mathbb{C} \end{cases} \]

Then

\[ y_2 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix} + (-2) \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} = w + z_2 \]

where

\[ z_2 = \begin{bmatrix} 4 \\ -1 \\ -2 \\ -1 \end{bmatrix} = (-2) \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \]

is obviously a solution of the homogeneous system since it is written as a linear combination of the vectors describing the null space of the coefficient matrix (or as a check, you could just evaluate the equations in the homogeneous system with \( z_2 \)).

Again

\[ y_3 = \begin{bmatrix} 7 \\ 8 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \\ -1 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} = w + z_3 \]

where

\[ z_3 = \begin{bmatrix} 7 \\ 7 \\ -1 \\ 2 \end{bmatrix} = (-1) \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \]

is obviously a solution of the homogeneous system since it is written as a linear combination of the vectors describing the null space of the coefficient matrix (or as a check, you could just evaluate the equations in the homogeneous system with \( z_2 \)).

Here’s another view of this theorem, in the context of this example. Grab two new solutions of the original system of equations, say

\[ y_4 = \begin{bmatrix} 11 \\ 0 \\ -3 \\ -1 \end{bmatrix} \quad y_5 = \begin{bmatrix} -4 \\ 2 \\ 4 \\ 2 \end{bmatrix} \]

and form their difference,

\[ u = \begin{bmatrix} 11 \\ 0 \\ -3 \\ -1 \end{bmatrix} - \begin{bmatrix} -4 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 15 \\ -2 \\ -7 \\ -3 \end{bmatrix} \]
It is no accident that \( u \) is a solution to the homogeneous system (check this!). In other words, the difference between any two solutions to a linear system of equations is an element of the null space of the coefficient matrix. This is an equivalent way to state Theorem PSPHS\cite{113}. (See Exercise MM.T50\cite{225}).

The ideas of this subsection will appear again in Chapter LT\cite{503} when we discuss pre-images of linear transformations (Definition PI\cite{515}).

Subsection URREF
Uniqueness of Reduced Row-Echelon Form

We are now in a position to establish that the reduced row-echelon form of a matrix is unique. Going forward, we will emphasize the point-of-view that a matrix is a collection of columns. But there are two occasions when we need to work carefully with the rows of a matrix. This is the first such occasion. We could define something called a row vector that would equal a given row of a matrix, and might be written as a horizontal list. Then we could define vector equality, the basic operations of vector addition and scalar multiplication, followed by a definition of a linear combination of row vectors. We will not incur the overhead of stating all these definitions, but will instead convert the rows of a matrix to column vectors and use our definitions that are already in place. This was our reason for delaying this proof until now. Remind yourself as you work through this proof that it only relies only on the definition of equivalent matrices, reduced row-echelon form and linear combinations. So in particular, we are not guilty of circular reasoning. Should we have defined vector operations and linear combinations just prior to discussing reduced row-echelon form, then the following proof of uniqueness could have been presented at that time. OK, here we go.

**Theorem RREFU**
Reduced Row-Echelon Form is Unique

Suppose that \( A \) is an \( m \times n \) matrix and that \( B \) and \( C \) are \( m \times n \) matrices that are row-equivalent to \( A \) and in reduced row-echelon form. Then \( B = C \).

**Proof**
Denote the pivot columns of \( B \) as \( D = \{d_1, d_2, d_3, \ldots, d_r\} \) and the pivot columns of \( C \) as \( D' = \{d'_1, d'_2, d'_3, \ldots, d'_r\} \) (Notation RREFA\cite{36}). We begin by showing that \( D = D' \).

For both \( B \) and \( C \), we can take the elements of a row of the matrix and use them to construct a column vector. We will denote these by \( b_i \) and \( c_i \), respectively, \( 1 \leq i \leq m \). Since \( B \) and \( C \) are both row-equivalent to \( A \), there is a sequence of row operations that will convert \( B \) to \( C \), and vice-versa, since row operations are reversible. If we can convert \( B \) into \( C \) via a sequence of row operations, then any row of \( C \) expressed as a column vector, say \( c_k \), is a linear combination of the column vectors derived from the rows of \( B \), \( \{b_1, b_2, b_3, \ldots, b_m\} \). Similarly, any row of \( B \) is a linear combination of the set of rows of \( C \). Our principal device in this proof is to carefully analyze individual entries of vector equalities between a single row of either \( B \) or \( C \) and a linear combination of the rows of the other matrix.

Let’s first show that \( d_1 = d'_1 \). Suppose that \( d_1 < d'_1 \). We can write the first row of \( B \) as a linear combination of the rows of \( C \), that is, there are scalars \( a_1, a_2, a_3, \ldots, a_m \).
such that

\[ \mathbf{b}_1 = a_1 \mathbf{c}_1 + a_2 \mathbf{c}_2 + a_3 \mathbf{c}_3 + \cdots + a_m \mathbf{c}_m \]

Consider the entry in location \( d_1 \) on both sides of this equality. Since \( B \) is in reduced row-echelon form (Definition RREF [35]) we find a one in \( \mathbf{b}_1 \) on the left. Since \( d_1 < d'_1 \), and \( C \) is in reduced row-echelon form (Definition RREF [35]) each vector \( \mathbf{c}_i \) has a zero in location \( d_1 \), and therefore the linear combination on the right also has a zero in location \( d_1 \). This is a contradiction, so we know that \( d_1 \geq d'_1 \). By an entirely similar argument, we could conclude that \( d_1 \leq d'_1 \). This means that \( d_1 = d'_1 \).

Suppose that we have determined that \( d_1 = d'_1 , d_2 = d'_2 , d_3 = d'_3 , \ldots , d_k = d'_k \). Let’s now show that \( d_{k+1} = d'_{k+1} \). To achieve a contradiction, suppose that \( d_{k+1} < d'_{k+1} \). Row \( k+1 \) of \( B \) is a linear combination of the rows of \( C \), so there are scalars \( a_1, a_2, a_3, \ldots, a_m \) such that

\[ \mathbf{b}_{k+1} = a_1 \mathbf{c}_1 + a_2 \mathbf{c}_2 + a_3 \mathbf{c}_3 + \cdots + a_m \mathbf{c}_m \]

Since \( B \) is in reduced row-echelon form (Definition RREF [35]), the entries of \( \mathbf{b}_{k+1} \) in locations \( d_1, d_2, d_3, \ldots, d_k \) are all zero. Since \( C \) is in reduced row-echelon form (Definition RREF [35]), location \( d_i \) of \( \mathbf{c}_i \) is one for each \( 1 \leq i \leq k \). The equality of these vectors in locations \( d_1, d_2, d_3, \ldots, d_k \) then implies that \( a_1 = 0, a_2 = 0, a_3 = 0, \ldots, a_k = 0 \).

Now consider location \( d_{k+1} \) in this vector equality. The vector \( \mathbf{b}_{k+1} \) on the left is one in this location since \( B \) is in reduced row-echelon form (Definition RREF [35]). Vectors \( \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \ldots, \mathbf{c}_k \), are multiplied by zero scalars in the linear combination on the right. The remaining vectors, \( \mathbf{c}_{k+1}, \mathbf{c}_{k+2}, \mathbf{c}_{k+3}, \ldots, \mathbf{c}_m \), each has a zero in location \( d_{k+1} \) since \( d_{k+1} < d'_{k+1} \) and \( C \) is in reduced row-echelon form (Definition RREF [35]). So the right hand side of the vector equality is zero in location \( d_{k+1} \), a contradiction. Thus \( d_{k+1} \geq d'_{k+1} \). By an entirely similar argument, we could conclude that \( d_{k+1} \leq d'_{k+1} \), and therefore \( d_{k+1} = d'_{k+1} \).

Now we establish that \( r = r' \). Suppose that \( r < r' \). By the arguments above we can show that \( d_1 = d'_1 , d_2 = d'_2 , d_3 = d'_3 , \ldots , d_r = d'_r \). Row \( r' \) of \( C \) is a linear combination of the \( r \) non-zero rows of \( B \), so there are scalars \( a_1, a_2, a_3, \ldots, a_r \) so that

\[ \mathbf{c}_{r'} = a_1 \mathbf{b}_1 + a_2 \mathbf{b}_2 + a_3 \mathbf{b}_3 + \cdots + a_r \mathbf{b}_r \]

Locations \( d_1, d_2, d_3, \ldots, d_r \) of \( \mathbf{c}_{r'} \) are all zero since \( r < r' \) and \( C \) is in reduced row-echelon form (Definition RREF [35]). For a given index \( i \), \( 1 \leq i \leq r \), the vectors \( \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots, \mathbf{b}_r \) have zeros in location \( d_i \), except that the vector \( \mathbf{b}_i \) is one in location \( d_i \) since \( B \) is in reduced row-echelon form (Definition RREF [35]). This consideration of location \( d_i \) implies that \( a_i = 0 \), \( 1 \leq i \leq r \). With all the scalars in the linear combination equal to zero, we conclude that \( \mathbf{c}_{r'} = \mathbf{0} \), contradicting the existence of a leading 1 in \( \mathbf{c}_{r'} \). So \( r \geq r' \). By a similar argument, we conclude that \( r \leq r' \) and therefore \( r = r' \). Thus \( D = D' \).

To finally show that \( B = C \), we will show that the rows of the two matrices are equal. Row \( k \) of \( C \), \( \mathbf{c}_k \), is a linear combination of the \( r \) non-zero rows of \( B \), so there are scalars \( a_1, a_2, a_3, \ldots, a_r \) such that

\[ \mathbf{c}_k = a_1 \mathbf{b}_1 + a_2 \mathbf{b}_2 + a_3 \mathbf{b}_3 + \cdots + a_r \mathbf{b}_r \]

Because \( C \) is in reduced row-echelon form (Definition RREF [35]), location \( d_i \) of \( \mathbf{c}_k \) is zero for \( 1 \leq i \leq r \), except in location \( d_k \) where the entry is one. In the linear combination on the right of the vector equality, the vectors \( \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots, \mathbf{b}_r \) have zeros in location
Section LC  Linear Combinations

$d_i$, except that $b_k$ has a one in location $d_i$, since $B$ is in reduced row-echelon form (Definition RREF [35]). This implies that $a_1 = 0$, $a_2 = 0$, \ldots, $a_{k-1} = 0$, $a_{k+1} = 0$, $a_{k+2} = 0$, \ldots, $a_r = 0$ and $a_k = 1$. Then the vector equality reduces to simply $c_k = b_k$. Since $k$ was arbitrary, $B$ and $C$ have equal rows and so are equal matrices. 

Subsection READ
Reading Questions

1. Earlier, a reading question asked you to solve the system of equations

\[
\begin{align*}
2x_1 + 3x_2 - x_3 &= 0 \\
x_1 + 2x_2 + x_3 &= 3 \\
x_1 + 3x_2 + 3x_3 &= 7
\end{align*}
\]

Use a linear combination to rewrite this system of equations as a vector equality.

2. Find a linear combination of the vectors

\[
S = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -5 \end{bmatrix} \right\}
\]

that equals the vector \[ \begin{bmatrix} 1 \\ -9 \\ 11 \end{bmatrix}. \]

3. The matrix below is the augmented matrix of a system of equations, row-reduced to reduced row-echelon form. Write the vector form of the solutions to the system.

\[
\begin{bmatrix}
1 & 3 & 0 & 6 & 0 & 9 \\
0 & 0 & 1 & -2 & 0 & -8 \\
0 & 0 & 0 & 0 & 1 & 3
\end{bmatrix}
\]
Subsection EXC Exercises

C21 Consider each archetype that is a system of equations. For individual solutions listed (both for the original system and the corresponding homogeneous system) express the vector of constants as a linear combination of the columns of the coefficient matrix, as guaranteed by Theorem SLSLC [100]. Verify this equality by computing the linear combination. For systems with no solutions, recognize that it is then impossible to write the vector of constants as a linear combination of the columns of the coefficient matrix. Note too, for homogeneous systems, that the solutions give rise to linear combinations that equal the zero vector.

Archetype A [721]
Archetype B [726]
Archetype C [731]
Archetype D [735]
Archetype E [739]
Archetype F [743]
Archetype G [748]
Archetype H [752]
Archetype I [757]
Archetype J [762]

Contributed by Robert Beezer Solution [121]

C22 Consider each archetype that is a system of equations. Write elements of the solution set in vector form, as guaranteed by Theorem VFSLS [107].

Archetype A [721]
Archetype B [726]
Archetype C [731]
Archetype D [735]
Archetype E [739]
Archetype F [743]
Archetype G [748]
Archetype H [752]
Archetype I [757]
Archetype J [762]

Contributed by Robert Beezer Solution [121]

C40 Find the vector form of the solutions to the system of equations below.

\[
\begin{align*}
2x_1 - 4x_2 + 3x_3 + x_5 &= 6 \\
x_1 - 2x_2 - 2x_3 + 14x_4 - 4x_5 &= 15 \\
x_1 - 2x_2 + x_3 + 2x_4 + x_5 &= -1 \\
-2x_1 + 4x_2 - 12x_4 + x_5 &= -7
\end{align*}
\]

Contributed by Robert Beezer Solution [121]
C41 Find the vector form of the solutions to the system of equations below.

\[-2x_1 - 1x_2 - 8x_3 + 8x_4 + 4x_5 - 9x_6 - 1x_7 - 1x_8 - 18x_9 = 3\]
\[3x_1 - 2x_2 + 5x_3 + 2x_4 - 2x_5 - 5x_6 + 1x_7 + 2x_8 + 15x_9 = 10\]
\[4x_1 - 2x_2 + 8x_3 + 2x_5 - 14x_6 - 2x_8 + 2x_9 = 36\]
\[-1x_1 + 2x_2 + 1x_3 - 6x_4 + 7x_6 - 1x_7 - 3x_9 = -8\]
\[3x_1 + 2x_2 + 13x_3 - 14x_4 - 1x_5 + 5x_6 - 1x_8 + 12x_9 = 15\]
\[-2x_1 + 2x_2 - 2x_3 - 4x_4 + 1x_5 + 6x_6 - 2x_7 - 2x_8 - 15x_9 = -7\]

Contributed by Robert Beezer Solution [121]

M10 Example TLC [97] asks if the vector

\[\mathbf{w} = \begin{bmatrix} 13 \\ 15 \\ 5 \\ -17 \\ 2 \\ 25 \end{bmatrix}\]

can be written as a linear combination of the four vectors

\[\mathbf{u}_1 = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 1 \\ 2 \\ 9 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 6 \\ 3 \\ 0 \\ -2 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -5 \\ 2 \\ 1 \\ -3 \\ 0 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 7 \\ 1 \\ 3 \end{bmatrix}\]

Can it? Can any vector in \(\mathbb{C}^6\) be written as a linear combination of the four vectors \(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\)?

Contributed by Robert Beezer Solution [122]

M11 At the end of Example VFS [104], the vector \(\mathbf{w}\) is claimed to be a solution to the linear system under discussion. Verify that \(\mathbf{w}\) really is a solution. Then determine the four scalars that express \(\mathbf{w}\) as a linear combination of \(\mathbf{c}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\).

Contributed by Robert Beezer Solution [122]

T30 Suppose that \(\mathbf{x}\) is a solution to \(\mathcal{L}S(A, b)\) and that \(\mathbf{z}\) is a solution to the homogeneous system \(\mathcal{L}S(A, 0)\). Prove that \(\mathbf{x} + \mathbf{z}\) is a solution to \(\mathcal{L}S(A, b)\).

Contributed by Robert Beezer Solution [122]
Subsection SOL
Solutions

C21 Contributed by Robert Beezer Statement 119
Solutions for Archetype A 721 and Archetype B 726 are described carefully in Example AALC 99 and Example ABLC 98.

C22 Contributed by Robert Beezer Statement 119
Solutions for Archetype D 735 and Archetype I 757 are described carefully in Example VFSAD 102 and Example VFSAI 110. The technique described in these examples is probably more useful than carefully deciphering the notation of Theorem VFSLS 107. The solution for each archetype is contained in its description. So now you can check-off the box for that item.

C40 Contributed by Robert Beezer Statement 119
Row-reduce the augmented matrix representing this system, to find
\[
\begin{bmatrix}
1 & -2 & 0 & 6 & 0 & 1 \\
0 & 0 & 1 & -4 & 0 & 3 \\
0 & 0 & 0 & 0 & 1 & -5 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
The system is consistent (no leading one in column 6, Theorem RCLS 54). \(x_2\) and \(x_4\) are the free variables. Now apply Theorem VFSLS 107 directly, or follow the three-step process of Example VFS 104, Example VFSAD 102, Example VFSAI 110, or Example VFSAL 111 to obtain
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix} = \begin{bmatrix} 1 \\
0 \\
3 \\
0 \\
-5 \\
\end{bmatrix} + \begin{bmatrix} 2 \\
1 \\
0 \\
0 \\
0 \\
\end{bmatrix} x_2 + \begin{bmatrix} -6 \\
0 \\
4 \\
0 \\
1 \\
\end{bmatrix} x_4
\]

C41 Contributed by Robert Beezer Statement 120
Row-reduce the augmented matrix representing this system, to find
\[
\begin{bmatrix}
1 & 0 & 3 & -2 & 0 & -1 & 0 & 0 & 3 & 6 \\
0 & 1 & 2 & -4 & 0 & 3 & 0 & 0 & 2 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 & -2 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
The system is consistent (no leading one in column 10, Theorem RCLS 54). \(F = \{3, 4, 6, 9, 10\}\), so the free variables are \(x_3, x_4, x_6\) and \(x_9\). Now apply Theorem VFSLS 107 directly, or follow the three-step process of Example VFS 104, Example VF-
SAD 102, Example VFSAI 110, or Example VFSAL 111 to obtain the solution set

\[
S = \begin{cases}
\begin{bmatrix}
6 \\
-1 \\
0 \\
0 \\
3 \\
0 \\
-2 \\
0
\end{bmatrix} + x_3 \begin{bmatrix}
-3 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
2 \\
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + x_6 \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + x_9 \begin{bmatrix}
-3 \\
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\end{cases}
\]

\(x_3, x_4, x_6, x_9 \in \mathbb{C}\)

M10 Contributed by Robert Beezer Statement 120

No, it is not possible to create \(w\) as a linear combination of the four vectors \(u_1, u_2, u_3, u_4\).

By creating the desired linear combination with unknowns as scalars, Theorem SLSLC 100 provides a system of equations that has no solution. This one computation is enough to show us that it is not possible to create all the vectors of \(\mathbb{C}^6\) through linear combinations of the four vectors \(u_1, u_2, u_3, u_4\).

M11 Contributed by Robert Beezer Statement 120

The coefficient of \(c\) is 1. The coefficients of \(u_1, u_2, u_3\) lie in the third, fourth and seventh entries of \(w\). Can you see why? (Hint: \(F = \{3, 4, 7, 8\}\), so the free variables are \(x_3, x_4\) and \(x_7\).)

T30 Contributed by Robert Beezer Statement 120

Write the columns of \(A\) as \(A_1, A_2, A_3, \ldots, A_n\). Then

\[
b = [x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n
\]

Theorem SLSLC 100

\[
= [x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n + 0
\]

Property ZC 91

\[
= [x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n + [z]_1 A_1 + [z]_2 A_2 + [z]_3 A_3 + \cdots + [z]_n A_n
\]

Theorem SLSLC 100

\[
= ([x]_1 + [z]_1) A_1 + ([x]_2 + [z]_2) A_2 + \cdots + ([x]_n + [z]_n) A_n
\]

Theorem VSPCV 91

\[
= [x + z]_1 A_1 + [x + z]_2 A_2 + \cdots + [x + z]_n A_n
\]

Definition CVA 89

This equation then allows us to employ Theorem SLSLC 100 and conclude that \(x + z\) is a solution to \(LS(A, b)\).
Section SS
Spanning Sets

In this section we will describe a compact way to indicate the elements of an infinite set of vectors, making use of linear combinations. This will give us a convenient way to describe the elements of a set of solutions to a linear system, or the elements of the null space of a matrix, or many other sets of vectors.

Subsection SSV
Span of a Set of Vectors

In Example VFSAL we saw the solution set of a homogeneous system described as all possible linear combinations of two particular vectors. This happens to be a useful way to construct or describe infinite sets of vectors, so we encapsulate this idea in a definition.

Definition SSCV
Span of a Set of Column Vectors

Given a set of vectors \( S = \{ u_1, u_2, u_3, \ldots, u_p \} \), their span, \( \langle S \rangle \), is the set of all possible linear combinations of \( u_1, u_2, u_3, \ldots, u_p \). Symbolically,

\[
\langle S \rangle = \{ \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_p u_p \mid \alpha_i \in \mathbb{C}, \ 1 \leq i \leq p \}
\]

(This definition contains Notation SSV.)

The span is just a set of vectors, though in all but one situation it is an infinite set. (Just when is it not infinite?) So we start with a finite collection of vectors \( S \) (of them to be precise), and use this finite set to describe an infinite set of vectors, \( \langle S \rangle \). Confusing the finite set \( S \) with the infinite set \( \langle S \rangle \) is one of the most pervasive problems in understanding introductory linear algebra. We will see this construction repeatedly, so let’s work through some examples to get comfortable with it. The most obvious question about a set is if a particular item of the correct type is in the set, or not.

Example ABS
A basic span

Consider the set of 5 vectors, \( S \), from \( \mathbb{C}^4 \)

\[
S = \begin{bmatrix}
1 & 2 & 7 & 1 & -1 \\
1 & 1 & 3 & 0 & \\
3 & 2 & 5 & -1 & 9 \\
1 & -1 & -5 & 2 & 0
\end{bmatrix}
\]

and consider the infinite set of vectors \( \langle S \rangle \) formed from all possible linear combinations of the elements of \( S \). Here are four vectors we definitely know are elements of \( \langle S \rangle \), since
we will construct them in accordance with Definition SSCV [123],

\[
\begin{align*}
\mathbf{w} &= (2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 7 \\ 3 \\ -5 \end{bmatrix} + (2) \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + (3) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 28 \\ 10 \end{bmatrix} \\
\mathbf{x} &= (5) \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + (-6) \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix} + (-3) \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix} + (4) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix} + (2) \begin{bmatrix} 0 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} -26 \\ 2 \\ 34 \end{bmatrix} \\
\mathbf{y} &= (1) \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} + (0) \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix} + (1) \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix} + (0) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix} + (1) \begin{bmatrix} -1 \\ 0 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 17 \\ -4 \end{bmatrix} \\
\mathbf{z} &= (0) \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} + (0) \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} + (0) \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix} + (0) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix} + (0) \begin{bmatrix} -1 \\ 0 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{align*}
\]

The purpose of a set is to collect objects with some common property, and to exclude objects without that property. So the most fundamental question about a set is if a given object is an element of the set or not. Let’s learn more about \( \langle S \rangle \) by investigating which vectors are an element of the set, and which are not.

First, is \( \mathbf{u} = \begin{bmatrix} -15 \\ -6 \\ 19 \\ 5 \end{bmatrix} \) an element of \( \langle S \rangle \)? We are asking if there are scalars \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \) such that

\[
\begin{align*}
\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 7 \\ 3 \\ -5 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \alpha_5 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} &= \mathbf{u} = \begin{bmatrix} 15 \\ -6 \\ 19 \end{bmatrix}
\end{align*}
\]

Applying Theorem SLSLC [100] we recognize the search for these scalars as a solution to a linear system of equations with augmented matrix

\[
\begin{bmatrix}
1 & 2 & 7 & 1 & -1 & -15 \\
1 & 3 & 1 & 0 & 0 & -6 \\
3 & 2 & 5 & -1 & 9 & 19 \\
1 & -1 & -5 & 2 & 0 & 5
\end{bmatrix}
\]

which row-reduces to

\[
\begin{bmatrix}
1 & 0 & -1 & 0 & 3 & 10 \\
0 & 1 & 4 & 0 & -1 & -9 \\
0 & 0 & 0 & 1 & -2 & -7 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

At this point, we see that the system is consistent (no a leading 1 in the last column, Theorem RCLS [54]), so we know there is a solution for the five scalars \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \).
This is enough evidence for us to say that \( u \in \langle S \rangle \). If we wished further evidence, we could compute an actual solution, say
\[
\begin{align*}
\alpha_1 &= 2 \\
\alpha_2 &= 1 \\
\alpha_3 &= -2 \\
\alpha_4 &= -3 \\
\alpha_5 &= 2
\end{align*}
\]
This particular solution allows us to write
\[
(2) \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + (-2) \begin{bmatrix} 7 \\ 3 \\ -5 \end{bmatrix} + (-3) \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + (2) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = u = \begin{bmatrix} -15 \\ -6 \\ 19 \end{bmatrix}
\]
making it even more obvious that \( u \in \langle S \rangle \).

Let’s do it again. Is \( v = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \) an element of \( \langle S \rangle \)? We are asking if there are scalars \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \) such that
\[
\alpha_1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 7 \\ 3 \\ -5 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \alpha_5 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = v = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}
\]
Applying Theorem SLSLC\([100]\) we recognize the search for these scalars as a solution to a linear system of equations with augmented matrix
\[
\begin{bmatrix} 1 & 2 & 7 & 1 & -1 & 3 \\ 1 & 1 & 3 & 1 & 0 & 1 \\ 3 & 2 & 5 & -1 & 9 & 2 \\ 1 & -1 & -5 & 2 & 0 & -1 \end{bmatrix}
\]
which row-reduces to
\[
\begin{bmatrix} 1 & 0 & -1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]
At this point, we see that the system is inconsistent (a leading 1 in the last column, Theorem RCLS\([54]\)), so we know there is not a solution for the five scalars \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \). This is enough evidence for us to say that \( v \notin \langle S \rangle \). End of story. ☑

**Example SCAA**

**Span of the columns of Archetype A**

Begin with the finite set of three vectors of size 3
\[
S = \{ u_1, u_2, u_3 \} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}
\]
and consider the infinite set \( \langle S \rangle \). The vectors of \( S \) could have been chosen to be anything, but for reasons that will become clear later, we have chosen the three columns of the
coefficient matrix in Archetype A [721]. First, as an example, note that
\[ v = (5) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-3) \begin{bmatrix} -1 \\ 1 \end{bmatrix} + (7) \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 22 \\ 14 \end{bmatrix} \]
is in \( \langle S \rangle \), since it is a linear combination of \( u_1, u_2, u_3 \). We write this succinctly as \( v \in \langle S \rangle \). There is nothing magical about the scalars \( \alpha_1 = 5, \alpha_2 = -3, \alpha_3 = 7 \), they could have been chosen to be anything. So repeat this part of the example yourself, using different values of \( \alpha_1, \alpha_2, \alpha_3 \). What happens if you choose all three scalars to be zero?

So we know how to quickly construct sample elements of the set \( \langle S \rangle \). A slightly different question arises when you are handed a vector of the correct size and asked if it is an element of \( \langle S \rangle \). For example, is \( w = \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix} \) in \( \langle S \rangle \)? More succinctly, \( w \in \langle S \rangle \)?

To answer this question, we will look for scalars \( \alpha_1, \alpha_2, \alpha_3 \) so that
\[ \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = w. \]
By Theorem SLSLC [100] solutions to this vector equality are solutions to the system of equations
\[
\begin{align*}
\alpha_1 - \alpha_2 + 2\alpha_3 &= 1 \\
2\alpha_1 + \alpha_2 + \alpha_3 &= 8 \\
\alpha_1 + \alpha_2 &= 5.
\end{align*}
\]
Building the augmented matrix for this linear system, and row-reducing, gives
\[
\begin{bmatrix}
1 & 0 & 1 & 3 \\
0 & 1 & -1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]
This system has infinitely many solutions (there’s a free variable in \( x_3 \)), but all we need is one solution vector. The solution,
\[ \alpha_1 = 2 \quad \alpha_2 = 3 \quad \alpha_3 = 1 \]
tells us that
\[ (2)u_1 + (3)u_2 + (1)u_3 = w \]
so we are convinced that \( w \) really is in \( \langle S \rangle \). Notice that there are an infinite number of ways to answer this question affirmatively. We could choose a different solution, this time choosing the free variable to be zero,
\[ \alpha_1 = 3 \quad \alpha_2 = 2 \quad \alpha_3 = 0 \]
shows us that
\[ (3)u_1 + (2)u_2 + (0)u_3 = w \]
Verifying the arithmetic in this second solution maybe makes it seem obvious that \( w \) is in this span? And of course, we now realize that there are an infinite number of ways
to realize \( \mathbf{w} \) as element of \( \langle S \rangle \). Let’s ask the same type of question again, but this time with \( \mathbf{y} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \), i.e. is \( \mathbf{y} \in \langle S \rangle \)?

So we’ll look for scalars \( \alpha_1, \alpha_2, \alpha_3 \) so that

\[
\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 = \mathbf{y}.
\]

By \textbf{Theorem SLSLC} [100] this linear combination becomes the system of equations

\[
\begin{align*}
\alpha_1 - \alpha_2 + 2\alpha_3 &= 2 \\
2\alpha_1 + \alpha_2 + \alpha_3 &= 4 \\
\alpha_1 + \alpha_2 &= 3.
\end{align*}
\]

Building the augmented matrix for this linear system, and row-reducing, gives

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

This system is inconsistent (there’s a leading 1 in the last column, \textbf{Theorem RCLS} [54]), so there are no scalars \( \alpha_1, \alpha_2, \alpha_3 \) that will create a linear combination of \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \) that equals \( \mathbf{y} \). More precisely, \( \mathbf{y} \notin \langle S \rangle \).

There are three things to observe in this example. (1) It is easy to construct vectors in \( \langle S \rangle \). (2) It is possible that some vectors are in \( \langle S \rangle \) (e.g. \( \mathbf{w} \)), while others are not (e.g. \( \mathbf{y} \)). (3) Deciding if a given vector is in \( \langle S \rangle \) leads to solving a linear system of equations and asking if the system is consistent.

With a computer program in hand to solve systems of linear equations, could you create a program to decide if a vector was, or wasn’t, in the span of a given set of vectors? Is this art or science?

This example was built on vectors from the columns of the coefficient matrix of \textbf{Archetype A} [721]. Study the determination that \( \mathbf{v} \in \langle S \rangle \) and see if you can connect it with some of the other properties of \textbf{Archetype A} [721].

Having analyzed \textbf{Archetype A} [721] in Example \textbf{SCAA} [125], we will of course subject \textbf{Archetype B} [726] to a similar investigation.

\textbf{Example SCAB}

\textbf{Span of the columns of Archetype B}

Begin with the finite set of three vectors of size 3 that are the columns of the coefficient matrix in \textbf{Archetype B} [726].

\[
R = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} = \left\{ \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix} \right\}
\]

and consider the infinite set \( V = \langle R \rangle \). First, as an example, note that

\[
\mathbf{x} = (2) \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix} + (4) \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ -10 \end{bmatrix}
\]
is in \( \langle R \rangle \), since it is a linear combination of \( v_1, v_2, v_3 \). In other words, \( x \in \langle R \rangle \). Try some different values of \( \alpha_1, \alpha_2, \alpha_3 \) yourself, and see what vectors you can create as elements of \( \langle R \rangle \).

Now ask if a given vector is an element of \( \langle R \rangle \). For example, is \( z = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix} \) in \( \langle R \rangle \)? Is \( z \in \langle R \rangle \)?

To answer this question, we will look for scalars \( \alpha_1, \alpha_2, \alpha_3 \) so that
\[
\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = z.
\]

By [Theorem SLSLC 100] this linear combination becomes the system of equations
\[
\begin{align*}
-7\alpha_1 - 6\alpha_2 - 12\alpha_3 &= -33 \\
5\alpha_1 + 5\alpha_2 + 7\alpha_3 &= 24 \\
\alpha_1 + 4\alpha_3 &= 5.
\end{align*}
\]

Building the augmented matrix for this linear system, and row-reducing, gives
\[
\begin{bmatrix}
1 & 0 & 0 & -3 \\
0 & 1 & 0 & 5 \\
0 & 0 & 1 & 2
\end{bmatrix}.
\]

This system has a unique solution,
\[
\alpha_1 = -3 \quad \alpha_2 = 5 \quad \alpha_3 = 2
\]
telling us that
\[
(-3)v_1 + (5)v_2 + (2)v_3 = z
\]
so we are convinced that \( z \) really is in \( \langle R \rangle \). Notice that in this case we have only one way to answer the question affirmatively since the solution is unique.

Let's ask about another vector, say is \( x = \begin{bmatrix} -7 \\ 8 \\ -3 \end{bmatrix} \) in \( \langle R \rangle \)? Is \( x \in \langle R \rangle \)?

We desire scalars \( \alpha_1, \alpha_2, \alpha_3 \) so that
\[
\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = x.
\]

By [Theorem SLSLC 100] this linear combination becomes the system of equations
\[
\begin{align*}
-7\alpha_1 - 6\alpha_2 - 12\alpha_3 &= -7 \\
5\alpha_1 + 5\alpha_2 + 7\alpha_3 &= 8 \\
\alpha_1 + 4\alpha_3 &= -3.
\end{align*}
\]

Building the augmented matrix for this linear system, and row-reducing, gives
\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & -1
\end{bmatrix}.
\]
This system has a unique solution,
\[ \alpha_1 = 1 \quad \alpha_2 = 2 \quad \alpha_3 = -1 \]
telling us that
\[ (1)v_1 + (2)v_2 + (-1)v_3 = x \]
so we are convinced that \( x \) really is in \( \langle R \rangle \). Notice that in this case we again have only one way to answer the question affirmatively since the solution is again unique.

We could continue to test other vectors for membership in \( \langle R \rangle \), but there is no point. A question about membership in \( \langle R \rangle \) inevitably leads to a system of three equations in the three variables \( \alpha_1, \alpha_2, \alpha_3 \) with a coefficient matrix whose columns are the vectors \( v_1, v_2, v_3 \). This particular coefficient matrix is nonsingular, so by Theorem NMUS \( 79 \), it is guaranteed to have a solution. (This solution is unique, but that’s not critical here.) So no matter which vector we might have chosen for \( z \), we would have been certain to discover that it was an element of \( \langle R \rangle \). Stated differently, every vector of size 3 is in \( \langle R \rangle \), or \( \langle R \rangle = \mathbb{C}^3 \).

Compare this example with Example SCAA \( 125 \), and see if you can connect \( z \) with some aspects of the write-up for Archetype B \( 726 \).

Subsection SSNS
Spanning Sets of Null Spaces

We saw in Example VFSAL \( 111 \) that when a system of equations is homogeneous the solution set can be expressed in the form described by Theorem VFSLS \( 107 \) where the vector \( c \) is the zero vector. We can essentially ignore this vector, so that the remainder of the typical expression for a solution looks like an arbitrary linear combination, where the scalars are the free variables and the vectors are \( u_1, u_2, u_3, \ldots, u_{n-r} \). Which sounds a lot like a span. This is the substance of the next theorem.

**Theorem SSNS**
Spanning Sets for Null Spaces

Suppose that \( A \) is an \( m \times n \) matrix, and \( B \) is a row-equivalent matrix in reduced row-echelon form with \( r \) nonzero rows. Let \( D = \{d_1, d_2, d_3, \ldots, d_r\} \) be the column indices where \( B \) has leading 1’s (pivot columns) and \( F = \{f_1, f_2, f_3, \ldots, f_{n-r}\} \) be the set of column indices where \( B \) does not have leading 1’s. Construct the \( n-r \) vectors \( z_j, 1 \leq j \leq n-r \) of size \( n \) as

\[
[z_j]_i = \begin{cases} 
1 & \text{if } i \in F, i = f_j \\
0 & \text{if } i \in F, i \neq f_j \\
-Bk_{f_j} & \text{if } i \in D, i = d_k 
\end{cases}
\]

Then the null space of \( A \) is given by
\[ \mathcal{N}(A) = \langle \{z_1, z_2, z_3, \ldots, z_{n-r}\} \rangle. \]
Proof Consider the homogeneous system with $A$ as a coefficient matrix, $\mathcal{L}S(A, \mathbf{0})$. Its set of solutions, $S$, is by Definition NSM [68], the null space of $A$, $\mathcal{N}(A)$. Let $B'$ denote the result of row-reducing the augmented matrix of this homogeneous system. Since the system is homogeneous, the final column of the augmented matrix will be all zeros, and after any number of row operations (Definition RO [33]), the column will still be all zeros. So $B'$ has a final column that is totally zeros.

Now apply Theorem VFSLS [107] to $B'$, after noting that our homogeneous system must be consistent (Theorem HSC [65]). The vector $c$ has zeros for each entry that corresponds to an index in $F$. For entries that correspond to an index in $D$, the value is $-B'_{k,n+1}$, but for $B'$ any entry in the final column (index $n+1$) is zero. So $c = 0$. The vectors $z_j$, $1 \leq j \leq n - r$ are identical to the vectors $u_j$, $1 \leq j \leq n - r$ described in Theorem SSNS [129]. Putting it all together and applying Definition SSCV [123] in the final step,

$$\mathcal{N}(A) = S = \{ c + x_{f_1}u_1 + x_{f_2}u_2 + x_{f_3}u_3 + \cdots + x_{f_{n-r}}u_{n-r} \mid x_{f_1}, x_{f_2}, x_{f_3}, \ldots, x_{f_{n-r}} \in \mathbb{C} \}$$

$$= \{ x_{f_1}z_1 + x_{f_2}z_2 + x_{f_3}z_3 + \cdots + x_{f_{n-r}}z_{n-r} \mid x_{f_1}, x_{f_2}, x_{f_3}, \ldots, x_{f_{n-r}} \in \mathbb{C} \}$$

$$= \langle \{ z_1, z_2, z_3, \ldots, z_{n-r} \} \rangle$$

Example SSNS
Spanning set of a null space
Find a set of vectors, $S$, so that the null space of the matrix $A$ below is the span of $S$, that is, $\langle S \rangle = \mathcal{N}(A)$.

$$A = \begin{bmatrix}
1 & 3 & 3 & -1 & -5 \\
2 & 5 & 7 & 1 & 1 \\
1 & 1 & 5 & 1 & 5 \\
-1 & -4 & -2 & 0 & 4 \\
\end{bmatrix}$$

The null space of $A$ is the set of all solutions to the homogeneous system $\mathcal{L}S(A, \mathbf{0})$. If we find the vector form of the solutions to this homogenous system (Theorem VFSLS [107]) then the vectors $u_j$, $1 \leq j \leq n - r$ in the linear combination are exactly the vectors $z_j$, $1 \leq j \leq n - r$ described in Theorem SSNS [129]. So we can mimic Example VFSAL [111] to arrive at these vectors (rather than being a slave to the formulas in the statement of the theorem).

Begin by row-reducing $A$. The result is

$$\begin{bmatrix}
1 & 3 & 3 & -1 & -5 \\
0 & -1 & -4 & -2 & 0 \\
0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

With $D = \{1, 2, 4\}$ and $F = \{3, 5\}$ we recognize that $x_3$ and $x_5$ are free variables and we can express each nonzero row as an expression for the dependent variables $x_1$, $x_2$, $x_4$ (respectively) in the free variables $x_3$ and $x_5$. With this we can write the vector form of
a solution vector as

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} =
\begin{bmatrix}
-6x_3 - 4x_5 \\
x_3 + 2x_5 \\
x_3 \\
-3x_5 \\
x_5
\end{bmatrix} = x_3 \begin{bmatrix}
-6 \\
1 \\
1 \\
0 \\
0
\end{bmatrix} + x_5 \begin{bmatrix}
-4 \\
2 \\
0 \\
-3 \\
1
\end{bmatrix}
\]

Then in the notation of Theorem SSNS [129],

\[
z_1 =
\begin{bmatrix}
-6 \\
1 \\
1 \\
0 \\
0
\end{bmatrix}
\quad z_2 =
\begin{bmatrix}
-4 \\
2 \\
0 \\
-3 \\
1
\end{bmatrix}
\]

and

\[
N(A) = \langle \{z_1, z_2\} \rangle = \langle \begin{bmatrix}
-6 \\
1 \\
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
-4 \\
2 \\
0 \\
-3 \\
1
\end{bmatrix} \rangle
\]

Example NSDS

Null space directly as a span

Let’s express the null space of \(A\) as the span of a set of vectors, applying Theorem SSNS [129] as economically as possible, without reference to the underlying homogeneous system of equations (in contrast to Example SSNS [130]).

\[
A = \begin{bmatrix}
2 & 1 & 5 & 1 & 5 & 1 \\
1 & 1 & 3 & 1 & 6 & -1 \\
-1 & 1 & -1 & 0 & 4 & -3 \\
-3 & 2 & -4 & -4 & -7 & 0 \\
3 & -1 & 5 & 2 & 2 & 3
\end{bmatrix}
\]

Theorem SSNS [129] creates vectors for the span by first row-reducing the matrix in question. The row-reduced version of \(A\) is

\[
B = \begin{bmatrix}
1 & 0 & 2 & 0 & -1 & 2 \\
0 & 1 & 1 & 0 & 3 & -1 \\
0 & 0 & 0 & 1 & 4 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

I usually find it easier to envision the construction of the homogenous system of equations represented by this matrix, solve for the dependent variables and then unravel the equations into a linear combination. But we can just as well mechanically follow the prescription of Theorem SSNS [129]. Here we go, in two big steps.

First, the indices of the non-pivot columns have indices \(F = \{3, 5, 6\}\), so we will construct the \(n - r = 6 - 3 = 3\) vectors with a pattern of zeros and ones corresponding to
the indices in $F$. This is the realization of the first two lines of the three-case definition of the vectors $z_j, 1 \leq j \leq n - r$.

$$z_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad z_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Each of these vectors arises due to the presence of a column that is not a pivot column. The remaining entries of each vector are the entries of the corresponding non-pivot column, negated, and distributed into the empty slots in order (these slots have indices in the set $D$ and correspond to pivot columns). This is the realization of the third line of the three-case definition of the vectors $z_j, 1 \leq j \leq n - r$.

$$z_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \quad z_3 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So, by Theorem SSNS [129], we have

$$\mathcal{N}(A) = \langle \{z_1, z_2, z_3\} \rangle = \left\{ \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

We know that the null space of $A$ is the solution set of the homogeneous system $\mathcal{L}(A, \mathbf{0})$, but nowhere in this application of Theorem SSNS [129] have we found occasion to reference the variables or equations of this system.

More advanced computational devices will compute the null space of a matrix. See: Computation NS.MMA [681]. Here’s an example that will simultaneously exercise the span construction and Theorem SSNS [129], while also pointing the way to the next section.

**Example SCAD**

**Span of the columns of Archetype D**

Begin with the set of four vectors of size 3

$$T = \{w_1, w_2, w_3, w_4\} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ -5 \end{bmatrix} \right\}$$

and consider the infinite set $W = \langle T \rangle$. The vectors of $T$ have been chosen as the four columns of the coefficient matrix in Archetype D [735]. Check that the vector

$$z_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$
is a solution to the homogeneous system \( L(S(D, 0) \) (it is the vector \( z_2 \) provided by the description of the null space of the coefficient matrix \( D \) from Theorem SSNS \[129\]). Applying Theorem SLSLC \[100\], we can write the linear combination,

\[
2w_1 + 3w_2 + 0w_3 + 1w_4 = 0
\]

which we can solve for \( w_4 \),

\[
w_4 = (-2)w_1 + (-3)w_2.
\]

This equation says that whenever we encounter the vector \( w_4 \), we can replace it with a specific linear combination of the vectors \( w_1 \) and \( w_2 \). So using \( w_4 \) in the set \( T \), along with \( w_1 \) and \( w_2 \), is excessive. An example of what we mean here can be illustrated by the computation,

\[
5w_1 + (-4)w_2 + 6w_3 + (-3)w_4 = 5w_1 + (-4)w_2 + 6w_3 + (-3)((-2)w_1 + (-3)w_2) \\
= 5w_1 + (-4)w_2 + 6w_3 + (6w_1 + 9w_2) \\
= 11w_1 + 5w_2 + 6w_3.
\]

So what began as a linear combination of the vectors \( w_1, w_2, w_3, w_4 \) has been reduced to a linear combination of the vectors \( w_1, w_2, w_3 \). A careful proof using our definition of set equality [Definition SE \[694\]] would now allow us to conclude that this reduction is possible for any vector in \( W \), so

\[
W = \langle \{w_1, w_2, w_3\} \rangle.
\]

So the span of our set of vectors, \( W \), has not changed, but we have described it by the span of a set of three vectors, rather than four. Furthermore, we can achieve yet another, similar, reduction.

Check that the vector

\[
z_1 = \begin{bmatrix}
-3 \\
-1 \\
1 \\
0
\end{bmatrix}
\]

is a solution to the homogeneous system \( L(S(D, 0) \) (it is the vector \( z_1 \) provided by the description of the null space of the coefficient matrix \( D \) from Theorem SSNS \[129\]). Applying Theorem SLSLC \[100\], we can write the linear combination,

\[
(-3)w_1 + (-1)w_2 + 1w_3 = 0
\]

which we can solve for \( w_3 \),

\[
w_3 = 3w_1 + 1w_2.
\]

This equation says that whenever we encounter the vector \( w_3 \), we can replace it with a specific linear combination of the vectors \( w_1 \) and \( w_2 \). So, as before, the vector \( w_3 \) is not needed in the description of \( W \), provided we have \( w_1 \) and \( w_2 \) available. In particular, a careful proof would show that

\[
W = \langle \{w_1, w_2\} \rangle.
\]

So \( W \) began life as the span of a set of four vectors, and we have now shown (utilizing solutions to a homogeneous system) that \( W \) can also be described as the span of a set of just two vectors. Convince yourself that we cannot go any further. In other words, it is
not possible to dismiss either $w_1$ or $w_2$ in a similar fashion and winnow the set down to just one vector.

What was it about the original set of four vectors that allowed us to declare certain vectors as surplus? And just which vectors were we able to dismiss? And why did we have to stop once we had two vectors remaining? The answers to these questions motivate “linear independence,” our next section and next definition, and so are worth considering carefully now.

It is possible to have your computational device crank out the vector form of the solution set to a linear system of equations. See: Computation VFSS.MMA 681.

Subsection READ

Reading Questions

1. Let $S$ be the set of three vectors below.

$$S = \begin{Bmatrix}
\begin{pmatrix}
1 \\
2 \\
-1
\end{pmatrix},
\begin{pmatrix}
3 \\
-4 \\
2
\end{pmatrix},
\begin{pmatrix}
4 \\
-2 \\
1
\end{pmatrix}
\end{Bmatrix}$$

Let $W = \langle S \rangle$ be the span of $S$. Is the vector $\begin{pmatrix}
-1 \\
8 \\
-4
\end{pmatrix}$ in $W$? Give an explanation of the reason for your answer.

2. Use $S$ and $W$ from the previous question. Is the vector $\begin{pmatrix}
6 \\
5 \\
-1
\end{pmatrix}$ in $W$? Give an explanation of the reason for your answer.

3. For the matrix $A$ below, find a set $S$ so that $\langle S \rangle = N(A)$, where $N(A)$ is the null space of $A$. (See Theorem SSNS 129.)

$$A = \begin{bmatrix}
1 & 3 & 1 & 9 \\
2 & 1 & -3 & 8 \\
1 & 1 & -1 & 5
\end{bmatrix}$$
Subsection EXC
Exercises

C22  For each archetype that is a system of equations, consider the corresponding homogeneous system of equations. Write elements of the solution set to these homogeneous systems in vector form, as guaranteed by Theorem VFSL [107]. Then write the null space of the coefficient matrix of each system as the span of a set of vectors, as described in Theorem SSNS [129].

Archetype A [721]
Archetype B [726]
Archetype C [731]
Archetype D [735] / Archetype E [739]
Archetype F [743]
Archetype G [748] / Archetype H [752]
Archetype I [757]
Archetype J [762]

Contributed by Robert Beezer  Solution [139]

C23  Archetype K [767] and Archetype L [771] are defined as matrices. Use Theorem SSNS [129] directly to find a set $S$ so that $\langle S \rangle$ is the null space of the matrix. Do not make any reference to the associated homogeneous system of equations in your solution.

Contributed by Robert Beezer  Solution [139]

C40  Suppose that $S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\}$. Let $W = \langle S \rangle$ and let $x = \begin{bmatrix} 5 \\ 8 \\ -12 \\ -5 \end{bmatrix}$. Is $x \in W$? If so, provide an explicit linear combination that demonstrates this.

Contributed by Robert Beezer  Solution [139]

C41  Suppose that $S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\}$. Let $W = \langle S \rangle$ and let $y = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 5 \end{bmatrix}$. Is $y \in W$?

If so, provide an explicit linear combination that demonstrates this.

Contributed by Robert Beezer  Solution [139]

C42  Suppose $R = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \right\}$. Is $y = \begin{bmatrix} 1 \\ -1 \\ -8 \\ -4 \\ -3 \end{bmatrix}$ in $\langle R \rangle$?

Contributed by Robert Beezer  Solution [140]
C43  Suppose \( R = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 3 \\ -2 \end{bmatrix} \right\} \). Is \( z = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 3 \\ 1 \end{bmatrix} \) in \( \langle R \rangle \)?

Contributed by Robert Beezer  Solution [141]

C44  Suppose that \( S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\} \). Let \( W = \langle S \rangle \) and let \( y = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} \). Is \( x \in W \)? If so, provide an explicit linear combination that demonstrates this.

Contributed by Robert Beezer  Solution [141]

C45  Suppose that \( S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \). Let \( W = \langle S \rangle \) and let \( w = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \). Is \( x \in W \)? If so, provide an explicit linear combination that demonstrates this.

Contributed by Robert Beezer  Solution [141]

C50  Let \( A \) be the matrix below.
(a) Find a set \( S \) so that \( \mathcal{N}(A) = \langle S \rangle \).
(b) If \( z = \begin{bmatrix} 3 \\ -5 \\ 1 \\ 2 \end{bmatrix} \), then show directly that \( z \in \mathcal{N}(A) \).
(c) Write \( z \) as a linear combination of the vectors in \( S \).

\[
A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 2 & 1 & 3 \\ -1 & 0 & 1 & 1 \end{bmatrix}
\]

Contributed by Robert Beezer  Solution [142]

C60  For the matrix \( A \) below, find a set of vectors \( S \) so that the span of \( S \) equals the null space of \( A \), \( \langle S \rangle = \mathcal{N}(A) \).

\[
A = \begin{bmatrix} 1 & 1 & 6 & -8 \\ 1 & -2 & 0 & 1 \\ -2 & 1 & -6 & 7 \end{bmatrix}
\]

Contributed by Robert Beezer  Solution [143]

M20  In Example SCAD [132] we began with the four columns of the coefficient matrix of Archetype D [735], and used these columns in a span construction. Then we methodically argued that we could remove the last column, then the third column, and create the same set by just doing a span construction with the first two columns. We claimed we could not go any further, and had removed as many vectors as possible. Provide a convincing argument for why a third vector cannot be removed.

Contributed by Robert Beezer
M21  In the spirit of Example SCAD 132, begin with the four columns of the coefficient matrix of Archetype C 731, and use these columns in a span construction to build the set $S$. Argue that $S$ can be expressed as the span of just three of the columns of the coefficient matrix (saying exactly which three) and in the spirit of Exercise SS.M20 136 argue that no one of these three vectors can be removed and still have a span construction create $S$.

Contributed by Robert Beezer  Solution 144

T10  Suppose that $v_1, v_2 \in \mathbb{C}^m$. Prove that

$$\langle\{v_1, v_2\}\rangle = \langle\{v_1, v_2, 5v_1 + 3v_2\}\rangle$$

Contributed by Robert Beezer  Solution 144

T20  Suppose that $S$ is a set of vectors from $\mathbb{C}^m$. Prove that the zero vector, $0$, is an element of $\langle S \rangle$.

Contributed by Robert Beezer  Solution 144

T21  Suppose that $S$ is a set of vectors from $\mathbb{C}^m$ and $x, y \in \langle S \rangle$. Prove that $x + y \in \langle S \rangle$.

Contributed by Robert Beezer

T22  Suppose that $S$ is a set of vectors from $\mathbb{C}^m$, $\alpha \in \mathbb{C}$, and $x \in \langle S \rangle$. Prove that $\alpha x \in \langle S \rangle$.

Contributed by Robert Beezer
Subsection SOL

Solutions

C22 Contributed by Robert Beezer Statement [135]
The vector form of the solutions obtained in this manner will involve precisely the vectors described in Theorem SSNS [129] as providing the null space of the coefficient matrix of the system as a span. These vectors occur in each archetype in a description of the null space. Studying Example VFSAL [111] may be of some help.

C23 Contributed by Robert Beezer Statement [135]
Study Example NSDS [131] to understand the correct approach to this question. The solution for each is listed in the Archetypes (Appendix A [717]) themselves.

C40 Contributed by Robert Beezer Statement [135]
Rephrasing the question, we want to know if there are scalars $\alpha_1$ and $\alpha_2$ such that

$$\begin{align*}
\alpha_1 \begin{pmatrix} 2 \\ -1 \\ 3 \\ 4 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 2 \\ -2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 5 \\ 8 \\ -12 \\ -5 \end{pmatrix}
\end{align*}$$

Theorem SLSLC [100] allows us to rephrase the question again as a quest for solutions to the system of four equations in two unknowns with an augmented matrix given by

$$\begin{pmatrix} 2 & 3 & 5 \\ -1 & 2 & 8 \\ 3 & -2 & -12 \\ 4 & 1 & -5 \end{pmatrix}$$

This matrix row-reduces to

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From the form of this matrix, we can see that $\alpha_1 = -2$ and $\alpha_2 = 3$ is an affirmative answer to our question. More convincingly,

$$(-2) \begin{pmatrix} 2 \\ -1 \\ 3 \\ 4 \end{pmatrix} + (3) \begin{pmatrix} 3 \\ 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ -12 \\ -5 \end{pmatrix}$$

C41 Contributed by Robert Beezer Statement [135]
Rephrasing the question, we want to know if there are scalars $\alpha_1$ and $\alpha_2$ such that

$$\begin{align*}
\alpha_1 \begin{pmatrix} 2 \\ -1 \\ 3 \\ 4 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 2 \\ -2 \\ 1 \end{pmatrix} &= \begin{pmatrix} 5 \\ 1 \\ 3 \\ 5 \end{pmatrix}
\end{align*}$$
Theorem SLSLC [100] allows us to rephrase the question again as a quest for solutions to the system of four equations in two unknowns with an augmented matrix given by

\[
\begin{bmatrix}
  2 & 3 & 5 \\
  -1 & 2 & 1 \\
  3 & -2 & 3 \\
  4 & 1 & 5 \\
\end{bmatrix}
\]

This matrix row-reduces to

\[
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  0 & 0 & 0 \\
\end{bmatrix}
\]

With a leading 1 in the last column of this matrix (Theorem RCLS [54]) we can see that the system of equations has no solution, so there are no values for \( \alpha_1 \) and \( \alpha_2 \) that will allow us to conclude that \( y \) is in \( W \). So \( y \notin W \).

**C42** Contributed by Robert Beezer Statement [135]
Form a linear combination, with unknown scalars, of \( R \) that equals \( y \),

\[
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
\end{bmatrix} = \begin{bmatrix}
  2 \\
  -1 \\
  3 \\
  4 \\
\end{bmatrix} + \begin{bmatrix}
  1 \\
  1 \\
  2 \\
  -1 \\
\end{bmatrix} + \begin{bmatrix}
  3 \\
  -1 \\
  0 \\
  -2 \\
\end{bmatrix} = \begin{bmatrix}
  1 \\
  -1 \\
  -8 \\
  -4 \\
\end{bmatrix}
\]

We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in \( \langle R \rangle \). By Theorem SLSLC [100] any such values will also be solutions to the linear system represented by the augmented matrix,

\[
\begin{bmatrix}
  2 & 1 & 3 & 1 \\
  -1 & 1 & -1 & -1 \\
  3 & 2 & 0 & -8 \\
  4 & 2 & 3 & -4 \\
  0 & -1 & -2 & -3 \\
\end{bmatrix}
\]

Row-reducing the matrix yields,

\[
\begin{bmatrix}
  1 & 0 & 0 & -2 \\
  0 & 1 & 0 & -1 \\
  0 & 0 & 1 & 2 \\
  0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

From this we see that the system of equations is consistent (Theorem RCLS [54]), and has a unique solution. This solution will provide a linear combination of the vectors in \( R \) that equals \( y \). So \( y \in R \).

**C43** Contributed by Robert Beezer Statement [136]
Form a linear combination, with unknown scalars, of $R$ that equals $z$,

$$a_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ -1 \\ 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 3 \end{bmatrix}$$

We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in $\langle R \rangle$. By Theorem SLSLC [100], any such values will also be solutions to the linear system represented by the augmented matrix,

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & 1 & -1 & 1 \\ 3 & 2 & 0 & 5 \\ 4 & 2 & 3 & 3 \\ 0 & -1 & -2 & 1 \end{bmatrix}$$

Row-reducing the matrix yields,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

With a leading 1 in the last column, the system is inconsistent (Theorem RCLS [54]), so there are no scalars $a_1, a_2, a_3$ that will create a linear combination of the vectors in $R$ that equal $z$. So $z \notin R$.

C44 Contributed by Robert Beezer Statement [136]

Form a linear combination, with unknown scalars, of $S$ that equals $y$,

$$a_1 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + a_4 \begin{bmatrix} -6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$

We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in $\langle S \rangle$. By Theorem SLSLC [100], any such values will also be solutions to the linear system represented by the augmented matrix,

$$\begin{bmatrix} -1 & 3 & 1 & -6 & -5 \\ 2 & 1 & 5 & 5 & 3 \\ 1 & 2 & 4 & 1 & 0 \end{bmatrix}$$

Row-reducing the matrix yields,

$$\begin{bmatrix} 1 & 0 & 2 & 3 & 2 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
From this we see that the system of equations is consistent (Theorem RCLS [54]), and has a infinitely many solutions. Any solution will provide a linear combination of the vectors in $R$ that equals $y$. So $y \in S$, for example,

$$\begin{pmatrix} -10 \\ 2 \\ 1 \end{pmatrix} + (-2) \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + (3) \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} + (2) \begin{pmatrix} -6 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix}$$

**C45** Contributed by Robert Beezer  Statement [136]

Form a linear combination, with unknown scalars, of $S$ that equals $w$,

$$a_1 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} + a_4 \begin{pmatrix} -6 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

We want to know if there are values for the scalars that make the vector equation true since that is the definition of membership in $\langle S \rangle$. By Theorem SLSLC [100] any such values will also be solutions to the linear system represented by the augmented matrix,

$$\begin{bmatrix} -1 & 3 & 1 & -6 & 2 \\ 2 & 1 & 5 & 5 & 1 \\ 1 & 2 & 4 & 1 & 3 \end{bmatrix}$$

Row-reducing the matrix yields,

$$\begin{bmatrix} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

With a leading 1 in the last column, the system is inconsistent (Theorem RCLS [54]), so there are no scalars $a_1, a_2, a_3, a_4$ that will create a linear combination of the vectors in $S$ that equal $w$. So $w \not\in \langle S \rangle$.

**C50** Contributed by Robert Beezer  Statement [136]

(a) Theorem SSNS [129] provides formulas for a set $S$ with this property, but first we must row-reduce $A$:

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_3$ and $x_4$ would be the free variables in the homogeneous system $\mathcal{L}S(A, 0)$ and Theorem SSNS [129] provides the set $S = \{z_1, z_2\}$ where

$$z_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \quad z_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

(b) Simply employ the components of the vector $z$ as the variables in the homogeneous system $\mathcal{L}S(A, 0)$. The three equations of this system evaluate as follows,

$$2(3) + 3(-5) + 1(1) + 4(2) = 0$$
$1(3) + 2(-5) + 1(1) + 3(2) = 0$
$-1(3) + 0(-5) + 1(1) + 1(2) = 0$

Since each result is zero, $z$ qualifies for membership in $\mathcal{N}(A)$.

(c) By Theorem SSNS [129] we know this must be possible (that is the moral of this exercise). Find scalars $\alpha_1$ and $\alpha_2$ so that

$$\alpha_1 z_1 + \alpha_2 z_2 = \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 1 \\ 2 \end{bmatrix} = z$$

Theorem SLSLC [100] allows us to convert this question into a question about a system of four equations in two variables. The augmented matrix of this system row-reduces to

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A solution is $\alpha_1 = 1$ and $\alpha_2 = 2$. (Notice too that this solution is unique!)

C60 Contributed by Robert Beezer Statement [136]

Theorem SSNS [129] says that if we find the vector form of the solutions to the homogeneous system $LS(A, 0)$, then the fixed vectors (one per free variable) will have the desired property. Row-reduce $A$, viewing it as the augmented matrix of a homogeneous system with an invisible columns of zeros as the last column,

$$\begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Moving to the vector form of the solutions (Theorem VFSLS [107]), with free variables $x_3$ and $x_4$, solutions to the consistent system (it is homogeneous, Theorem HSC [65]) can be expressed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Then with $S$ given by

$$S = \left\{ \begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Theorem SSNS [129] guarantees that

$$\mathcal{N}(A) = \langle S \rangle = \left\{ \begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$
M21  Contributed by Robert Beezer  Statement [137]
If the columns of the coefficient matrix from Archetype C [731] are named \( u_1, u_2, u_3, u_4 \) then we can discover the equation

\[
(-2)u_1 + (-3)u_2 + u_3 + u_4 = 0
\]

by building a homogeneous system of equations and viewing a solution to the system as scalars in a linear combination via Theorem SLSLC [100]. This particular vector equation can be rearranged to read

\[
u_1 = (2)u_1 + (3)u_2 + (-1)u_3
\]

This can be interpreted to mean that \( u_4 \) is unnecessary in \( \langle\{u_1, u_2, u_3, u_4\}\rangle \), so that

\[
\langle\{u_1, u_2, u_3, u_4\}\rangle = \langle\{u_1, u_2, u_3\}\rangle
\]

If we try to repeat this process and find a linear combination of \( u_1, u_2, u_3 \) that equals the zero vector, we will fail. The required homogeneous system of equations (via Theorem SLSLC [100]) has only a trivial solution, which will not provide the kind of equation we need to remove one of the three remaining vectors.

T10  Contributed by Robert Beezer  Statement [137]
This is an equality of sets, so Definition SE [694] applies.
First show that \( X = \langle\{v_1, v_2\}\rangle \subseteq \langle\{v_1, v_2, 5v_1 + 3v_2\}\rangle = Y \).
Choose \( x \in X \). Then \( x = a_1v_1 + a_2v_2 \) for some scalars \( a_1 \) and \( a_2 \). Then,

\[
x = a_1v_1 + a_2v_2 = a_1v_1 + a_2v_2 + 0(5v_1 + 3v_2)
\]

which qualifies \( x \) for membership in \( Y \), as it is a linear combination of \( v_1, v_2, 5v_1 + 3v_2 \).
Now show the opposite inclusion, \( Y = \langle\{v_1, v_2, 5v_1 + 3v_2\}\rangle \subseteq \langle\{v_1, v_2\}\rangle = X \).
Choose \( y \in Y \). Then there are scalars \( a_1, a_2, a_3 \) such that

\[
y = a_1v_1 + a_2v_2 + a_3(5v_1 + 3v_2)
\]

Rearranging, we obtain,

\[
y = a_1v_1 + a_2v_2 + a_3(5v_1 + 3v_2) = a_1v_1 + a_2v_2 + 5a_3v_1 + 3a_3v_2
\]

\[
= a_1v_1 + 5a_3v_1 + a_2v_2 + 3a_3v_2
\]

\[
= (a_1 + 5a_3)v_1 + (a_2 + 3a_3)v_2
\]

This is an expression for \( y \) as a linear combination of \( v_1 \) and \( v_2 \), earning \( y \) membership in \( X \). Since \( X \) is a subset of \( Y \), and vice versa, we see that \( X = Y \), as desired.

T20  Contributed by Robert Beezer  Statement [137]
No matter what the elements of the set \( S \) are, we can choose the scalars in a linear combination to all be zero. Suppose that \( S = \{v_1, v_2, v_3, \ldots, v_p\} \). Then compute

\[
0v_1 + 0v_2 + 0v_3 + \cdots + 0v_p = 0 + 0 + 0 + \cdots + 0 = 0
\]

But what if we choose \( S \) to be the empty set? The convention is that the empty sum in Definition SSCV [123] evaluates to “zero,” in this case this is the zero vector.
Section LI
Linear Independence

Subsection LISV
Linearly Independent Sets of Vectors

Theorem SLSLC [100] tells us that a solution to a homogeneous system of equations is a linear combination of the columns of the coefficient matrix that equals the zero vector. We used just this situation to our advantage (twice!) in Example SCAD [132] where we reduced the set of vectors used in a span construction from four down to two, by declaring certain vectors as surplus. The next two definitions will allow us to formalize this situation.

Definition RLDCV
Relation of Linear Dependence for Column Vectors

Given a set of vectors $S = \{u_1, u_2, u_3, \ldots, u_n\}$, a true statement of the form

$$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_n u_n = 0$$

is a relation of linear dependence on $S$. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \leq i \leq n$, then we say it is the trivial relation of linear dependence on $S$.

Definition LICV
Linear Independence of Column Vectors

The set of vectors $S = \{u_1, u_2, u_3, \ldots, u_n\}$ is linearly dependent if there is a relation of linear dependence on $S$ that is not trivial. In the case where the only relation of linear dependence on $S$ is the trivial one, then $S$ is a linearly independent set of vectors.

Example LDS
Linearly dependent set in $\mathbb{C}^5$

Consider the set of $n = 4$ vectors from $\mathbb{C}^5$,

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$
To determine linear independence we first form a relation of linear dependence,

\[
\alpha_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -1 \\ 5 \\ 2 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} -6 \\ 7 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}.
\]

We know that \(\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0\) is a solution to this equation, but that is of no interest whatsoever. That is always the case, no matter what four vectors we might have chosen. We are curious to know if there are other, nontrivial, solutions. Theorem SLSLC tells us that we can find such solutions as solutions to the homogeneous system \(\mathcal{L}S(A, \mathbf{0})\) where the coefficient matrix has these four vectors as columns,

\[
A = \begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix}.
\]

Row-reducing this coefficient matrix yields,

\[
\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

We could solve this homogeneous system completely, but for this example all we need is one nontrivial solution. Setting the lone free variable to any nonzero value, such as \(x_4 = 1\), yields the nontrivial solution

\[
x = \begin{bmatrix} 2 \\ -4 \\ 3 \\ 1 \end{bmatrix}.
\]

completing our application of Theorem SLSLC, we have

\[
2 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} + (-4) \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -6 \\ 7 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}.
\]

This is a relation of linear dependence on \(S\) that is not trivial, so we conclude that \(S\) is linearly dependent.
Consider the set of \( n = 4 \) vectors from \( \mathbb{C}^5 \),

\[
T = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 7 \\ 1 \end{bmatrix} \right\}.
\]

To determine linear independence we first form a relation of linear dependence,

\[
\alpha_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \\ 6 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 5 \\ 2 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} -6 \\ 7 \\ 1 \end{bmatrix} = 0.
\]

We know that \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \) is a solution to this equation, but that is of no interest whatsoever. That is always the case, no matter what four vectors we might have chosen. We are curious to know if there are other, nontrivial, solutions. Theorem SLSLC \([100]\) tells us that we can find such solutions as solution to the homogeneous system \( LS(B, 0) \) where the coefficient matrix has these four vectors as columns,

\[
B = \begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}.
\]

Row-reducing this coefficient matrix yields,

\[
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

From the form of this matrix, we see that there are no free variables, so the solution is unique, and because the system is homogeneous, this unique solution is the trivial solution. So we now know that there is but one way to combine the four vectors of \( T \) into a relation of linear dependence, and that one way is the easy and obvious way. In this situation we say that the set, \( T \), is linearly independent.

\[\text{Example LDS \([145]\) and Example LIS \([147]\) relied on solving a homogeneous system of equations to determine linear independence. We can codify this process in a time-saving theorem.}\]

**Theorem LIVHS**

**Linearly Independent Vectors and Homogeneous Systems**

Suppose that \( A \) is an \( m \times n \) matrix and \( S = \{A_1, A_2, A_3, \ldots, A_n\} \) is the set of vectors in \( \mathbb{C}^m \) that are the columns of \( A \). Then \( S \) is a linearly independent set if and only if the homogeneous system \( LS(A, 0) \) has a unique solution.
Suppose that $\mathcal{L}(A, 0)$ has a unique solution. Since it is a homogeneous system, this solution must be the trivial solution $x = 0$. By Theorem SLSLC, this means that the only relation of linear dependence on $S$ is the trivial one. So $S$ is linearly independent.

($\Rightarrow$) We will prove the contrapositive. Suppose that $\mathcal{L}(A, 0)$ does not have a unique solution. Since it is a homogeneous system, it is consistent (Theorem HSC), and so must have infinitely many solutions (Theorem PSSLS). One of these infinitely many solutions must be nontrivial (in fact, almost all of them are), so choose one. By Theorem SLSLC this nontrivial solution will give a nontrivial relation of linear dependence on $S$, so we can conclude that $S$ is a linearly dependent set.

Since Theorem LIVHS is an equivalence, we can use it to determine the linear independence or dependence of any set of column vectors, just by creating a corresponding matrix and analyzing the row-reduced form. Let's illustrate this with two more examples.

**Example LIHS**

**Linearly independent, homogeneous system**

Is the set of vectors

\[
S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -4 \\ 5 \\ 1 \end{bmatrix} \right\}
\]

linearly independent or linearly dependent?

Theorem LIVHS suggests we study the matrix whose columns are the vectors in $S$,

\[
A = \begin{bmatrix} 2 & 6 & 4 \\ -1 & 2 & 3 \\ 3 & -1 & -4 \\ 4 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix}
\]

Specifically, we are interested in the size of the solution set for the homogeneous system $\mathcal{L}(A, 0)$. Row-reducing $A$, we obtain

\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

Now, $r = 3$, so there are $n - r = 3 - 3 = 0$ free variables and we see that $\mathcal{L}(A, 0)$ has a unique solution (Theorem HSC, Theorem FVCS). By Theorem LIVHS, the set $S$ is linearly independent.

**Example LDHS**

**Linearly dependent, homogeneous system**


Is the set of vectors
\[ S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -4 \\ -1 \\ 2 \end{bmatrix} \right\} \]
linearly independent or linearly dependent?

**Theorem LIVHS** suggests we study the matrix whose columns are the vectors in \( S \),
\[ A = \begin{bmatrix} 2 & 6 & 4 \\ -1 & 2 & 3 \\ 3 & -1 & -4 \\ 4 & 3 & -1 \\ 2 & 4 & 2 \end{bmatrix} \]

Specifically, we are interested in the size of the solution set for the homogeneous system \( \mathcal{L}S(A, 0) \). Row-reducing \( A \), we obtain
\[ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

Now, \( r = 2 \), so there are \( n - r = 3 - 2 = 1 \) free variables and we see that \( \mathcal{L}S(A, 0) \) has infinitely many solutions (Theorem HSC, Theorem FVCS). By **Theorem LIVHS**, the set \( S \) is linearly dependent.

As an equivalence, **Theorem LIVHS** gives us a straightforward way to determine if a set of vectors is linearly independent or dependent.

Review **Example LIHS** and **Example LDHS**. They are very similar, differing only in the last two slots of the third vector. This resulted in slightly different matrices when row-reduced, and slightly different values of \( r \), the number of nonzero rows. Notice, too, that we are less interested in the actual solution set, and more interested in its form or size. These observations allow us to make a slight improvement in **Theorem LIVHS**.

**Theorem LIVRN**

**Linearly Independent Vectors, \( r \) and \( n \)**

Suppose that \( A \) is an \( m \times n \) matrix and \( S = \{ A_1, A_2, A_3, \ldots, A_n \} \) is the set of vectors in \( \mathbb{C}^m \) that are the columns of \( A \). Let \( B \) be a matrix in reduced row-echelon form that is row-equivalent to \( A \) and let \( r \) denote the number of non-zero rows in \( B \). Then \( S \) is linearly independent if and only if \( n = r \).

**Proof** **Theorem LIVHS** says the linear independence of \( S \) is equivalent to the homogeneous linear system \( \mathcal{L}S(A, 0) \) having a unique solution. Since \( \mathcal{L}S(A, 0) \) is consistent (Theorem HSC), we can apply **Theorem CSRN** to see that the solution is unique exactly when \( n = r \).

So now here’s an example of the most straightforward way to determine if a set of column vectors in linearly independent or linearly dependent. While this method can be quick and easy, don’t forget the logical progression from the definition of linear independence through homogeneous system of equations which makes it possible.
Example LDRN
Linearly dependent, \( r < n \)

Is the set of vectors

\[
S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ -6 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 4 \\ 3 \\ 2 \end{bmatrix} \right\}
\]

linearly independent or linearly dependent? Theorem LIVHS \[147\] suggests we place these vectors into a matrix as columns and analyze the row-reduced version of the matrix,

\[
\begin{bmatrix}
2 & 9 & 1 & -3 & 6 \\
-1 & -6 & 1 & 1 & -2 \\
3 & -2 & 1 & 4 & 1 \\
1 & 3 & 0 & 2 & 4 \\
0 & 2 & 0 & 1 & 3 \\
3 & 1 & 1 & 2 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Now we need only compute that \( r = 4 < 5 = n \) to recognize, via Theorem LIVHS \[147\] that \( S \) is a linearly dependent set. Boom!

Example LLDS
Large linearly dependent set in \( \mathbb{C}^4 \)

Consider the set of \( n = 9 \) vectors from \( \mathbb{C}^4 \),

\[
R = \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \\ 6 \\ -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -3 \\ -1 \\ 6 \\ 2 \\ 9 \\ -2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \\ 0 \\ -3 \\ -2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 2 \\ -6 \\ 1 \\ 1 \\ -3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -5 \\ -1 \\ 2 \\ 4 \\ 1 \\ -3 \\ 1 \\ 3 \end{bmatrix} \right\}
\]

To employ Theorem LIVHS \[147\], we form a \( 4 \times 9 \) coefficient matrix, \( C \),

\[
C = \begin{bmatrix}
-1 & 7 & 1 & 0 & 5 & 2 & 3 & 1 & -6 \\
3 & 1 & 2 & 4 & -2 & 1 & 0 & 1 & -1 \\
1 & -3 & -1 & 2 & 4 & -6 & -3 & 5 & 1 \\
2 & 6 & -2 & 9 & 3 & 4 & 1 & 3 & 1
\end{bmatrix}
\]

To determine if the homogeneous system \( \mathcal{L}S(C, 0) \) has a unique solution or not, we would normally row-reduce this matrix. But in this particular example, we can do better. Theorem HMVEI \[67\] tells us that since the system is homogeneous with \( n = 9 \) variables in \( m = 4 \) equations, and \( n > m \), there must be infinitely many solutions. Since there is not a unique solution, Theorem LIVHS \[147\] says the set is linearly dependent.

The situation in Example LLDS \[150\] is slick enough to warrant formulating as a theorem.

Theorem MVSLD
More Vectors than Size implies Linear Dependence

Suppose that \( S = \{u_1, u_2, u_3, \ldots, u_n\} \) is the set of vectors in \( \mathbb{C}^m \), and that \( n > m \). Then \( S \) is a linearly dependent set.

\[\square\]
Subsection LI.LINM Linear Independence and Nonsingular Matrices

Proof Form the $m \times n$ coefficient matrix $A$ that has the column vectors $u_i$, $1 \leq i \leq n$ as its columns. Consider the homogeneous system $\mathcal{L}(A, 0)$. By Theorem HMVEI [67] this system has infinitely many solutions. Since the system does not have a unique solution, Theorem LIVHS [147] says the columns of $A$ form a linearly dependent set, which is the desired conclusion.

Subsection LINM Linear Independence and Nonsingular Matrices

We will now specialize to sets of $n$ vectors from $\mathbb{C}^n$. This will put Theorem MVSLD [150] off-limits, while Theorem LIVHS [147] will involve square matrices. Let's begin by contrasting Archetype A [721] and Archetype B [726].

Example LDCAA Linearly dependent columns in Archetype A

Archetype A [721] is a system of linear equations with coefficient matrix,

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Do the columns of this matrix form a linearly independent or dependent set? By Example S [76] we know that $A$ is singular. According to the definition of nonsingular matrices, Definition NM [75], the homogeneous system $\mathcal{L}(A, 0)$ has infinitely many solutions. So by Theorem LIVHS [147], the columns of $A$ form a linearly dependent set.

Example LICAB Linearly independent columns in Archetype B

Archetype B [726] is a system of linear equations with coefficient matrix,

$$B = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}.$$

Do the columns of this matrix form a linearly independent or dependent set? By Example NM [76] we know that $B$ is nonsingular. According to the definition of nonsingular matrices, Definition NM [75], the homogeneous system $\mathcal{L}(A, 0)$ has a unique solution. So by Theorem LIVHS [147], the columns of $B$ form a linearly independent set.

That Archetype A [721] and Archetype B [726] have opposite properties for the columns of their coefficient matrices is no accident. Here's the theorem, and then we will update our equivalences for nonsingular matrices, Theorem NME1 [81].

Theorem NMLIC Nonsingular Matrices have Linearly Independent Columns

Suppose that $A$ is a square matrix. Then $A$ is nonsingular if and only if the columns of $A$ form a linearly independent set.

Proof This is a proof where we can chain together equivalences, rather than proving the two halves separately.

$A$ nonsingular $\iff \mathcal{L}(A, 0)$ has a unique solution Definition NM [75]
Here’s an update to Theorem NME1 [81].

**Theorem NME2**

**Nonsingular Matrix Equivalences, Round 2**

Suppose that $A$ is a square matrix. The following are equivalent.

1. $A$ is nonsingular.
2. $A$ row-reduces to the identity matrix.
3. The null space of $A$ contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
4. The linear system $\mathcal{L}(A, b)$ has a unique solution for every possible choice of $b$.
5. The columns of $A$ form a linearly independent set.

**Proof** Theorem NMLIC [151] is yet another equivalence for a nonsingular matrix, so we can add it to the list in Theorem NME1 [81].

**Subsection NSSLI**

**Null Spaces, Spans, Linear Independence**

In Subsection SS.SSNS [129] we proved Theorem SSNS [129] which provided $n - r$ vectors that could be used with the span construction to build the entire null space of a matrix. As we have hinted in Example SCAD [132], and as we will see again going forward, linearly dependent sets carry redundant vectors with them when used in building a set as a span. Our aim now is to show that the vectors provided by Theorem SSNS [129] form a linearly independent set, so in one sense they are as efficient as possible a way to describe the null space. Notice that the vectors $z_j$, $1 \leq j \leq n - r$ first appear in the vector form of solutions to arbitrary linear systems (Theorem VFSLS [107]). The exact same vectors appear again in the span construction in the conclusion of Theorem SSNS [129]. Since this second theorem specializes to homogeneous systems the only real difference is that the vector $c$ in Theorem VFSLS [107] is the zero vector for a homogeneous system. Finally, Theorem BNS [154] will now show that these same vectors are a linearly independent set. We’ll set the stage for the proof of this theorem with a moderately large example. Study the example carefully, as it will make it easier to understand the proof.

**Example LINSB**

**Linearly independence of null space basis**

Suppose that we are interested in the null space of the a $3 \times 7$ matrix, $A$, which row-reduces to

$$B = \begin{bmatrix} 1 & 0 & -2 & 4 & 0 & 3 & 9 \\ 0 & 1 & 5 & 6 & 0 & 7 & 1 \\ 0 & 0 & 0 & 1 & 8 & -5 \end{bmatrix}$$
The set \( F = \{3, 4, 6, 7\} \) is the set of indices for our four free variables that would be used in a description of the solution set for the homogeneous system \( \text{homosystem}_A \). Applying Theorem SSNS we can begin to construct a set of four vectors whose span is the null space of \( A \), a set of vectors we will reference as \( T \).

\[
\mathcal{N}(A) = \langle T \rangle = \langle \{z_1, z_2, z_3, z_4\} \rangle = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\}
\]

So far, we have constructed as much of these individual vectors as we can, based just on the knowledge of the contents of the set \( F \). This has allowed us to determine the entries in slots 3, 4, 6 and 7, while we have left slots 1, 2 and 5 blank. Without doing any more, let’s ask if \( T \) is linearly independent? Begin with a relation of linear dependence on \( T \), and see what we can learn about the scalars,

\[
0 = \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 + \alpha_4 z_4
\]

Applying Definition CVE to the two ends of this chain of equalities, we see that \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \). So the only relation of linear dependence on the set \( T \) is a trivial one. By Definition LICV the set \( T \) is linearly independent. The important feature of this example is how the “pattern of zeros and ones” in the four vectors led to the conclusion of linear independence.

The proof of Theorem BNS is really quite straightforward, and relies on the “pattern of zeros and ones” that arise in the vectors \( z_i \), \( 1 \leq i \leq n - r \) in the entries that correspond to the free variables. Play along with Example LINSB as you study the proof. Also, take a look at Example VFSAD, Example VFSAI and Example VFSAL, especially at the conclusion of Step 2 (temporarily ignore the construction of the constant vector, \( c \)). This proof is also a good first example of how to prove a conclusion that states a set is linearly independent.

**Theorem BNS**
Basis for Null Spaces

Suppose that \( A \) is an \( m \times n \) matrix, and \( B \) is a row-equivalent matrix in reduced row-echelon form with \( r \) nonzero rows. Let \( D = \{d_1, d_2, d_3, \ldots, d_r\} \) and \( F = \{f_1, f_2, f_3, \ldots, f_{n-r}\} \) be the sets of column indices where \( B \) does and does not (respectively) have leading 1's. Construct the \( n-r \) vectors \( z_j, 1 \leq j \leq n-r \) of size \( n \) as

\[
[z_j]_i = \begin{cases} 
1 & \text{if } i \in F, i = f_j \\
0 & \text{if } i \in F, i \neq f_j \\
-B_{k,f_j} & \text{if } i \in D, i = d_k
\end{cases}
\]

Define the set \( S = \{z_1, z_2, z_3, \ldots, z_{n-r}\} \). Then

1. \( \mathcal{N}(A) = \langle S \rangle \).
2. \( S \) is a linearly independent set.

\[ \square \]

**Proof** Notice first that the vectors \( z_j, 1 \leq j \leq n-r \) are exactly the same as the \( n-r \) vectors defined in **Theorem SSNS** [129]. Also, the hypotheses of **Theorem SSNS** [129] are the same as the hypotheses of the theorem we are currently proving. So it is then simply the conclusion of **Theorem SSNS** [129] that tells us that \( \mathcal{N}(A) = \langle S \rangle \). That was the easy half, but the second part is not much harder. What is new here is the claim that \( S \) is a linearly independent set.

To prove the linear independence of a set, we need to start with a relation of linear dependence and somehow conclude that the scalars involved must all be zero, i.e. that the relation of linear dependence only happens in the trivial fashion. So to establish the linear independence of \( S \), we start with

\[
\alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 + \cdots + \alpha_{n-r} z_{n-r} = 0.
\]

For each \( j, 1 \leq j \leq n-r \), consider the equality of the individual entries of the vectors on both sides of this equality in position \( f_j \),

\[
0 = [0]_{f_j} = [\alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3 + \cdots + \alpha_{n-r} z_{n-r}]_{f_j} = \begin{bmatrix} \alpha_1 z_1 \end{bmatrix}_{f_j} + \begin{bmatrix} \alpha_2 z_2 \end{bmatrix}_{f_j} + \begin{bmatrix} \alpha_3 z_3 \end{bmatrix}_{f_j} + \cdots + \begin{bmatrix} \alpha_{n-r} z_{n-r} \end{bmatrix}_{f_j} \quad \text{Definition CVE} \ 88
\]

\[
= \alpha_1 [z_1]_{f_j} + \alpha_2 [z_2]_{f_j} + \alpha_3 [z_3]_{f_j} + \cdots + \alpha_{n-r} [z_{n-r}]_{f_j} = \begin{bmatrix} \alpha_j z_j \end{bmatrix}_{f_j} + \begin{bmatrix} \alpha_{j+1} z_{j+1} \end{bmatrix}_{f_j} + \cdots + \begin{bmatrix} \alpha_{n-r} z_{n-r} \end{bmatrix}_{f_j} \quad \text{Definition CVA} \ 89
\]

\[
= \alpha_j z_j
\quad \text{Definition CVSM} \ 89
\]

So for all \( j, 1 \leq j \leq n-r \), we have \( \alpha_j = 0 \), which is the conclusion that tells us that the only relation of linear dependence on \( S = \{z_1, z_2, z_3, \ldots, z_{n-r}\} \) is the trivial one, hence the set is linearly independent, as desired. \[ \square \]
Example NSLIL
Null space spanned by linearly independent set, Archetype L

In Example VFSAL [111] we previewed Theorem SSNS [129] by finding a set of two vectors such that their span was the null space for the matrix in Archetype L [771]. Writing the matrix as \( L \), we have

\[
\mathcal{N}(L) = \left\langle \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\} \right\rangle.
\]

Solving the homogeneous system \( LS(L, 0) \) resulted in recognizing \( x_4 \) and \( x_5 \) as the free variables. So look in entries 4 and 5 of the two vectors above and notice the pattern of zeros and ones that provides the linear independence of the set.

Subsection READ
Reading Questions

1. Let \( S \) be the set of three vectors below.

\[
S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \right\}
\]

Is \( S \) linearly independent or linearly dependent? Explain why.

2. Let \( S \) be the set of three vectors below.

\[
S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -4 \end{bmatrix} \right\}
\]

Is \( S \) linearly independent or linearly dependent? Explain why.

3. Based on your answer to the previous question, is the matrix below singular or nonsingular? Explain.

\[
\begin{bmatrix}
1 & 3 & 4 \\
-1 & 2 & 3 \\
0 & 2 & -4
\end{bmatrix}
\]
Subsection EXC
Exercises

Determine if the sets of vectors in Exercises C20–C25 are linearly independent or linearly dependent.

C20 \[
\begin{bmatrix}
1 \\
-2 \\
1
\end{bmatrix},
\begin{bmatrix}
2 \\
-1 \\
3
\end{bmatrix},
\begin{bmatrix}
1 \\
5 \\
0
\end{bmatrix}
\]
Contributed by Robert Beezer Solution 161

C21 \[
\begin{bmatrix}
-1 \\
2 \\
4 \\
2
\end{bmatrix},
\begin{bmatrix}
3 \\
3 \\
-1 \\
3
\end{bmatrix},
\begin{bmatrix}
7 \\
3 \\
-6 \\
4
\end{bmatrix}
\]
Contributed by Robert Beezer Solution 161

C22 \[
\begin{bmatrix}
1 \\
5 \\
1
\end{bmatrix},
\begin{bmatrix}
6 \\
-1 \\
2
\end{bmatrix},
\begin{bmatrix}
9 \\
-3 \\
8
\end{bmatrix},
\begin{bmatrix}
2 \\
8 \\
-1
\end{bmatrix},
\begin{bmatrix}
3 \\
-1 \\
0
\end{bmatrix}
\]
Contributed by Robert Beezer Solution 161

C23 \[
\begin{bmatrix}
1 \\
-2 \\
2 \\
5 \\
3
\end{bmatrix},
\begin{bmatrix}
3 \\
3 \\
1 \\
2 \\
-4
\end{bmatrix},
\begin{bmatrix}
2 \\
1 \\
-1 \\
1 \\
1
\end{bmatrix}
\]
Contributed by Robert Beezer Solution 161

C24 \[
\begin{bmatrix}
1 \\
2 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
3 \\
2 \\
2 \\
3
\end{bmatrix},
\begin{bmatrix}
4 \\
4 \\
2 \\
3
\end{bmatrix}
\]
Contributed by Robert Beezer Solution 161

C25 \[
\begin{bmatrix}
2 \\
1 \\
3 \\
-1
\end{bmatrix},
\begin{bmatrix}
4 \\
-2 \\
1 \\
2
\end{bmatrix},
\begin{bmatrix}
10 \\
-7 \\
10 \\
4
\end{bmatrix}
\]
Contributed by Robert Beezer Solution 162

C30 For the matrix \(B\) below, find a set \(S\) that is linearly independent and spans the null space of \(B\), that is, \(N(B) = \langle S \rangle\).

\[
B = \begin{bmatrix}
-3 & 1 & -2 & 7 \\
-1 & 2 & 1 & 4 \\
1 & 1 & 2 & -1
\end{bmatrix}
\]
Contributed by Robert Beezer Solution 162
C31 For the matrix $A$ below, find a linearly independent set $S$ so that the null space of $A$ is spanned by $S$, that is, $\mathcal{N}(A) = \langle S \rangle$.

$$A = \begin{bmatrix}
-1 & -2 & 2 & 1 & 5 \\
1 & 2 & 1 & 1 & 5 \\
3 & 6 & 1 & 2 & 7 \\
2 & 4 & 0 & 1 & 2
\end{bmatrix}$$

Contributed by Robert Beezer  
Solution 163

C50 Consider each archetype that is a system of equations and consider the solutions listed for the homogeneous version of the archetype. (If only the trivial solution is listed, then assume this is the only solution to the system.) From the solution set, determine if the columns of the coefficient matrix form a linearly independent or linearly dependent set. In the case of a linearly dependent set, use one of the sample solutions to provide a nontrivial relation of linear dependence on the set of columns of the coefficient matrix (Definition RLD 345). Indicate when Theorem MVSLD 150 applies and connect this with the number of variables and equations in the system of equations.

Archetype A 721  
Archetype B 726  
Archetype C 731  
Archetype D 735  
Archetype E 739  
Archetype F 743  
Archetype G 748  
Archetype H 752  
Archetype I 757  
Archetype J 762  

Contributed by Robert Beezer

C51 For each archetype that is a system of equations consider the homogeneous version. Write elements of the solution set in vector form (Theorem VFSLS 107) and from this extract the vectors $z_j$ described in Theorem BNS 154. These vectors are used in a span construction to describe the null space of the coefficient matrix for each archetype. What does it mean when we write a null space as $\langle \{ \} \rangle$?

Archetype A 721  
Archetype B 726  
Archetype C 731  
Archetype D 735  
Archetype E 739  
Archetype F 743  
Archetype G 748  
Archetype H 752  
Archetype I 757  
Archetype J 762  

Contributed by Robert Beezer

C52 For each archetype that is a system of equations consider the homogeneous version. Sample solutions are given and a linearly independent spanning set is given for the null space of the coefficient matrix. Write each of the sample solutions individually as a linear combination of the vectors in the spanning set for the null space of the coefficient matrix.
C60 For the matrix $A$ below, find a set of vectors $S$ so that (1) $S$ is linearly independent, and (2) the span of $S$ equals the null space of $A$, $\langle S \rangle = \mathcal{N}(A)$. (See Exercise SS.C60 136.)

$$A = \begin{bmatrix} 1 & 1 & 6 & -8 \\ 1 & -2 & 0 & 1 \\ -2 & 1 & -6 & 7 \end{bmatrix}$$

Contributed by Robert Beezer Solution 163

M50 Consider the set of vectors from $\mathbb{C}^3$, $W$, given below. Find a set $T$ that contains three vectors from $W$ and such that $W = \langle T \rangle$.

$$W = \langle \{v_1, v_2, v_3, v_4, v_5\} \rangle = \langle \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} , \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ -3 \end{bmatrix} \right\} \rangle$$

Contributed by Robert Beezer Solution 164

T10 Prove that if a set of vectors contains the zero vector, then the set is linearly dependent. (Ed. “The zero vector is death to linearly independent sets.”)

Contributed by Martin Jackson

T15 Suppose that $\{v_1, v_2, v_3, \ldots, v_n\}$ is a set of vectors. Prove that

$$\{v_1 - v_2, v_2 - v_3, v_3 - v_4, \ldots, v_n - v_1\}$$

is a linearly dependent set.

Contributed by Robert Beezer Solution 164

T20 Suppose that $\{v_1, v_2, v_3, v_4\}$ is a linearly independent set in $\mathbb{C}^{35}$. Prove that

$$\{v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4\}$$

is a linearly independent set.

Contributed by Robert Beezer Solution 165

T50 Suppose that $A$ is matrix with linearly independent columns and the linear system $LS(A, b)$ is consistent. Show that this system has a unique solution. (Notice that we are not requiring $A$ to be square.)

Contributed by Robert Beezer Solution 165
Subsection SOL
Solutions

C20 Contributed by Robert Beezer Statement [157]
With three vectors from $\mathbb{C}^3$, we can form a square matrix by making these three vectors the columns of a matrix. We do so, and row-reduce to obtain,

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

the $3 \times 3$ identity matrix. So by Theorem NME2 [152] the original matrix is nonsingular and its columns are therefore a linearly independent set.

C21 Contributed by Robert Beezer Statement [157]
Theorem LIVRN [149] says we can answer this question by putting these vectors into a matrix as columns and row-reducing. Doing this we obtain,

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

With $n = 3$ (3 vectors, 3 columns) and $r = 3$ (3 leading 1’s) we have $n = r$ and the theorem says the vectors are linearly independent.

C22 Contributed by Robert Beezer Statement [157]
Five vectors from $\mathbb{C}^3$. Theorem MVSLD [150] says the set is linearly dependent. Boom.

C23 Contributed by Robert Beezer Statement [157]
Theorem LIVRN [149] suggests we analyze a matrix whose columns are the vectors of $S$,

\[
A = \begin{bmatrix}
1 & 3 & 2 & 1 \\
-2 & 3 & 1 & 0 \\
2 & 1 & 2 & 1 \\
5 & 2 & -1 & 2 \\
3 & -4 & 1 & 2
\end{bmatrix}
\]

Row-reducing the matrix $A$ yields,

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

We see that $r = 4 = n$, where $r$ is the number of nonzero rows and $n$ is the number of columns. By Theorem LIVRN [149], the set $S$ is linearly independent.

C24 Contributed by Robert Beezer Statement [157]
Theorem LIVRN [149] suggests we analyze a matrix whose columns are the vectors from
the set,

\[
A = \begin{bmatrix}
1 & 3 & 4 & -1 \\
2 & 2 & 4 & 2 \\
-1 & -1 & -2 & -1 \\
0 & 2 & 2 & -2 \\
1 & 2 & 3 & 0
\end{bmatrix}
\]

Row-reducing the matrix \( A \) yields,

\[
\begin{bmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

We see that \( r = 2 \neq 4 = n \), where \( r \) is the number of nonzero rows and \( n \) is the number of columns. By [Theorem LIVRN](#), the set \( S \) is linearly dependent.

C25 Contributed by Robert Beezer Statement [157]

Theorem LIVRN [149] suggests we analyze a matrix whose columns are the vectors from the set,

\[
A = \begin{bmatrix}
2 & 4 & 10 \\
1 & -2 & -7 \\
3 & 1 & 0 \\
-1 & 3 & 10 \\
2 & 2 & 4
\end{bmatrix}
\]

Row-reducing the matrix \( A \) yields,

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 3 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

We see that \( r = 2 \neq 3 = n \), where \( r \) is the number of nonzero rows and \( n \) is the number of columns. By [Theorem LIVRN](#), the set \( S \) is linearly dependent.

C30 Contributed by Robert Beezer Statement [157]

The requested set is described by [Theorem BNS](#). It is easiest to find by using the procedure of [Example VFSAL](#). Begin by row-reducing the matrix, viewing it as the coefficient matrix of a homogeneous system of equations. We obtain,

\[
\begin{bmatrix}
1 & 0 & 1 & -2 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Now build the vector form of the solutions to this homogeneous system ([Theorem VF-SLS](#)). The free variables are \( x_3 \) and \( x_4 \), corresponding to the columns without leading 1’s,

\[
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4
\end{bmatrix} = x_3 \begin{bmatrix}
-1 \\ -1 \\ 1 \\ 0
\end{bmatrix} + x_4 \begin{bmatrix}
2 \\ -1 \\ 0 \\ 1
\end{bmatrix}
\]
The desired set $S$ is simply the constant vectors in this expression, and these are the vectors $z_1$ and $z_2$ described by Theorem BNS 154.

$$S = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

To apply Theorem BNS 154 or Theorem VFSLS 107 we first row-reduce the matrix, resulting in

$$B = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So we see that $n - r = 5 - 3 = 2$ and $F = \{2, 5\}$, so the vector form of a generic solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ -6 \\ 4 \\ 1 \end{bmatrix}$$

So we have

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -6 \\ 4 \\ 1 \end{bmatrix} \right\}$$

Theorem BNS 154 says that if we find the vector form of the solutions to the homogeneous system $L_S(A, 0)$, then the fixed vectors (one per free variable) will have the desired properties. Row-reduce $A$, viewing it as the augmented matrix of a homogeneous system with an invisible columns of zeros as the last column,

$$\begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Moving to the vector form of the solutions (Theorem VFSLS 107), with free variables $x_3$ and $x_4$, solutions to the consistent system (it is homogeneous, Theorem HSC 65) can be expressed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$
Then with $S$ given by

$$S = \begin{bmatrix} -4 & -2 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 3 & 0 & 1 \end{bmatrix}$$

Theorem BNS guarantees the set has the desired properties.

M50 Contributed by Robert Beezer Statement

We want to first find some relations of linear dependence on \{v_1, v_2, v_3, v_4, v_5\} that will allow us to “kick out” some vectors, in the spirit of Example SCAD. To find relations of linear dependence, we formulate a matrix $A$ whose columns are $v_1, v_2, v_3, v_4, v_5$. Then we consider the homogeneous system of equations $LS(A, 0)$ by row-reducing its coefficient matrix (remember that if we formulated the augmented matrix we would just add a column of zeros). After row-reducing, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

From this we that solutions can be obtained employing the free variables $x_4$ and $x_5$. With appropriate choices we will be able to conclude that vectors $v_4$ and $v_5$ are unnecessary for creating $W$ via a span. By Theorem SLSLC the choice of free variables below lead to solutions and linear combinations, which are then rearranged.

\[ x_4 = 1, x_5 = 0 \Rightarrow (-2)v_1 + (-1)v_2 + (0)v_3 + (1)v_4 + (0)v_5 = 0 \Rightarrow v_4 = 2v_1 + v_2 \]
\[ x_4 = 0, x_5 = 1 \Rightarrow (1)v_1 + (2)v_2 + (0)v_3 + (0)v_4 + (1)v_5 = 0 \Rightarrow v_5 = -v_1 - 2v_2 \]

Since $v_4$ and $v_5$ can be expressed as linear combinations of $v_1$ and $v_2$ we can say that $v_4$ and $v_5$ are not needed for the linear combinations used to build $W$ (a claim that we could establish carefully with a pair of set equality arguments). Thus

\[ W = \langle \{v_1, v_2, v_3\} \rangle = \left\langle \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\} \right\rangle \]

That the $\{v_1, v_2, v_3\}$ is linearly independent set can be established quickly with Theorem LIVRN. There are other answers to this question, but notice that any nontrivial linear combination of $v_1, v_2, v_3, v_4, v_5$ will have a zero coefficient on $v_3$, so this vector can never be eliminated from the set used to build the span.

T15 Contributed by Robert Beezer Statement

Consider the following linear combination

\[ 1(v_1 - v_2) + 1(v_2 - v_3) + 1(v_3 - v_4) + \cdots + 1(v_n - v_1) = v_1 - v_2 + v_2 - v_3 + v_3 - v_4 + \cdots + v_n - v_1 = v_1 + 0 + 0 + \cdots + 0 - v_1 = 0 \]
This is a nontrivial relation of linear dependence (Definition RLDCV [145]), so by Definition LICV [145] the set is linearly dependent.

T20 Contributed by Robert Beezer Statement [159]

Our hypothesis and our conclusion use the term linear independence, so it will get a workout. To establish linear independence, we begin with the definition (Definition LICV [145]) and write a relation of linear dependence (Definition RLDCV [145]),

\[ \alpha_1 (v_1) + \alpha_2 (v_1 + v_2) + \alpha_3 (v_1 + v_2 + v_3) + \alpha_4 (v_1 + v_2 + v_3 + v_4) = 0 \]

Using the distributive and commutative properties of vector addition and scalar multiplication (Theorem VSPCV [91]) this equation can be rearranged as

\[ (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) v_1 + (\alpha_2 + \alpha_3 + \alpha_4) v_2 + (\alpha_3 + \alpha_4) v_3 + (\alpha_4) v_4 = 0 \]

However, this is a relation of linear dependence (Definition RLDCV [145]) on a linearly independent set, \( \{v_1, v_2, v_3, v_4\} \) (this was our lone hypothesis). By the definition of linear independence (Definition LICV [145]) the scalars must all be zero. This is the homogeneous system of equations,

\[ \begin{align*}
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0 \\
\alpha_2 + \alpha_3 + \alpha_4 &= 0 \\
\alpha_3 + \alpha_4 &= 0 \\
\alpha_4 &= 0
\end{align*} \]

Row-reducing the coefficient matrix of this system (or back-solving) gives the conclusion

\[ \alpha_1 = 0 \quad \alpha_2 = 0 \quad \alpha_3 = 0 \quad \alpha_4 = 0 \]

This means, by Definition LICV [145], that the original set

\[ \{v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_2 + v_3 + v_4\} \]

is linearly independent.

T50 Contributed by Robert Beezer Statement [159]

Let \( A = [A_1 | A_2 | A_3 | \ldots | A_n] \). \( \mathcal{L}(A, b) \) is consistent, so we know the system has at least one solution (Definition CS [51]). We would like to show that there are no more than one solution to the system. Employing Technique U [709], suppose that \( x \) and \( y \) are two solution vectors for \( \mathcal{L}(A, b) \). By Theorem SLSLC [100] we know we can write,

\[ \begin{align*}
b &= [x_1] A_1 + [x_2] A_2 + [x_3] A_3 + \cdots + [x_n] A_n \\
\end{align*} \]

Then

\[ 0 = b - b = ([x_1] A_1 + [x_2] A_2 + [x_3] A_3 + \cdots + [x_n] A_n) - ([y_1] A_1 + [y_2] A_2 + [y_3] A_3 + \cdots + [y_n] A_n) = ([x_1] - [y_1]) A_1 + ([x_2] - [y_2]) A_2 + \cdots + ([x_n] - [y_n]) A_n \]

This is a relation of linear dependence (Definition RLDCV [145]) on a linearly independent set (the columns of \( A \)). So the scalars must all be zero,

\[ [x_1] - [y_1] = 0 \quad [x_2] - [y_2] = 0 \quad \ldots \quad [x_n] - [y_n] = 0 \]

Rearranging these equations yields the statement that \( [x_i] = [y_i] \), for \( 1 \leq i \leq n \). However, this is exactly how we define vector equality (Definition CVE [88]), so \( x = y \).
Section LDS  
Linear Dependence and Spans

In any linearly dependent set there is always one vector that can be written as a linear combination of the others. This is the substance of the upcoming Theorem DLDS \[167\]. Perhaps this will explain the use of the word “dependent.” In a linearly dependent set, at least one vector “depends” on the others (via a linear combination).

Indeed, because Theorem DLDS \[167\] is an equivalence (Technique E \[704\]) some authors use this condition as a definition (Technique D \[698\]) of linear dependence. Then linear independence is defined as the logical opposite of linear dependence. Of course, we have chosen to take Definition LICV \[145\] as our definition, and then present Theorem DLDS \[167\] as a theorem.

Subsection LDSS  
Linearly Dependent Sets and Spans

If we use a linearly dependent set to construct a span, then we can always create the same infinite set with a starting set that is one vector smaller in size. We will illustrate this behavior in Example RSC5 \[168\]. However, this will not be possible if we build a span from a linearly independent set. So in a certain sense, using a linearly independent set to formulate a span is the best possible way to go about it — there aren’t any extra vectors being used to build up all the necessary linear combinations. OK, here’s the theorem, and then the example.

Theorem DLDS  
Dependency in Linearly Dependent Sets

Suppose that \( S = \{u_1, u_2, u_3, \ldots, u_n\} \) is a set of vectors. Then \( S \) is a linearly dependent set if and only if there is an index \( t, 1 \leq t \leq n \) such that \( u_t \) is a linear combination of the vectors \( u_1, u_2, u_3, \ldots, u_{t-1}, u_{t+1}, \ldots, u_n \).

\[ \implies \]

\[ (\Rightarrow) \] Suppose that \( S \) is linearly dependent, so there is a nontrivial relation of linear dependence,

\[ \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_n u_n = 0. \]

Since the \( \alpha_i \) cannot all be zero, choose one, say \( \alpha_t \), that is nonzero. Then,

\[ -\alpha_t u_t = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_{t-1} u_{t-1} + \alpha_{t+1} u_{t+1} + \cdots + \alpha_n u_n \]

and we can multiply by \( \frac{1}{\alpha_t} \) since \( \alpha_t \neq 0 \),

\[ u_t = \frac{-\alpha_1}{\alpha_t} u_1 + \frac{-\alpha_2}{\alpha_t} u_2 + \frac{-\alpha_3}{\alpha_t} u_3 + \cdots + \frac{-\alpha_{t-1}}{\alpha_t} u_{t-1} + \frac{-\alpha_{t+1}}{\alpha_t} u_{t+1} + \cdots + \frac{-\alpha_n}{\alpha_t} u_n. \]

Since the values of \( \frac{\alpha_i}{\alpha_t} \) are again scalars, we have expressed \( u_t \) as the desired linear combination.
Suppose that the vector \( u_t \) is a linear combination of the other vectors in \( S \). Write this linear combination as

\[
\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \cdots + \beta_{t-1} u_{t-1} + \beta_{t+1} u_{t+1} + \cdots + \beta_n u_n = u_t
\]

and move \( u_t \) to the other side of the equality

\[
\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \cdots + \beta_{t-1} u_{t-1} + (-1)u_t + \beta_{t+1} u_{t+1} + \cdots + \beta_n u_n = 0.
\]

Then the scalars \( \beta_1, \beta_2, \beta_3, \ldots, \beta_{t-1}, \beta_t = -1, \beta_{t+1}, \ldots, \beta_n \) provide a nontrivial linear combination of the vectors in \( S \), thus establishing that \( S \) is a linearly dependent set.

This theorem can be used, sometimes repeatedly, to whittle down the size of a set of vectors used in a span construction. We have seen some of this already in Example SCAD [132], but in the next example we will detail some of the subtleties.

**Example RSC5**

**Reducing a span in \( \mathbb{C}^5 \)**

Consider the set of \( n = 4 \) vectors from \( \mathbb{C}^5 \),

\[
R = \{ v_1, v_2, v_3, v_4 \} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -7 \\ 6 \\ -11 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \\ 1 \\ 6 \end{bmatrix} \right\}
\]

and define \( V = \langle R \rangle \).

To employ Theorem LIVHS [147], we form a \( 5 \times 4 \) coefficient matrix, \( D \),

\[
D = \begin{bmatrix}
1 & 2 & 0 & 4 \\
2 & 1 & -7 & 1 \\
-1 & 3 & 6 & 2 \\
3 & 1 & -11 & 1 \\
2 & 2 & -2 & 6
\end{bmatrix}
\]

and row-reduce to understand solutions to the homogeneous system \( \mathcal{L}S(D, 0) \),

\[
\begin{bmatrix}
1 & 0 & 0 & 4 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

We can find infinitely many solutions to this system, most of them nontrivial, and we choose any one we like to build a relation of linear dependence on \( R \). Let’s begin with \( x_4 = 1 \), to find the solution

\[
\begin{bmatrix}
-4 \\
0 \\
-1 \\
1
\end{bmatrix}
\]

So we can write the relation of linear dependence,

\[ (-4)v_1 + 0v_2 + (-1)v_3 + 1v_4 = 0. \]
Theorem DLDS guarantees that we can solve this relation of linear dependence for some vector in $R$, but the choice of which one is up to us. Notice however that $v_2$ has a zero coefficient. In this case, we cannot choose to solve for $v_2$. Maybe some other relation of linear dependence would produce a nonzero coefficient for $v_2$ if we just had to solve for this vector. Unfortunately, this example has been engineered to always produce a zero coefficient here, as you can see from solving the homogeneous system. Every solution has $x_2 = 0$.

OK, if we are convinced that we cannot solve for $v_2$, let’s instead solve for $v_3$,

$$v_3 = (-4)v_1 + 0v_2 + 1v_4 = (-4)v_1 + 1v_4.$$  

We now claim that this particular equation will allow us to write

$$V = \langle R \rangle = \langle \{v_1, v_2, v_3, v_4\} \rangle = \langle \{v_1, v_2, v_4\} \rangle$$

in essence declaring $v_3$ as surplus for the task of building $V$ as a span. This claim is an equality of two sets, so we will use Definition SE to establish it carefully. Let $R' = \{v_1, v_2, v_4\}$ and $V' = \langle R' \rangle$. We want to show that $V = V'$.

First show that $V' \subseteq V$. Since every vector of $R'$ is in $R$, any vector we can construct in $V'$ as a linear combination of vectors from $R'$ can also be constructed as a vector in $V$ by the same linear combination of the same vectors in $R$. That was easy, now turn it around.

Next show that $V \subseteq V'$. Choose any $v$ from $V$. Then there are scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ so that

$$v = \alpha_1v_1 + \alpha_2v_2 + \alpha_3v_3 + \alpha_4v_4$$

$$= \alpha_1v_1 + \alpha_2v_2 + \alpha_3((-4)v_1 + 1v_4) + \alpha_4v_4$$

$$= \alpha_1v_1 + \alpha_2v_2 + ((-4\alpha_3)v_1 + \alpha_3v_4) + \alpha_4v_4$$

$$= (\alpha_1 - 4\alpha_3)v_1 + \alpha_2v_2 + (\alpha_3 + \alpha_4)v_4.$$  

This equation says that $v$ can then be written as a linear combination of the vectors in $R'$ and hence qualifies for membership in $V'$. So $V \subseteq V'$ and we have established that $V = V'$.

If $R'$ was also linearly dependent (its not), we could reduce the set even further. Notice that we could have chosen to eliminate any one of $v_1, v_3$ or $v_4$, but somehow $v_2$ is essential to the creation of $V$ since it cannot be replaced by any linear combination of $v_1, v_3$ or $v_4$.

\[\Box\]

### Subsection COV

**Casting Out Vectors**

In Example RSC5, we used four vectors to create a span. With a relation of linear dependence in hand, we were able to “toss-out” one of these four vectors and create the same span from a subset of just three vectors from the original set of four. We did have to take some care as to just which vector we tossed-out. In the next example, we will be more methodical about just how we choose to eliminate vectors from a linearly dependent set while preserving a span.
Example COV
Casting out vectors

We begin with a set \( S \) containing seven vectors from \( \mathbb{C}^4 \),

\[
S = \begin{Bmatrix}
\begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \end{bmatrix}, \\
\begin{bmatrix} 4 \\ 8 \\ 0 \\ -4 \end{bmatrix}, \\
\begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \\
\begin{bmatrix} -1 \\ -3 \\ -4 \\ 4 \end{bmatrix}, \\
\begin{bmatrix} 0 \\ 9 \\ -4 \\ 8 \end{bmatrix}, \\
\begin{bmatrix} 7 \\ -13 \\ 12 \\ -8 \end{bmatrix}, \\
\begin{bmatrix} 7 \\ -9 \\ 37 \end{bmatrix}
\end{Bmatrix}
\]

and define \( W = \langle S \rangle \). The set \( S \) is obviously linearly dependent by Theorem MVSLD\[150\], since we have \( n = 7 \) vectors from \( \mathbb{C}^4 \). So we can slim down \( S \) some, and still create \( W \) as the span of a smaller set of vectors. As a device for identifying relations of linear dependence among the vectors of \( S \), we place the seven column vectors of \( S \) into a matrix as columns,

\[
A = [A_1|A_2|A_3|\ldots|A_7] = \begin{bmatrix}
1 & 4 & 0 & -1 & 0 & 7 & -9 \\
2 & 8 & -1 & 3 & 9 & -13 & 7 \\
0 & 0 & 2 & -3 & -4 & 12 & -8 \\
-1 & -4 & 2 & 4 & 8 & -31 & 37
\end{bmatrix}
\]

By Theorem SLSLC\[100\] a nontrivial solution to \( \mathcal{L}S(A, 0) \) will give us a nontrivial relation of linear dependence (Definition RLDCV\[145\]) on the columns of \( A \) (which are the elements of the set \( S \)). The row-reduced form for \( A \) is the matrix

\[
B = \begin{bmatrix}
1 & 4 & 0 & 0 & 2 & 1 & -3 \\
0 & 0 & 1 & 0 & 1 & -3 & 5 \\
0 & 0 & 0 & 1 & 2 & -6 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

so we can easily create solutions to the homogeneous system \( \mathcal{L}S(A, 0) \) using the free variables \( x_2, x_5, x_6, x_7 \). Any such solution will correspond to a relation of linear dependence on the columns of \( I \). These solutions will allow us to solve for one column vector as a linear combination of some others, in the spirit of Theorem DLDSD\[167\], and remove that vector from the set. We’ll set about forming these linear combinations methodically. Set the free variable \( x_2 \) to one, and set the other free variables to zero. Then a solution to \( \mathcal{L}S(A, 0) \) is

\[
x = \begin{bmatrix}
-4 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

which can be used to create the linear combination

\[
(-4)A_1 + 1A_2 + 0A_3 + 0A_4 + 0A_5 + 0A_6 + 0A_7 = 0
\]

This can then be arranged and solved for \( A_2 \), resulting in \( A_2 \) expressed as a linear combination of \( \{A_1, A_3, A_4\} \),

\[
A_2 = 4A_1 + 0A_3 + 0A_4
\]
This means that \( \mathbf{A}_2 \) is surplus, and we can create \( W \) just as well with a smaller set with this vector removed,

\[
W = \langle \{ \mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7 \} \rangle
\]

Technically, this set equality for \( W \) requires a proof, in the spirit of [Example RSC5][168], but we will bypass this requirement here, and in the next few paragraphs.

Now, set the free variable \( x_5 \) to one, and set the other free variables to zero. Then a solution to \( \mathcal{L}S(I, 0) \) is

\[
x = \begin{bmatrix}
-2 \\
0 \\
-1 \\
-2 \\
1 \\
0 \\
0
\end{bmatrix}
\]

which can be used to create the linear combination

\[
(-2) \mathbf{A}_1 + 0 \mathbf{A}_2 + (-1) \mathbf{A}_3 + (-2) \mathbf{A}_4 + 1 \mathbf{A}_5 + 0 \mathbf{A}_6 + 0 \mathbf{A}_7 = 0
\]

This can then be arranged and solved for \( \mathbf{A}_5 \), resulting in \( \mathbf{A}_5 \) expressed as a linear combination of \( \{ \mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4 \} \),

\[
\mathbf{A}_5 = 2 \mathbf{A}_1 + 1 \mathbf{A}_3 + 2 \mathbf{A}_4
\]

This means that \( \mathbf{A}_5 \) is surplus, and we can create \( W \) just as well with a smaller set with this vector removed,

\[
W = \langle \{ \mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_6, \mathbf{A}_7 \} \rangle
\]

Do it again, set the free variable \( x_6 \) to one, and set the other free variables to zero. Then a solution to \( \mathcal{L}S(I, 0) \) is

\[
x = \begin{bmatrix}
-1 \\
0 \\
3 \\
6 \\
1 \\
0
\end{bmatrix}
\]

which can be used to create the linear combination

\[
(-1) \mathbf{A}_1 + 0 \mathbf{A}_2 + 3 \mathbf{A}_3 + 6 \mathbf{A}_4 + 0 \mathbf{A}_5 + 1 \mathbf{A}_6 + 0 \mathbf{A}_7 = 0
\]

This can then be arranged and solved for \( \mathbf{A}_6 \), resulting in \( \mathbf{A}_6 \) expressed as a linear combination of \( \{ \mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4 \} \),

\[
\mathbf{A}_6 = 1 \mathbf{A}_1 + (-3) \mathbf{A}_3 + (-6) \mathbf{A}_4
\]

This means that \( \mathbf{A}_6 \) is surplus, and we can create \( W \) just as well with a smaller set with this vector removed,

\[
W = \langle \{ \mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_7 \} \rangle
\]
Set the free variable $x_7$ to one, and set the other free variables to zero. Then a solution to $LS(I, 0)$ is

$$x = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

which can be used to create the linear combination

$$3A_1 + 0A_2 + (-5)A_3 + (-6)A_4 + 0A_5 + 0A_6 + 1A_7 = 0$$

This can then be arranged and solved for $A_7$, resulting in $A_7$ expressed as a linear combination of $\{A_1, A_3, A_4\}$,

$$A_7 = (-3)A_1 + 5A_3 + 6A_4$$

This means that $A_7$ is surplus, and we can create $W$ just as well with a smaller set with this vector removed,

$$W = \langle\{A_1, A_3, A_4\}\rangle$$

You might think we could keep this up, but we have run out of free variables. And not coincidentally, the set $\{A_1, A_3, A_4\}$ is linearly independent (check this!). It should be clear how each free variable was used to eliminate the corresponding column from the set used to span the column space, as this will be the essence of the proof of the next theorem. The column vectors in $S$ were not chosen entirely at random, they are the columns of Archetype I \[757\]. See if you can mimic this example using the columns of Archetype J \[762\]. Go ahead, we’ll go grab a cup of coffee and be back before you finish up.

For extra credit, notice that the vector

$$b = \begin{bmatrix} 3 \\ 9 \\ 1 \end{bmatrix}$$

is the vector of constants in the definition of Archetype I \[757\]. Since the system $LS(I, b)$ is consistent, we know by Theorem SLSLC \[100\] that $b$ is a linear combination of the columns of $A$, or stated equivalently, $b \in W$. This means that $b$ must also be a linear combination of just the three columns $A_1, A_3, A_4$. Can you find such a linear combination? Did you notice that there is just a single (unique) answer? Hmmm.

Example COV \[170\] deserves your careful attention, since this important example motivates the following very fundamental theorem.

**Theorem BS**

**Basis of a Span**

Suppose that $S = \{v_1, v_2, v_3, \ldots, v_n\}$ is a set of column vectors. Define $W = \langle S \rangle$ and let $A$ be the matrix whose columns are the vectors from $S$. Let $B$ be the reduced row-echelon form of $A$, with $D = \{d_1, d_2, d_3, \ldots, d_r\}$ the set of column indices corresponding to the pivot columns of $B$. Then
1. \( T = \{ \mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \ldots, \mathbf{v}_{d_r} \} \) is a linearly independent set.

2. \( W = \langle T \rangle \).

**Proof** To prove that \( T \) is linearly independent, begin with a relation of linear dependence on \( T \),

\[
0 = \alpha_1 \mathbf{v}_{d_1} + \alpha_2 \mathbf{v}_{d_2} + \alpha_3 \mathbf{v}_{d_3} + \ldots + \alpha_r \mathbf{v}_{d_r}
\]

and we will try to conclude that the only possibility for the scalars \( \alpha_i \) is that they are all zero. Denote the non-pivot columns of \( B \) by \( F = \{ f_1, f_2, f_3, \ldots, f_{n-r} \} \). Then we can preserve the equality by adding a big fat zero to the linear combination,

\[
0 = \alpha_1 \mathbf{v}_{d_1} + \alpha_2 \mathbf{v}_{d_2} + \alpha_3 \mathbf{v}_{d_3} + \ldots + \alpha_r \mathbf{v}_{d_r} + 0 \mathbf{v}_{f_1} + 0 \mathbf{v}_{f_2} + 0 \mathbf{v}_{f_3} + \ldots + 0 \mathbf{v}_{f_{n-r}}
\]

By **Theorem SLSLC** [100], the scalars in this linear combination (suitably reordered) are a solution to the homogeneous system \( \mathcal{L}S(A, 0) \). But notice that this is the solution obtained by setting each free variable to zero. If we consider the description of a solution vector in the conclusion of **Theorem VFSLs** [107], in the case of a homogeneous system, then we see that if all the free variables are set to zero the resulting solution vector is trivial (all zeros). So it must be that \( \alpha_i = 0, 1 \leq i \leq r \). This implies by **Definition LICV** [145] that \( T \) is a linearly independent set.

The second conclusion of this theorem is an equality of sets ( **Definition SE** [694]). Since \( T \) is a subset of \( S \), any linear combination of elements of the set \( T \) can also be viewed as a linear combination of elements of the set \( S \). So \( \langle T \rangle \subseteq \langle S \rangle = W \). It remains to prove that \( W = \langle S \rangle \subseteq \langle T \rangle \).

For each \( k, 1 \leq k \leq n-r \), form a solution \( \mathbf{x} \) to \( \mathcal{L}S(A, 0) \) by setting the free variables as follows:

\[
x_{f_1} = 0, \quad x_{f_2} = 0, \quad x_{f_3} = 0, \quad \ldots, \quad x_{f_k} = 1, \quad \ldots, \quad x_{f_{n-r}} = 0
\]

By **Theorem VFSLs** [107], the remainder of this solution vector is given by,

\[
x_{d_1} = -[B]_{1,f_k} \quad x_{d_2} = -[B]_{2,f_k} \quad x_{d_3} = -[B]_{3,f_k} \quad \ldots \quad x_{d_r} = -[B]_{r,f_k}
\]

From this solution, we obtain a relation of linear dependence on the columns of \( A \),

\[-[B]_{1,f_k} \mathbf{v}_{d_1} - [B]_{2,f_k} \mathbf{v}_{d_2} - [B]_{3,f_k} \mathbf{v}_{d_3} - \ldots - [B]_{r,f_k} \mathbf{v}_{d_r} + 1 \mathbf{v}_{f_k} = 0
\]

which can be arranged as the equality

\[
\mathbf{v}_{f_k} = [B]_{1,f_k} \mathbf{v}_{d_1} + [B]_{2,f_k} \mathbf{v}_{d_2} + [B]_{3,f_k} \mathbf{v}_{d_3} + \ldots + [B]_{r,f_k} \mathbf{v}_{d_r}
\]

Now, suppose we take an arbitrary element, \( \mathbf{w} \), of \( W = \langle S \rangle \) and write it as a linear combination of the elements of \( S \), but with the terms organized according to the indices in \( D \) and \( F \),

\[
\mathbf{w} = \alpha_1 \mathbf{v}_{d_1} + \alpha_2 \mathbf{v}_{d_2} + \alpha_3 \mathbf{v}_{d_3} + \ldots + \alpha_r \mathbf{v}_{d_r} + \beta_1 \mathbf{v}_{f_1} + \beta_2 \mathbf{v}_{f_2} + \beta_3 \mathbf{v}_{f_3} + \ldots + \beta_{n-r} \mathbf{v}_{f_{n-r}}
\]

From the above, we can replace each \( \mathbf{v}_{f_j} \) by a linear combination of the \( \mathbf{v}_{d_i} \),

\[
\mathbf{w} = \alpha_1 \mathbf{v}_{d_1} + \alpha_2 \mathbf{v}_{d_2} + \alpha_3 \mathbf{v}_{d_3} + \ldots + \alpha_r \mathbf{v}_{d_r} + 
\]
With repeated applications of several of the properties of Theorem VSPCV \[91\] we can rearrange this expression as,

\[
\begin{align*}
\beta_1 \left( [B]_{1,f_1} v_{d_1} + [B]_{2,f_1} v_{d_2} + [B]_{3,f_1} v_{d_3} + \ldots + [B]_{r,f_1} v_{d_r} \right) + \\
\beta_2 \left( [B]_{1,f_2} v_{d_1} + [B]_{2,f_2} v_{d_2} + [B]_{3,f_2} v_{d_3} + \ldots + [B]_{r,f_2} v_{d_r} \right) + \\
\beta_3 \left( [B]_{1,f_3} v_{d_1} + [B]_{2,f_3} v_{d_2} + [B]_{3,f_3} v_{d_3} + \ldots + [B]_{r,f_3} v_{d_r} \right) + \\
\quad \vdots \\
\beta_{n-r} \left( [B]_{1,f_{n-r}} v_{d_1} + [B]_{2,f_{n-r}} v_{d_2} + [B]_{3,f_{n-r}} v_{d_3} + \ldots + [B]_{r,f_{n-r}} v_{d_r} \right)
\end{align*}
\]

This mess expresses the vector \( \mathbf{w} \) as a linear combination of the vectors in

\[
T = \{ v_{d_1}, v_{d_2}, v_{d_3}, \ldots, v_{d_r} \}
\]

thus saying that \( \mathbf{w} \in \langle T \rangle \). Therefore, \( W = \langle S \rangle \subseteq \langle T \rangle \). \( \blacksquare \)

In Example COV \[170\], we tossed-out vectors one at a time. But in each instance, we rewrote the offending vector as a linear combination of those vectors that corresponded to the pivot columns of the reduced row-echelon form of the matrix of columns. In the proof of Theorem BS \[172\], we accomplish this reduction in one big step. In Example COV \[170\] we arrived at a linearly independent set at exactly the same moment that we ran out of free variables to exploit. This was not a coincidence, it is the substance of our conclusion of linear independence in Theorem BS \[172\].

Here’s a straightforward application of Theorem BS \[172\].

Example RSSC4
Reducing a span in \( \mathbb{C}^4 \)

Begin with a set of five vectors from \( \mathbb{C}^4 \),

\[
S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \\ 1 \end{bmatrix} \right\}
\]

and let \( W = \langle S \rangle \). To arrive at a (smaller) linearly independent set, follow the procedure described in Theorem BS \[172\]. Place the vectors from \( S \) into a matrix as columns, and row-reduce,

\[
\begin{bmatrix}
1 & 2 & 2 & 7 & 0 \\
1 & 2 & 0 & 1 & 2 \\
2 & 4 & -1 & -1 & 5 \\
1 & 2 & 1 & 4 & 1 \\
\end{bmatrix}
\xrightarrow{\text{rref}}
\begin{bmatrix}
1 & 2 & 0 & 1 & 2 \\
0 & 0 & 1 & 3 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Columns 1 and 3 are the pivot columns \( (D = \{1, 3\}) \) so the set

\[
T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}
\]

is linearly independent and \( \langle T \rangle = \langle S \rangle = W \). Boom!

Since the reduced row-echelon form of a matrix is unique \( (\text{Theorem RREFU} \ [116]) \), the procedure of \( \text{Theorem BS} \ [172] \) leads us to a unique set \( T \). However, there is a wide variety of possibilities for sets \( T \) that are linearly independent and which can be employed in a span to create \( W \). Without proof, we list two other possibilities:

\[
T' = \left\{ \begin{bmatrix} 2 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}
\]

\[
T^* = \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \\ 0 \end{bmatrix} \right\}
\]

Can you prove that \( T' \) and \( T^* \) are linearly independent sets and \( W = \langle S \rangle = \langle T' \rangle = \langle T^* \rangle \)?

**Example RES**

**Reworking elements of a span**

Begin with a set of five vectors from \( \mathbb{C}^4 \),

\[
R = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -8 \\ -9 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -10 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\}
\]

It is easy to create elements of \( X = \langle R \rangle \) — we will create one at random,

\[
y = 6 \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix} + (\text{-7}) \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -8 \\ -9 \\ -4 \\ -2 \end{bmatrix} + 6 \begin{bmatrix} 3 \\ -1 \\ -1 \\ -2 \end{bmatrix} + 2 \begin{bmatrix} -10 \\ -1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 1 \\ -3 \end{bmatrix}
\]

We know we can replace \( R \) by a smaller set (since it is obviously linearly dependent by \( \text{Theorem MVSLD} \ [150] \) that will create the same span. Here goes,

\[
\begin{bmatrix} 2 & -1 & -8 & 3 & -10 \\ 1 & -1 & 1 & -1 \\ 3 & 0 & -1 & -1 \\ 2 & 1 & -4 & -2 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -3 & 0 & -1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

So, if we collect the first, second and fourth vectors from \( R \),

\[
P = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ -2 \end{bmatrix} \right\}
\]
then $P$ is linearly independent and $\langle P \rangle = \langle R \rangle = X$ by Theorem BS [172]. Since we built $y$ as an element of $\langle R \rangle$ it must also be an element of $\langle P \rangle$. Can we write $y$ as a linear combination of just the three vectors in $P$? The answer is, of course, yes. But let’s compute an explicit linear combination just for fun. By Theorem SLSLC [100] we can get such a linear combination by solving a system of equations with the column vectors of $R$ as the columns of a coefficient matrix, and $y$ as the vector of constants. Employing an augmented matrix to solve this system,

$$
\begin{bmatrix}
2 & -1 & 3 & 9 \\
1 & 1 & 1 & 2 \\
3 & 0 & -1 & 1 \\
2 & 1 & -2 & -3
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

So we see, as expected, that

$$
\begin{bmatrix}
2 \\
1 \\
3 \\
2
\end{bmatrix} + (-1)\begin{bmatrix}
-1 \\
1 \\
0 \\
1
\end{bmatrix} + 2\begin{bmatrix}
3 \\
1 \\
-1 \\
-2
\end{bmatrix} = \begin{bmatrix}
9 \\
2 \\
1 \\
-3
\end{bmatrix} = y
$$

A key feature of this example is that the linear combination that expresses $y$ as a linear combination of the vectors in $P$ is unique. This is a consequence of the linear independence of $P$. The linearly independent set $P$ is smaller than $R$, but still just (barely) big enough to create elements of the set $X = \langle R \rangle$. There are many, many ways to write $y$ as a linear combination of the five vectors in $R$ (the appropriate system of equations to verify this claim has two free variables in the description of the solution set), yet there is precisely one way to write $y$ as a linear combination of the three vectors in $P$. 

Subsection READ

Reading Questions

1. Let $S$ be the linearly dependent set of three vectors below.

$$
S = \left\{ \begin{bmatrix} 1 \\ 10 \\ 100 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1000 \end{bmatrix}, \begin{bmatrix} 5 \\ 23 \\ 203 \\ 2003 \end{bmatrix} \right\}
$$

Write one vector from $S$ as a linear combination of the other two (you should be able to do this on sight, rather than doing some computations). Convert this expression into a relation of linear dependence on $S$.

2. Explain why the word “dependent” is used in the definition of linear dependence.

3. Suppose that $Y = \langle P \rangle = \langle Q \rangle$, where $P$ is a linearly dependent set and $Q$ is linearly independent. Would you rather use $P$ or $Q$ to describe $Y$? Why?
Subsection EXC Exercises

C20 Let $T$ be the set of columns of the matrix $B$ below. Define $W = \langle T \rangle$. Find a set $R$ so that (1) $R$ has 3 vectors, (2) $R$ is a subset of $T$, and (3) $W = \langle R \rangle$.

$$B = \begin{bmatrix} -3 & 1 & -2 & 7 \\ -1 & 2 & 1 & 4 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

Contributed by Robert Beezer Solution

C40 Verify that the set $R' = \{v_1, v_2, v_4\}$ at the end of Example RSC5 is linearly independent.

Contributed by Robert Beezer

C50 Consider the set of vectors from $\mathbb{C}^3$, $W$, given below. Find a linearly independent set $T$ that contains three vectors from $W$ and such that $\langle W \rangle = \langle T \rangle$.

$$W = \{v_1, v_2, v_3, v_4, v_5\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix} \right\}$$

Contributed by Robert Beezer Solution

C51 Given the set $S$ below, find a linearly independent set $T$ so that $\langle T \rangle = \langle S \rangle$.

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} \right\}$$

Contributed by Robert Beezer Solution

C55 Let $T$ be the set of vectors $T = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} \right\}$. Find two different subsets of $T$, named $R$ and $S$, so that $R$ and $S$ each contain three vectors, and so that $\langle R \rangle = \langle T \rangle$ and $\langle S \rangle = \langle T \rangle$. Prove that both $R$ and $S$ are linearly independent.

Contributed by Robert Beezer Solution

C70 Reprise Example RES by creating a new version of the vector $y$. In other words, form a new, different linear combination of the vectors in $R$ to create a new vector $y$ (but do not simplify the problem too much by choosing any of the five new scalars to be zero). Then express this new $y$ as a combination of the vectors in $P$.

Contributed by Robert Beezer

M10 At the conclusion of Example RSSC4 two alternative solutions, sets $T'$ and $T^*$, are proposed. Verify these claims by proving that $\langle T \rangle = \langle T' \rangle$ and $\langle T \rangle = \langle T^* \rangle$.

Contributed by Robert Beezer
Suppose that \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are any two vectors from \( \mathbb{C}^m \). Prove the following set equality.

\[
\langle \{\mathbf{v}_1, \mathbf{v}_2\} \rangle = \langle \{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\} \rangle
\]

Contributed by Robert Beezer  Solution [180]
Subsection SOL
Solutions

C20 Contributed by Robert Beezer Statement 177

Let \( T = \{w_1, w_2, w_3, w_4\} \). The vector
\[
\begin{bmatrix}
2 \\
-1 \\
0 \\
1
\end{bmatrix}
\]
is a solution to the homogeneous system with the matrix \( B \) as the coefficient matrix (check this!). By Theorem SLSLC 100 it provides the scalars for a linear combination of the columns of \( B \) (the vectors in \( T \)) that equals the zero vector, a relation of linear dependence on \( T \),

\[
2w_1 + (-1)w_2 + (1)w_4 = 0
\]

We can rearrange this equation by solving for \( w_4 \),

\[
w_4 = (-2)w_1 + w_2
\]

This equation tells us that the vector \( w_4 \) is superfluous in the span construction that creates \( W \). So \( W = \langle\{w_1, w_2, w_3\}\rangle \). The requested set is \( R = \{w_1, w_2, w_3\} \).

C50 Contributed by Robert Beezer Statement 177
To apply Theorem BS 172, we formulate a matrix \( A \) whose columns are \( v_1, v_2, v_3, v_4, v_5 \). Then we row-reduce \( A \). After row-reducing, we obtain

\[
\begin{bmatrix}
1 & 0 & 0 & 2 & -1 \\
0 & 1 & 0 & 1 & -2 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

From this we that the pivot columns are \( D = \{1, 2, 3\} \). Thus

\[ T = \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\} \]

is a linearly independent set and \( \langle T \rangle = W \). Compare this problem with Exercise LI.M50 159.

C55 Contributed by Robert Beezer Statement 177
Let \( A \) be the matrix whose columns are the vectors in \( T \). Then row-reduce \( A \),

\[
A \xrightarrow{\text{RREF}} B = \begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

From Theorem BS 172 we can form \( R \) by choosing the columns of \( A \) that correspond to the pivot columns of \( B \). Theorem BS 172 also guarantees that \( R \) will be linearly independent.

\[
R = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} \right\}
\]
That was easy. To find $S$ will require a bit more work. From $B$ we can obtain a solution to $LS(A, 0)$, which by \textbf{Theorem SLSLC} \[100\] will provide a nontrivial relation of linear dependence on the columns of $A$, which are the vectors in $T$. To wit, choose the free variable $x_4$ to be 1, then $x_1 = -2$, $x_2 = 1$, $x_3 = -1$, and so

$$
\begin{pmatrix}
-2 \\
 1 \\
 2
\end{pmatrix}
+ (1) \begin{pmatrix}
 3 \\
 0 \\
 1
\end{pmatrix}
+ (-1) \begin{pmatrix}
 4 \\
 2 \\
 3
\end{pmatrix}
+ (1) \begin{pmatrix}
 3 \\
 0 \\
 6
\end{pmatrix}
= \begin{pmatrix}
 0 \\
 0 \\
 0
\end{pmatrix}
$$

this equation can be rewritten with the second vector staying put, and the other three moving to the other side of the equality,

$$
\begin{pmatrix}
 3 \\
 0 \\
 1
\end{pmatrix}
= (2) \begin{pmatrix}
 1 \\
 -1 \\
 2
\end{pmatrix}
+ (1) \begin{pmatrix}
 4 \\
 2 \\
 3
\end{pmatrix}
+ (-1) \begin{pmatrix}
 3 \\
 0 \\
 6
\end{pmatrix}
$$

We could have chosen other vectors to stay put, but may have then needed to divide by a nonzero scalar. This equation is enough to conclude that the second vector in $T$ is “surplus” and can be replaced (see the careful argument in \textbf{Example RSC5} \[168\]). So set

$$
S = \left\{ \begin{pmatrix}
 1 \\
 -1 \\
 2
\end{pmatrix},
\begin{pmatrix}
 4 \\
 2 \\
 3
\end{pmatrix},
\begin{pmatrix}
 3 \\
 0 \\
 6
\end{pmatrix}\right\}
$$

and then $\langle S \rangle = \langle T \rangle$. $T$ is also a linearly independent set, which we can show directly. Make a matrix $C$ whose columns are the vectors in $S$. Row-reduce $B$ and you will obtain the identity matrix $I_3$. By \textbf{Theorem LIVRN} \[149\], the set $S$ is linearly independent.

\textbf{C51} Contributed by Robert Beezer Statement \[177\]

\textbf{Theorem BS} \[172\] says we can make a matrix with these four vectors as columns, row-reduce, and just keep the columns with indices in the set $D$. Here we go, forming the relevant matrix and row-reducing,

$$
\begin{pmatrix}
 2 & 3 & 1 & 5 \\
-1 & 0 & 1 & -1 \\
 2 & 1 & -1 & 3
\end{pmatrix}
\xrightarrow{\text{RREF}}
\begin{pmatrix}
 1 & 0 & -1 & 1 \\
 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 0
\end{pmatrix}
$$

Analyzing the row-reduced version of this matrix, we see that the first two columns are pivot columns, so $D = \{1, 2\}$. \textbf{Theorem BS} \[172\] says we need only “keep” the first two columns to create a set with the requisite properties,

$$
T = \left\{ \begin{pmatrix}
 2 \\
 -1 \\
 2
\end{pmatrix},
\begin{pmatrix}
 3 \\
 0 \\
 1
\end{pmatrix}\right\}
$$

\textbf{T40} Contributed by Robert Beezer Statement \[178\]

This is an equality of sets, so \textbf{Definition SE} \[694\] applies.

The “easy” half first. Show that $X = \langle \{v_1 + v_2, v_1 - v_2\} \rangle \subseteq \langle \{v_1, v_2\} \rangle = Y$. Choose $x \in X$. Then $x = a_1(v_1 + v_2) + a_2(v_1 - v_2)$ for some scalars $a_1$ and $a_2$. Then,

$$
x = a_1(v_1 + v_2) + a_2(v_1 - v_2)
$$
\[ a_1 v_1 + a_1 v_2 + a_2 v_1 + (-a_2) v_2 \]
\[ = (a_1 + a_2) v_1 + (a_1 - a_2) v_2 \]

which qualifies \( x \) for membership in \( Y \), as it is a linear combination of \( v_1, v_2 \).

Now show the opposite inclusion, \( Y = \langle \{v_1, v_2\} \rangle \subseteq \langle \{v_1 + v_2, v_1 - v_2\} \rangle = X \).

Choose \( y \in Y \). Then there are scalars \( b_1, b_2 \) such that \( y = b_1 v_1 + b_2 v_2 \). Rearranging, we obtain,

\[ y = b_1 v_1 + b_2 v_2 \]
\[ = \frac{b_1}{2} [(v_1 + v_2) + (v_1 - v_2)] + \frac{b_2}{2} [(v_1 + v_2) - (v_1 - v_2)] \]
\[ = \frac{b_1 + b_2}{2} (v_1 + v_2) + \frac{b_1 - b_2}{2} (v_1 - v_2) \]

This is an expression for \( y \) as a linear combination of \( v_1 + v_2 \) and \( v_1 - v_2 \), earning \( y \) membership in \( X \). Since \( X \) is a subset of \( Y \), and vice versa, we see that \( X = Y \), as desired.
In this section we define a couple more operations with vectors, and prove a few theorems. These definitions and results are not central to what follows, but we will make use of them frequently throughout the remainder of the course on various occasions. Because we have chosen to use \( \mathbb{C} \) as our set of scalars, this subsection is a bit more, uh, . . . complex than it would be for the real numbers. We’ll explain as we go along how things get easier for the real numbers \( \mathbb{R} \). If you haven’t already, now would be a good time to review some of the basic properties of arithmetic with complex numbers described in Section CNO \([687]\). With that done, we can extend the basics of complex number arithmetic to our study of vectors in \( \mathbb{C}^m \).

Subsection CAV
Complex arithmetic and vectors

We know how the addition and multiplication of complex numbers is employed in defining the operations for vectors in \( \mathbb{C}^m \) (Definition CVA \([89]\) and Definition CVSM \([89]\)). We can also extend the idea of the conjugate to vectors.

**Definition CCCV**
Complex Conjugate of a Column Vector
Suppose that \( \mathbf{u} \) is a vector from \( \mathbb{C}^m \). Then the conjugate of the vector, \( \overline{\mathbf{u}} \), is defined by

\[
\overline{\mathbf{u}}_i = \overline{\mathbf{u}}_i \quad 1 \leq i \leq m
\]

(This definition contains Notation CCCV.)

With this definition we can show that the conjugate of a column vector behaves as we would expect with regard to vector addition and scalar multiplication.

**Theorem CRVA**
Conjugation Respects Vector Addition
Suppose \( \mathbf{x} \) and \( \mathbf{y} \) are two vectors from \( \mathbb{C}^m \). Then

\[
\mathbf{x} + \overline{\mathbf{y}} = \mathbf{x} + \overline{\mathbf{y}}
\]

**Proof** Apply the definition of vector addition (Definition CVA \([89]\)) and the definition of the conjugate of a vector (Definition CCCV \([183]\)), and in each component apply the similar property for complex numbers (Theorem CCRA \([690]\)).

**Theorem CRSM**
Conjugation Respects Vector Scalar Multiplication
Suppose \( \mathbf{x} \) is a vector from \( \mathbb{C}^m \), and \( \alpha \in \mathbb{C} \) is a scalar. Then
\[
\overline{\alpha \mathbf{x}} = \overline{\alpha} \overline{\mathbf{x}}
\]

**Proof**  Apply the definition of scalar multiplication (Definition CVSM [89]) and the definition of the conjugate of a vector (Definition CCCV [183]), and in each component apply the similar property for complex numbers (Theorem CCRM [690]).

These two theorems together tell us how we can “push” complex conjugation through linear combinations.

**Subsection IP**

**Inner products**

**Definition IP**

**Inner Product**

Given the vectors \( \mathbf{u}, \mathbf{v} \in \mathbb{C}^m \) the **inner product** of \( \mathbf{u} \) and \( \mathbf{v} \) is the scalar quantity in \( \mathbb{C} \),
\[
\langle \mathbf{u}, \mathbf{v} \rangle = [u_1 \overline{v}_1] + [u_2 \overline{v}_2] + [u_3 \overline{v}_3] + \cdots + [u_m \overline{v}_m] = \sum_{i=1}^{m} |u_i| |\overline{v}_i|
\]

(This definition contains Notation IP.)

This operation is a bit different in that we begin with two vectors but produce a scalar. Computing one is straightforward.

**Example CSIP**

**Computing some inner products**

The scalar product of
\[
\mathbf{u} = \begin{bmatrix} 2 + 3i \\ 5 + 2i \\ -3 + i \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 + 2i \\ -4 + 5i \\ 0 + 5i \end{bmatrix}
\]
is
\[
\langle \mathbf{u}, \mathbf{v} \rangle = (2 + 3i)(1 + 2i) + (5 + 2i)(-4 + 5i) + (3 + i)(0 + 5i) = (8 - i) + (-10 - 33i) + (5 + 15i) = 3 - 19i
\]

The scalar product of
\[
\mathbf{w} = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 2 \\ 8 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}
\]
is
\[ \langle \mathbf{w}, \mathbf{x} \rangle = 2(3)+4(1)+(-3)(0)+2(-1)+8(-2) = -8. \]

In the case where the entries of our vectors are all real numbers (as in the second part of Example CSIP [184]), the computation of the inner product may look familiar and be known to you as a dot product or scalar product. So you can view the inner product as a generalization of the scalar product to vectors from \( \mathbb{C}^n \) (rather than \( \mathbb{R}^n \)).

Also, note that we have chosen to conjugate the entries of the second vector listed in the inner product, while many authors choose to conjugate entries from the first component. It really makes no difference which choice is made, it just requires that subsequent definitions and theorems are consistent with the choice. You can study the conclusion of Theorem IPAC [186] as an explanation of the magnitude of the difference that results from this choice. But be careful as you read other treatments of the inner product or its use in applications, and be sure you know ahead of time which choice has been made.

There are several quick theorems we can now prove, and they will each be useful later.

**Theorem IPVA**

**Inner Product and Vector Addition**

Suppose \( \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^n \). Then

1. \( \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle \)
2. \( \langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle \)

**Proof** The proofs of the two parts are very similar, with the second one requiring just a bit more effort due to the conjugation that occurs. We will prove part 2 and you can prove part 1 (Exercise O.T10 [195]).

\[
\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \sum_{i=1}^{m} [\mathbf{u}]_i [\mathbf{v} + \mathbf{w}]_i \\
= \sum_{i=1}^{m} [\mathbf{u}]_i ([\mathbf{v}]_i + [\mathbf{w}]_i) \\
= \sum_{i=1}^{m} [\mathbf{u}]_i [\mathbf{v}]_i + [\mathbf{u}]_i [\mathbf{w}]_i \\
= \sum_{i=1}^{m} [\mathbf{u}]_i [\mathbf{v}]_i + \sum_{i=1}^{m} [\mathbf{u}]_i [\mathbf{w}]_i \\
= \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle
\]

\[ \square \]
Theorem IPSM

Inner Product and Scalar Multiplication

Suppose \( u, v \in \mathbb{C}^m \) and \( \alpha \in \mathbb{C} \). Then

1. \( \langle \alpha u, v \rangle = \alpha \langle u, v \rangle \)
2. \( \langle u, \alpha v \rangle = \overline{\alpha} \langle u, v \rangle \)

\( \square \)

Proof

The proofs of the two parts are very similar, with the second one requiring just a bit more effort due to the conjugation that occurs. We will prove part 2 and you can prove part 1 (Exercise O.T11 [195]).

\[
\langle u, \alpha v \rangle = \sum_{i=1}^{m} |u|_i |\alpha v|_i = \sum_{i=1}^{m} |u|_i |\overline{\alpha} v|_i = \sum_{i=1}^{m} |u|_i |\alpha v|_i = |\alpha| \sum_{i=1}^{m} |u|_i |v|_i = |\alpha| \langle u, v \rangle
\]

\( \square \)

Theorem IPAC

Inner Product is Anti-Commutative

Suppose that \( u \) and \( v \) are vectors in \( \mathbb{C}^m \). Then \( \langle u, v \rangle = \overline{\langle v, u \rangle} \).

\( \square \)

Proof

\[
\langle u, v \rangle = \sum_{i=1}^{m} |u|_i |v|_i = \sum_{i=1}^{m} |\overline{u}|_i |v|_i = \sum_{i=1}^{m} |u|_i |\overline{v}|_i = \overline{\left( \sum_{i=1}^{m} |u|_i |v|_i \right)} = \overline{\langle v, u \rangle}
\]

\( \square \)
If treating linear algebra in a more geometric fashion, the length of a vector occurs naturally, and is what you would expect from its name. With complex numbers, we will define a similar function. Recall that if $c$ is a complex number, then $|c|$ denotes its modulus (Definition MCN [690]).

**Definition NV**

**Norm of a Vector**

The norm of the vector $u$ is the scalar quantity in $\mathbb{C}$

$$||u|| = \sqrt{|[u]_1|^2 + |[u]_2|^2 + |[u]_3|^2 + \cdots + |[u]_m|^2} = \sqrt{\sum_{i=1}^{m} |[u]_i|^2}$$

(This definition contains Notation NV.)

Computing a norm is also easy to do.

**Example CNSV**

**Computing the norm of some vectors**

The norm of

$$u = \begin{bmatrix} 3 + 2i \\ 1 - 6i \\ 2 + 4i \\ 2 + i \end{bmatrix}$$

is

$$||u|| = \sqrt{|3 + 2i|^2 + |1 - 6i|^2 + |2 + 4i|^2 + |2 + i|^2} = \sqrt{13 + 37 + 20 + 5} = \sqrt{75} = 5\sqrt{3}.$$ 

The norm of

$$v = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 4 \\ -3 \end{bmatrix}$$

is

$$||v|| = \sqrt{|3|^2 + |-1|^2 + |2|^2 + |4|^2 + |-3|^2} = \sqrt{3^2 + 1^2 + 2^2 + 4^2 + 3^2} = \sqrt{39}.$$ 

Notice how the norm of a vector with real number entries is just the length of the vector. Inner products and norms are related by the following theorem.

**Theorem IPN**

**Inner Products and Norms**

Suppose that $u$ is a vector in $\mathbb{C}^m$. Then $||u||^2 = \langle u, u \rangle$. 

□
Proof

\[ \|u\|^2 = \left( \sum_{i=1}^{m} |u_i|^2 \right)^2 \]

\[ = \sum_{i=1}^{m} |u_i|^2 \]

\[ = \sum_{i=1}^{m} |u_i|^2 \quad \text{Definition MCN 690} \]

\[ = \langle u, u \rangle \quad \text{Definition IP 184} \]

When our vectors have entries only from the real numbers \textbf{Theorem IPN 187} says that the dot product of a vector with itself is equal to the length of the vector squared.

\textbf{Theorem PIP}

\textbf{Positive Inner Products}

Suppose that \( u \) is a vector in \( \mathbb{C}^m \). Then \( \langle u, u \rangle \geq 0 \) with equality if and only if \( u = 0 \).

\textbf{Proof} From the proof of \textbf{Theorem IPN 187} we see that

\[ \langle u, u \rangle = |u_1|^2 + |u_2|^2 + |u_3|^2 + \cdots + |u_m|^2 \]

Since each modulus is squared, every term is positive, and the sum must also be positive. (Notice that in general the inner product is a complex number and cannot be compared with zero, but in the special case of \( \langle u, u \rangle \) the result is a real number.) The phrase, “with equality if and only if” means that we want to show that the statement \( \langle u, u \rangle = 0 \) (i.e. with equality) is equivalent (“if and only if”) to the statement \( u = 0 \).

If \( u = 0 \), then it is a straightforward computation to see that \( \langle u, u \rangle = 0 \). In the other direction, assume that \( \langle u, u \rangle = 0 \). As before, \( \langle u, u \rangle \) is a sum of moduli. So we have

\[ 0 = \langle u, u \rangle = |u_1|^2 + |u_2|^2 + |u_3|^2 + \cdots + |u_m|^2 \]

Now we have a sum of squares equaling zero, so each term must be zero. Then by similar logic, \( |u_i| = 0 \) will imply that \( u_i = 0 \), since \( 0 + 0i \) is the only complex number with zero modulus. Thus every entry of \( u \) is zero and so \( u = 0 \), as desired.

Notice that \textbf{Theorem PIP 188} contains three implications: \( u \) is any vector \( \Rightarrow \langle u, u \rangle \geq 0 \), \( u = 0 \Rightarrow \langle u, u \rangle = 0 \), and \( \langle u, u \rangle = 0 \Rightarrow u = 0 \). The results contained in \textbf{Theorem PIP 188} are summarized by saying “the inner product is \textbf{positive definite}.”

\textbf{Subsection OV}

\textbf{Orthogonal Vectors}

“Orthogonal” is a generalization of “perpendicular.” You may have used mutually perpendicular vectors in a physics class, or you may recall from a calculus class that perpendicular vectors have a zero dot product. We will now extend these ideas into the realm of higher dimensions and complex scalars.
Definition OV
Orthogonal Vectors
A pair of vectors, \( u \) and \( v \), from \( \mathbb{C}^m \) are orthogonal if their inner product is zero, that is, \( \langle u, v \rangle = 0 \).

Example TOV
Two orthogonal vectors
The vectors
\[
\begin{pmatrix}
2 + 3i \\
4 - 2i \\
1 + i \\
1 + i
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 - i \\
2 + 3i \\
4 - 6i \\
1
\end{pmatrix}
\]
are orthogonal since
\[
\langle u, v \rangle = (2 + 3i)(1 + i) + (4 - 2i)(2 - 3i) + (1 + i)(4 + 6i) + (1 + i)(1)
= (-1 + 5i) + (2 - 16i) + (-2 + 10i) + (1 + i)
= 0 + 0i.
\]

We extend this definition to whole sets by requiring vectors to be pairwise orthogonal. Despite using the same word, careful thought about what objects you are using will eliminate any source of confusion.

Definition OSV
Orthogonal Set of Vectors
Suppose that \( S = \{u_1, u_2, u_3, \ldots, u_n\} \) is a set of vectors from \( \mathbb{C}^m \). Then the set \( S \) is orthogonal if every pair of different vectors from \( S \) is orthogonal, that is, \( \langle u_i, u_j \rangle = 0 \) whenever \( i \neq j \).

The next example is trivial in some respects, but is still worthy of discussion since it is the prototypical orthogonal set.

Example SUVOS
Standard Unit Vectors are an Orthogonal Set
The standard unit vectors are the columns of the identity matrix (Definition SUV [234]). Computing the inner product of two distinct vectors, \( e_i, e_j, i \neq j \), gives,
\[
\langle e_i, e_j \rangle = 0\bar{0} + 0\bar{0} + \cdots + 1\bar{0} + \cdots + 0\bar{0} + \cdots + 0\bar{0} + 0\bar{0}
= 0(0) + 0(0) + \cdots + 1(0) + \cdots + 0(1) + \cdots + 0(0) + 0(0)
= 0.
\]

Example AOS
An orthogonal set
The set
\[
\{x_1, x_2, x_3, x_4\} = \left\{ \begin{pmatrix} 1 + i \\ 6 + 5i \\ 1 - i \\ i \end{pmatrix}, \begin{pmatrix} 1 + 5i \\ 6 + 5i \\ -7 - i \\ 1 - 6i \end{pmatrix}, \begin{pmatrix} -7 + 34i \\ -8 - 23i \\ -10 + 22i \\ 30 + 13i \end{pmatrix}, \begin{pmatrix} -2 - 4i \\ 6 + i \\ 4 + 3i \\ 6 - i \end{pmatrix} \right\}
\]

Version 0.92
is an orthogonal set. Since the inner product is anti-commutative (Theorem IPAC [186])
we can test pairs of different vectors in any order. If the result is zero, then it will also
be zero if the inner product is computed in the opposite order. This means there are six
pairs of different vectors to use in an inner product computation. We’ll do two and you
can practice your inner products on the other four.

\[
\langle x_1, x_3 \rangle = (1 + i)(-7 - 34i) + (1)(-8 + 23i) + (1 - i)(-10 - 22i) + (i)(30 - 13i) \\
= (27 - 41i) + (-8 + 23i) + (-32 - 12i) + (13 + 30i) \\
= 0 + 0i
\]

and

\[
\langle x_2, x_4 \rangle = (1 + 5i)(-2 + 4i) + (6 + 5i)(6 - i) + (-7 - i)(4 - 3i) + (1 - 6i)(6 + i) \\
= (-22 - 6i) + (41 + 24i) + (-31 + 17i) + (12 - 35i) \\
= 0 + 0i
\]

So far, this section has seen lots of definitions, and lots of theorems establishing un-
surprising consequences of those definitions. But here is our first theorem that suggests
that inner products and orthogonal vectors have some utility. It is also one of our first
illustrations of how to arrive at linear independence as the conclusion of a theorem.

**Theorem OSLI**

**Orthogonal Sets are Linearly Independent**

Suppose that \( S = \{u_1, u_2, u_3, \ldots, u_n\} \) is an orthogonal set of nonzero vectors. Then
\( S \) is linearly independent. □

**Proof** To prove linear independence of a set of vectors, we can appeal to the defi-
nition (Definition LICV [145]) and begin with a relation of linear dependence (Defini-
tion RLDCV [145]),

\[
\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_n u_n = 0.
\]

Then, for every \( 1 \leq i \leq n \), we have

\[
0 = 0 \langle u_i, u_i \rangle \\
= \langle 0u_i, u_i \rangle \\
= \langle 0, u_i \rangle \\
= \langle \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_n u_n, u_i \rangle \\
= \langle \alpha_1 u_1, u_i \rangle + \langle \alpha_2 u_2, u_i \rangle + \langle \alpha_3 u_3, u_i \rangle + \cdots + \langle \alpha_n u_n, u_i \rangle \\
= \alpha_1 \langle u_1, u_i \rangle + \alpha_2 \langle u_2, u_i \rangle + \alpha_3 \langle u_3, u_i \rangle + \cdots + \alpha_n \langle u_n, u_i \rangle \\
= \alpha_1 \langle u_1, u_i \rangle + \alpha_2 (0) + \alpha_3 (0) + \cdots + \alpha_n \langle u_n, u_i \rangle \\
= \alpha_i \langle u_i, u_i \rangle
\]

So we have \( 0 = \alpha_i \langle u_i, u_i \rangle \). However, since \( u_i \neq 0 \) (the hypothesis said our vectors were
nonzero), Theorem PIP [188] says that \( \langle u_i, u_i \rangle > 0 \). So we must conclude that \( \alpha_i = 0 \)
for all \( 1 \leq i \leq n \). But this says that \( S \) is a linearly independent set since the only way to
form a relation of linear dependence is the trivial way, with all the scalars zero. Boom!
The Gram-Schmidt Procedure is really a theorem. It says that if we begin with a linearly independent set of \( p \) vectors, \( S \), then we can do a number of calculations with these vectors and produce an orthogonal set of \( p \) vectors, \( T \), so that \( \langle S \rangle = \langle T \rangle \). Given the large number of computations involved, it is indeed a procedure to do all the necessary computations, and it is best employed on a computer. However, it also has value in proofs where we may on occasion wish to replace a linearly independent set by an orthogonal set.

This is our first ocassion to use the technique of “mathematical induction” for a proof, a technique we will see again several times, especially in Chapter D. So study the simple example described in Technique I first.

**Theorem GSPCV**

*Gram-Schmidt Procedure, Column Vectors*

Suppose that \( S = \{v_1, v_2, v_3, \ldots, v_p\} \) is a linearly independent set of vectors in \( \mathbb{C}^m \). Define the vectors \( u_i \), \( 1 \leq i \leq p \) by

\[
 u_i = v_i - \frac{\langle v_i, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_i, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle v_i, u_3 \rangle}{\langle u_3, u_3 \rangle} u_3 - \cdots - \frac{\langle v_i, u_{i-1} \rangle}{\langle u_{i-1}, u_{i-1} \rangle} u_{i-1}
\]

Then if \( T = \{u_1, u_2, u_3, \ldots, u_p\} \), then \( T \) is an orthogonal set of non-zero vectors, and \( \langle T \rangle = \langle S \rangle \).

**Proof** We will prove the result by using induction on \( p \) (Technique I). To begin, we prove that \( T \) has the desired properties when \( p = 1 \). In this case \( u_1 = v_1 \) and \( T = \{u_1\} = \{v_1\} = S \). Because \( S \) and \( T \) are equal, \( \langle S \rangle = \langle T \rangle \). Equally trivial, \( T \) is an orthogonal set. If \( u_1 = 0 \), then \( S \) would be a linearly dependent set, a contradiction.

Now suppose that the theorem is true for any set of \( p - 1 \) linearly independent vectors. Let \( S = \{v_1, v_2, v_3, \ldots, v_p\} \) be a linearly independent set of \( p \) vectors. Then \( S' = \{v_1, v_2, v_3, \ldots, v_{p-1}\} \) is also linearly independent. So we can apply the theorem to \( S' \) and construct the vectors \( T' = \{u_1, u_2, u_3, \ldots, u_{p-1}\} \). \( T' \) is therefore an orthogonal set of nonzero vectors and \( \langle S' \rangle = \langle T' \rangle \). Define

\[
 u_p = v_p - \frac{\langle v_p, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_p, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle v_p, u_3 \rangle}{\langle u_3, u_3 \rangle} u_3 - \cdots - \frac{\langle v_p, u_{p-1} \rangle}{\langle u_{p-1}, u_{p-1} \rangle} u_{p-1}
\]

and let \( T = T' \cup \{u_p\} \). We need to now show that \( T \) has several properties by building on what we know about \( T' \). But first notice that the above equation has no problems with the denominators (\( \langle u_i, u_i \rangle \)) being zero, since the \( u_i \) are from \( T' \), which is composed of nonzero vectors.

We show that \( \langle T \rangle = \langle S \rangle \), by first establishing that \( \langle T \rangle \subseteq \langle S \rangle \). Suppose \( x \in \langle T \rangle \), so

\[
 x = a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_p u_p
\]

The term \( a_p u_p \) is a linear combination of vectors from \( T' \) and the vector \( v_p \), while the remaining terms are a linear combination of vectors from \( T' \). Since \( \langle T' \rangle = \langle S' \rangle \), any term that is a multiple of a vector from \( T' \) can be rewritten as a linear combination of vectors from \( S' \). The remaining term \( a_p v_p \) is a multiple of a vector in \( S \). So we see that \( x \) can be rewritten as a linear combination of vectors from \( S \), i.e. \( x \in \langle S \rangle \).
To show that \( \langle S \rangle \subseteq \langle T \rangle \), begin with \( y \in \langle S \rangle \), so
\[
y = a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_pv_p
\]
Rearrange our defining equation for \( u_p \) by solving for \( v_p \). Then the term \( a_pv_p \) is a multiple of a linear combination of elements of \( T \). The remaining terms are a linear combination of \( v_1, v_2, v_3, \ldots, v_{p-1} \), hence an element of \( \langle S' \rangle = \langle T' \rangle \). Thus these remaining terms can be written as a linear combination of the vectors in \( T' \). So \( y \) is a linear combination of vectors from \( T \), i.e. \( y \in \langle T \rangle \).

The elements of \( T' \) are nonzero, but what about \( u_p \)? Suppose to the contrary that \( u_p = 0 \),
\[
0 = u_p = v_p - \left( \frac{v_p, u_1}{u_1, u_1} \right) u_1 - \left( \frac{v_p, u_2}{u_2, u_2} \right) u_2 - \left( \frac{v_p, u_3}{u_3, u_3} \right) u_3 - \cdots - \left( \frac{v_p, u_{p-1}}{u_{p-1}, u_{p-1}} \right) u_{p-1}
\]
\[
v_p = \left( \frac{v_p, u_1}{u_1, u_1} \right) u_1 + \left( \frac{v_p, u_2}{u_2, u_2} \right) u_2 + \left( \frac{v_p, u_3}{u_3, u_3} \right) u_3 + \cdots + \left( \frac{v_p, u_{p-1}}{u_{p-1}, u_{p-1}} \right) u_{p-1}
\]
Since \( \langle S' \rangle = \langle T' \rangle \) we can write the vectors \( u_1, u_2, u_3, \ldots, u_{p-1} \) on the right side of this equation in terms of the vectors \( v_1, v_2, v_3, \ldots, v_{p-1} \) and we then have the vector \( v_p \) expressed as a linear combination of the other \( p - 1 \) vectors in \( S \), implying that \( S \) is a linearly dependent set \( \text{[Theorem DLDS \[167\]]} \), contrary to our lone hypothesis about \( S \).

Finally, it is a simple matter to establish that \( T \) is an orthogonal set, though it will not appear so simple looking. Think about your objects as you work through the following — what is a vector and what is a scalar. Since \( T' \) is an orthogonal set by induction, most pairs of elements in \( T \) are orthogonal. We just need to test inner products between \( u_p \) and \( u_i \), for \( 1 \leq i \leq p - 1 \). Here we go, using summation notation,
\[
\langle u_p, u_i \rangle = \left( v_p - \sum_{k=1}^{p-1} \left( \frac{v_p, u_k}{u_k, u_k} \right) u_k \right) \cdot u_i
\]
\[
= \langle v_p, u_i \rangle - \sum_{k=1}^{p-1} \left( \frac{v_p, u_k}{u_k, u_k} \right) \langle u_k, u_i \rangle \quad \text{[Theorem IPVA \[185\] ]}
\]
\[
= \langle v_p, u_i \rangle - \sum_{k=1}^{p-1} \left( \frac{v_p, u_k}{u_k, u_k} \right) \langle u_k, u_i \rangle \quad \text{[Theorem IPVA \[185\] ]}
\]
\[
= \langle v_p, u_i \rangle - \sum_{k=1}^{p-1} \left( \frac{v_p, u_k}{u_k, u_k} \right) \langle u_k, u_i \rangle \quad \text{[Theorem IPSM \[186\] ]}
\]
\[
= \langle v_p, u_i \rangle - \frac{\langle v_p, u_i \rangle}{\langle u_i, u_i \rangle} \langle u_i, u_i \rangle - \sum_{k \neq i} \frac{\langle v_p, u_k \rangle}{\langle u_k, u_k \rangle} \langle u_k, u_i \rangle \quad T' \text{ orthogonal}
\]
\[
= \langle v_p, u_i \rangle - \langle v_p, u_i \rangle - \sum_{k \neq i} 0 = 0
\]

**Example GSTV**

**Gram-Schmidt of three vectors**

We will illustrate the Gram-Schmidt process with three vectors. Begin with the
linearly independent (check this!) set

\[ S = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} = \left\{ \begin{bmatrix} 1 & 1+i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ i \end{bmatrix} \right\} \]

Then

\[ \mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 + i \end{bmatrix} \]

\[ \mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 = \frac{1}{4} \begin{bmatrix} -2 - 3i \\ 1 - i \\ 2 + 5i \end{bmatrix} \]

\[ \mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle} \mathbf{u}_2 = \frac{1}{11} \begin{bmatrix} -3 - i \\ 1 + 3i \\ -1 - i \end{bmatrix} \]

and

\[ T = \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} = \left\{ \begin{bmatrix} 1 \\ 1 + i \\ 1 \end{bmatrix}, \frac{1}{4} \begin{bmatrix} -2 - 3i \\ 1 - i \\ 2 + 5i \end{bmatrix}, \frac{1}{11} \begin{bmatrix} -3 - i \\ 1 + 3i \\ -1 - i \end{bmatrix} \right\} \]

is an orthogonal set (which you can check) of nonzero vectors and \( \langle T \rangle = \langle S \rangle \) (all by Theorem GSPCV [191]). Of course, as a by-product of orthogonality, the set \( T \) is also linearly independent (Theorem OSLI [190]).

One final definition related to orthogonal vectors.

**Definition ONS**

**OrthoNormal Set**

Suppose \( S = \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \} \) is an orthogonal set of vectors such that \( \| \mathbf{u}_i \| = 1 \) for all \( 1 \leq i \leq n \). Then \( S \) is an **orthonormal** set of vectors.

Once you have an orthogonal set, it is easy to convert it to an orthonormal set — multiply each vector by the reciprocal of its norm, and the resulting vector will have norm 1. This scaling of each vector will not affect the orthogonality properties (apply Theorem IPSM [186]).

**Example ONTV**

**Orthonormal set, three vectors**

The set

\[ T = \{ \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \} = \left\{ \begin{bmatrix} 1 \\ 1 + i \\ 1 \end{bmatrix}, \frac{1}{4} \begin{bmatrix} -2 - 3i \\ 1 - i \\ 2 + 5i \end{bmatrix}, \frac{1}{11} \begin{bmatrix} -3 - i \\ 1 + 3i \\ -1 - i \end{bmatrix} \right\} \]

from Example GSTV [192] is an orthogonal set. We compute the norm of each vector,

\[ \| \mathbf{u}_1 \| = 2 \quad \| \mathbf{u}_2 \| = \frac{1}{2} \sqrt{11} \quad \| \mathbf{u}_3 \| = \frac{\sqrt{2}}{\sqrt{11}} \]
Converting each vector to a norm of 1, yields an orthonormal set,

\[ w_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 + i \\ 1 \end{bmatrix} \]

\[ w_2 = \frac{1}{\sqrt{11}} \begin{bmatrix} -2 - 3i \\ 1 - i \\ 2 + 5i \end{bmatrix} = \frac{1}{2\sqrt{11}} \begin{bmatrix} -2 - 3i \\ 1 - i \\ 2 + 5i \end{bmatrix} \]

\[ w_3 = \frac{1}{\sqrt{11}} \begin{bmatrix} -3 - i \\ 1 + 3i \\ -1 - i \end{bmatrix} = \frac{1}{\sqrt{22}} \begin{bmatrix} -3 - i \\ 1 + 3i \\ -1 - i \end{bmatrix} \]

Example ONFV
Orthonormal set, four vectors

As an exercise convert the linearly independent set

\[ S = \left\{ \begin{bmatrix} 1 + i \\ 1 \\ 1 - i \end{bmatrix}, \begin{bmatrix} i \\ 1 + i \\ -1 \end{bmatrix}, \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}, \begin{bmatrix} -1 - i \\ i \\ -1 \end{bmatrix} \right\} \]

to an orthogonal set via the Gram-Schmidt Process (Theorem GSPCV [191]) and then scale the vectors to norm 1 to create an orthonormal set. You should get the same set you would if you scaled the orthogonal set of Example AOS [189] to become an orthonormal set.

It is crazy to do all but the simplest and smallest instances of the Gram-Schmidt procedure by hand. Well, OK, maybe just once or twice to get a good understanding of Theorem GSPCV [191]. After that, let a machine do the work for you. That’s what they are for. See: Computation GSP.MMA [682]. Over the course of the next couple of chapters we will discover that orthonormal sets have some very nice properties (in addition to being linearly independent).

Subsection READ
Reading Questions

1. Is the set

\[ \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ -2 \end{bmatrix} \right\} \]

an orthogonal set? Why?

2. What is the distinction between an orthogonal set and an orthonormal set?

3. What is nice about the output of the Gram-Schmidt process?
Subsection EXC
Exercises

C20 Complete Example AOS I89 by verifying that the four remaining inner products are zero.
Contributed by Robert Beezer

C21 Verify that the set $T$ created in Example GSTV 192 by the Gram-Schmidt Procedure is an orthogonal set.
Contributed by Robert Beezer

T10 Prove part 1 of the conclusion of Theorem IPVA I85.
Contributed by Robert Beezer

T11 Prove part 1 of the conclusion of Theorem IPSM I86.
Contributed by Robert Beezer
Chapter M
Matrices

We have made frequent use of matrices for solving systems of equations, and we have begun to investigate a few of their properties, such as the null space and nonsingularity. In this chapter, we will take a more systematic approach to the study of matrices.

Section MO
Matrix Operations

In this section we will back up and start simple. First a definition of a totally general set of matrices.

Definition VSM
Vector Space of $m \times n$ Matrices
The vector space $M_{mn}$ is the set of all $m \times n$ matrices with entries from the set of complex numbers.
(This definition contains Notation VSM.)

Subsection MEASM
Matrix equality, addition, scalar multiplication

Just as we made, and used, a careful definition of equality for column vectors, so too, we have precise definitions for matrices.

Definition ME
Matrix Equality
The $m \times n$ matrices $A$ and $B$ are equal, written $A = B$ provided $[A]_{ij} = [B]_{ij}$ for all $1 \leq i \leq m, 1 \leq j \leq n$.
(This definition contains Notation ME.)

So equality of matrices translates to the equality of complex numbers, on an entry-by-entry basis. Notice that we now have yet another definition that uses the symbol “=” for shorthand. Whenever a theorem has a conclusion saying two matrices are equal (think about your objects), we will consider appealing to this definition as a way of
formulating the top-level structure of the proof. We will now define two operations on the set $M_{mn}$. Again, we will overload a symbol (‘+’) and a convention (juxtaposition for scalar multiplication).

**Definition MA**

**Matrix Addition**

Given the $m \times n$ matrices $A$ and $B$, define the sum of $A$ and $B$ as an $m \times n$ matrix, written $A + B$, according to

$$[A + B]_{ij} = [A]_{ij} + [B]_{ij} \quad 1 \leq i \leq m, 1 \leq j \leq n$$

(This definition contains Notation MA.)

So matrix addition takes two matrices of the same size and combines them (in a natural way!) to create a new matrix of the same size. Perhaps this is the “obvious” thing to do, but it doesn’t relieve us from the obligation to state it carefully.

**Example MA**

**Addition of two matrices in $M_{23}$**

If

$$A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -7 \end{bmatrix} \quad B = \begin{bmatrix} 6 & 2 & -4 \\ 3 & 5 & 2 \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -7 \end{bmatrix} + \begin{bmatrix} 6 & 2 & -4 \\ 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 + 6 & -3 + 2 & 4 + (-4) \\ 1 + 3 & 0 + 5 & -7 + 2 \end{bmatrix} = \begin{bmatrix} 8 & -1 & 0 \\ 4 & 5 & -5 \end{bmatrix}$$

Our second operation takes two objects of different types, specifically a number and a matrix, and combines them to create another matrix. As with vectors, in this context we call a number a **scalar** in order to emphasize that it is not a matrix.

**Definition MSM**

**Matrix Scalar Multiplication**

Given the $m \times n$ matrix $A$ and the scalar $\alpha \in \mathbb{C}$, the scalar multiple of $A$ is an $m \times n$ matrix, written $\alpha A$ and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij} \quad 1 \leq i \leq m, 1 \leq j \leq n$$

(This definition contains Notation MSM.)

Notice again that we have yet another kind of multiplication, and it is again written putting two symbols side-by-side. Computationally, scalar matrix multiplication is very easy.

**Example MSM**

**Scalar multiplication in $M_{32}$**

If

$$A = \begin{bmatrix} 2 & 8 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}$$
and $\alpha = 7$, then

$$\alpha A = 7 \begin{bmatrix} 2 & 8 \\ -3 & 5 \\ 0 & 1 \end{bmatrix} = 7 \begin{bmatrix} 2(0) & 8(0) \\ 7(0) & 7(5) \\ 7(0) & 7(1) \end{bmatrix} = 7 \begin{bmatrix} 14 & 56 \\ -21 & 35 \\ 0 & 7 \end{bmatrix} \red{\square}$$

Subsection VSP
Vector Space Properties

With definitions of matrix addition and scalar multiplication we can now state, and prove, several properties of each operation, and some properties that involve their interplay. We now collect ten of them here for later reference.

**Theorem VSPM**
Vector Space Properties of Matrices
Suppose that $M_{mn}$ is the set of all $m \times n$ matrices (Definition VSM [197]) with addition and scalar multiplication as defined in Definition MA [198] and Definition MSM [198]. Then

- **ACM** Additive Closure, Matrices
  If $A, B \in M_{mn}$, then $A + B \in M_{mn}$.

- **SCM** Scalar Closure, Matrices
  If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.

- **CM** Commutativity, Matrices
  If $A, B \in M_{mn}$, then $A + B = B + A$.

- **AAM** Additive Associativity, Matrices
  If $A, B, C \in M_{mn}$, then $A + (B + C) = (A + B) + C$.

- **ZM** Zero Vector, Matrices
  There is a matrix, $\mathcal{O}$, called the zero matrix, such that $A + \mathcal{O} = A$ for all $A \in M_{mn}$.

- **AIM** Additive Inverses, Matrices
  If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.

- **SMAM** Scalar Multiplication Associativity, Matrices
  If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha \beta)A$.

- **DMAM** Distributivity across Matrix Addition, Matrices
  If $\alpha \in \mathbb{C}$ and $A, B \in M_{mn}$, then $\alpha(A + B) = \alpha A + \alpha B$.

- **DSAM** Distributivity across Scalar Addition, Matrices
  If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.

- **OM** One, Matrices
  If $A \in M_{mn}$, then $1A = A$. 

Version 0.92
Proof While some of these properties seem very obvious, they all require proof. However, the proofs are not very interesting, and border on tedious. We’ll prove one version of distributivity very carefully, and you can test your proof-building skills on some of the others. We’ll give our new notation for matrix entries a workout here. Compare the style of the proofs here with those given for vectors in Theorem VSPCV [91] — while the objects here are more complicated, our notation makes the proofs cleaner.

To prove Property DSAM [199], \((\alpha + \beta)A = \alpha A + \beta A\), we need to establish the equality of two matrices (see Technique GS [702]). Definition ME [197] says we need to establish the equality of their entries, one-by-one. How do we do this, when we do not even know how many entries the two matrices might have? This is where Notation ME [197] comes into play. Ready? Here we go.

For any \(i\) and \(j\), \(1 \leq i \leq m, 1 \leq j \leq n\),

\[
[\,(\alpha + \beta)A]\,_{ij} = (\alpha + \beta)\,_{ij} \quad \text{Definition MSM [198]}
\]
\[
= \alpha \,_{ij} + \beta \,_{ij} \quad \text{Distributivity in } \mathbb{C}
\]
\[
= [\alpha A]\,_{ij} + [\beta A]\,_{ij} \quad \text{Definition MSM [198]}
\]
\[
= [\alpha A + \beta A]\,_{ij} \quad \text{Definition MA [198]}
\]

There are several things to notice here. (1) Each equals sign is an equality of numbers. (2) The two ends of the equation, being true for any \(i\) and \(j\), allow us to conclude the equality of the matrices by Definition ME [197]. (3) There are several plus signs, and several instances of juxtaposition. Identify each one, and state exactly what operation is being represented by each.

For now, note the similarities between Theorem VSPM [199] about matrices and Theorem VSPCV [91] about vectors.

The zero matrix described in this theorem, \(O\), is what you would expect — a matrix full of zeros.

Definition ZM
Zero Matrix

The \(m \times n\) \textbf{zero matrix} is written as \(O = O_{m \times n}\) and defined by \([O]\,_{ij} = 0\), for all \(1 \leq i \leq m, 1 \leq j \leq n\).

(This definition contains Notation ZM.)

Subsection TSM
Transposes and Symmetric Matrices

We describe one more common operation we can perform on matrices. Informally, to transpose a matrix is to build a new matrix by swapping its rows and columns.

Definition TM
Transpose of a Matrix

Given an \(m \times n\) matrix \(A\), its \textbf{transpose} is the \(n \times m\) matrix \(A^t\) given by

\[
[A^t]\,_{ij} = [A]\,_{ji}, \quad 1 \leq i \leq n, 1 \leq j \leq m.
\]
Example TM

Transpose of a $3 \times 4$ matrix

Suppose

$$D = \begin{bmatrix} 3 & 7 & 2 & -3 \\ -1 & 4 & 2 & 8 \\ 0 & 3 & -2 & 5 \end{bmatrix}.$$  

We could formulate the transpose, entry-by-entry, using the definition. But it is easier to just systematically rewrite rows as columns (or vice-versa). The form of the definition given will be more useful in proofs. So we have

$$D^t = \begin{bmatrix} 3 & -1 & 0 \\ 7 & 4 & 3 \\ 2 & 2 & -2 \\ -3 & 8 & 5 \end{bmatrix}.$$  

It will sometimes happen that a matrix is equal to its transpose. In this case, we will call a matrix symmetric. These matrices occur naturally in certain situations, and also have some nice properties, so it is worth stating the definition carefully. Informally a matrix is symmetric if we can “flip” it about the main diagonal (upper-left corner, running down to the lower-right corner) and have it look unchanged.

Definition SYM

Symmetric Matrix

The matrix $A$ is symmetric if $A = A^t$.

Example SYM

A symmetric $5 \times 5$ matrix

The matrix

$$E = \begin{bmatrix} 2 & 3 & -9 & 5 & 7 \\ 3 & 1 & 6 & -2 & -3 \\ -9 & 6 & 0 & -1 & 9 \\ 5 & -2 & 4 & -8 & -3 \\ 7 & -3 & 9 & -8 & -3 \end{bmatrix}$$

is symmetric.

You might have noticed that Definition SYM did not specify the size of the matrix $A$, as has been our custom. That’s because it wasn’t necessary. An alternative would have been to state the definition just for square matrices, but this is the substance of the next proof. Before reading the next proof, we want to offer you some advice about how to become more proficient at constructing proofs. Perhaps you can apply this advice to the next theorem. Have a peek at Technique P now.

Theorem SMS

Symmetric Matrices are Square

Suppose that $A$ is a symmetric matrix. Then $A$ is square.
**Proof** We start by specifying $A$’s size, without assuming it is square, since we are trying to prove that, so we can’t also assume it. Suppose $A$ is an $m \times n$ matrix. Because $A$ is symmetric, we know by Definition SM that $A = A^t$. So, in particular, Definition ME requires that $A$ and $A^t$ must have the same size. The size of $A^t$ is $n \times m$. Because $A$ has $m$ rows and $A^t$ has $n$ rows, we conclude that $m = n$, and hence $A$ must be square by Definition SQM.

We finish this section with three easy theorems, but they illustrate the interplay of our three new operations, our new notation, and the techniques used to prove matrix equalities.

**Theorem TMA**
**Transpose and Matrix Addition**
Suppose that $A$ and $B$ are $m \times n$ matrices. Then $(A + B)^t = A^t + B^t$.

**Proof** The statement to be proved is an equality of matrices, so we work entry-by-entry and use Definition ME. Think carefully about the objects involved here, and the many uses of the plus sign.

$$
(A + B)^t = A^t + B^t
$$

Since the matrices $(A + B)^t$ and $A^t + B^t$ agree at each entry, Definition ME tells us the two matrices are equal.

**Theorem TMSM**
**Transpose and Matrix Scalar Multiplication**
Suppose that $\alpha \in \mathbb{C}$ and $A$ is an $m \times n$ matrix. Then $(\alpha A)^t = \alpha A^t$.

**Proof** The statement to be proved is an equality of matrices, so we work entry-by-entry and use Definition ME. Think carefully about the objects involved here, the many uses of juxtaposition.

$$
(\alpha A)^t = \alpha A^t
$$

Since the matrices $(\alpha A)^t$ and $\alpha A^t$ agree at each entry, Definition ME tells us the two matrices are equal.

**Theorem TT**
**Transpose of a Transpose**
Suppose that $A$ is an $m \times n$ matrix. Then $(A^t)^t = A$.

**Proof** We again want to prove an equality of matrices, so we work entry-by-entry and use Definition ME.

$$
(A^t)^t = A^t
$$
As we did with vectors (Definition CCCV [183]), we can define what it means to take the conjugate of a matrix.

**Definition CCM**

**Complex Conjugate of a Matrix**

Suppose $A$ is an $m \times n$ matrix. Then the conjugate of $A$, written $\overline{A}$ is an $m \times n$ matrix defined by

$$\overline{A}_{ij} = \overline{A}_{ij}$$

(This definition contains Notation CCM.)

**Example CCM**

**Complex conjugate of a matrix**

If

$$A = \begin{bmatrix} 2 - i & 3 & 5 + 4i \\ -3 + 6i & 2 - 3i & 0 \end{bmatrix}$$

then

$$\overline{A} = \begin{bmatrix} 2 + i & 3 & 5 - 4i \\ -3 - 6i & 2 + 3i & 0 \end{bmatrix}$$

The interplay between the conjugate of a matrix and the two operations on matrices is what you might expect.

**Theorem CRMA**

**Conjugation Respects Matrix Addition**

Suppose that $A$ and $B$ are $m \times n$ matrices. Then $\overline{A + B} = \overline{A} + \overline{B}$. □

**Proof**

$$[A + B]_{ij} = \overline{[A + B]_{ij}}$$

Definition CCM 203

$$= \overline{A}_{ij} + \overline{B}_{ij}$$

Definition MA 198

$$= \overline{A}_{ij} + \overline{B}_{ij}$$

Theorem CCRA 690

$$= \overline{A}_{ij} + \overline{B}_{ij}$$

Definition CCM 203

$$= \overline{A + B}_{ij}$$

Definition MA 198

Since the matrices $\overline{A + B}$ and $\overline{A} + \overline{B}$ are equal in each entry, Definition ME 197 says that $A + B = \overline{A + B}$. □
Theorem CRMSM
Conjugation Respects Matrix Scalar Multiplication

Suppose that \( \alpha \in \mathbb{C} \) and \( A \) is an \( m \times n \) matrix. Then \( \alpha A = \overline{\alpha A} \).

\[ \begin{align*}
[\alpha A]_{ij} &= \overline{[\alpha A]_{ij}} \\
&= \alpha [A]_{ij} \\
&= \overline{[\alpha A]_{ij}} \\
&= \overline{[A]_{ji}} \\
&= \overline{[\overline{A}]_{ji}} \\
&= [\overline{\alpha A}]_{ij}
\end{align*} \]

Since the matrices \( \alpha A \) and \( \overline{\alpha A} \) are equal in each entry, Definition ME 197 says that \( \alpha A = \overline{\alpha A} \).

Finally, we will need the following result about matrix conjugation and transposes later.

Theorem MCT
Matrix Conjugation and Transposes

Suppose that \( A \) is an \( m \times n \) matrix. Then \( (A^t) = (\overline{A})^t \).

\[ \begin{align*}
\left[ (A^t) \right]_{ij} &= \overline{\left[ A^t \right]_{ij}} \\
&= \overline{[A]_{ji}} \\
&= \overline{[\overline{A}]_{ji}} \\
&= \overline{\left[ (\overline{A})^t \right]_{ij}} \end{align*} \]

Since the matrices \( (A^t) \) and \( (\overline{A})^t \) are equal in each entry, Definition ME 197 says that \( (A^t) = (\overline{A})^t \).

Subsection READ
Reading Questions

1. Perform the following matrix computation.
\[
\begin{pmatrix}
2 & -2 & 8 & 1 \\
4 & 5 & -1 & 3 \\
7 & -3 & 0 & 2
\end{pmatrix} + (-2) \begin{pmatrix}
2 & 7 & 1 & 2 \\
3 & -1 & 0 & 5 \\
1 & 7 & 3 & 3
\end{pmatrix}
\]

2. Theorem VSPM 199 reminds you of what previous theorem? How strong is the similarity?
3. Compute the transpose of the matrix below.

\[
\begin{bmatrix}
6 & 8 & 4 \\
-2 & 1 & 0 \\
9 & -5 & 6
\end{bmatrix}
\]
In Chapter V we defined the operations of vector addition and vector scalar multiplication in Definition CVA and Definition CVSM. These two operations formed the underpinnings of the remainder of the chapter. We have now defined similar operations for matrices in Definition MA and Definition MSM. You will have noticed the resulting similarities between Theorem VSPCV and Theorem VSPM.

In Exercises M20–M25, you will be asked to extend these similarities to other fundamental definitions and concepts we first saw in Chapter V. This sequence of problems was suggested by Martin Jackson.

**M20** Suppose \( S = \{B_1, B_2, B_3, \ldots, B_p\} \) is a set of matrices from \( M_{mn} \). Formulate appropriate definitions for the following terms and give an example of the use of each.

1. A linear combination of elements of \( S \).
2. A relation of linear dependence on \( S \), both trivial and non-trivial.
3. \( S \) is a linearly independent set.
4. \( \langle S \rangle \).

Contributed by Robert Beezer

**M21** Show that the set \( S \) is linearly independent in \( M_{2,2} \).

\[
S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]

Contributed by Robert Beezer

**M22** Determine if the set

\[
S = \left\{ \begin{bmatrix} -2 & 3 & 4 \\ -1 & 3 & -2 \end{bmatrix}, \begin{bmatrix} 4 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -2 & -2 \\ 2 & 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 2 & -2 \\ 0 & -1 & -2 \end{bmatrix} \right\}
\]

is linearly independent in \( M_{2,3} \).

Contributed by Robert Beezer

**M23** Determine if the matrix \( A \) is in the span of \( S \). In other words, is \( A \in \langle S \rangle \)? If so write \( A \) as a linear combination of the elements of \( S \).

\[
A = \begin{bmatrix} -13 & 24 & 2 \\ -8 & -2 & -20 \end{bmatrix}
\]

\[
S = \left\{ \begin{bmatrix} -2 & 3 & 4 \\ -1 & 3 & -2 \end{bmatrix}, \begin{bmatrix} 4 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -2 & -2 \\ 2 & 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 2 & -2 \\ 0 & -1 & -2 \end{bmatrix} \right\}
\]
M24 Suppose $Y$ is the set of all $3 \times 3$ symmetric matrices (Definition SYM [201]). Find a set $T$ so that $T$ is linearly independent and $\langle T \rangle = Y$.

M25 Define a subset of $M_{3,3}$ by

$$U_{33} = \left\{ A \in M_{3,3} \mid [A]_{ij} = 0 \text{ whenever } i > j \right\}$$

Find a set $R$ so that $R$ is linearly independent and $\langle R \rangle = U_{33}$.

T13 Prove Property CM [199] of Theorem VSPM [199]. Write your proof in the style of the proof of Property DSAM [199] given in this section.

T17 Prove Property SMAM [199] of Theorem VSPM [199]. Write your proof in the style of the proof of Property DSAM [199] given in this section.

T18 Prove Property DMAM [199] of Theorem VSPM [199]. Write your proof in the style of the proof of Property DSAM [199] given in this section.
For all \( A, B \in M_{mn} \) and for all \( 1 \leq i \leq m, 1 \leq j \leq n \),

\[
[A + B]_{ij} = [A]_{ij} + [B]_{ij} \quad \text{Definition MA 198}
\]

\[
= [B]_{ij} + [A]_{ij} \quad \text{Commutativity in } \mathbb{C}
\]

\[
= [B + A]_{ij} \quad \text{Definition MA 198}
\]

With equality of each entry of the matrices \( A + B \) and \( B + A \) being equal \text{Definition ME 197} tells us the two matrices are equal.
Section MM
Matrix Multiplication

We know how to add vectors and how to multiply them by scalars. Together, these operations give us the possibility of making linear combinations. Similarly, we know how to add matrices and how to multiply matrices by scalars. In this section we mix all these ideas together and produce an operation known as matrix multiplication. This will lead to some results that are both surprising and central. We begin with a definition of how to multiply a vector by a matrix.

Subsection MVP
Matrix-Vector Product

We have repeatedly seen the importance of forming linear combinations of the columns of a matrix. As one example of this, the oft-used Theorem SLSLC, said that every solution to a system of linear equations gives rise to a linear combination of the column vectors of the coefficient matrix that equals the vector of constants. This theorem, and others, motivate the following central definition.

Definition MVP
Matrix-Vector Product
Suppose $A$ is an $m \times n$ matrix with columns $A_1, A_2, A_3, \ldots, A_n$ and $u$ is a vector of size $n$. Then the matrix-vector product of $A$ with $u$ is the linear combination

$$Au = [u]_1 A_1 + [u]_2 A_2 + [u]_3 A_3 + \cdots + [u]_n A_n$$

(This definition contains Notation MVP.)

So, the matrix-vector product is yet another version of “multiplication,” at least in the sense that we have yet again overloaded juxtaposition of two symbols as our notation. Remember your objects, an $m \times n$ matrix times a vector of size $n$ will create a vector of size $m$. So if $A$ is rectangular, then the size of the vector changes. With all the linear combinations we have performed so far, this computation should now seem second nature.

Example MTV
A matrix times a vector
Consider

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 & 4 \\ -3 & 2 & 0 & 1 & -2 \\ 1 & 6 & -3 & -1 & 5 \end{bmatrix}, \quad u = \begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \\ -1 \end{bmatrix}$$
Then
\[
Au = 2 \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} + (-2) \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 6 \end{bmatrix}.
\]

This definition now makes it possible to represent systems of linear equations compactly in terms of an operation.

**Theorem SLEMM**

**Systems of Linear Equations as Matrix Multiplication**

Solutions to the linear system \( L \mathbf{S}(A, b) \) are the solutions for \( x \) in the vector equation \( Ax = b \). □

**Proof** This theorem says that two sets (of solutions) are equal. So we need to show that one set of solutions is a subset of the other, and vice versa (recall Definition SE [694]). Let \( A_1, A_2, A_3, \ldots, A_n \) be the columns of \( A \). Both of these set inclusions then follow from the following chain of equivalences,

\[
x \text{ is a solution to } L \mathbf{S}(A, b) \iff [x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n = b \quad \text{Theorem SLSLC [100]} \\
\iff x \text{ is a solution to } Ax = b \quad \text{Definition MVP [211]}
\]

**Example MNSLE**

**Matrix notation for systems of linear equations**

Consider the system of linear equations from Example NSLE [32].

\[
\begin{align*}
2x_1 + 4x_2 - 3x_3 + 5x_4 + x_5 &= 9 \\
3x_1 + x_2 + x_4 - 3x_5 &= 0 \\
-2x_1 + 7x_2 - 5x_3 + 2x_4 + 2x_5 &= -3
\end{align*}
\]

has coefficient matrix
\[
A = \begin{bmatrix} 2 & 4 & -3 & 5 & 1 \\ 3 & 1 & 0 & 1 & -3 \\ -2 & 7 & -5 & 2 & 2 \end{bmatrix}
\]

and vector of constants
\[
b = \begin{bmatrix} 9 \\ 0 \\ -3 \end{bmatrix}
\]

and so will be described compactly by the vector equation \( Ax = b \). □

The matrix-vector product is a very natural computation. We have motivated it by its connections with systems of equations, but here is a another example.

**Example MBC**

**Money’s best cities**

Every year *Money* magazine selects several cities in the United States as the “best”
cities to live in, based on a wide array of statistics about each city. This is an example of how the editors of *Money* might arrive at a single number that consolidates the statistics about a city. We will analyze Los Angeles, Chicago and New York City, based on four criteria: average high temperature in July (Fahrenheit), number of colleges and universities in a 30-mile radius, number of toxic waste sites in the Superfund clean-up program and a personal crime index based on FBI statistics (average = 100, smaller is safer). It should be apparent how to generalize the example to a greater number of cities and a greater number of statistics.

We begin by building a table of statistics. The rows will be labeled with the cities, and the columns with statistical categories. These values are from *Money*’s website in early 2005.

<table>
<thead>
<tr>
<th>City</th>
<th>Temp</th>
<th>Colleges</th>
<th>Superfund</th>
<th>Crime</th>
</tr>
</thead>
<tbody>
<tr>
<td>Los Angeles</td>
<td>77</td>
<td>28</td>
<td>93</td>
<td>254</td>
</tr>
<tr>
<td>Chicago</td>
<td>84</td>
<td>38</td>
<td>85</td>
<td>363</td>
</tr>
<tr>
<td>New York</td>
<td>84</td>
<td>99</td>
<td>1</td>
<td>193</td>
</tr>
</tbody>
</table>

Conceivably these data might reside in a spreadsheet. Now we must combine the statistics for each city. We could accomplish this by weighting each category, scaling the values and summing them. The sizes of the weights would depend upon the numerical size of each statistic generally, but more importantly, they would reflect the editors opinions or beliefs about which statistics were most important to their readers. Is the crime index more important than the number of colleges and universities? Of course, there is no right answer to this question.

Suppose the editors finally decide on the following weights to employ: temperature, 0.23; colleges, 0.46; Superfund, −0.05; crime, −0.20. Notice how negative weights are used for undesirable statistics. Then, for example, the editors would compute for Los Angeles,

\[
(0.23)(77) + (0.46)(28) + (-0.05)(93) + (-0.20)(254) = -24.86
\]

This computation might remind you of an inner product, but we will produce the computations for all of the cities as a matrix-vector product. Write the table of raw statistics as a matrix

\[
T = \begin{bmatrix}
77 & 28 & 93 & 254 \\
84 & 38 & 85 & 363 \\
84 & 99 & 1 & 193
\end{bmatrix}
\]

and the weights as a vector

\[
w = \begin{bmatrix}
0.23 \\
0.46 \\
-0.05 \\
-0.20
\end{bmatrix}
\]

then the matrix-vector product (Definition MVP[211]) yields

\[
Tw = (0.23) \begin{bmatrix}
77 \\
84 \\
84
\end{bmatrix} + (0.46) \begin{bmatrix}
28 \\
38 \\
99
\end{bmatrix} + (-0.05) \begin{bmatrix}
93 \\
85 \\
1
\end{bmatrix} + (-0.20) \begin{bmatrix}
254 \\
363 \\
193
\end{bmatrix} = \begin{bmatrix}
-24.86 \\
-40.05 \\
26.21
\end{bmatrix}
\]

This vector contains a single number for each of the cities being studied, so the editors would rank New York best, Los Angeles next, and Chicago third. Of course, the mayor’s
offices in Chicago and Los Angeles are free to counter with a different set of weights that cause their city to be ranked best. These alternative weights would be chosen to play to each cities’ strengths, and minimize their problem areas.

If a spreadsheet were used to make these computations, a row of weights would be entered somewhere near the table of data and the formulas in the spreadsheet would effect a matrix-vector product. This example is meant to illustrate how “linear” computations (addition, multiplication) can be organized as a matrix-vector product.

Another example would be the matrix of numerical scores on examinations and exercises for students in a class. The rows would correspond to students and the columns to exams and assignments. The instructor could then assign weights to the different exams and assignments, and via a matrix-vector product, compute a single score for each student.

Later (much later) we will need the following theorem, which is really a technical lemma (see Technique LC [716]). Since we are in a position to prove it now, we will. But you can safely skip it now, if you promise to come back later to study the proof when the theorem is employed.

**Theorem EMMVP**

**Equal Matrices and Matrix-Vector Products**

Suppose that $A$ and $B$ are $m \times n$ matrices such that $Ax = Bx$ for every $x \in \mathbb{C}^n$. Then $A = B$. □

**Proof** Since $Ax = Bx$ for all $x \in \mathbb{C}^n$, choose $x$ to be a vector of all zeros, with a lone 1 in the $i$-th slot. Then

$$Ax = [A_1 \vert A_2 \vert A_3 \vert \ldots \vert A_n] \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0A_1 + 0A_2 + 0A_3 + \cdots + 0A_{i-1} + 1A_i + 0A_{i+1} + \cdots + 0A_n \quad \text{Definition MVP 211}
$$

$$= A_i
$$

Similarly, $Bx = B_i$, so $A_i = B_i$, $1 \leq i \leq n$ and so all the columns of $A$ and $B$ are equal. Then our definition of column vector equality (Definition CVE 88) establishes that the individual entries of $A$ and $B$ in each column are equal. So by Definition ME 197 the matrices $A$ and $B$ are equal. □

The hypotheses of this theorem could be weakened to suppose only the equality of the matrix-vector products for just the standard unit vectors (Definition SUV 234) or any other basis (Definition B 363) of $\mathbb{C}^n$. However, when we apply this theorem we will only need this weaker form.
Subsection MM
Matrix Multiplication

We now define how to multiply two matrices together. Stop for a minute and think about how you might define this new operation.

Many books would present this definition much earlier in the course. However, we have taken great care to delay it as long as possible and to present as many ideas as practical based mostly on the notion of linear combinations. Towards the conclusion of the course, or when you perhaps take a second course in linear algebra, you may be in a position to appreciate the reasons for this. For now, understand that matrix multiplication is a central definition and perhaps you will appreciate its importance more by having saved it for later.

Definition MM
Matrix Multiplication
Suppose $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix with columns $B_1, B_2, B_3, \ldots, B_p$. Then the matrix product of $A$ with $B$ is the $m \times p$ matrix where column $i$ is the matrix-vector product $AB_i$. Symbolically,

$$AB = A \begin{bmatrix} B_1 \mid B_2 \mid B_3 \mid \ldots \mid B_p \end{bmatrix} = [AB_1 \mid AB_2 \mid AB_3 \mid \ldots \mid AB_p].$$

Example PTM
Product of two matrices
Set

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 & 6 \\ 0 & -4 & 1 & 2 & 3 \\ -5 & 1 & 2 & -3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 & 2 & 1 \\ -1 & 4 & 3 & 2 \\ 1 & 1 & 2 & 3 \\ 6 & 4 & -1 & 2 \\ 1 & -2 & 3 & 0 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & -1 & 6 & 1 \\ 1 & 4 & 3 & 2 \\ 6 & 4 & -1 & 2 \\ 1 & -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 28 & 17 & 20 & 10 \\ 20 & -13 & -3 & -1 \\ -18 & -44 & 12 & -3 \end{bmatrix}.$$
But it gets weirder than that. Many of your old ideas about “multiplication” won’t apply to matrix multiplication, but some still will. So make no assumptions, and don’t do anything until you have a theorem that says you can. Even if the sizes are right, matrix multiplication is not commutative — order matters.

Example MMNC
Matrix Multiplication is not commutative

Set
\[
A = \begin{bmatrix}
  1 & 3 \\
-1 & 2
\end{bmatrix} \quad B = \begin{bmatrix}
  4 & 0 \\
  5 & 1
\end{bmatrix}.
\]

Then we have two square, $2 \times 2$ matrices, so Definition MM \[215\] allows us to multiply them in either order. We find
\[
AB = \begin{bmatrix}
  19 & 3 \\
  6 & 2
\end{bmatrix} \quad BA = \begin{bmatrix}
  4 & 12 \\
  4 & 17
\end{bmatrix}
\]
and $AB \neq BA$. Not even close. It should not be hard for you to construct other pairs of matrices that do not commute (try a couple of $3 \times 3$’s). Can you find a pair of non-identical matrices that do commute? ☐

Matrix multiplication is fundamental, so it is a natural procedure for any computational device. See: Computation MM.MMA \[683\].

Subsection MMEE
Matrix Multiplication, Entry-by-Entry

While certain “natural” properties of multiplication don’t hold, many more do. In the next subsection, we’ll state and prove the relevant theorems. But first, we need a theorem that provides an alternate means of multiplying two matrices. In many texts, this would be given as the definition of matrix multiplication. We prefer to turn it around and have the following formula as a consequence of our definition. It will prove useful for proofs of matrix equality, where we need to examine products of matrices, entry-by-entry.

Theorem EMP

Entries of Matrix Products

Suppose $A$ is an $m \times n$ matrix and $B = \begin{bmatrix} \end{bmatrix}$ is an $n \times p$ matrix. Then for $1 \leq i \leq m$, $1 \leq j \leq p$, the individual entries of $AB$ are given by
\[
[AB]_{ij} = [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \cdots + [A]_{in} [B]_{nj} = \sum_{k=1}^{n} [A]_{ik} [B]_{kj}
\]

Proof Denote the columns of $A$ as the vectors $A_1, A_2, A_3, \ldots, A_n$ and the columns of $B$ as the vectors $B_1, B_2, B_3, \ldots, B_p$. Then for $1 \leq i \leq m$, $1 \leq j \leq p$,
\[
[AB]_{ij} = [AB]_{i},
\]

Definition MM \[215\]
Subsection MM.MMEE  Matrix Multiplication, Entry-by-Entry  217

\[
\begin{align*}
&= [B]_1 A_1 + [B]_2 A_2 + [B]_3 A_3 + \cdots + [B]_n A_n, \quad \text{Definition MVP} \ [211] \\
&= [B]_1 A_1 + [B]_2 A_2 + [B]_3 A_3 + \cdots + [B]_n A_n, \quad \text{Definition CVA} \ [89] \\
&= [B]_1 A_1 + [B]_2 A_2 + [B]_3 A_3 + \cdots + [B]_n A_n, \quad \text{Definition CVSM} \ [89] \\
&= [B]_{ij} A_{i1} + [B]_{j2} A_{i2} + [B]_{j3} A_{i3} + \cdots + [B]_{jn} A_{in}, \quad \text{Notation ME} \ [197] \\
&= [A]_{i1} B_{1j} + [A]_{i2} B_{2j} + [A]_{i3} B_{3j} + \cdots + [A]_{in} B_{nj}, \quad \text{Commutativity in } C
\end{align*}
\]

\[
\sum_{k=1}^{n} [A]_{ik} B_{kj}
\]

\[=\]

\[\begin{bmatrix}
1 & 2 & -1 & 4 & 6 \\
0 & -4 & 1 & 2 & 3 \\
-5 & 1 & 2 & -3 & 4
\end{bmatrix}
\]

\[=\]

\[\begin{bmatrix}
1 & 6 & 2 & 1 \\
-1 & 4 & 3 & 2 \\
1 & 1 & 2 & 3 \\
6 & 4 & -1 & 2 \\
1 & -2 & 3 & 0
\end{bmatrix}
\]

Then suppose we just wanted the entry of \(AB\) in the second row, third column:

\[
[AB]_{23} = [A]_{21} B_{13} + [A]_{22} B_{23} + [A]_{23} B_{33} + [A]_{24} B_{43} + [A]_{25} B_{53}
\]

\[
= (0)(2) + (-4)(3) + (1)(2) + (2)(-1) + (3)(3) = -3
\]

Notice how there are 5 terms in the sum, since 5 is the common dimension of the two matrices (column count for \(A\), row count for \(B\)). In the conclusion of \textbf{Theorem EMP} \[216\], it would be the index \(k\) that would run from 1 to 5 in this computation. Here’s a bit more practice.

The entry of third row, first column:

\[
[AB]_{31} = [A]_{31} B_{11} + [A]_{32} B_{21} + [A]_{33} B_{31} + [A]_{34} B_{41} + [A]_{35} B_{51}
\]

\[
=(-5)(1) + (1)(-1) + (2)(1) + (-3)(6) + (4)(1) = -18
\]

To get some more practice on your own, complete the computation of the other 10 entries of this product. Construct some other pairs of matrices (of compatible sizes) and compute their product two ways. First use \textbf{Definition MM} \[215\]. Since linear combinations are straightforward for you now, this should be easy to do and to do correctly. Then do it again, using \textbf{Theorem EMP} \[216\]. Since this process may take some practice, use your first computation to check your work.

\textbf{Theorem EMP} \[216\] is the way most people compute matrix products by hand. It will also be very useful for the theorems we are going to prove shortly. However, the definition (\textbf{Definition MM} \[215\]) is frequently the most useful for its connections with deeper ideas like the null space and the upcoming column space.
Subsection PMM
Properties of Matrix Multiplication

In this subsection, we collect properties of matrix multiplication and its interaction with the zero matrix (Definition ZM [200]), the identity matrix (Definition IM [76]), matrix addition (Definition MA [198]), scalar matrix multiplication (Definition MSM [198]), the inner product (Definition IP [184]), conjugation (Theorem MMCC [221]), and the transpose (Definition TM [200]). Whew! Here we go. These are great proofs to practice with, so try to concoct the proofs before reading them, they’ll get progressively more complicated as we go.

Theorem MMZM
Matrix Multiplication and the Zero Matrix
Suppose \(A\) is an \(m \times n\) matrix. Then
1. \(AO_{n \times p} = O_{m \times p}\)
2. \(O_{p \times m}A = O_{p \times n}\)

Proof We’ll prove (1) and leave (2) to you. Entry-by-entry,

\[
[AO_{n \times p}]_{ij} = \sum_{k=1}^{n} [A]_{ik} [O_{n \times p}]_{kj} = \sum_{k=1}^{n} [A]_{ik} 0 = 0.
\]

So every entry of the product is the scalar zero, i.e. the result is the zero matrix. □

Theorem MMIM
Matrix Multiplication and Identity Matrix
Suppose \(A\) is an \(m \times n\) matrix. Then
1. \(AI_{n} = A\)
2. \(I_{m}A = A\)

Proof Again, we’ll prove (1) and leave (2) to you. Entry-by-entry,

\[
[AI_{n}]_{ij} = \sum_{k=1}^{n} [A]_{ik} [I_{n}]_{kj} = [A]_{ij} [I_{n}]_{jj} + \sum_{k=1,k \neq j}^{n} [A]_{ik} [I_{n}]_{kj} = [A]_{ij} (1) + \sum_{k=1,k \neq j}^{n} [A]_{ik} (0) = [A]_{ij} + \sum_{k=1,k \neq j}^{n} 0 = [A]_{ij}.
\]
So the matrices $A$ and $AI_n$ are equal, entry-by-entry, and by the definition of matrix equality (Definition ME [197]) we can say they are equal matrices. ■

It is this theorem that gives the identity matrix its name. It is a matrix that behaves with matrix multiplication like the scalar 1 does with scalar multiplication. To multiply by the identity matrix is to have no effect on the other matrix.

**Theorem MMDAA**
Matrix Multiplication Distributes Across Addition

Suppose $A$ is an $m \times n$ matrix and $B$ and $C$ are $n \times p$ matrices and $D$ is a $p \times s$ matrix. Then
1. $A(B + C) = AB + AC$  
2. $(B + C)D = BD + CD$

**Proof** We’ll do (1), you do (2). Entry-by-entry,

$$[A(B + C)]_{ij} = \sum_{k=1}^{n} [A]_{ik} [B + C]_{kj}$$

**Theorem EMP** [216]

$$= \sum_{k=1}^{n} [A]_{ik} ([B]_{kj} + [C]_{kj})$$

**Definition MA** [198]

$$= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} + [A]_{ik} [C]_{kj}$$

Distributivity in $C$

$$= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} + \sum_{k=1}^{n} [A]_{ik} [C]_{kj}$$

Commutativity in $C$

$$= [AB]_{ij} + [AC]_{ij}$$

**Theorem EMP** [216]

$$= [AB + AC]_{ij}$$

**Definition MA** [198]

So the matrices $A(B + C)$ and $AB + AC$ are equal, entry-by-entry, and by the definition of matrix equality (Definition ME [197]) we can say they are equal matrices. ■

**Theorem MMSMM**
Matrix Multiplication and Scalar Matrix Multiplication

Suppose $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix. Let $\alpha$ be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$. ■

**Proof** These are equalities of matrices. We’ll do the first one, the second is similar and will be good practice for you.

$$[\alpha(AB)]_{ij} = \alpha [AB]_{ij}$$

**Definition MSM** [198]

$$= \alpha \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$$

**Theorem EMP** [216]

$$= \sum_{k=1}^{n} \alpha [A]_{ik} [B]_{kj}$$

Distributivity in $C$

$$= \sum_{k=1}^{n} [\alpha A]_{ik} [B]_{kj}$$

**Definition MSM** [198]

$$= [(\alpha A)B]_{ij}$$

**Theorem EMP** [216]

Version 0.92
So the matrices $\alpha(AB)$ and $(\alpha A)B$ are equal, entry-by-entry, and by the definition of matrix equality (Definition ME [197]) we can say they are equal matrices.

**Theorem MMA**

**Matrix Multiplication is Associative**

Suppose $A$ is an $m \times n$ matrix, $B$ is an $n \times p$ matrix and $D$ is a $p \times s$ matrix. Then $A(BD) = (AB)D$.

**Proof** A matrix equality, so we'll go entry-by-entry, no surprise there.

\[
[A(BD)]_{ij} = \sum_{k=1}^{n} [A]_{ik} [BD]_{kj} \quad \text{Theorem EMP [216]}
\]
\[
= \sum_{k=1}^{n} [A]_{ik} \left( \sum_{\ell=1}^{p} [B]_{k\ell} [D]_{\ell j} \right) \quad \text{Theorem EMP [216]}
\]
\[
= \sum_{k=1}^{n} \sum_{\ell=1}^{p} [A]_{ik} [B]_{k\ell} [D]_{\ell j} \quad \text{Distributivity in } \mathbb{C}
\]

We can switch the order of the summation since these are finite sums,

\[
= \sum_{\ell=1}^{p} \sum_{k=1}^{n} [A]_{ik} [B]_{k\ell} [D]_{\ell j} \quad \text{Commutativity in } \mathbb{C}
\]

As $[D]_{\ell j}$ does not depend on the index $k$, we can factor it out of the inner sum,

\[
= \sum_{\ell=1}^{p} [D]_{\ell j} \left( \sum_{k=1}^{n} [A]_{ik} [B]_{k\ell} \right) \quad \text{Distributivity in } \mathbb{C}
\]
\[
= \sum_{\ell=1}^{p} [D]_{\ell j} [AB]_{i\ell} \quad \text{Theorem EMP [216]}
\]
\[
= \sum_{\ell=1}^{p} [AB]_{i\ell} [D]_{\ell j} \quad \text{Commutativity in } \mathbb{C}
\]
\[
= [(AB)D]_{ij} \quad \text{Theorem EMP [216]}
\]

So the matrices $(AB)D$ and $A(BD)$ are equal, entry-by-entry, and by the definition of matrix equality (Definition ME [197]) we can say they are equal matrices.

**Theorem MMIP**

**Matrix Multiplication and Inner Products**

If we consider the vectors $u, v \in \mathbb{C}^m$ as $m \times 1$ matrices then

\[
\langle u, v \rangle = u^t v
\]

**Proof**

\[
\langle u, v \rangle = \sum_{k=1}^{m} [u]_k [v]_k \quad \text{Definition IP [184]}
\]
\[
\sum_{k=1}^{m} [u]_{k1} [v]_{k1}
\]
Column vectors as matrices
\[
\sum_{k=1}^{m} [u']_{1k} [v]_{k1}
\]
Definition TM [200]
\[
\sum_{k=1}^{m} [u']_{1k} [v]_{k1}
\]
Definition CCCV [183]
\[
[u'v]_{11}
\]
Theorem EMP [216]

To finish we just blur the distinction between a \(1 \times 1\) matrix \((u'v)\) and its lone entry. □

**Theorem MMCC**

**Matrix Multiplication and Complex Conjugation**

Suppose \(A\) is an \(m \times n\) matrix and \(B\) is an \(n \times p\) matrix. Then \(\overline{AB} = \overline{A}\overline{B}\). □

**Proof** To obtain this matrix equality, we will work entry-by-entry,

\[
[AB]_{ij} = \overline{[AB]_{ij}}
\]
Definition CM [30]
\[
= \sum_{k=1}^{n} [A]_{ik} [B]_{kj}
\]
Theorem EMP [216]
\[
= \sum_{k=1}^{n} [A]_{ik} \overline{[B]_{kj}}
\]
Theorem CCRA [690]
\[
= \sum_{k=1}^{n} \overline{[A]_{ik}} [\overline{B}]_{kj}
\]
Theorem CCRM [690]
\[
= \sum_{k=1}^{n} \overline{[A]_{ik}} [\overline{B}]_{kj}
\]
Definition CCM [203]
\[
= [\overline{A}\overline{B}]_{ij}
\]
Theorem EMP [216]

So the matrices \(\overline{AB}\) and \(\overline{A}\overline{B}\) are equal, entry-by-entry, and by the definition of matrix equality (Definition ME [197]) we can say they are equal matrices. □

One more theorem in this style, and its a good one. If you’ve been practicing with the previous proofs you should be able to do this one yourself.

**Theorem MMT**

**Matrix Multiplication and Transposes**

Suppose \(A\) is an \(m \times n\) matrix and \(B\) is an \(n \times p\) matrix. Then \((AB)^t = B^tA^t\). □

**Proof** This theorem may be surprising but if we check the sizes of the matrices involved, then maybe it will not seem so far-fetched. First, \(AB\) has size \(m \times p\), so its transpose has size \(p \times m\). The product of \(B^t\) with \(A^t\) is a \(p \times n\) matrix times an \(n \times m\) matrix, also resulting in a \(p \times m\) matrix. So at least our objects are compatible for equality (and would not be, in general, if we didn’t reverse the order of the operation).

Here we go again, entry-by-entry,

\[
[(AB)^t]_{ij} = [AB]_{ji}
\]
Definition TM [200]
\[ = \sum_{k=1}^{n} [A]_{jk} [B]_{ki} \quad \text{Theorem EMP 216} \]
\[ = \sum_{k=1}^{n} [B]_{ki} [A]_{jk} \quad \text{Commutativity in } \mathbb{C} \]
\[ = \sum_{k=1}^{n} [B^t]_{ik} [A^t]_{kj} \quad \text{Definition TM 200} \]
\[ = [B^t A^t]_{ij} \quad \text{Theorem EMP 216} \]

So the matrices \((AB)^t\) and \(B^t A^t\) are equal, entry-by-entry, and by the definition of matrix equality (Definition ME 197) we can say they are equal matrices.

This theorem seems odd at first glance, since we have to switch the order of \(A\) and \(B\). But if we simply consider the sizes of the matrices involved, we can see that the switch is necessary for this reason alone. That the individual entries of the products then come along to be equal is a bonus.

Notice how none of these proofs above relied on writing out huge general matrices with lots of ellipses ("...") and trying to formulate the equalities a whole matrix at a time. This messy business is a "proof technique" to be avoided at all costs.

These theorems, along with Theorem VSPM 199, give you the "rules" for how matrices interact with the various operations we have defined. Use them and use them often. But don’t try to do anything with a matrix that you don’t have a rule for. Together, we would informally call all these operations, and the attendant theorems, “the algebra of matrices.” Notice, too, that every column vector is just a \(n \times 1\) matrix, so these theorems apply to column vectors also. Finally, these results may make us feel that the definition of matrix multiplication is not so unnatural.

### Subsection READ

#### Reading Questions

1. Form the matrix vector product of

\[
\begin{bmatrix}
2 & 3 & -1 & 0 \\
1 & -2 & 7 & 3 \\
1 & 5 & 3 & 2
\end{bmatrix}
\]

with

\[
\begin{bmatrix}
2 \\
-3 \\
0 \\
5
\end{bmatrix}
\]

2. Multiply together the two matrices below (in the order given).

\[
\begin{bmatrix}
2 & 3 & -1 & 0 \\
1 & -2 & 7 & 3 \\
1 & 5 & 3 & 2
\end{bmatrix}
\begin{bmatrix}
2 & 6 \\
-3 & -4 \\
0 & 2 \\
3 & -1
\end{bmatrix}
\]

3. Rewrite the system of linear equations below as a vector equality and using a matrix-vector product. (This question does not ask for a solution to the system.)
But it does ask you to express the system of equations in a new form using tools from this section.)

\begin{align*}
2x_1 + 3x_2 - x_3 &= 0 \\
x_1 + 2x_2 + x_3 &= 3 \\
x_1 + 3x_2 + 3x_3 &= 7
\end{align*}
Subsection EXC
Exercises

C20  Compute the product of the two matrices below, $AB$. Do this using the definitions of the matrix-vector product (Definition MVP [211]) and the definition of matrix multiplication (Definition MM [215]).

$$A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 5 & -3 & 4 \\ 2 & 0 & 2 & -3 \end{bmatrix}$$

Contributed by Robert Beezer  Solution [227]

T10  Suppose that $A$ is a square matrix and there is a vector, $b$, such that $LS(A,b)$ has a unique solution. Prove that $A$ is nonsingular. Give a direct proof (perhaps appealing to Theorem PSPHS [113]) rather than just negating a sentence from the text discussing a similar situation.

Contributed by Robert Beezer  Solution [227]

T20  Prove the second part of Theorem MMZM [218].

Contributed by Robert Beezer  Solution [227]

T21  Prove the second part of Theorem MMIM [218].

Contributed by Robert Beezer  Solution [227]

T22  Prove the second part of Theorem MMDAA [219].

Contributed by Robert Beezer  Solution [227]

T23  Prove the second part of Theorem MMSMM [219].

Contributed by Robert Beezer  Solution [227]

T31  Suppose that $A$ is an $m \times n$ matrix and $x$, $y \in \mathcal{N}(A)$. Prove that $x + y \in \mathcal{N}(A)$.

Contributed by Robert Beezer  Solution [227]

T32  Suppose that $A$ is an $m \times n$ matrix, $\alpha \in \mathbb{C}$, and $x \in \mathcal{N}(A)$. Prove that $\alpha x \in \mathcal{N}(A)$.

Contributed by Robert Beezer  Solution [227]

T40  Suppose that $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix. Prove that the null space of $B$ is a subset of the null space of $AB$, that is $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$. Provide an example where the opposite is false, in other words give an example where $\mathcal{N}(AB) \nsubseteq \mathcal{N}(B)$.

Contributed by Robert Beezer  Solution [228]

T41  Suppose that $A$ is an $n \times n$ nonsingular matrix and $B$ is an $n \times p$ matrix. Prove that the null space of $B$ is equal to the null space of $AB$, that is $\mathcal{N}(B) = \mathcal{N}(AB)$. (Compare with Exercise MM.T40 [225].)

Contributed by Robert Beezer  Solution [228]

T50  Suppose $u$ and $v$ are any two solutions of the linear system $LS(A,b)$. Prove that $u - v$ is an element of the null space of $A$, that is, $u - v \in \mathcal{N}(A)$.

Contributed by Robert Beezer  Solution [228]
**T51** Give a new proof of Theorem PSPHS 113 replacing applications of Theorem SLSLC 100 with matrix-vector products (Theorem SLEMM 212).

Contributed by Robert Beezer Solution 228

**T52** Suppose that $x, y \in \mathbb{C}^n$, $b \in \mathbb{C}^m$ and $A$ is an $m \times n$ matrix. If $x$, $y$ and $x + y$ are each a solution to the linear system $\mathcal{L}S(A, b)$, what interesting can you say about $b$? Form an implication with the existence of the three solutions as the hypothesis and an interesting statement about $\mathcal{L}S(A, b)$ as the conclusion, and then give a proof.

Contributed by Robert Beezer Solution 229
Subsection SOL
Solutions

C20 Contributed by Robert Beezer Statement 225

By Definition MM 215,

\[ AB = \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} \]

Repeated applications of Definition MVP 211 give

\[ = 1 \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ -2 \\ -2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + (-3) \begin{bmatrix} 3 \\ -2 \end{bmatrix} \]

\[ = \begin{bmatrix} 12 & 10 & 4 & -7 \\ 5 & -5 & 9 & -13 \\ -2 & 10 & -10 & 14 \end{bmatrix} \]

T10 Contributed by Robert Beezer Statement 225

Since \( L_S(A, b) \) has at least one solution, we can apply Theorem PSPHS 113. Because the solution is assumed to be unique, the null space of \( A \) must be trivial. Then Theorem NMTNS 78 implies that \( A \) is nonsingular.

The converse of this statement is a trivial application of Theorem NMUS 79. That said, we could extend our NSMxx series of theorems with an added equivalence for nonsingularity, “Given a single vector of constants, \( b \), the system \( L_S(A, b) \) has a unique solution.”

T23 Contributed by Robert Beezer Statement 225

We’ll run the proof entry-by-entry.

\[ [\alpha(AB)]_{ij} = \alpha [AB]_{ij} \]

Definition MSM 198

\[ = \alpha \sum_{k=1}^{n} [A]_{ik} [B]_{kj} \]

Theorem EMP 216

\[ = \sum_{k=1}^{n} \alpha [A]_{ik} [B]_{kj} \]

Distributivity in \( \mathbb{C} \)

\[ = \sum_{k=1}^{n} [A]_{ik} \alpha [B]_{kj} \]

Commutativity in \( \mathbb{C} \)

\[ = \sum_{k=1}^{n} [A]_{ik} [\alpha B]_{kj} \]

Definition MSM 198

\[ = [A(\alpha B)]_{ij} \]

Theorem EMP 216

So the matrices \( \alpha(AB) \) and \( A(\alpha B) \) are equal, entry-by-entry, and by the definition of matrix equality (Definition ME 197) we can say they are equal matrices.
To prove that one set is a subset of another, we start with an element of the smaller set and see if we can determine that it is a member of the larger set (Definition SSET 693). Suppose $x \in \mathcal{N}(B)$. Then we know that $Bx = 0$ by Definition NSM 68. Consider

$$(AB)x = A(Bx) = A0 = 0$$

This establishes that $x \in \mathcal{N}(AB)$, so $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$.

To show that the inclusion does not hold in the opposite direction, choose $B$ to be any nonsingular matrix of size $n$. Then $\mathcal{N}(B) = \{0\}$ by Theorem NMTNS 78. Let $A$ be the square zero matrix, $O$, of the same size. Then $AB = OB = O$ by Theorem MMZM 218 and therefore $\mathcal{N}(AB) = \mathbb{C}^n$, and is not a subset of $\mathcal{N}(B) = \{0\}$.

From the solution to Exercise MM.T40 we know that $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$. So to establish the set equality (Definition SE 694) we need to show that $\mathcal{N}(AB) \subseteq \mathcal{N}(B)$.

Suppose $x \in \mathcal{N}(AB)$. Then we know that $ABx = 0$ by Definition NSM 68. Consider

$$Bx = I_nBx = (A^{-1}A)Bx = A^{-1}(AB)x = 0$$

This establishes that $x \in \mathcal{N}(B)$, so $\mathcal{N}(AB) \subseteq \mathcal{N}(B)$ and combined with the solution to Exercise MM.T40 we have $\mathcal{N}(B) = \mathcal{N}(AB)$ when $A$ is nonsingular.

We will work with the vector equality representations of the relevant systems of equations, as described by Theorem SLEMM 212.

$(\Leftarrow)$ Suppose $y = w + z$ and $z \in \mathcal{N}(A)$. Then

$$Ay = A(w + z) = Aw + Az = b + 0 = b$$

$demonstrating that $y$ is a solution.

$(\Rightarrow)$ Suppose $y$ is a solution to $LS(A, b)$. Then

$$A(y - w) = Ay - Aw = b - b = 0$$

Version 0.92
which says that $y - w \in \mathcal{N}(A)$. In other words, $y - w = z$ for some vector $z \in \mathcal{N}(A)$. Rewritten, this is $y = w + z$, as desired.

\textbf{T52} Contributed by Robert Beezer Statement 226

$\mathcal{L}(A, b)$ must be homogeneous. To see this consider that

\[
\begin{align*}
 b &= Ax \\
 &= Ax + 0 \\
 &= Ax + Ay - Ay \\
 &= A(x + y) - Ay \\
 &= b - b \\
 &= 0
\end{align*}
\]

By Definition HS 65 we see that $\mathcal{L}(A, b)$ is homogeneous.
Section MISLE  Matrix Inverses and Systems of Linear Equations

We begin with a familiar example, performed in a novel way.

Example SABMI
Solutions to Archetype B with a matrix inverse
Archetype B \[726\] is the system of \(m = 3\) linear equations in \(n = 3\) variables,

\[-7x_1 - 6x_2 - 12x_3 = -33\]
\[5x_1 + 5x_2 + 7x_3 = 24\]
\[x_1 + 4x_3 = 5\]

By Theorem SLEMM \[212\] we can represent this system of equations as

\[Ax = b\]

where

\[A = \begin{bmatrix}
-7 & -6 & -12 \\
5 & 5 & 7 \\
1 & 0 & 4
\end{bmatrix}, \quad x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}, \quad b = \begin{bmatrix}
-33 \\
24 \\
5
\end{bmatrix}\]

We’ll pull a rabbit out of our hat and present the \(3 \times 3\) matrix \(B\),

\[B = \begin{bmatrix}
-10 & -12 & -9 \\
\frac{13}{2} & 8 & \frac{11}{2} \\
\frac{13}{2} & 3 & \frac{11}{2}
\end{bmatrix}\]

and note that

\[BA = \begin{bmatrix}
-10 & -12 & -9 \\
\frac{13}{2} & 8 & \frac{11}{2} \\
\frac{13}{2} & 3 & \frac{11}{2}
\end{bmatrix} \begin{bmatrix}
-7 & -6 & -12 \\
5 & 5 & 7 \\
1 & 0 & 4
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Now apply this computation to the problem of solving the system of equations,

\[x = I_3x \quad \text{Theorem MMIM} \quad 218\]
\[= (BA)x \quad \text{Substitution}\]
\[= B(Ax) \quad \text{Theorem MMA} \quad 220\]
\[= Bb \quad \text{Substitution}\]

So we have

\[x = Bb = \begin{bmatrix}
\frac{13}{2} & 8 & \frac{11}{2} \\
\frac{13}{2} & 3 & \frac{11}{2}
\end{bmatrix} \begin{bmatrix}
-33 \\
24 \\
5
\end{bmatrix} = \begin{bmatrix}
-3 \\
5 \\
2
\end{bmatrix}\]

So with the help and assistance of \(B\) we have been able to determine a solution to the system represented by \(Ax = b\) through judicious use of matrix multiplication. We know
by [Theorem NMUS][79] that since the coefficient matrix in this example is nonsingular, there would be a unique solution, no matter what the choice of \( b \). The derivation above amplifies this result, since we were forced to conclude that \( x = Bb \) and the solution couldn’t be anything else. You should notice that this argument would hold for any particular value of \( b \).

The matrix \( B \) of the previous example is called the inverse of \( A \). When \( A \) and \( B \) are combined via matrix multiplication, the result is the identity matrix, which can be inserted “in front” of \( x \) as the first step in finding the solution. This is entirely analogous to how we might solve a single linear equation like \( 3x = 12 \).

\[
x = 1x = \left( \frac{1}{3} (3) \right) x = \frac{1}{3} (3x) = \frac{1}{3} (12) = 4
\]

Here we have obtained a solution by employing the “multiplicative inverse” of \( 3 \), \( 3^{-1} = \frac{1}{3} \). This works fine for any scalar multiple of \( x \), except for zero, since zero does not have a multiplicative inverse. For matrices, it is more complicated. Some matrices have inverses, some do not. And when a matrix does have an inverse, just how would we compute it? In other words, just where did that matrix \( B \) in the last example come from? Are there other matrices that might have worked just as well?

**Subsection IM**

**Inverse of a Matrix**

**Definition MI**

**Matrix Inverse**

Suppose \( A \) and \( B \) are square matrices of size \( n \) such that \( AB = I_n \) and \( BA = I_n \). Then \( A \) is invertible and \( B \) is the inverse of \( A \). In this situation, we write \( B = A^{-1} \).

(This definition contains Notation MI.)

Notice that if \( B \) is the inverse of \( A \), then we can just as easily say \( A \) is the inverse of \( B \), or \( A \) and \( B \) are inverses of each other.

Not every square matrix has an inverse. In **Example SABMI** [231] the matrix \( B \) is the inverse the coefficient matrix of **Archetype B** [726]. To see this it only remains to check that \( AB = I_3 \). What about **Archetype A** [721]? It is an example of a square matrix without an inverse.

**Example MWIAA**

**A matrix without an inverse, Archetype A**

Consider the coefficient matrix from **Archetype A** [721],

\[
A = \begin{bmatrix}
1 & -1 & 2 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]

Suppose that \( A \) is invertible and does have an inverse, say \( B \). Choose the vector of constants

\[
b = \begin{bmatrix}
1 \\
3 \\
2
\end{bmatrix}
\]
and consider the system of equations $LS(A, b)$. Just as in Example SABMI [231], this vector equation would have the unique solution $x = Bb$.

However, the system $LS(A, b)$ is inconsistent. Form the augmented matrix $[A | b]$ and row-reduce to

$$
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

which allows to recognize the inconsistency by Theorem RCLS [54].

So the assumption of $A$’s inverse leads to a logical inconsistency (the system can’t be both consistent and inconsistent), so our assumption is false. $A$ is not invertible.

Its possible this example is less than satisfying. Just where did that particular choice of the vector $b$ come from anyway? Stay tuned for an application of the future Theorem CSCS [262] in Example CSAA [266].

Let’s look at one more matrix inverse before we embark on a more systematic study.

**Example MI**

**Matrix Inverse**

Consider the matrix,

$$
A = \begin{bmatrix}
1 & 2 & 1 & 2 & 1 \\
-2 & -3 & 0 & -5 & -1 \\
1 & 1 & 0 & 2 & 1 \\
-2 & -3 & -1 & -3 & -2 \\
-1 & -3 & -1 & -3 & 1 \\
\end{bmatrix}
$$

And the matrix

$$
B = \begin{bmatrix}
-3 & 3 & 6 & -1 & -2 \\
0 & -2 & -5 & -1 & 1 \\
1 & 2 & 4 & 1 & -1 \\
1 & 0 & 1 & 1 & 0 \\
1 & -1 & -2 & 0 & 1 \\
\end{bmatrix}
$$

Then

$$
AB = \begin{bmatrix}
1 & 2 & 1 & 2 & 1 \\
-2 & -3 & 0 & -5 & -1 \\
1 & 1 & 0 & 2 & 1 \\
-2 & -3 & -1 & -3 & -2 \\
-1 & -3 & -1 & -3 & 1 \\
\end{bmatrix}
\begin{bmatrix}
-3 & 3 & 6 & -1 & -2 \\
0 & -2 & -5 & -1 & 1 \\
1 & 2 & 4 & 1 & -1 \\
1 & 0 & 1 & 1 & 0 \\
1 & -1 & -2 & 0 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

and

$$
BA = \begin{bmatrix}
-3 & 3 & 6 & -1 & -2 \\
0 & -2 & -5 & -1 & 1 \\
1 & 2 & 4 & 1 & -1 \\
1 & 0 & 1 & 1 & 0 \\
1 & -1 & -2 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 1 & 2 & 1 \\
-2 & -3 & 0 & -5 & -1 \\
1 & 1 & 0 & 2 & 1 \\
-2 & -3 & -1 & -3 & -2 \\
-1 & -3 & -1 & -3 & 1 \\
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

so by Definition MI [232], we can say that $A$ is invertible and write $B = A^{-1}$. 

We will now concern ourselves less with whether or not an inverse of a matrix exists, but instead with how you can find one when it does exist. In Section MINM [249] we will have some theorems that allow us to more quickly and easily determine when a matrix is invertible.
Subsection CIM
Computing the Inverse of a Matrix

We will have occasion in this subsection (and later) to reference the following frequently used vectors, so we will make a useful definition now.

Definition SUV
Standard Unit Vectors
Let $e_j \in \mathbb{C}^m$ denote the column vector that is column $j$ of the $m \times m$ identity matrix $I_m$. Then the set

$$\{e_1, e_2, e_3, \ldots, e_m\} = \{e_j \mid 1 \leq j \leq m\}$$

is the set of standard unit vectors in $\mathbb{C}^m$. △

We will make reference to these vectors often. Notice that $e_j$ is a column vector full of zeros, with a lone 1 in the $j$-th position, so an alternate definition is

$$[e_j]_i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

We’ve seen that the matrices from Archetype B [726] and Archetype K [767] both have inverses, but these inverse matrices have just dropped from the sky. How would we compute an inverse? And just when is a matrix invertible, and when is it not? Writing a putative inverse with $n^2$ unknowns and solving the resultant $n^2$ equations is one approach. Applying this approach to $2 \times 2$ matrices can get us somewhere, so just for fun, let’s do it.

Theorem TTMI
Two-by-Two Matrix Inverse
Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then $A$ is invertible if and only if $ad - bc \neq 0$. When $A$ is invertible, we have

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$
Working on the matrices on both ends of this equation, we will multiply the top row by $c$ and the bottom row by $a$.

\[
\begin{bmatrix}
c & 0 \\
0 & a
\end{bmatrix}
= \begin{bmatrix}
ace + bcf & acf + bch \\
ace + adg & acf + adh
\end{bmatrix}
\]

We are assuming that $ad = bc$, so we can replace two occurrences of $ad$ by $bc$ in the bottom row of the right matrix.

\[
\begin{bmatrix}
c & 0 \\
0 & a
\end{bmatrix}
= \begin{bmatrix}
ace + bcf & acf + bch \\
ace + bcf & acf + bch
\end{bmatrix}
\]

The matrix on the right now has two rows that are identical, and therefore the same must be true of the matrix on the left. Given the form of the matrix on the left, identical rows implies that $a = 0$ and $c = 0$.

With this information, the product $AB$ becomes

\[
\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} = I_2 = AB = \begin{bmatrix}
ae + bg & af + bh \\
c + dg & cf + dh
\end{bmatrix}
= \begin{bmatrix}
bg & bh \\
dg & dh
\end{bmatrix}
\]

So $bg = dh = 1$ and thus $b, g, d, h$ are all nonzero. But then $bh$ and $dg$ (the “other corners”) must also be nonzero, so this is (finally) a contradiction. So our assumption was false and we see that $ad - bc \neq 0$ whenever $A$ has an inverse.

There are several ways one could try to prove this theorem, but there is a continual temptation to divide by one of the eight entries involved ($a$ through $f$), but we can never be sure if these numbers are zero or not. This could lead to an analysis by cases, which is messy, messy, messy. Note how the above proof never divides, but always multiplies, and how zero/nonzero considerations are handled. Pay attention to the expression $ad - bc$, we will see it again in a while.

This theorem is cute, and it is nice to have a formula for the inverse, and a condition that tells us when we can use it. However, this approach becomes impractical for larger matrices, even though it is possible to demonstrate that, in theory, there is a general formula. (Think for a minute about extending this result to just $3 \times 3$ matrices. For starters, we need 18 letters!) Instead, we will work column-by-column. Let’s first work an example that will motivate the main theorem and remove some of the previous mystery.

Example CMI

Computing a Matrix Inverse

Consider the matrix defined in Example MI \[233\] as,

\[
A = \begin{bmatrix}
1 & 2 & 1 & 2 & 1 \\
-2 & -3 & 0 & -5 & -1 \\
1 & 1 & 0 & 2 & 1 \\
-2 & -3 & -1 & -3 & -2 \\
-1 & -3 & -1 & -3 & 1
\end{bmatrix}
\]

For its inverse, we desire a matrix $B$ so that $AB = I_5$. Emphasizing the structure of the columns and employing the definition of matrix multiplication Definition MM \[215\],

\[
AB = I_5 \quad A[B_1|B_2|B_3|B_4|B_5] = [e_1|e_2|e_3|e_4|e_5]
\]
Row-reduce the augmented matrix of the linear system $\mathcal{L}S(A, e_1)$,

\[
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 & 1 \\
-2 & -3 & 0 & -5 & -1 & 0 \\
1 & 1 & 0 & 2 & 1 & 0 \\
-2 & -3 & -1 & -3 & -2 & 0 \\
-1 & -3 & -1 & -3 & 1 & 1 \\
\end{pmatrix}
\rightarrow \text{RREF}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -3 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\quad \text{so} \quad B_1 = \begin{pmatrix}
-3 \\
0 \\
1 \\
1 \\
1 \\
\end{pmatrix}
\]

Row-reduce the augmented matrix of the linear system $\mathcal{L}S(A, e_2)$,

\[
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 & 0 \\
-2 & -3 & 0 & -5 & -1 & 1 \\
1 & 1 & 0 & 2 & 1 & 0 \\
-2 & -3 & -1 & -3 & -2 & 0 \\
-1 & -3 & -1 & -3 & 1 & 0 \\
\end{pmatrix}
\rightarrow \text{RREF}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 3 \\
0 & 1 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 4 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -2 \\
\end{pmatrix}
\quad \text{so} \quad B_2 = \begin{pmatrix}
3 \\
2 \\
4 \\
1 \\
-2 \\
\end{pmatrix}
\]

Row-reduce the augmented matrix of the linear system $\mathcal{L}S(A, e_3)$,

\[
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 & 0 \\
-2 & -3 & 0 & -5 & -1 & 0 \\
1 & 1 & 0 & 2 & 1 & 0 \\
-2 & -3 & -1 & -3 & -2 & 0 \\
-1 & -3 & -1 & -3 & 1 & 0 \\
\end{pmatrix}
\rightarrow \text{RREF}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 6 \\
0 & 1 & 0 & 0 & 0 & 5 \\
0 & 0 & 1 & 0 & 0 & 4 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -2 \\
\end{pmatrix}
\quad \text{so} \quad B_3 = \begin{pmatrix}
6 \\
5 \\
4 \\
1 \\
-2 \\
\end{pmatrix}
\]

Row-reduce the augmented matrix of the linear system $\mathcal{L}S(A, e_4)$,

\[
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 & 0 \\
-2 & -3 & 0 & -5 & -1 & 0 \\
1 & 1 & 0 & 2 & 1 & 0 \\
-2 & -3 & -1 & -3 & -2 & 1 \\
-1 & -3 & -1 & -3 & 1 & 0 \\
\end{pmatrix}
\rightarrow \text{RREF}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\quad \text{so} \quad B_4 = \begin{pmatrix}
-1 \\
-1 \\
1 \\
1 \\
0 \\
\end{pmatrix}
\]

Row-reduce the augmented matrix of the linear system $\mathcal{L}S(A, e_5)$,

\[
\begin{pmatrix}
1 & 2 & 1 & 2 & 1 & 0 \\
-2 & -3 & 0 & -5 & -1 & 0 \\
1 & 1 & 0 & 2 & 1 & 0 \\
-2 & -3 & -1 & -3 & -2 & 0 \\
-1 & -3 & -1 & -3 & 1 & 1 \\
\end{pmatrix}
\rightarrow \text{RREF}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -2 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\quad \text{so} \quad B_5 = \begin{pmatrix}
-2 \\
1 \\
-1 \\
0 \\
1 \\
\end{pmatrix}
\]
We can now collect our 5 solution vectors into the matrix $B$,

$$B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 & B_5 \end{bmatrix}$$

By this method, we know that $AB = I_5$. Check that $BA = I_5$, and then we will know that we have the inverse of $A$.

Notice how the five systems of equations in the preceding example were all solved by exactly the same sequence of row operations. Wouldn’t it be nice to avoid this obvious duplication of effort? Our main theorem for this section follows, and it mimics this previous example, while also avoiding all the overhead.

**Theorem CINM**

**Computing the Inverse of a Nonsingular Matrix**

Suppose $A$ is a nonsingular square matrix of size $n$. Create the $n \times 2n$ matrix $M$ by placing the $n \times n$ identity matrix $I_n$ to the right of the matrix $A$. Let $N$ be a matrix that is row-equivalent to $M$ and in reduced row-echelon form. Finally, let $J$ be the matrix formed from the final $n$ columns of $N$. Then $AJ = I_n$.

**Proof**  $A$ is nonsingular, so by [Theorem NMRRI][77] there is a sequence of row operations that will convert $A$ into $I_n$. It is this same sequence of row operations that will convert $M$ into $N$, since having the identity matrix in the first $n$ columns of $N$ is sufficient to guarantee that $N$ is in reduced row-echelon form.

If we consider the systems of linear equations, $LS(A, e_i)$, $1 \leq i \leq n$, we see that the aforementioned sequence of row operations will also bring the augmented matrix of each of these systems into reduced row-echelon form. Furthermore, the unique solution to $LS(A, e_i)$ appears in column $n + 1$ of the row-reduced augmented matrix of the system and is identical to column $n + i$ of $N$. Let $N_1, N_2, N_3, \ldots, N_{2n}$ denote the columns of $N$. So we find,

$$AJ = A[N_{n+1} | N_{n+2} | N_{n+3} | \ldots | N_{n+n}]$$

$$= [AN_{n+1} | AN_{n+2} | AN_{n+3} | \ldots | AN_{n+n}]$$

$$= [e_1 | e_2 | e_3 | \ldots | e_n]$$

$$= I_n$$

as desired.

We have to be just a bit careful here about both what this theorem says and what it doesn’t say. If $A$ is a nonsingular matrix, then we are guaranteed a matrix $B$ such that
$AB = I_n$, and the proof gives us a process for constructing $B$. However, the definition of the inverse of a matrix (Definition MI [232]) requires that $BA = I_n$ also. So at this juncture we must compute the matrix product in the “opposite” order before we claim $B$ as the inverse of $A$. However, we’ll soon see that this is always the case, in Theorem OSIS [250], so the title of this theorem is not inaccurate.

What if $A$ is singular? At this point we only know that Theorem CINM [237] cannot be applied. The question of $A$’s inverse is still open. (But see Theorem NI [251] in the next section.) We’ll finish by computing the inverse for the coefficient matrix of Archetype B [726], the one we just pulled from a hat in Example SABMI [231]. There are more examples in the Archetypes (Appendix A [717]) to practice with, though notice that it is silly to ask for the inverse of a rectangular matrix (the sizes aren’t right) and not every square matrix has an inverse (remember Example MWIAA [232]?).

Example CMIAB

Computing a Matrix Inverse, Archetype B

Archetype B [726] has a coefficient matrix given as

$$B = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$$

Exercising Theorem CINM [237] we set

$$M = \begin{bmatrix} -7 & -6 & -12 & 1 & 0 & 0 \\ 5 & 5 & 7 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{bmatrix}.$$ 

which row reduces to

$$N = \begin{bmatrix} 1 & 0 & 0 & -10 & -12 & -9 \\ 0 & 1 & 0 & \frac{13}{2} & 8 & \frac{11}{2} \\ 0 & 0 & 1 & \frac{5}{2} & 3 & \frac{5}{2} \end{bmatrix}.$$ 

So

$$B^{-1} = \begin{bmatrix} -10 & -12 & -9 \\ \frac{13}{2} & 8 & \frac{11}{2} \\ \frac{5}{2} & 3 & \frac{5}{2} \end{bmatrix}$$

once we check that $B^{-1}B = I_3$ (the product in the opposite order is a consequence of the theorem).

While we can use a row-reducing procedure to compute any needed inverse, most computational devices have a built-in procedure to compute the inverse of a matrix straightaway. See: Computation MI.MMA [684].

Subsection PMI

Properties of Matrix Inverses

The inverse of a matrix enjoys some nice properties. We collect a few here. First, a matrix can have but one inverse.
**Theorem MIU**  
**Matrix Inverse is Unique**

Suppose the square matrix $A$ has an inverse. Then $A^{-1}$ is unique. □

**Proof**  
As described in [Technique U](#), we will assume that $A$ has two inverses. The hypothesis tells there is at least one. Suppose then that $B$ and $C$ are both inverses for $A$. Then, repeated use of [Definition MI](#) and [Theorem MMIM](#) plus one application of [Theorem MMA](#) gives:

$$
B = BI_n = B(AC) = (BA)C = I_n C = C
$$

So we conclude that $B$ and $C$ are the same, and cannot be different. So any matrix that acts like an inverse, must be the inverse. ■

When most of us dress in the morning, we put on our socks first, followed by our shoes. In the evening we must then first remove our shoes, followed by our socks. Try to connect the conclusion of the following theorem with this everyday example.

**Theorem SS**  
**Socks and Shoes**

Suppose $A$ and $B$ are invertible matrices of size $n$. Then $(AB)^{-1} = B^{-1}A^{-1}$ and $AB$ is an invertible matrix. □

**Proof**  
At the risk of carrying our everyday analogies too far, the proof of this theorem is quite easy when we compare it to the workings of a dating service. We have a statement about the inverse of the matrix $AB$, which for all we know right now might not even exist. Suppose $AB$ was to sign up for a dating service with two requirements for a compatible date. Upon multiplication on the left, and on the right, the result should be the identity matrix. In other words, $AB$’s ideal date would be its inverse.

Now along comes the matrix $B^{-1}A^{-1}$ (which we know exists because our hypothesis says both $A$ and $B$ are invertible and we can form the product of these two matrices), also looking for a date. Let’s see if $B^{-1}A^{-1}$ is a good match for $AB$. First they meet at a non-committal neutral location, say a coffee shop, for quiet conversation:

$$
(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_n B = B^{-1}B = I_n
$$

The first date having gone smoothly, a second, more serious, date is arranged, say dinner and a show:

$$
(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_n A^{-1} = AA^{-1}
$$
\[ = I_n \]  
Definition MI 232

So the matrix \( B^{-1}A^{-1} \) has met all of the requirements to be \( AB \)'s inverse (date) and with the ensuing marriage proposal we can announce that \( (AB)^{-1} = B^{-1}A^{-1} \). ■

**Theorem MIMI**  
Matrix Inverse of a Matrix Inverse  
Suppose \( A \) is an invertible matrix. Then \( A^{-1} \) is invertible and \( (A^{-1})^{-1} = A \). □

**Proof**  
As with the proof of Theorem SS 239, we examine if \( A^{-1} \) is a suitable inverse for \( A \) (by definition, the opposite is true).

\[ AA^{-1} = I_n \]  
Definition MI 232

and

\[ A^{-1}A = I_n \]  
Definition MI 232

The matrix \( A \) has met all the requirements to be the inverse of \( A^{-1} \), and so is invertible and we can write \( A = (A^{-1})^{-1} \). ■

**Theorem MIT**  
Matrix Inverse of a Transpose  
Suppose \( A \) is an invertible matrix. Then \( A^t \) is invertible and \( (A^t)^{-1} = (A^{-1})^t \). □

**Proof**  
As with the proof of Theorem SS 239, we see if \( (A^{-1})^t \) is a suitable inverse for \( A^t \). Apply Theorem MMT 221 to see that

\[
(A^{-1})^t A^t = (AA^{-1})^t \\
\quad = I_n^t \\
\quad = I_n
\]  
Theorem MMT 221

Definition MI 232

\[ I_n \] is symmetric

and

\[
A^t (A^{-1})^t = (A^{-1}A)^t \\
\quad = I_n^t \\
\quad = I_n
\]  
Theorem MMT 221

Definition MI 232

\[ I_n \] is symmetric

The matrix \( (A^{-1})^t \) has met all the requirements to be the inverse of \( A^t \), and so is invertible and we can write \( (A^t)^{-1} = (A^{-1})^t \). ■

**Theorem MISM**  
Matrix Inverse of a Scalar Multiple  
Suppose \( A \) is an invertible matrix and \( \alpha \) is a nonzero scalar. Then \( (\alpha A)^{-1} = \frac{1}{\alpha} A^{-1} \) and \( \alpha A \) is invertible.

**Proof**  
As with the proof of Theorem SS 239, we see if \( \frac{1}{\alpha} A^{-1} \) is a suitable inverse for \( \alpha A \).

\[
\left(\frac{1}{\alpha} A^{-1}\right) (\alpha A) = \left(\frac{1}{\alpha}\right) (AA^{-1}) \]  
Theorem MMSMM 219
Scalar multiplicative inverses

\[ = I_n \]

\[ = I_n \]  

Property OM \[199\]

and

\[
(\alpha A) \left( \frac{1}{\alpha} A^{-1} \right) = \left( \alpha \frac{1}{\alpha} \right) (A^{-1}A)
\]

\[ = I_n \]  

Scalar multiplicative inverses

\[ = I_n \]  

Property OM \[199\]

The matrix \(\frac{1}{\alpha}A^{-1}\) has met all the requirements to be the inverse of \(\alpha A\), so we can write \((\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}\).

\[ \blacksquare \]

Notice that there are some likely theorems that are missing here. For example, it would be tempting to think that \((A + B)^{-1} = A^{-1} + B^{-1}\), but this is false. Can you find a counterexample (see Exercise MISLE.T10 \[244\])?

Subsection READ

Reading Questions

1. Compute the inverse of the matrix below.

\[
\begin{bmatrix}
4 & 10 \\
2 & 6
\end{bmatrix}
\]

2. Compute the inverse of the matrix below.

\[
\begin{bmatrix}
2 & 3 & 1 \\
1 & -2 & -3 \\
-2 & 4 & 6
\end{bmatrix}
\]

3. Explain why Theorem SS \[239\] has the title it does. (Do not just state the theorem, explain the choice of the title making reference to the theorem itself.)
Subsection EXC
Exercises

C21 Verify that $B$ is the inverse of $A$.

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ -2 & -1 & 2 & -3 \\ 1 & 1 & 0 & 2 \\ -1 & 2 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 2 & 0 & -1 \\ 8 & 4 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ -6 & -3 & 1 & 1 \end{bmatrix}$$

Contributed by Robert Beezer  Solution 245

C22 Recycle the matrices $A$ and $B$ from Exercise MISLE.C21 and set

$$c = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \quad d = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Employ the matrix $B$ to solve the two linear systems $LS(A, c)$ and $LS(A, d)$.

Contributed by Robert Beezer  Solution 245

C23 If it exists, find the inverse of the $2 \times 2$ matrix

$$A = \begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$$

and check your answer. (See Theorem TTMI.)

Contributed by Robert Beezer

C24 If it exists, find the inverse of the $2 \times 2$ matrix

$$A = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$$

and check your answer. (See Theorem TTMI.)

Contributed by Robert Beezer

C25 At the conclusion of Example CMI, verify that $BA = I_5$ by computing the matrix product.

Contributed by Robert Beezer

C26 Let

$$D = \begin{bmatrix} 1 & -1 & 3 & -2 & 1 \\ -2 & 3 & -5 & 3 & 0 \\ 1 & -1 & 4 & -2 & 2 \\ -1 & 4 & -1 & 0 & 4 \\ 1 & 0 & 5 & -2 & 5 \end{bmatrix}$$

Compute the inverse of $D$, $D^{-1}$, by forming the $5 \times 10$ matrix $[D | I_5]$ and row-reducing (Theorem CINM). Then use a calculator to compute $D^{-1}$ directly.

Contributed by Robert Beezer  Solution 245
C27  Let

\[
E = \begin{bmatrix}
1 & -1 & 3 & -2 & 1 \\
-2 & 3 & -5 & 3 & -1 \\
1 & -1 & 4 & -2 & 2 \\
-1 & 4 & -1 & 0 & 2 \\
1 & 0 & 5 & -2 & 4
\end{bmatrix}
\]

Compute the inverse of \( E \), \( E^{-1} \), by forming the 5 \times 10 matrix \([E \mid I_5]\) and row-reducing (Theorem CINM [237]). Then use a calculator to compute \( E^{-1} \) directly.

Contributed by Robert Beezer  Solution [245]

C28  Let

\[
C = \begin{bmatrix}
1 & 1 & 3 & 1 \\
-2 & -1 & -4 & -1 \\
1 & 4 & 10 & 2 \\
-2 & 0 & -4 & 5
\end{bmatrix}
\]

Compute the inverse of \( C \), \( C^{-1} \), by forming the 4 \times 8 matrix \([C \mid I_4]\) and row-reducing (Theorem CINM [237]). Then use a calculator to compute \( C^{-1} \) directly.

Contributed by Robert Beezer  Solution [245]

C40  Find all solutions to the system of equations below, making use of the matrix inverse found in Exercise MISLE.C28 [244].

\[
\begin{align*}
x_1 + x_2 + 3x_3 + x_4 &= -4 \\
-2x_1 - x_2 - 4x_3 - x_4 &= 4 \\
x_1 + 4x_2 + 10x_3 + 2x_4 &= -20 \\
-2x_1 - 4x_3 + 5x_4 &= 9
\end{align*}
\]

Contributed by Robert Beezer  Solution [246]

C41  Use the inverse of a matrix to find all the solutions to the following system of equations.

\[
\begin{align*}
x_1 + 2x_2 - x_3 &= -3 \\
2x_1 + 5x_2 - x_3 &= -4 \\
-x_1 - 4x_2 &= 2
\end{align*}
\]

Contributed by Robert Beezer  Solution [246]

C42  Use a matrix inverse to solve the linear system of equations.

\[
\begin{align*}
x_1 - x_2 + 2x_3 &= 5 \\
x_1 - 2x_3 &= -8 \\
2x_1 - x_2 - x_3 &= -6
\end{align*}
\]

Contributed by Robert Beezer  Solution [246]

T10  Construct an example to demonstrate that \((A + B)^{-1} = A^{-1} + B^{-1}\) is not true for all square matrices \( A \) and \( B \) of the same size.

Contributed by Robert Beezer  Solution [247]
Subsection SOL

Solutions

C21 Contributed by Robert Beezer Statement 243
Check that both matrix products (Definition MM 215) $AB$ and $BA$ equal the $4 \times 4$ identity matrix $I_4$ (Definition IM 76).

C22 Contributed by Robert Beezer Statement 243
Represent each of the two systems by a vector equality, $A\mathbf{x} = \mathbf{c}$ and $A\mathbf{y} = \mathbf{d}$. Then in the spirit of Example SABMI 231, solutions are given by

$$\mathbf{x} = B\mathbf{c} = \begin{bmatrix} 8 \\ 21 \\ -5 \\ -16 \end{bmatrix} \quad \mathbf{y} = B\mathbf{d} = \begin{bmatrix} 5 \\ 10 \\ 0 \\ -7 \end{bmatrix}$$

Notice how we could solve many more systems having $A$ as the coefficient matrix, and how each such system has a unique solution. You might check your work by substituting the solutions back into the systems of equations, or forming the linear combinations of the columns of $A$ suggested by Theorem SLSLC 100.

C26 Contributed by Robert Beezer Statement 243
The inverse of $D$ is

$$D^{-1} = \begin{bmatrix} -7 & -6 & -3 & 2 & 1 \\ -7 & -4 & 2 & 2 & -1 \\ -5 & -2 & 3 & 1 & -1 \\ -6 & -3 & 1 & 1 & 0 \\ 4 & 2 & -2 & -1 & 1 \end{bmatrix}$$

C27 Contributed by Robert Beezer Statement 244
The matrix $E$ has no inverse, though we do not yet have a theorem that allows us to reach this conclusion. However, when row-reducing the matrix $[E | I_5]$, the first 5 columns will not row-reduce to the $5 \times 5$ identity matrix, so we are at a loss on how we might compute the inverse. When requesting that your calculator compute $E^{-1}$, it should give some indication that $E$ does not have an inverse.

C28 Contributed by Robert Beezer Statement 244
Employ Theorem CINM 237.

$$\begin{bmatrix} 1 & 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ -2 & -1 & -4 & -1 & 0 & 1 & 0 & 0 \\ 1 & 4 & 10 & 2 & 0 & 0 & 1 & 0 \\ -2 & 0 & -4 & 5 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 38 & 18 & -5 & -2 \\ 0 & 1 & 0 & 0 & 0 & 96 & 47 & -12 & -5 \\ 0 & 0 & 1 & 0 & -39 & -19 & 5 & 2 \\ 0 & 0 & 0 & 1 & -16 & -8 & 2 & 1 \end{bmatrix}$$

And therefore we see that $C$ is nonsingular ($C$ row-reduces to the identity matrix, Theorem NMRRI 77) and by Theorem CINM 237,

$$C^{-1} = \begin{bmatrix} 38 & 18 & -5 & -2 \\ 96 & 47 & -12 & -5 \\ -39 & -19 & 5 & 2 \\ -16 & -8 & 2 & 1 \end{bmatrix}$$
C40  Contributed by Robert Beezer  Statement 244

View this system as \( L \mathbf{S}(C, \mathbf{b}) \), where \( C \) is the 4\( \times \)4 matrix from Exercise MISLE.C28 and \( \mathbf{b} = \begin{bmatrix} -4 \\ 4 \\ -20 \\ 9 \end{bmatrix} \). Since \( C \) was seen to be nonsingular in Exercise MISLE.C28, Theorem SNCM says the solution, which is unique by Theorem NMUS, is given by

\[
C^{-1} \mathbf{b} = \begin{bmatrix} 38 & 18 & -5 & -2 \\ 96 & 47 & -12 & -5 \\ -39 & -19 & 5 & 2 \\ -16 & -8 & 2 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 4 \\ -20 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \\ 1 \end{bmatrix}
\]

Notice that this solution can be easily checked in the original system of equations.

C41  Contributed by Robert Beezer  Statement 244

The coefficient matrix of this system of equations is

\[
A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ -1 & -4 & 0 \end{bmatrix}
\]

and the vector of constants is \( \mathbf{b} = \begin{bmatrix} -3 \\ -4 \\ 2 \end{bmatrix} \). So by Theorem SLEMM we can convert the system to the form \( A \mathbf{x} = \mathbf{b} \). Row-reducing this matrix yields the identity matrix so by Theorem NMRRI we know \( A \) is nonsingular. This allows us to apply Theorem SNCM to find the unique solution as

\[
\mathbf{x} = A^{-1} \mathbf{b} = \begin{bmatrix} -4 & 4 & 3 \\ 1 & -1 & -1 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}
\]

Remember, you can check this solution easily by evaluating the matrix-vector product \( A \mathbf{x} \) (Definition MVP).

C42  Contributed by Robert Beezer  Statement 244

We can reformulate the linear system as a vector equality with a matrix-vector product via Theorem SLEMM. The system is then represented by \( A \mathbf{x} = \mathbf{b} \) where

\[
A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 5 \\ -8 \\ -6 \end{bmatrix}
\]

According to Theorem SNCM, if \( A \) is nonsingular then the (unique) solution will be given by \( A^{-1} \mathbf{b} \). We attempt the computation of \( A^{-1} \) through Theorem CINM, or with our favorite computational device and obtain,

\[
A^{-1} = \begin{bmatrix} 2 & 3 & -2 \\ 3 & 5 & -4 \\ 1 & 1 & -1 \end{bmatrix}
\]
So by Theorem NI, we know $A$ is nonsingular, and so the unique solution is

$$A^{-1}b = \begin{bmatrix} 2 & 3 & -2 \\ 3 & 5 & -4 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -8 \\ -6 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$$

T10 Contributed by Robert Beezer Statement

Let $D$ be any $2 \times 2$ matrix that has an inverse (Theorem TTMI can help you construct such a matrix, $I_2$ is a simple choice). Set $A = D$ and $B = (-1)D$. While $A^{-1}$ and $B^{-1}$ both exist, what is $(A + B)^{-1}$? Can the proposed statement be a theorem?
We saw in [Theorem CINM 237] that if a square matrix $A$ is nonsingular, then there is a matrix $B$ so that $AB = I_n$. In other words, $B$ is halfway to being an inverse of $A$. We will see in this section that $B$ automatically fulfills the second condition ($BA = I_n$). Example MWIAA 232 showed us that the coefficient matrix from [Archetype A 721] had no inverse. Not coincidentally, this coefficient matrix is singular. We’ll make all these connections precise now. Not many examples or definitions in this section, just theorems.

**Subsection NMI**

**Nonsingular Matrices are Invertible**

We need a couple of technical results for starters. Some books would call these minor, but essential, results “lemmas.” We’ll just call ’em theorems. See [Technique LC 716] for more on the distinction.

The first of these technical results is interesting in that the hypothesis says something about a product of two square matrices and the conclusion then says the same thing about each individual matrix in the product.

**Theorem NPNT**

**Nonsingular Product has Nonsingular Terms**

Suppose that $A$ and $B$ are square matrices of size $n$ and the product $AB$ is nonsingular. Then $A$ and $B$ are both nonsingular.

**Proof** We’ll do the proof in two parts, each as a proof by contradiction (Technique CD 708). Establishing that $B$ is nonsingular is the easier part, so we will do it first, but in reality, we will need to know that $B$ is nonsingular when we prove that $A$ is nonsingular.

You can also think of this proof as being a study of four possible conclusions in the table below. One of the four rows must happen (the list is exhaustive). In the proof we learn that the first three rows lead to contradictions, and so are impossible. That leaves the fourth row as a certainty, which is our desired conclusion.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>Singular</td>
<td>Singular</td>
<td>1</td>
</tr>
<tr>
<td>Nonsingular</td>
<td>Singular</td>
<td>1</td>
</tr>
<tr>
<td>Singular</td>
<td>Nonsingular</td>
<td>2</td>
</tr>
<tr>
<td>Nonsingular</td>
<td>Nonsingular</td>
<td></td>
</tr>
</tbody>
</table>

Case 1. Suppose $B$ is singular. Then there is a nonzero vector $z$ that is a solution to $LS(B, 0)$. So

$$(AB)z = A(Bz) = A0$$

$z$ solution to $LS(B, 0)$, [Theorem SLEMM 212]
Because \( z \) is a nonzero solution to \( \mathcal{L}S(AB, 0) \), we conclude that \( AB \) is singular (Definition NM [75]). This is a contradiction, so \( B \) is nonsingular, as desired.

**Case 2.** Suppose \( A \) is singular. Then there is a nonzero vector \( y \) that is a solution to \( \mathcal{L}S(A, 0) \). Now use this vector \( y \) and consider the linear system \( \mathcal{L}S(B, y) \). Since we know \( B \) is nonsingular (from Case 1), the system has a unique solution, which we will call \( w \). We claim \( w \) is not the zero vector either. Assuming the opposite, suppose that \( w = 0 \). Then

\[
y = Bw = 0
\]

Substitution, \( w = 0 \)

contrary to \( y \) being nonzero. So \( w \neq 0 \). The pieces are in place, so here we go,

\[
(AB)w = A(Bw) = Ay = 0
\]

So \( w \) is a nonzero solution to \( \mathcal{L}S(AB, 0) \), and thus we can say that \( AB \) is singular (Definition NM [75]). This is a contradiction, so \( A \) is nonsingular, as desired.

This is a powerful result, because it allows us to begin with a hypothesis that something complicated (the matrix product \( AB \)) has the property of being nonsingular, and we can then conclude that the simpler constituents \( A \) and \( B \) individually) then also have the property of being nonsingular. If we had thought that the matrix product was an artificial construction, results like this would make us begin to think twice.

The contrapositive of this result is equally interesting. It says that if either \( A \) or \( B \) (or both) is a singular matrix, then the product \( AB \) is also singular. Notice how the negation of the theorem’s conclusion \( A \) and \( B \) both nonsingular) becomes the statement “at least one of \( A \) and \( B \) is singular.” (See Technique CP [706].)

**Theorem OSIS**

**One-Sided Inverse is Sufficient**

Suppose \( A \) and \( B \) are square matrices of size \( n \) such that \( AB = I_n \). Then \( BA = I_n \).

**Proof** The matrix \( I_n \) is nonsingular (since it row-reduces easily to \( I_n \), Theorem NM-RRI [77]). So \( A \) and \( B \) are nonsingular by Theorem NPNT [249], so in particular \( B \) is nonsingular. We can therefore apply Theorem CINM [237] to assert the existence of a matrix \( C \) so that \( BC = I_n \). This application of Theorem CINM [237] could be a bit confusing, mostly because of the names of the matrices involved. \( B \) is nonsingular, so there must be a “right-inverse” for \( B \), and we’re calling it \( C \).

Now

\[
BA = (BA)I_n = (BA)(BC) \quad \text{Theorem MMIM [218]}
\]

\[
C \quad \text{“right-inverse” of } B
\]
Subsection MINM.NMI  Nonsingular Matrices are Invertible  251

\[ B(AB)C = B I_n C = BC = I_n \]  

Theorem MMA 220  Hypothesis  Theorem MMIM 218  Theorem CINM 237.  

C “right-inverse” of B  

which is the desired conclusion. 

So Theorem OSIS 250 tells us that if A is nonsingular, then the matrix B guaranteed by Theorem CINM 237 will be both a “right-inverse” and a “left-inverse” for A, so A is invertible and \( A^{-1} = B \).

So if you have a nonsingular matrix, A, you can use the procedure described in Theorem CINM 237 to find an inverse for A. If A is singular, then the procedure in Theorem CINM 237 will fail as the first \( n \) columns of M will not row-reduce to the identity matrix. However, we can say a bit more. When A is singular, then A does not have an inverse (which is very different from saying that the procedure in Theorem CINM 237 fails to find an inverse). This may feel like we are splitting hairs, but it’s important that we do not make unfounded assumptions. These observations motivate the next theorem.

**Theorem NI**  

**Nonsingularity is Invertibility**  

Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible. □

**Proof**  \((\Leftarrow)\) Suppose A is invertible, and suppose that \( x \) is any solution to the homogeneous system \( LS(A, 0) \). Then

\[
\begin{align*}
    x &= I_n x \\
    &= (A^{-1}A) x \\
    &= A^{-1} (Ax) \\
    &= A^{-1}0 \\
    &= 0
\end{align*}
\]

Theorem MMIM 218  Definition MI 232  Theorem MMA 220  x solution to \( LS(A, 0) \), Theorem SLEMM 212  Theorem MMZM 218

So the only solution to \( LS(A, 0) \) is the zero vector, so by Definition NM 75, A is nonsingular.  

\((\Rightarrow)\) Suppose now that A is nonsingular. By Theorem CINM 237 we find B so that \( AB = I_n \). Then Theorem OSIS 250 tells us that \( BA = I_n \). So B is A’s inverse, and by construction, A is invertible. □

So for a square matrix, the properties of having an inverse and of having a trivial null space are one and the same. Can’t have one without the other.

**Theorem NME3**  

**Nonsingular Matrix Equivalences, Round 3**  

Suppose that A is a square matrix of size \( n \). The following are equivalent.

1. A is nonsingular.

2. A row-reduces to the identity matrix.

3. The null space of A contains only the zero vector, \( \mathcal{N}(A) = \{0\} \).
4. The linear system $\mathcal{L}S(A, b)$ has a unique solution for every possible choice of $b$.

5. The columns of $A$ are a linearly independent set.

6. $A$ is invertible.

**Proof** We can update our list of equivalences for nonsingular matrices (Theorem NME2 [152]) with the equivalent condition from Theorem NI [251].

In the case that $A$ is a nonsingular coefficient matrix of a system of equations, the inverse allows us to very quickly compute the unique solution, for any vector of constants.

**Theorem SNCM**  
**Solution with Nonsingular Coefficient Matrix**

Suppose that $A$ is nonsingular. Then the unique solution to $\mathcal{L}S(A, b)$ is $A^{-1}b$.

**Proof** By Theorem NMUS [79] we know already that $\mathcal{L}S(A, b)$ has a unique solution for every choice of $b$. We need to show that the expression stated is indeed a solution (the solution). That’s easy, just “plug it in” to the corresponding vector equation representation (Theorem SLEMM [212]),

$$A\ (A^{-1}b) = (AA^{-1})\ b$$

$$= I_n b$$

$$= b$$

Since $Ax = b$ is true when we substitute $A^{-1}b$ for $x$, $A^{-1}b$ is a (the!) solution to $\mathcal{L}S(A, b)$.

**Subsection UM**  
**Unitary Matrices**

**Definition UM**  
**Unitary Matrices**

Suppose that $Q$ is a square matrix of size $n$ such that $(Q)^t Q = I_n$. Then we say $Q$ is unitary.

This condition may seem rather far-fetched at first glance. Would there be any matrix that behaved this way? Well, yes, here’s one.

**Example UM3**

Unitary matrix of size 3

$$Q = \begin{bmatrix}
\frac{1+i}{\sqrt{2}} & \frac{3+2i}{\sqrt{10}} & \frac{2+2i}{\sqrt{2}} \\
\frac{1-i}{\sqrt{2}} & \frac{2+2i}{\sqrt{10}} & \frac{-3+i}{\sqrt{2}} \\
\frac{3+i}{\sqrt{5}} & \frac{-3+2i}{\sqrt{10}} & \frac{-\sqrt{2}}{2}
\end{bmatrix}$$

The computations get a bit tiresome, but if you work your way through $(Q)^t Q$, you will arrive at the $3 \times 3$ identity matrix $I_3$.  

Version 0.92
Unitary matrices do not have to look quite so gruesome. Here’s a larger one that is a bit more pleasing.

**Example UPM**

**Unitary permutation matrix**

The matrix

\[
P = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

is unitary as can be easily checked. Notice that it is just a rearrangement of the columns of the $5 \times 5$ identity matrix, $I_5$ (Definition IM [76]).

An interesting exercise is to build another $5 \times 5$ unitary matrix, $R$, using a different rearrangement of the columns of $I_5$. Then form the product $PR$. This will be another unitary matrix (Exercise MINM.T10 [257]). If you were to build all $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ matrices of this type you would have a set that remains closed under matrix multiplication. It is an example of another algebraic structure known as a *group* since together the set and the one operation (matrix multiplication here) is closed, associative, has an identity ($I_5$), and inverses (Theorem UMI [253]). Notice though that the operation in this group is not commutative!

If a matrix $A$ has only real number entries (we say it is a *real matrix*) then the defining property of being unitary simplifies to $A^t A = I_n$. In this case we, and everybody else, calls the matrix *orthogonal*, so you may often encounter this term in your other reading.

Unitary matrices have easily computed inverses. They also have columns that form orthonormal sets. Here are the theorems that show us that unitary matrices are not as strange as they might initially appear.

**Theorem UMI**

**Unitary Matrices are Invertible**

Suppose that $Q$ is a unitary matrix of size $n$. Then $Q$ is nonsingular, and $Q^{-1} = (\overline{Q})^t$.

**Proof** By Definition UM [252], we know that $(\overline{Q})^t Q = I_n$. The matrix $I_n$ is nonsingular (since it row-reduces easily to $I_n$, Theorem NMRRI [77]). So by Theorem NPNT [249], $Q$ and $(\overline{Q})^t$ are both nonsingular.

The equation $(\overline{Q})^t Q = I_n$ gets us halfway to an inverse of $Q$, and Theorem OSIS [250] tells us that $Q(\overline{Q})^t = I_n$ also. So $Q$ and $(\overline{Q})^t$ are inverses of each other (Definition MI [232]).

**Theorem CUMOS**

**Columns of Unitary Matrices are Orthonormal Sets**

Suppose that $A$ is a square matrix of size $n$ with columns $S = \{A_1, A_2, A_3, \ldots, A_n\}$. Then $A$ is a unitary matrix if and only if $S$ is an orthonormal set.

**Proof** The proof revolves around recognizing that a typical entry of the product $(\overline{A})^t A$ is an inner product of columns of $A$. Here are the details to support this claim.

\[
[(\overline{A})^t A]_{ij} = \sum_{k=1}^{n} [(\overline{A})^t]_{ik} [A]_{kj}
\]

Theorem EMP [216]
\[
\sum_{k=1}^{n} [\overline{A}]_{ki} [A]_{kj} = \sum_{k=1}^{n} [A]_{kj} [\overline{A}]_{ki} = \sum_{k=1}^{n} [A]_{kj} [A]_{ki} = \sum_{k=1}^{n} [A]_{kj} [A]_{ki} = \langle A_j, A_i \rangle \]

We now employ this equality in a chain of equivalences,

\[
S = \{A_1, A_2, A_3, \ldots, A_n\} \text{ is an orthonormal set}
\]

\[\iff \langle A_j, A_i \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \quad \text{Definition ONS} \quad 193\]

\[\iff [A]_{ij} = [I_n]_{ij}, \ 1 \leq i \leq n, \ 1 \leq j \leq n \quad \text{Definition IM} \quad 76\]

\[\iff (A)^t A = I_n \quad \text{Definition ME} \quad 197\]

\[\iff A \text{ is a unitary matrix} \quad \text{Definition UM} \quad 252\]

Example OSMC
Orthonormal Set from Matrix Columns

The matrix

\[
Q = \begin{bmatrix}
\frac{1+i}{\sqrt{5}} & \frac{3+2i}{\sqrt{55}} & \frac{2+2i}{\sqrt{22}} \\
\frac{1-i}{\sqrt{5}} & \frac{2+2i}{\sqrt{55}} & \frac{-3+2i}{\sqrt{22}} \\
\frac{1}{\sqrt{5}} & \frac{-3+2i}{\sqrt{55}} & \frac{-2}{\sqrt{22}}
\end{bmatrix}
\]

from Example OM3 is a unitary matrix. By Theorem CUMOS, its columns

\[
\left\{ \begin{bmatrix} \frac{1+i}{\sqrt{5}} \\ \frac{1-i}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \ \begin{bmatrix} \frac{3+2i}{\sqrt{55}} \\ \frac{2+2i}{\sqrt{55}} \\ \frac{-3+2i}{\sqrt{55}} \end{bmatrix}, \ \begin{bmatrix} \frac{2+2i}{\sqrt{22}} \\ \frac{-3+2i}{\sqrt{22}} \\ \frac{-2}{\sqrt{22}} \end{bmatrix} \right\}
\]

form an orthonormal set. You might find checking the six inner products of pairs of these vectors easier than doing the matrix product \((Q)^t Q\). Or, because the inner product is anti-commutative, you only need check three inner products (see Exercise MINM.T12).

When using vectors and matrices that only have real number entries, orthogonal matrices are those matrices with inverses that equal their transpose. Similarly, the inner product is the familiar dot product. Keep this special case in mind as you read the next theorem.
Theorem UMPIP
Unitary Matrices Preserve Inner Products
Suppose that $Q$ is a unitary matrix of size $n$ and $u$ and $v$ are two vectors from $C^n$. Then

$$\langle Qu, Qv \rangle = \langle u, v \rangle \quad \text{and} \quad \|Qv\| = \|v\|$$

\[
\begin{align*}
\langle Qu, Qv \rangle &= (Qu)^tQv \\
&= u^tQ^tQv \\
&= u^t\overline{Q}^tQv \\
&= u^t\overline{(Q^{-1})}^tQv \\
&= u^t\overline{(Q)}^tQv \\
&= u^t\overline{I}_n v \\
&= u^tv \\
&= \langle u, v \rangle
\end{align*}
\]

The second conclusion is just a specialization of the first conclusion.

$$
\|Qv\| = \sqrt{\|Qv\|^2} = \sqrt{\langle Qv, Qv \rangle} = \sqrt{\langle v, v \rangle} = \sqrt{\|v\|^2} = \|v\| 
$$

Definition A
Adjoint
If $A$ is a square matrix, then its **adjoint** is $A^H = (\overline{A})^t$.

Sometimes a matrix is equal to its adjoint. One of the most common situations where this occurs is when a matrix has only real number entries. Then we are simply talking about symmetric matrices (Definition SYM [201]).

Definition HM
Hermitian Matrix
The square matrix $A$ is **Hermitian** (or **self-adjoint**) if $A = (\overline{A})^t$.

Again, the real matrices that are Hermitian is exactly the set of symmetric matrices. In Section PEE [467] we will uncover some amazing properties of Hermitian matrices, so
run back here then to remind yourself of this definition. This will all be preparation for Theorem ODHM [??] in Subsection SD.OD [494].

A final reminder: the terms “dot product,” “orthogonal matrix” and “symmetric matrix” used in reference to vectors or matrices with real number entries correspond to the terms inner product, unitary matrix and Hermitian matrix when we generalize to include complex number entries.

Subsection READ

Reading Questions

1. Show how to use the inverse of a matrix to solve the system of equations below and state the resulting solution.

\[ 4x_1 + 10x_2 = 12 \]
\[ 2x_1 + 6x_2 = 4 \]

2. In the reading questions for Section MISLE [231] you were asked to find the inverse of the \(3 \times 3\) matrix below.

\[
\begin{bmatrix}
2 & 3 & 1 \\
1 & -2 & -3 \\
-2 & 4 & 6
\end{bmatrix}
\]

Because the matrix was not nonsingular, you had no theorems at that point that would allow you to compute the inverse. Explain why you now know that the inverse does not exist (which is different than not being able to compute it) by quoting the relevant theorem’s acronym.

3. Is the matrix \(A\) orthogonal? Why?

\[
A = \begin{bmatrix}
\frac{1}{\sqrt{22}} (4 + 2i) & \frac{1}{\sqrt{374}} (5 + 3i) \\
\frac{1}{\sqrt{22}} (-1 - i) & \frac{1}{\sqrt{374}} (12 + 14i)
\end{bmatrix}
\]
Subsection EXC
Exercises

C40    Solve the system of equations below using the inverse of a matrix.

    \[ x_1 + x_2 + 3x_3 + x_4 = 5 \]
    \[ -2x_1 - x_2 - 4x_3 - x_4 = -7 \]
    \[ x_1 + 4x_2 + 10x_3 + 2x_4 = 9 \]
    \[ -2x_1 - 4x_3 + 5x_4 = 9 \]

Contributed by Robert Beezer   Solution 259

M20    Construct an example of a 4 × 4 unitary matrix.
Contributed by Robert Beezer   Solution 259

T10    Suppose that \( Q \) and \( P \) are unitary matrices of size \( n \). Prove that \( QP \) is a unitary matrix.
Contributed by Robert Beezer

T11    Prove that Hermitian matrices (Definition HM 255) have real entries on the diagonal. More precisely, suppose that \( A \) is a Hermitian matrix of size \( n \). Then \( [A]_{ii} \in \mathbb{R}, 1 \leq i \leq n \).
Contributed by Robert Beezer

T12    Suppose that we are checking if a square matrix of size \( n \) is unitary. Show that a straightforward application of Theorem CUMOS 253 requires the computation of \( n^2 \) inner products when the matrix is unitary, and fewer when the matrix is not orthogonal. Then show that this maximum number of inner products can be reduced to \( \frac{1}{2}n(n+1) \) in light of Theorem IPAC 186.
Contributed by Robert Beezer
Subsection SOL
Solutions

C40 Contributed by Robert Beezer Statement 257
The coefficient matrix and vector of constants for the system are
\[
\begin{bmatrix}
1 & 1 & 3 & 1 \\
-2 & -1 & -4 & -1 \\
1 & 4 & 10 & 2 \\
-2 & 0 & -4 & 5
\end{bmatrix}
\hspace{1cm}
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{bmatrix}
\]

Then Theorem SNCM 252 says the unique solution is
\[
A^{-1}b = \begin{bmatrix}
38 & 18 & -5 & -2 \\
96 & 47 & -12 & -5 \\
-39 & -19 & 5 & 2 \\
-16 & -8 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
5 \\
-7 \\
9 \\
9
\end{bmatrix}
= \begin{bmatrix}
1 \\
-2 \\
1 \\
3
\end{bmatrix}
\]

M20 Contributed by Robert Beezer Statement 257
The 4 \times 4 identity matrix, \(I_4\), would be one example (Definition IM 76). Any of the 23 other rearrangements of the columns of \(I_4\) would be a simple, but less trivial, example. See Example UPM 253.
Section CRS
Column and Row Spaces

Theorem SLSLC showed us that there is a natural correspondence between solutions to linear systems and linear combinations of the columns of the coefficient matrix. This idea motivates the following important definition.

Definition CSM
Column Space of a Matrix

Suppose that $A$ is an $m \times n$ matrix with columns $\{A_1, A_2, A_3, \ldots, A_n\}$. Then the column space of $A$, written $C(A)$, is the subset of $\mathbb{C}^m$ containing all linear combinations of the columns of $A$,

$$C(A) = \langle \{A_1, A_2, A_3, \ldots, A_n\} \rangle$$

(This definition contains Notation CSM.)

Some authors refer to the column space of a matrix as the range, but we will reserve this term for use with linear transformations (Definition RLT).

Subsection CSSE
Column spaces and systems of equations

Upon encountering any new set, the first question we ask is what objects are in the set, and which objects are not? Here’s an example of one way to answer this question, and it will motivate a theorem that will then answer the question precisely.

Example CSMCS
Column space of a matrix and consistent systems

Archetype D and Archetype E are linear systems of equations, with an identical $3 \times 4$ coefficient matrix, which we call $A$ here. However, Archetype D is consistent, while Archetype E is not. We can explain this difference by employing the column space of the matrix $A$.

The column vector of constants, $b$, in Archetype D is

$$b = \begin{bmatrix} 8 \\ -12 \\ 4 \end{bmatrix}$$

One solution to $LS(A, b)$, as listed, is

$$x = \begin{bmatrix} 7 \\ 8 \\ 1 \\ 3 \end{bmatrix}$$
By Theorem SLSLC [100], we can summarize this solution as a linear combination of the columns of $A$ that equals $b$,

$$
7 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + 8 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} -7 \\ -6 \\ -5 \end{bmatrix} = \begin{bmatrix} 8 \\ -12 \\ 4 \end{bmatrix} = b.
$$

This equation says that $b$ is a linear combination of the columns of $A$, and then by Definition CSM [261], we can say that $b \in \mathcal{C}(A)$.

On the other hand, Archetype E [739] is the linear system $\mathcal{L}(A, c)$, where the vector of constants is

$$
c = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}
$$

and this system of equations is inconsistent. This means $c \notin \mathcal{C}(A)$, for if it were, then it would equal a linear combination of the columns of $A$ and Theorem SLSLC [100] would lead us to a solution of the system $\mathcal{L}(A, c)$. ⊠

So if we fix the coefficient matrix, and vary the vector of constants, we can sometimes find consistent systems, and sometimes inconsistent systems. The vectors of constants that lead to consistent systems are exactly the elements of the column space. This is the content of the next theorem, and since it is an equivalence, it provides an alternate view of the column space.

**Theorem CSCS**

**Column Spaces and Consistent Systems**

Suppose $A$ is an $m \times n$ matrix and $b$ is a vector of size $m$. Then $b \in \mathcal{C}(A)$ if and only if $\mathcal{L}(A, b)$ is consistent. □

**Proof**  $(\Rightarrow)$ Suppose $b \in \mathcal{C}(A)$. Then we can write $b$ as some linear combination of the columns of $A$. By Theorem SLSLC [100], we can use the scalars from this linear combination to form a solution to $\mathcal{L}(A, b)$, so this system is consistent.

$(\Leftarrow)$ If $\mathcal{L}(A, b)$ is consistent, there is a solution that may be used with Theorem SLSLC [100] to write $b$ as a linear combination of the columns of $A$. This qualifies $b$ for membership in $\mathcal{C}(A)$. ■

This theorem tells us that asking if the system $\mathcal{L}(A, b)$ is consistent is exactly the same question as asking if $b$ is in the column space of $A$. Or equivalently, it tells us that the column space of the matrix $A$ is precisely those vectors of constants, $b$, that can be paired with $A$ to create a system of linear equations $\mathcal{L}(A, b)$ that is consistent.

Employing Theorem SLEMM [212] we can form the chain of equivalences

$$
b \in \mathcal{C}(A) \iff \mathcal{L}(A, b) \text{ is consistent} \iff Ax = b \text{ for some } x
$$

Thus, an alternative (and popular) definition of the column space of an $m \times n$ matrix $A$ is

$$
\mathcal{C}(A) = \{ y \in \mathbb{C}^m \mid y = Ax \text{ for some } x \in \mathbb{C}^n \}
$$

We recognize this as saying create all the matrix vector products possible with the matrix $A$ by letting $x$ range over all of the possibilities. By Definition MVP [211], we see that this means take all possible linear combinations of the columns of $A$ — precisely the definition of the column space (Definition CSM [261]) we have chosen.
Given a vector \( b \) and a matrix \( A \) it is now very mechanical to test if \( b \in \mathcal{C}(A) \). Form the linear system \( L\mathcal{S}(A, b) \), row-reduce the augmented matrix, \([A | b]\), and test for consistency with Theorem RCLS\(^*\). Here’s an example of this procedure.

Example MCSM
Membership in the column space of a matrix
Consider the column space of the \( 3 \times 4 \) matrix \( A \),

\[
A = \begin{bmatrix}
3 & 2 & 1 & -4 \\
-1 & 1 & -2 & 3 \\
2 & -4 & 6 & -8
\end{bmatrix}
\]

We first show that \( v = \begin{bmatrix} 18 \\ -6 \\ 12 \end{bmatrix} \) is in the column space of \( A \), \( v \in \mathcal{C}(A) \). Theorem CSCS\(^*\) says we need only check the consistency of \( L\mathcal{S}(A, v) \). Form the augmented matrix and row-reduce,

\[
\begin{bmatrix}
3 & 2 & 1 & -4 & 18 \\
-1 & 1 & -2 & 3 & -6 \\
2 & -4 & 6 & -8 & 12
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 1 & -2 & 6 \\
0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Without a leading 1 in the final column, Theorem RCLS\(^*\) tells us the system is consistent and therefore by Theorem CSCS\(^*\), \( v \in \mathcal{C}(A) \).

If we wished to demonstrate explicitly that \( v \) is a linear combination of the columns of \( A \), we can find a solution (any solution) of \( L\mathcal{S}(A, v) \) and use Theorem SLSLC\(^*\) to construct the desired linear combination. For example, set the free variables to \( x_3 = 2 \) and \( x_4 = 1 \). Then a solution has \( x_2 = 1 \) and \( x_1 = 6 \). Then by Theorem SLSLC\(^*\),

\[
v = \begin{bmatrix} 18 \\ -6 \\ 12 \end{bmatrix} = 6 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix} + 1 \begin{bmatrix} -4 \\ 3 \\ -8 \end{bmatrix}
\]

Now we show that \( w = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \) is not in the column space of \( A \), \( w \notin \mathcal{C}(A) \). Theorem CSCS\(^*\) says we need only check the consistency of \( L\mathcal{S}(A, w) \). Form the augmented matrix and row-reduce,

\[
\begin{bmatrix}
3 & 2 & 1 & -4 & 2 \\
-1 & 1 & -2 & 3 & 1 \\
2 & -4 & 6 & -8 & -3
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 1 & -2 & 0 \\
0 & 1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

With a leading 1 in the final column, Theorem RCLS\(^*\) tells us the system is inconsistent and therefore by Theorem CSCS\(^*\), \( w \notin \mathcal{C}(A) \).

Subsection CSSOC
Column space spanned by original columns

So we have a foolproof, automated procedure for determining membership in \( \mathcal{C}(A) \). While this works just fine a vector at a time, we would like to have a more useful description
of the set \( C(A) \) as a whole. The next example will preview the first of two fundamental results about the column space of a matrix.

**Example CSTW**

**Column space, two ways**

Consider the \( 5 \times 7 \) matrix \( A \),

\[
\begin{bmatrix}
2 & 4 & 1 & -1 & 1 & 4 & 4 \\
1 & 2 & 1 & 0 & 2 & 4 & 7 \\
0 & 0 & 1 & 4 & 1 & 8 & 7 \\
1 & 2 & -1 & 2 & 1 & 9 & 6 \\
-2 & -4 & 1 & 3 & -1 & -2 & -2 \\
\end{bmatrix}
\]

According to the definition ([Definition CSM][261]), the column space of \( A \) is

\[
C(A) = \langle \begin{bmatrix}
2 \\
1 \\
0 \\
1 \\
-2
\end{bmatrix}, \begin{bmatrix}
4 \\
2 \\
0 \\
2 \\
-4
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
3
\end{bmatrix}, \begin{bmatrix}
-1 \\
1 \\
4 \\
2 \\
3
\end{bmatrix}, \begin{bmatrix}
1 \\
0 \\
4 \\
9 \\
-2
\end{bmatrix}, \begin{bmatrix}
4 \\
4 \\
1 \\
2 \\
4
\end{bmatrix}, \begin{bmatrix}
4 \\
7 \\
1 \\
6 \\
-2
\end{bmatrix} \rangle
\]

While this is a concise description of an infinite set, we might be able to describe the span with fewer than seven vectors. This is the substance of [Theorem BS][172]. So we take these seven vectors and make them the columns of matrix, which is simply the original matrix \( A \) again. Now we row-reduce,

\[
\begin{bmatrix}
2 & 4 & 1 & -1 & 1 & 4 & 4 \\
1 & 2 & 1 & 0 & 2 & 4 & 7 \\
0 & 0 & 1 & 4 & 1 & 8 & 7 \\
1 & 2 & -1 & 2 & 1 & 9 & 6 \\
-2 & -4 & 1 & 3 & -1 & -2 & -2 \\
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 2 & 0 & 0 & 0 & 3 & 1 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The pivot columns are \( D = \{1, 3, 4, 5\} \), so we can create the set

\[
T = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\}
\]

and know that \( C(A) = \langle T \rangle \) and \( T \) is a linearly independent set of columns from the set of columns of \( A \).

We will now formalize the previous example, which will make it trivial to determine a linearly independent set of vectors that will span the column space of a matrix, and is constituted of just columns of \( A \).

**Theorem BCS**

**Basis of the Column Space**

Suppose that \( A \) is an \( m \times n \) matrix with columns \( A_1, A_2, A_3, \ldots, A_n \), and \( B \) is a row-equivalent matrix in reduced row-echelon form with \( r \) nonzero rows. Let \( D = \{d_1, d_2, d_3, \ldots, d_r\} \) be the set of column indices where \( B \) has leading 1’s. Let \( T = \{A_{d_1}, A_{d_2}, A_{d_3}, \ldots, A_{d_r}\} \). Then
1. $T$ is a linearly independent set.

2. $\mathcal{C}(A) = \langle T \rangle$.

**Proof** Definition CSM 261 describes the column space as the span of the set of columns of $A$. Theorem BS 172 tells us that we can reduce the set of vectors used in a span. If we apply Theorem BS 172 to $\mathcal{C}(A)$, we would collect the columns of $A$ into a matrix (which would just be $A$ again) and bring the matrix to reduced row-echelon form, which is the matrix $B$ in the statement of the theorem. In this case, the conclusions of Theorem BS 172 applied to $A$, $B$ and $\mathcal{C}(A)$ are exactly the conclusions we desire. ■

This is a nice result since it gives us a handful of vectors that describe the entire column space (through the span), and we believe this set is as small as possible because we cannot create any more relations of linear dependence to trim it down further. Furthermore, we defined the column space (Definition CSM 261) as all linear combinations of the columns of the matrix, and the elements of the set $S$ are still columns of the matrix (we won’t be so lucky in the next two constructions of the column space).

Procedurally this theorem is extremely easy to apply. Row-reduce the original matrix, identify $r$ columns with leading 1’s in this reduced matrix, and grab the corresponding columns of the original matrix. But it is still important to study the proof of Theorem BS 172 and its motivation in Example COV 170 which lie at the root of this theorem. We’ll trot through an example all the same.

**Example CSOCD**

Column space, original columns, Archetype D

Let’s determine a compact expression for the entire column space of the coefficient matrix of the system of equations that is Archetype D 735. Notice that in Example CSMCS 261 we were only determining if individual vectors were in the column space or not, now we are describing the entire column space.

To start with the application of Theorem BCS 264, call the coefficient matrix $A$

$$A = \begin{bmatrix} 2 & 1 & 7 & -7 \\ -3 & 4 & -5 & -6 \\ 1 & 1 & 4 & -5 \end{bmatrix}.$$

and row-reduce it to reduced row-echelon form,

$$B = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are leading 1’s in columns 1 and 2, so $D = \{1, 2\}$. To construct a set that spans $\mathcal{C}(A)$, just grab the columns of $A$ indicated by the set $D$, so

$$\mathcal{C}(A) = \left\langle \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \right\} \right\rangle.$$

That’s it.
In Example CSMCS 261 we determined that the vector
\[ \mathbf{c} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \]
was not in the column space of \( A \). Try to write \( \mathbf{c} \) as a linear combination of the first two columns of \( A \). What happens?

Also in Example CSMCS 261 we determined that the vector
\[ \mathbf{b} = \begin{bmatrix} 8 \\ -12 \\ 4 \end{bmatrix} \]
was in the column space of \( A \). Try to write \( \mathbf{b} \) as a linear combination of the first two columns of \( A \). What happens? Did you find a unique solution to this question? Hmmm.

\[ \square \]

Subsection CSNM
Column Space of a Nonsingular Matrix

Let’s specialize to square matrices and contrast the column spaces of the coefficient matrices in Archetype A 721 and Archetype B 726.

Example CSAA
Column space of Archetype A

The coefficient matrix in Archetype A 721 is
\[ A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \]
which row-reduces to
\[ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} . \]

Columns 1 and 2 have leading 1’s, so by Theorem BCS 264 we can write
\[ \mathcal{C}(A) = \langle \{A_1, A_2\} \rangle = \left\langle \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\rangle . \]

We want to show in this example that \( \mathcal{C}(A) \neq \mathbb{C}^3 \). So take, for example, the vector
\[ \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} . \]
Then there is no solution to the system \( \mathcal{L}S(A, \mathbf{b}) \), or equivalently, it is not possible to write \( \mathbf{b} \) as a linear combination of \( A_1 \) and \( A_2 \). Try one of these two computations yourself. (Or try both!). Since \( \mathbf{b} \notin \mathcal{C}(A) \), the column space of \( A \) cannot
be all of $\mathbb{C}^3$. So by varying the vector of constants, it is possible to create inconsistent systems of equations with this coefficient matrix (the vector $b$ being one such example).

In Example MWIAA we wished to show that the coefficient matrix from Archetype A was not invertible as a first example of a matrix without an inverse. Our device there was to find an inconsistent linear system with $A$ as the coefficient matrix. The vector of constants in that example was $b$, deliberately chosen outside the column space of A. 

Example CSAB

Column space of Archetype B

The coefficient matrix in Archetype B, call it $B$ here, is known to be nonsingular (see Example NM). By Theorem NMUS, the linear system $LS(B, b)$ has a (unique) solution for every choice of $b$. Then says that $b \in C(B)$ for all $b \in \mathbb{C}^3$. Stated differently, there is no way to build an inconsistent system with the coefficient matrix $B$, but then we knew that already from Theorem NMUS. 

Example CSAB and Example CSAA together motivate the following equivalence, which says that nonsingular matrices have column spaces that are as big as possible.

Theorem CSNM

Column Space of a Nonsingular Matrix

Suppose $A$ is a square matrix of size $n$. Then $A$ is nonsingular if and only if $C(A) = \mathbb{C}^n$.

Proof  ($\Rightarrow$) Suppose $A$ is nonsingular. We wish to establish the set equality $C(A) = \mathbb{C}^n$. 

By Definition CSM, $C(A) \subseteq \mathbb{C}^n$. To show that $\mathbb{C}^n \subseteq C(A)$ choose $b \in \mathbb{C}^n$. By Theorem NMUS, we know the linear system $LS(A, b)$ has a (unique) solution and therefore is consistent. Then says that $b \in C(A)$. So by Definition SE, $C(A) = \mathbb{C}^n$.

($\Leftarrow$) If $e_i$ is column $i$ of the $n \times n$ identity matrix and by hypothesis $C(A) = \mathbb{C}^n$, then $e_i \in C(A)$ for $1 \leq i \leq n$. By Theorem CSCS, the system $LS(A, e_i)$ is consistent for $1 \leq i \leq n$. Let $b_i$ denote any one particular solution to $LS(A, e_i)$, $1 \leq i \leq n$.

Define the $n \times n$ matrix $B = [b_1\mid b_2\mid b_3\mid \ldots \mid b_n]$. Then

$$AB = A[b_1\mid b_2\mid b_3\mid \ldots \mid b_n]$$

$$= [Ab_1\mid Ab_2\mid Ab_3\mid \ldots \mid Ab_n]$$

$$= [e_1\mid e_2\mid e_3\mid \ldots \mid e_n]$$

$$= I_n$$

So the matrix $B$ is a "right-inverse" for $A$. By Theorem NMRRI, $I_n$ is a nonsingular matrix, so by Theorem NPNT, both $A$ and $B$ are nonsingular. Thus, in particular, $A$ is nonsingular. (Travis Osborne contributed to this proof.)

With this equivalence for nonsingular matrices we can update our list.

Theorem NME4

Nonsingular Matrix Equivalences, Round 4

Suppose that $A$ is a square matrix of size $n$. The following are equivalent.

Version 0.92
1. $A$ is nonsingular.
2. $A$ row-reduces to the identity matrix.
3. The null space of $A$ contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
4. The linear system $\mathcal{L}\mathcal{S}(A, b)$ has a unique solution for every possible choice of $b$.
5. The columns of $A$ are a linearly independent set.
6. $A$ is invertible.
7. The column space of $A$ is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

\[\square\]

**Proof** Since Theorem CSNM is an equivalence, we can add it to the list in Theorem NME3.

**Subsection RSM**

**Row Space of a Matrix**

The rows of a matrix can be viewed as vectors, since they are just lists of numbers, arranged horizontally. So we will transpose a matrix, turning rows into columns, so we can then manipulate rows as column vectors. As a result we will be able to make some new connections between row operations and solutions to systems of equations. OK, here is the second primary definition of this section.

**Definition RSM**

**Row Space of a Matrix**

Suppose $A$ is an $m \times n$ matrix. Then the row space of $A$, $\mathcal{R}(A)$, is the column space of $A^t$, i.e. $\mathcal{R}(A) = \mathcal{C}(A^t)$.

(This definition contains Notation RSM.)

Informally, the row space is the set of all linear combinations of the rows of $A$. However, we write the rows as column vectors, thus the necessity of using the transpose to make the rows into columns. Additionally, with the row space defined in terms of the column space, all of the previous results of this section can be applied to row spaces.

Notice that if $A$ is a rectangular $m \times n$ matrix, then $\mathcal{C}(A) \subseteq \mathbb{C}^m$, while $\mathcal{R}(A) \subseteq \mathbb{C}^n$ and the two sets are not comparable since they do not even hold objects of the same type. However, when $A$ is square of size $n$, both $\mathcal{C}(A)$ and $\mathcal{R}(A)$ are subsets of $\mathbb{C}^n$, though usually the sets will not be equal (but see Exercise CRS.M20).

**Example RSAI**

**Row space of Archetype I**

The coefficient matrix in Archetype I is

\[
I = \begin{bmatrix}
1 & 4 & 0 & -1 & 0 & 7 & -9 \\
2 & 8 & -1 & 3 & 9 & -13 & 7 \\
0 & 0 & 2 & -3 & -4 & 12 & -8 \\
-1 & -4 & 2 & 4 & 8 & -31 & 37
\end{bmatrix}.
\]
To build the row space, we transpose the matrix,

\[
I^t = \begin{bmatrix}
1 & 2 & 0 & -1 \\
4 & 8 & 0 & -4 \\
0 & -1 & 2 & 2 \\
-1 & 3 & -3 & 4 \\
0 & 9 & -4 & 8 \\
7 & -13 & 12 & -31 \\
-9 & 7 & -8 & 37
\end{bmatrix}
\]

Then the columns of this matrix are used in a span to build the row space,

\[
\mathcal{R}(I) = \mathcal{C}(I^t) = \langle \begin{bmatrix} 1 \\ 4 \\ -1 \\ 7 \\ -9 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 3 \\ -13 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 9 \\ 12 \\ -8 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -3 \\ -31 \\ 37 \end{bmatrix} \rangle.
\]

However, we can use Theorem BCS [264] to get a slightly better description. First, row-reduce \( I^t \),

\[
\begin{bmatrix}
1 & 0 & 0 & \frac{-31}{7} \\
0 & 1 & 0 & \frac{12}{7} \\
0 & 0 & 1 & \frac{13}{7} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Since there are leading 1’s in columns with indices \( D = \{1, 2, 3\} \), the column space of \( I^t \) can be spanned by just the first three columns of \( I^t \),

\[
\mathcal{R}(I) = \mathcal{C}(I^t) = \langle \begin{bmatrix} 1 \\ 4 \\ -1 \\ 7 \\ -9 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 3 \\ -13 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 9 \\ 12 \\ -8 \end{bmatrix} \rangle.
\]

The row space would not be too interesting if it was simply the column space of the transpose. However, when we do row operations on a matrix we have no effect on the many linear combinations that can be formed with the rows of the matrix. This is stated more carefully in the following theorem.

**Theorem REMRS**

**Row-Equivalent Matrices have equal Row Spaces**

Suppose \( A \) and \( B \) are row-equivalent matrices. Then \( \mathcal{R}(A) = \mathcal{R}(B) \). □
**Proof** Two matrices are row-equivalent (Definition REM [33]) if one can be obtained from another by a sequence of (possibly many) row operations. We will prove the theorem for two matrices that differ by a single row operation, and then this result can be applied repeatedly to get the full statement of the theorem. The row spaces of $A$ and $B$ are spans of the columns of their transposes. For each row operation we perform on a matrix, we can define an analogous operation on the columns. Perhaps we should call these column operations. Instead, we will still call them row operations, but we will apply them to the columns of the transposes.

Refer to the columns of $A^t$ and $B^t$ as $A_i$ and $B_i$, $1 \leq i \leq m$. The row operation that switches rows will just switch columns of the transposed matrices. This will have no effect on the possible linear combinations formed by the columns.

Suppose that $B^t$ is formed from $A^t$ by multiplying column $A_i$ by $\alpha \neq 0$. In other words, $B_i = \alpha A_i$, and $B_i = A_i$ for all $i \neq t$. We need to establish that two sets are equal, $C(A^t) = C(B^t)$. We will take a generic element of one and show that it is contained in the other.

\[
\begin{align*}
\beta_1 B_1 + \beta_2 B_2 + \cdots + \beta_t B_t + \cdots + \beta_n B_m \\
= \beta_1 A_1 + \beta_2 A_2 + \cdots + \beta_t (\alpha A_t) + \cdots + \beta_m A_m \\
= \beta_1 A_1 + \beta_2 A_2 + \cdots + (\alpha \beta_t) A_t + \cdots + \beta_m A_m
\end{align*}
\]

says that $C(B^t) \subseteq C(A^t)$. Similarly,

\[
\begin{align*}
\gamma_1 A_1 + \gamma_2 A_2 + \cdots + \gamma_t A_t + \cdots + \gamma_m A_m \\
= \gamma_1 A_1 + \gamma_2 A_2 + \cdots + \gamma_t (\alpha A_t) + \cdots + \gamma_m A_m \\
= \gamma_1 A_1 + \gamma_2 A_2 + \cdots + \gamma_t \left( \frac{\gamma_t}{\alpha} A_t + \cdots + \gamma_m A_m \right)
\end{align*}
\]

says that $C(A^t) \subseteq C(B^t)$. So $R(A) = C(A^t) = C(B^t) = R(B)$ when a single row operation of the second type is performed.

Suppose now that $B^t$ is formed from $A^t$ by replacing $A_t$ with $\alpha A_s + A_t$ for some $\alpha \in \mathbb{C}$ and $s \neq t$. In other words, $B_t = \alpha A_s + A_t$, and $B_i = A_i$ for $i \neq t$.

\[
\begin{align*}
\beta_1 B_1 + \beta_2 B_2 + \cdots + \beta_t B_t + \cdots + \beta_n B_m \\
= \beta_1 A_1 + \beta_2 A_2 + \cdots + \beta_t (\alpha A_s + A_t) + \cdots + \beta_m A_m \\
= \beta_1 A_1 + \beta_2 A_2 + \cdots + (\beta_t \alpha) A_t + \cdots + \beta_m A_m
\end{align*}
\]

says that $C(B^t) \subseteq C(A^t)$. Similarly,

\[
\begin{align*}
\gamma_1 A_1 + \gamma_2 A_2 + \cdots + \gamma_s A_s + \cdots + \gamma_t A_t + \cdots + \gamma_m A_m \\
= \gamma_1 A_1 + \gamma_2 A_2 + \cdots + \gamma_s A_s + \cdots + \gamma_t (\alpha A_s + A_t) + \cdots + \gamma_m A_m \\
= \gamma_1 A_1 + \gamma_2 A_2 + \cdots + \gamma_t (\alpha A_t + \gamma_s A_s + \cdots + \gamma_m A_m)
\end{align*}
\]
Subsection CRS.RSM  Row Space of a Matrix  271

says that $C(A^t) \subseteq C(B^t)$. So $\mathcal{R}(A) = C(A^t) = C(B^t) = \mathcal{R}(B)$ when a single row operation of the third type is performed.

So the row space of a matrix is preserved by each row operation, and hence row spaces of row-equivalent matrices are equal sets. ■

Example RSREM
Row spaces of two row-equivalent matrices

In Example TREM [34] we saw that the matrices

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 5 & 2 & -2 & 3 \\ 1 & 1 & 0 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 & 0 & 6 \\ 3 & 0 & -2 & -9 \\ 2 & -1 & 3 & 4 \end{bmatrix}$$

are row-equivalent by demonstrating a sequence of two row operations that converted $A$ into $B$. Applying Theorem REMRS [269] we can say

$$\mathcal{R}(A) = \langle \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 6 \end{bmatrix} \rangle = \mathcal{R}(B)$$

Theorem REMRS [269] is at its best when one of the row-equivalent matrices is in reduced row-echelon form. The vectors that correspond to the zero rows can be ignored (who needs the zero vector when building a span?, see Exercise LI.T10 [159]). The echelon pattern insures that the nonzero rows yield vectors that are linearly independent. Here’s the theorem.

Theorem BRS
Basis for the Row Space

Suppose that $A$ is a matrix and $B$ is a row-equivalent matrix in reduced row-echelon form. Let $S$ be the set of nonzero columns of $B^t$. Then

1. $\mathcal{R}(A) = \langle S \rangle$.
2. $S$ is a linearly independent set.

Proof From Theorem REMRS [269] we know that $\mathcal{R}(A) = \mathcal{R}(B)$. If $B$ has any zero rows, these correspond to columns of $B^t$ that are the zero vector. We can safely toss out the zero vector in the span construction, since it can be recreated from the nonzero vectors by a linear combination where all the scalars are zero. So $\mathcal{R}(A) = \langle S \rangle$.

Suppose $B$ has $r$ nonzero rows and let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ denote the column indices of $B$ that have a leading one in them. Denote the $r$ column vectors of $B^t$, the vectors in $S$, as $B_1, B_2, B_3, \ldots, B_r$. To show that $S$ is linearly independent, start with a relation of linear dependence

$$\alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 + \cdots + \alpha_r B_r = 0$$

Now consider this vector equality in location $d_i$. Since $B$ is in reduced row-echelon form, the entries of column $d_i$ of $B$ are all zero, except for a (leading) 1 in row $i$. Thus, in $B^t$, row $d_i$ is all zeros, excepting a 1 in column $i$. So, for $1 \leq i \leq r$,

$$0 = [0]_{d_i}$$

Definition ZCV [30]
\[
= [\alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 + \cdots + \alpha_r B_r]_{d_i}
\]
\[
= [\alpha_1 B_1]_{d_i} + [\alpha_2 B_2]_{d_i} + [\alpha_3 B_3]_{d_i} + \cdots + [\alpha_r B_r]_{d_i}
\]
\[
= \alpha_1 [B_1]_{d_i} + \alpha_2 [B_2]_{d_i} + \alpha_3 [B_3]_{d_i} + \cdots + \alpha_r [B_r]_{d_i}
\]
\[
= \alpha_1(0) + \alpha_2(0) + \alpha_3(0) + \cdots + \alpha_i(1) + \cdots + \alpha_r(0)
\]
\[
= \alpha_i
\]

So we conclude that \(\alpha_i = 0\) for all \(1 \leq i \leq r\), establishing the linear independence of \(S\) (Definition LICV [145]).

**Example IAS**

**Improving a span**

Suppose in the course of analyzing a matrix (its column space, its null space, its . . .) we encounter the following set of vectors, described by a span

\[
X = \left\langle \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 1 & 2 \\ 6 & -1 \\ 6 & 6 \end{bmatrix} \right\rangle
\]

Let \(A\) be the matrix whose rows are the vectors in \(X\), so by design \(X = R(A)\),

\[
A = \begin{bmatrix} 1 & 2 & 1 & 6 & 6 \\ 3 & -1 & 2 & -1 & 6 \\ 1 & -1 & 0 & -1 & -2 \\ -3 & 2 & -3 & 6 & -10 \end{bmatrix}
\]

Row-reduce \(A\) to form a row-equivalent matrix in reduced row-echelon form,

\[
B = \begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

Then Theorem BRS [271] says we can grab the nonzero columns of \(B^t\) and write

\[
X = R(A) = R(B) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix} \right\rangle
\]

These three vectors provide a much-improved description of \(X\). There are fewer vectors, and the pattern of zeros and ones in the first three entries makes it easier to determine membership in \(X\). And all we had to do was row-reduce the right matrix and toss out a zero row. Next to row operations themselves, this is probably the most powerful computational technique at your disposal as it quickly provides a much improved description of a span, any span.

Theorem BRS [271] and the techniques of Example IAS [272] will provide yet another description of the column space of a matrix. First we state a triviality as a theorem, so we can reference it later.
Theorem CSRST
Column Space, Row Space, Transpose
Suppose \( A \) is a matrix. Then \( \mathcal{C}(A) = \mathcal{R}(A^t) \).

Proof
\[
\mathcal{C}(A) = \mathcal{C}\left( (A^t)^t \right) = \mathcal{R}(A^t)
\]

So to find another expression for the column space of a matrix, build its transpose, row-reduce it, toss out the zero rows, and convert the nonzero rows to column vectors to yield an improved set for the span construction. We’ll do \([\text{Archetype I} 757]\), then you do \([\text{Archetype J} 762]\).

Example CSROI
Column space from row operations, Archetype I
To find the column space of the coefficient matrix of \([\text{Archetype I} 757]\), we proceed as follows. The matrix is
\[
I = \begin{bmatrix}
1 & 4 & 0 & -1 & 0 & 7 & -9 \\
2 & 8 & -1 & 3 & 9 & -13 & 7 \\
0 & 0 & 2 & -3 & -4 & 12 & -8 \\
-1 & -4 & 2 & 4 & 8 & -31 & 37 \\
\end{bmatrix}
\]
The transpose is
\[
\begin{bmatrix}
1 & 2 & 0 & -1 \\
4 & 8 & 0 & -4 \\
0 & -1 & 2 & 2 \\
-1 & 3 & -3 & 4 \\
0 & 9 & -4 & 8 \\
7 & -13 & 12 & -31 \\
-9 & 7 & -8 & 37 \\
\end{bmatrix}
\]
Row-reduced this becomes,
\[
\begin{bmatrix}
1 & 0 & 0 & -\frac{31}{7} \\
0 & 1 & 0 & \frac{12}{7} \\
0 & 0 & 1 & \frac{13}{7} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Now, using \([\text{Theorem CSRST} 273]\) and \([\text{Theorem BRS} 271]\)
\[
\mathcal{C}(I) = \mathcal{R}(I^t) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle.
\]
This is a very nice description of the column space. Fewer vectors than the 7 involved in the definition, and the pattern of the zeros and ones in the first 3 slots can be used
to advantage. For example, Archetype I \(757\) is presented as a consistent system of equations with a vector of constants

\[
b = \begin{bmatrix} 3 \\ 9 \\ 1 \\ 4 \end{bmatrix}.
\]

Since \(\mathcal{L}S(I, b)\) is consistent, Theorem CSCS 262 tells us that \(b \in \mathcal{C}(I)\). But we could see this quickly with the following computation, which really only involves any work in the 4th entry of the vectors as the scalars in the linear combination are \textit{dictated} by the first three entries of \(b\).

\[
b = \begin{bmatrix} 3 \\ 9 \\ 1 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\frac{31}{7} \end{bmatrix} + 9 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 12 \frac{1}{7} \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 13 \frac{1}{7} \end{bmatrix}
\]

Can you now rapidly construct several vectors, \(b\), so that \(\mathcal{L}S(I, b)\) is consistent, and several more so that the system is inconsistent? \(\Box\)

Subsection READ
Reading Questions

1. Write the column space of the matrix below as the span of a set of three vectors.

\[
\begin{bmatrix}
1 & 3 & 1 & 3 \\
2 & 0 & 1 & 1 \\
-1 & 2 & 1 & 0
\end{bmatrix}
\]

2. Suppose that \(A\) is an \(n \times n\) nonsingular matrix. What can you say about its column space?

3. Is the vector \(\begin{bmatrix} 0 \\ 5 \\ 2 \\ 3 \end{bmatrix}\) in the row space of the following matrix? Why or why not?

\[
\begin{bmatrix}
1 & 3 & 1 & 3 \\
2 & 0 & 1 & 1 \\
-1 & 2 & 1 & 0
\end{bmatrix}
\]
Subsection EXC Exercises

C30 Example CSOCD [265] expresses the column space of the coefficient matrix from Archetype D [735] (call the matrix $A$ here) as the span of the first two columns of $A$. In Example CSMCS [261] we determined that the vector

$$c = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

was not in the column space of $A$ and that the vector

$$b = \begin{bmatrix} 8 \\ -12 \\ 4 \end{bmatrix}$$

was in the column space of $A$. Attempt to write $c$ and $b$ as linear combinations of the two vectors in the span construction for the column space in Example CSOCD [265] and record your observations.

Contributed by Robert Beezer Solution [279]

C31 For the matrix $A$ below find a set of vectors $T$ meeting the following requirements: (1) the span of $T$ is the column space of $A$, that is, $\langle T \rangle = \mathcal{C}(A)$, (2) $T$ is linearly independent, and (3) the elements of $T$ are columns of $A$.

$$A = \begin{bmatrix} 2 & 1 & 4 & -1 & 2 \\ 1 & -1 & 5 & 1 & 1 \\ -1 & 2 & -7 & 0 & 1 \\ 2 & -1 & 8 & -1 & 2 \end{bmatrix}$$

Contributed by Robert Beezer Solution [279]

C32 In Example CSAA [266], verify that the vector $b$ is not in the column space of the coefficient matrix.

Contributed by Robert Beezer

C33 Find a linearly independent set $S$ so that the span of $S$, $\langle S \rangle$, is row space of the matrix $B$, and $S$ is linearly independent.

$$B = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 2 & 3 & -4 \end{bmatrix}$$

Contributed by Robert Beezer Solution [279]

C34 For the $3 \times 4$ matrix $A$ and the column vector $y \in \mathbb{C}^4$ given below, determine if $y$ is in the row space of $A$. In other words, answer the question: $y \in \mathcal{R}(A)$? (15 points)

$$A = \begin{bmatrix} -2 & 6 & 7 & -1 \\ 7 & -3 & 0 & -3 \\ 8 & 0 & 7 & 6 \end{bmatrix} \quad y = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -2 \end{bmatrix}$$
Contributed by Robert Beezer  Solution 279

**C35** For the matrix $A$ below, find two different linearly independent sets whose spans equal the column space of $A$, $C(A)$, such that
(a) the elements are each columns of $A$.
(b) the set is obtained by a procedure that is substantially different from the procedure you use in part (a).

$$A = \begin{bmatrix} 3 & 5 & 1 & -2 \\ 1 & 2 & 3 & 3 \\ -3 & -4 & 7 & 13 \end{bmatrix}$$

Contributed by Robert Beezer  Solution 280

**C40** The following archetypes are systems of equations. For each system, write the vector of constants as a linear combination of the vectors in the span construction for the column space provided by Theorem BCS 264 (these vectors are listed for each of these archetypes).

Archetype A 721
Archetype B 726
Archetype C 731
Archetype D 735
Archetype E 739
Archetype F 743
Archetype G 748
Archetype H 752
Archetype I 757
Archetype J 762

Contributed by Robert Beezer

**C42** The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute a set of column vectors such that (1) the vectors are columns of the matrix, (2) the set is linearly independent, and (3) the span of the set is the column space of the matrix. See Theorem BCS 264.

Archetype A 721
Archetype B 726
Archetype C 731
Archetype D 735
Archetype E 739
Archetype F 743
Archetype G 748
Archetype H 752
Archetype I 757
Archetype J 762
Archetype K 767
Archetype L 771

Contributed by Robert Beezer
C50 The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute a set of column vectors such that (1) the set is linearly independent, and (2) the span of the set is the row space of the matrix. See Theorem BRS [271].

Archetype A [721]
Archetype B [726]
Archetype C [731]
Archetype D [735]
Archetype E [739]
Archetype F [743]
Archetype G [748]
Archetype H [752]
Archetype I [757]
Archetype J [762]
Archetype K [767]
Archetype L [771]

Contributed by Robert Beezer

C51 The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute the column space as the span of a linearly independent set as follows: transpose the matrix, row-reduce, toss out zero rows, convert rows into column vectors. See Example CSROI [273].

Archetype A [721]
Archetype B [726]
Archetype C [731]
Archetype D [735]
Archetype E [739]
Archetype F [743]
Archetype G [748]
Archetype H [752]
Archetype I [757]
Archetype J [762]
Archetype K [767]
Archetype L [771]

Contributed by Robert Beezer

C52 The following archetypes are systems of equations. For each different coefficient matrix build two new vectors of constants. The first should lead to a consistent system and the second should lead to an inconsistent system. Descriptions of the column space as spans of linearly independent sets of vectors with “nice patterns” of zeros and ones might be most useful and instructive in connection with this exercise. (See the end of Example CSROI [273].)

Archetype A [721]
Archetype B [726]
Archetype C [731]
Archetype D [735]
Archetype E [739]
Archetype F [743]
Archetype G [748]
Archetype H [752]
Archetype I [757]
For the matrix $E$ below, find vectors $b$ and $c$ so that the system $LS(E, b)$ is consistent and $LS(E, c)$ is inconsistent.

$$E = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 3 & -1 & 0 & 2 \\ 4 & 1 & 1 & 6 \end{bmatrix}$$

This is not a legitimate procedure, and therefore is not a theorem. Construct an example to show that the procedure will not in general create the column space of $A$.

Suppose that $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix. Prove that the column space of $AB$ is a subset of the column space of $A$, that is $C(AB) \subseteq C(A)$. Provide an example where the opposite is false, in other words give an example where $C(A) \nsubseteq C(AB)$. (Compare with Exercise MM.T40.)

Suppose that $A$ is an $m \times n$ matrix and $B$ is an $n \times n$ nonsingular matrix. Prove that the column space of $A$ is equal to the column space of $AB$, that is $C(A) = C(AB)$. (Compare with Exercise MM.T41 and Exercise CRS.T40.)

Suppose that $A$ is an $m \times n$ matrix and $B$ is an $n \times m$ matrix where $AB$ is a nonsingular matrix. Prove that

1. $N(B) = \{0\}$
2. $C(B) \cap N(A) = \{0\}$

Discuss the case when $m = n$ in connection with Theorem NPNT.
Subsection SOL

Solutions

C30  Contributed by Robert Beezer Statement 275

In each case, begin with a vector equation where one side contains a linear combination of the two vectors from the span construction that gives the column space of $A$ with unknowns for scalars, and then use Theorem SLSLC 100 to set up a system of equations. For $c$, the corresponding system has no solution, as we would expect.

For $b$ there is a solution, as we would expect. What is interesting is that the solution is unique. This is a consequence of the linear independence of the set of two vectors in the span construction. If we wrote $b$ as a linear combination of all four columns of $A$, then there would be infinitely many ways to do this.

C31  Contributed by Robert Beezer Statement 275

Theorem BCS 264 is the right tool for this problem. Row-reduce this matrix, identify the pivot columns and then grab the corresponding columns of $A$ for the set $T$. The matrix $A$ row-reduces to

$$\begin{bmatrix}
1 & 0 & 3 & 0 & 0 \\
0 & 1 & -2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}$$

So $D = \{1, 2, 4, 5\}$ and then

$$T = \{A_1, A_2, A_4, A_5\} = \left\{ \begin{bmatrix} 2 & 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \end{bmatrix} \right\}$$

has the requested properties.

C33  Contributed by Robert Beezer Statement 275

Theorem BRS 271 is the most direct route to a set with these properties. Row-reduce, toss zero rows, keep the others. You could also transpose the matrix, then look for the column space by row-reducing the transpose and applying Theorem BCS 264. We’ll do the former,

$$B \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & -1 & 2 \\
0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$$

So the set $S$ is

$$S = \left\{ \begin{bmatrix} 1 & 0 \\
0 & -1 \\
2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\
1 & 1 \\
-1 \end{bmatrix} \right\}$$

C34  Contributed by Robert Beezer Statement 275

$$y \in \mathcal{R}(A) \iff y \in \mathcal{C}(A^t)$$

Definition RSM 268
\[ \iff \mathcal{L}(A^t, y) \text{ is consistent} \quad \text{Theorem CPCS 262} \]

The augmented matrix \([A^t \mid y]\) row reduces to

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

and with a leading 1 in the final column Theorem RCLS 54 tells us the linear system is inconsistent and so \(y \not\in \mathcal{R}(A)\).

C35 Contributed by Robert Beezer Statement 276
(a) By Theorem BCS 264 we can row-reduce \(A\), identify pivot columns with the set \(D\), and “keep” those columns of \(A\) and we will have a set with the desired properties.

\[ A \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & -13 & -19 \\
0 & 1 & 8 & 11 \\
0 & 0 & 0 & 0
\end{bmatrix} \]

So we have the set of pivot columns \(D = \{1, 2\}\) and we “keep” the first two columns of \(A\),

\[ \left\{ \begin{bmatrix} 3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -4 \end{bmatrix} \right\} \]

(b) We can view the column space as the row space of the transpose (Theorem CSRST 273). We can get a basis of the row space of a matrix quickly by bringing the matrix to reduced row-echelon form and keeping the nonzero rows as column vectors (Theorem BRS 271). Here goes.

\[ A^t \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & -2 \\
0 & 1 & 3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \]

Taking the nonzero rows and tilting them up as columns gives us

\[ \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\} \]

An approach based on the matrix \(L\) from extended echelon form (Definition EEF 287) and Theorem FS 290 will work as well as an alternative approach.

M10 Contributed by Robert Beezer Statement 278
Any vector from \(\mathbb{C}^3\) will lead to a consistent system, and therefore there is no vector that will lead to an inconsistent system.

How do we convince ourselves of this? First, row-reduce \(E\),

\[ E \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix} \]
If we augment $E$ with any vector of constants, and row-reduce the augmented matrix, we will never find a leading 1 in the final column, so by Theorem RCLS\[54\] the system will always be consistent.

Said another way, the column space of $E$ is all of $\mathbb{C}^3$, $C(E) = \mathbb{C}^3$. So by Theorem CSCS\[262\] any vector of constants will create a consistent system (and none will create an inconsistent system).

M20 Contributed by Robert Beezer Statement 278
The $2 \times 2$ matrix
$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$
has $C(A) = N(A) = \langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rangle$.

M21 Contributed by Robert Beezer Statement 278
Begin with a matrix $A$ (of any size) that does not have any zero rows, but which when row-reduced to $B$ yields at least one row of zeros. Such a matrix should be easy to construct (or find, like say from Archetype A\[721\]).

$C(A)$ will contain some vectors whose final slot (entry $m$) is non-zero, however, every column vector from the matrix $B$ will have a zero in slot $m$ and so every vector in $C(B)$ will also contain a zero in the final slot. This means that $C(A) \neq C(B)$, since we have vectors in $C(A)$ that cannot be elements of $C(B)$.

T40 Contributed by Robert Beezer Statement 278
Choose $x \in C(AB)$. Then by Theorem CSCS\[262\] there is a vector $w$ that is a solution to $LS(AB, x)$. Define the vector $y$ by $y = Bw$. We’re set,

\[
Ay = A(Bw) \quad \text{Definition of } y \\
= (AB)w \quad \text{Theorem MMA\[220\]} \\
= x \quad w \text{ solution to } LS(AB, x)
\]

This says that $LS(A, x)$ is a consistent system, and by Theorem CSCS\[262\], we see that $x \in C(A)$ and therefore $C(AB) \subseteq C(A)$.

For an example where $C(A) \not\subseteq C(AB)$ choose $A$ to be any nonzero matrix and choose $B$ to be a zero matrix. Then $C(A) \neq \{0\}$ and $C(AB) = C(O) = \{0\}$.

T41 Contributed by Robert Beezer Statement 278
From the solution to Exercise CRS.T40\[278\] we know that $C(AB) \subseteq C(A)$. So to establish the set equality (Definition SE\[694\]) we need to show that $C(A) \subseteq C(AB)$.

Choose $x \in C(A)$. By Theorem CSCS\[262\] the linear system $LS(A, x)$ is consistent, so let $y$ be one such solution. Because $B$ is nonsingular, and linear system using $B$ as a coefficient matrix will have a solution (Theorem NMUS\[79\]). Let $w$ be the unique solution to the linear system $LS(B, y)$. All set, here we go,

\[
(AB)w = A(Bw) \quad \text{Theorem MMA\[220\]} \\
= Ay \quad w \text{ solution to } LS(B, y) \\
= x \quad y \text{ solution to } LS(A, x)
\]
This says that the linear system $LS(AB, x)$ is consistent, so by $\text{Theorem CSCS} \ 262$, $x \in C(AB)$. So $C(A) \subseteq C(AB)$.

**T45** Contributed by Robert Beezer Statement $\ 278$

First, $0 \in \mathcal{N}(B)$ trivially. Now suppose that $x \in \mathcal{N}(B)$. Then

$$ABx = A(Bx) \quad \text{Theorem MMA} \ 220$$
$$= Ax \quad x \in \mathcal{N}(B)$$
$$= 0 \quad \text{Theorem MMZM} \ 218$$

Since we have assumed $AB$ is nonsingular, $\text{Definition NM} \ 75$ implies that $x = 0$.

Second, $0 \in C(B)$ and $0 \in \mathcal{N}(A)$ trivially, and so the zero vector is in the intersection as well $\text{Definition SI} \ 695$). Now suppose that $y \in C(B) \cap \mathcal{N}(A)$. Because $y \in C(B)$, $\text{Theorem CSCS} \ 262$ says the system $LS(B, y)$ is consistent. Let $x \in \mathbb{C}^n$ be one solution to this system. Then

$$ABx = A(Bx) \quad \text{Theorem MMA} \ 220$$
$$= Ay \quad x \text{ solution to } LS(B, y)$$
$$= 0 \quad y \in \mathcal{N}(A)$$

Since we have assumed $AB$ is nonsingular, $\text{Definition NM} \ 75$ implies that $x = 0$. Then $y = Bx = B0 = 0$.

When $AB$ is nonsingular and $m = n$ we know that the first condition, $\mathcal{N}(B) = \{0\}$, means that $B$ is nonsingular $\text{Theorem NMTNS} \ 78$). Because $B$ is nonsingular $\text{Theorem CSNM} \ 267$ implies that $C(B) = \mathbb{C}^m$. In order to have the second condition fulfilled, $C(B) \cap \mathcal{N}(A) = \{0\}$, we must realize that $\mathcal{N}(A) = \{0\}$. However, a second application of $\text{Theorem NMTNS} \ 78$ shows that $A$ must be nonsingular. This reproduces $\text{Theorem NPNT} \ 249$. 
Section FS
Four Subsets

There are four natural subsets associated with a matrix. We have met three already: the null space, the column space and the row space. In this section we will introduce a fourth, the left null space. The objective of this section is to describe one procedure that will allow us to find linearly independent sets that span each of these four sets of column vectors. Along the way, we will make a connection with the inverse of a matrix, so Theorem FS \[290\] will tie together most all of this chapter (and the entire course so far).

Subsection LNS
Left Null Space

Definition LNS
Left Null Space
Suppose $A$ is an $m \times n$ matrix. Then the left null space is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$.

(This definition contains Notation LNS.)

The left null space will not feature prominently in the sequel, but we can explain its name and connect it to row operations. Suppose $y \in \mathcal{L}(A)$. Then by Definition LNS \[283\], $A^t y = 0$. We can then write

$$0^t = (A^t y)^t = y^t (A^t)^t = y^t A$$

The product $y^t A$ can be viewed as the components of $y$ acting as the scalars in a linear combination of the rows of $A$. And the result is a “row vector”, $0^t$ that is totally zeros. When we apply a sequence of row operations to a matrix, each row of the resulting matrix is some linear combination of the rows. These observations tell us that the vectors in the left null space are scalars that record a sequence of row operations that result in a row of zeros in the row-reduced version of the matrix. We will see this idea more explicitly in the course of proving Theorem FS \[290\].

Example LNS
Left null space
We will find the left null space of

$$A = \begin{bmatrix} 1 & -3 & 1 \\ -2 & 1 & 1 \\ 1 & 5 & 1 \\ 9 & -4 & 0 \end{bmatrix}$$
We transpose $A$ and row-reduce, 

$$A^t = \begin{bmatrix}
1 & -2 & 1 & 9 \\
-3 & 1 & 5 & -4 \\
1 & 1 & 1 & 0
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & 1
\end{bmatrix}$$

Applying Definition LNS \([283]\) and Theorem BNS \([154]\) we have 

$$\mathcal{L}(A) = \mathcal{N}(A^t) = \left\{ \begin{bmatrix}
-2 \\
3 \\
-1 \\
1
\end{bmatrix} \right\}$$

If you row-reduce $A$ you will discover one zero row in the reduced row-echelon form. This zero row is created by a sequence of row operations, which in total amounts to a linear combination, with scalars $a_1 = -2$, $a_2 = 3$, $a_3 = -1$ and $a_4 = 1$, on the rows of $A$ and which results in the zero vector (check this!). So the components of the vector describing the left null space of $A$ provide a relation of linear dependence on the rows of $A$. 

### Subsection CRS
**Computing Column Spaces**

We have three ways to build the column space of a matrix. First, we can use just the definition, Definition CSM \([261]\), and express the column space as a span of the columns of the matrix. A second approach gives us the column space as the span of some of the columns of the matrix, but this set is linearly independent (Theorem BCS \([264]\)). Finally, we can transpose the matrix, row-reduce the transpose, kick out zero rows, and transpose the remaining rows back into column vectors. Theorem CSRST \([273]\) and Theorem BRS \([271]\) tell us that the resulting vectors are linearly independent and their span is the column space of the original matrix.

We will now demonstrate a fourth method by way of a rather complicated example. Study this example carefully, but realize that its main purpose is to motivate a theorem that simplifies much of the apparent complexity. So other than an instructive exercise or two, the procedure we are about to describe will not be a usual approach to computing a column space.

**Example CSANS**
**Column space as null space**

Let’s find the column space of the matrix $A$ below with a new approach.

$$A = \begin{bmatrix}
10 & 0 & 3 & 8 & 7 \\
-16 & -1 & -4 & -10 & -13 \\
-6 & 1 & -3 & -6 & -6 \\
0 & 2 & -2 & -3 & -2 \\
3 & 0 & 1 & 2 & 3 \\
-1 & -1 & 1 & 1 & 0
\end{bmatrix}$$

By Theorem CSCS \([262]\) we know that the column vector $b$ is in the column space of $A$ if and only if the linear system $\mathcal{L}(A, b)$ is consistent. So let’s try to solve this system...
in full generality, using a vector of variables for the vector of constants. In other words, which vectors $b$ lead to consistent systems? Begin by forming the augmented matrix $[A \mid b]$ with a general version of $b$,

$$[A \mid b] = \begin{bmatrix} 10 & 0 & 3 & 8 & 7 & b_1 \\ -16 & -1 & -4 & -10 & -13 & b_2 \\ -6 & 1 & -3 & -6 & -6 & b_3 \\ 0 & 2 & -2 & -3 & -2 & b_4 \\ 3 & 0 & 1 & 2 & 3 & b_5 \\ -1 & -1 & 1 & 1 & 0 & b_6 \end{bmatrix}$$

To identify solutions we will row-reduce this matrix and bring it to reduced row-echelon form. Despite the presence of variables in the last column, there is nothing to stop us from doing this. Except our numerical routines on calculators can’t be used, and even some of the symbolic algebra routines do some unexpected maneuvers with this computation. So do it by hand. Yes, it is a bit of work. But worth it. We’ll still be here when you get back. Notice along the way that the row operations are exactly the same ones you would do if you were just row-reducing the coefficient matrix alone, say in connection with a homogeneous system of equations. The column with the $b_i$ acts as a sort of bookkeeping device. There are many different possibilities for the result, depending on what order you choose to perform the row operations, but shortly we’ll all be on the same page. Here’s one possibility (you can find this same result by doing additional row operations with the fifth and sixth rows to remove any occurences of $b_1$ and $b_2$ from the first four rows of your result):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 & b_3 - b_4 + 2b_5 - b_6 \\ 0 & 1 & 0 & 0 & -3 & -2b_3 + 3b_4 - 3b_5 + 3b_6 \\ 0 & 0 & 1 & 0 & 1 & b_3 + b_4 + 3b_5 + 3b_6 \\ 0 & 0 & 0 & 1 & -2 & -2b_3 + b_4 - 4b_5 \\ 0 & 0 & 0 & 0 & 0 & b_1 + 3b_3 - b_4 + 3b_5 + b_6 \\ 0 & 0 & 0 & 0 & 0 & b_2 - 2b_3 + b_4 + b_5 - b_6 \end{bmatrix}$$

Our goal is to identify those vectors $b$ which make $LS(A, b)$ consistent. By Theorem RCLS [54] we know that the consistent systems are precisely those without a leading 1 in the last column. Are the expressions in the last column of rows 5 and 6 equal to zero, or are they leading 1’s? The answer is: maybe. It depends on $b$. With a nonzero value for either of these expressions, we would scale the row and produce a leading 1. So we get a consistent system, and $b$ is in the column space, if and only if these two expressions are both simultaneously zero. In other words, members of the column space of $A$ are exactly those vectors $b$ that satisfy

$$b_1 + 3b_3 - b_4 + 3b_5 + b_6 = 0$$
$$b_2 - 2b_3 + b_4 + b_5 - b_6 = 0$$

Hmmm. Looks suspiciously like a homogeneous system of two equations with six variables. If you’ve been playing along (and we hope you have) then you may have a slightly different system, but you should have just two equations. Form the coefficient matrix and row-reduce (notice that the system above has a coefficient matrix that is already in reduced row-echelon form). We should all be together now with the same matrix,

$$L = \begin{bmatrix} 1 & 0 & 3 & -1 & 3 & 1 \\ 0 & 1 & -2 & 1 & 1 & -1 \end{bmatrix}$$
So, \( C(A) = \mathcal{N}(L) \) and we can apply Theorem BNS to obtain a linearly independent set to use in a span construction,

\[
C(A) = \mathcal{N}(L) = \left\{ \begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}
\]

Whew! As a postscript to this central example, you may wish to convince yourself that the four vectors above really are elements of the column space? Do they create consistent systems with \( A \) as coefficient matrix? Can you recognize the constant vector in your description of these solution sets?

OK, that was so much fun, let’s do it again. But simpler this time. And we’ll all get the same results all the way through. Doing row operations by hand with variables can be a bit error prone, so let’s see if we can improve the process some. Rather than row-reduce a column vector \( b \) full of variables, let’s write \( b = I_6 b \) and we will row-reduce the matrix \( I_6 \) and when we finish row-reducing, then we will compute the matrix-vector product. You should first convince yourself that we can operate like this (see Exercise XX-commutingops-todo on commuting operations). Rather than augmenting \( A \) with \( b \), we will instead augment it with \( I_6 \) (does this feel familiar?),

\[
M = \begin{bmatrix} 10 & 0 & 3 & 8 & 7 & 1 & 0 & 0 & 0 & 0 & 0 \\ -16 & -1 & -4 & -10 & -13 & 0 & 1 & 0 & 0 & 0 & 0 \\ -6 & 1 & -3 & -6 & -6 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & -3 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

We want to row-reduce the left-hand side of this matrix, but we will apply the same row operations to the right-hand side as well. And once we get the left-hand side in reduced row-echelon form, we will continue on to put leading 1’s in the final two rows, as well as clearing out the columns containing those two additional leading 1’s. It is these additional row operations that will ensure that we all get to the same place, since the reduced row-echelon form is unique (Theorem RREFU),

\[
N = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 0 & -3 & 0 & 0 & -2 & 3 & -3 & 3 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & -2 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & -1 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 1 & 1 & -1 \end{bmatrix}
\]

We are after the final six columns of this matrix, which we will multiply by \( b \)

\[
J = \begin{bmatrix} 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & -2 & 3 & -3 & 3 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & -2 & 1 & -4 & 0 \\ 1 & 0 & 3 & -1 & 3 & 1 \\ 0 & 1 & -2 & 1 & 1 & -1 \end{bmatrix}
\]
Subsection FS.EEF  Extended echelon form  287

so

\[
Jb = \begin{bmatrix}
0 & 0 & 1 & -1 & 2 & -1 \\
0 & 0 & -2 & 3 & -3 & 3 \\
0 & 0 & 1 & 1 & 3 & 3 \\
0 & 0 & -2 & 1 & -4 & 0 \\
1 & 0 & 3 & -1 & 3 & 1 \\
0 & 1 & -2 & 1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
b_5 \\
b_6
\end{bmatrix} = \begin{bmatrix}
b_3 - b_4 + 2b_5 - b_6 \\
-2b_3 + 3b_4 - 3b_5 + 3b_6 \\
b_3 + b_4 + 3b_5 + 3b_6 \\
-2b_3 + b_4 - 4b_5 \\
b_1 + 3b_3 - b_4 + 3b_5 + b_6 \\
b_2 - 2b_3 + b_4 + b_5 - b_6
\end{bmatrix}
\]

So by applying to the identity matrix the same row operations that row-reduce \(A\) (which we could do with a calculator once \(I_6\) is placed alongside of \(A\)), we can then arrive at the result of row-reducing a column of symbols where the vector of constants usually resides. Since the row-reduced version of \(A\) has two zero rows, for a consistent system we require that

\[
b_1 + 3b_3 - b_4 + 3b_5 + b_6 = 0
\]

\[
b_2 - 2b_3 + b_4 + b_5 - b_6 = 0
\]

Now we are exactly back where we were on the first go-round. Notice that we obtain the matrix \(L\) as simply the last two rows and last six columns of \(N\).

This example motivates the remainder of this section, so it is worth careful study. You might attempt to mimic the second approach with the coefficient matrices of Archetype I [757] and Archetype J [762]. We will see shortly that the matrix \(L\) contains more information about \(A\) than just the column space.

Subsection EEF

Extended echelon form

The final matrix that we row-reduced in Example CSANS [284] should look familiar in most respects to the procedure we used to compute the inverse of a nonsingular matrix, Theorem CINM [237]. We will now generalize that procedure to matrices that are not necessarily nonsingular, or even square. First a definition.

**Definition EEF**

**Extended Echelon Form**

Suppose \(A\) is an \(m \times n\) matrix. Add \(m\) new columns to \(A\) that together equal an \(m \times m\) identity matrix to form an \(m \times (n + m)\) matrix \(M\). Use row operations to bring \(M\) to reduced row-echelon form and call the result \(N\). \(N\) is the extended reduced row-echelon form of \(A\), and we will standardize on names for five submatrices \((B, C, J, K, L)\) of \(N\).

Let \(B\) denote the \(m \times n\) matrix formed from the first \(n\) columns of \(N\) and let \(J\) denote the \(m \times m\) matrix formed from the last \(m\) columns of \(N\). Suppose that \(B\) has \(r\) nonzero rows. Further partition \(N\) by letting \(C\) denote the \(r \times n\) matrix formed from all of the non-zero rows of \(B\). Let \(K\) be the \(r \times m\) matrix formed from the first \(r\) rows of \(J\), while \(L\) will be the \((m - r) \times m\) matrix formed from the bottom \(m - r\) rows of \(J\). Pictorially,

\[
M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix}
C & K \\
0 & L
\end{bmatrix}
\]
Example SEEF

Submatrices of extended echelon form

We illustrate Definition EEF \[287\] with the matrix \( A \),

\[
A = \begin{bmatrix}
1 & -1 & -2 & 7 & 1 & 6 \\
-6 & 2 & -4 & -18 & -3 & -26 \\
4 & -1 & 4 & 10 & 2 & 17 \\
3 & -1 & 2 & 9 & 1 & 12
\end{bmatrix}
\]

Augmenting with the \( 4 \times 4 \) identity matrix, \( M = \)

\[
\begin{bmatrix}
1 & -1 & -2 & 7 & 1 & 6 & 1 & 0 & 0 & 0 \\
-6 & 2 & -4 & -18 & -3 & -26 & 0 & 1 & 0 & 0 \\
4 & -1 & 4 & 10 & 2 & 17 & 0 & 0 & 1 & 0 \\
3 & -1 & 2 & 9 & 1 & 12 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

and row-reducing, we obtain

\[
N = \begin{bmatrix}
\color{red}{1} & 0 & 2 & 1 & 0 & 3 & 0 & 1 & 1 & 1 \\
0 & \color{red}{1} & 4 & -6 & 0 & -1 & 0 & 2 & 3 & 0 \\
0 & 0 & 0 & 0 & \color{red}{1} & 2 & 0 & -1 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & \color{red}{1} & 2 & 2 & 1
\end{bmatrix}
\]

So we then obtain

\[
B = \begin{bmatrix}
\color{red}{1} & 0 & 2 & 1 & 0 & 3 \\
0 & \color{red}{1} & 4 & -6 & 0 & -1 \\
0 & 0 & 0 & 0 & \color{red}{1} & 2 \\
0 & 0 & 0 & 0 & 0 & \color{red}{1}
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
\color{red}{1} & 0 & 2 & 1 & 0 & 3 \\
0 & \color{red}{1} & 4 & -6 & 0 & -1 \\
0 & 0 & 0 & 0 & \color{red}{1} & 2 \\
0 & 0 & 0 & 0 & 0 & \color{red}{1}
\end{bmatrix}
\]

\[
J = \begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 2 & 3 & 0 \\
0 & -1 & 0 & -2 \\
\color{red}{1} & 2 & 2 & 1
\end{bmatrix}
\]

\[
K = \begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 2 & 3 & 0 \\
0 & -1 & 0 & -2 \\
\color{red}{1}
\end{bmatrix}
\]

\[
L = \begin{bmatrix}
\color{red}{1} & 2 & 2 & 1
\end{bmatrix}
\]

You can observe (or verify) the properties of the following theorem with this example.

**Theorem PEEF**

**Properties of Extended Echelon Form**

Suppose that \( A \) is an \( m \times n \) matrix and that \( N \) is its extended echelon form. Then

1. \( J \) is nonsingular.

Version 0.92
2. $B = JA$.

3. If $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$, then $Ax = y$ if and only if $Bx = Jy$.

4. $C$ is in reduced row-echelon form, has no zero rows and has $r$ pivot columns.

5. $L$ is in reduced row-echelon form, has no zero rows and has $m - r$ pivot columns.

\[\square\]

**Proof**  $J$ is the result of applying a sequence of row operations to $I_m$, as such $J$ and $I_m$ are row-equivalent. $LS(I_m, 0)$ has only the zero solution, since $I_m$ is nonsingular (Theorem NMRRI [77]). Thus, $LS(J, 0)$ also has only the zero solution (Theorem REMES [34], Definition ESYS [15]) and $J$ is therefore nonsingular (Definition NSM [68]).

To prove the second part of this conclusion, first convince yourself that row operations and the matrix-vector are commutative operations. By this we mean the following. Suppose that $F$ is an $m \times n$ matrix that is row-equivalent to the matrix $G$. Apply to the column vector $Fw$ the same sequence of row operations that converts $F$ to $G$. Then the result is $Gw$. So we can do row operations on the matrix, then do a matrix-vector product, or do a matrix-vector product and then do row operations on a column vector, and the result will be the same either way. Since matrix multiplication is defined by a collection of matrix-vector products $\cdot$, if we apply to the matrix product $FH$ the same sequence of row operations that converts $F$ to $G$ then the result will equal $GH$. Now apply these observations to $A$.

Write $AI_n = I_mA$ and apply the row operations that convert $M$ to $N$. $A$ is converted to $B$, while $I_m$ is converted to $J$, so we have $BI_n = JA$. Simplifying the left side gives the desired conclusion.

For the third conclusion, we now establish the two equivalences

$$Ax = y \iff JAx = Jy \iff Bx = Jy$$

The forward direction of the first equivalence is accomplished by multiplying both sides of the matrix equality by $J$, while the backward direction is accomplished by multiplying by the inverse of $J$ (which we know exists by Theorem N [251] since $J$ is nonsingular). The second equivalence is obtained simply by the substitutions given by $JA = B$.

The first $r$ rows of $N$ are in reduced row-echelon form, since any contiguous collection of rows taken from a matrix in reduced row-echelon form will form a matrix that is again in reduced row-echelon form. Since the matrix $C$ is formed by removing the last $n$ entries of each these rows, the remainder is still in reduced row-echelon form. By its construction, $C$ has no zero rows. $C$ has $r$ rows and each contains a leading 1, so there are $r$ pivot columns in $C$.

The final $m - r$ rows of $N$ are in reduced row-echelon form, since any contiguous collection of rows taken from a matrix in reduced row-echelon form will form a matrix that is again in reduced row-echelon form. Since the matrix $L$ is formed by removing the first $n$ entries of each these rows, and these entries are all zero (they form the zero rows of $B$), the remainder is still in reduced row-echelon form. $L$ is the final $m - r$ rows of the nonsingular matrix $J$, so none of these rows can be totally zero, or $J$ would not row-reduce to the identity matrix. $L$ has $m - r$ rows and each contains a leading 1, so there are $m - r$ pivot columns in $L$.  

\[\square\]
Notice that in the case where $A$ is a nonsingular matrix we know that the reduced row-echelon form of $A$ is the identity matrix (Theorem NMRRI 77), so $B = I_n$. Then the second conclusion above says $JA = B = I_n$, so $J$ is the inverse of $A$. Thus this theorem generalizes Theorem CINM 237, though the result is a “left-inverse” of $A$ rather than a “right-inverse.”

The third conclusion of Theorem PEEF 288 is the most telling. It says that $x$ is a solution to the linear system $\mathcal{L}S(A, y)$ if and only if $x$ is a solution to the linear system $\mathcal{L}S(B, Jy)$. Or said differently, if we row-reduce the augmented matrix $[A| x]$ we will get the augmented matrix $[B| Jy]$. The matrix $J$ tracks the cumulative effect of the row operations that converts $A$ to reduced row-echelon form, here effectively applying them to the vector of constants in a system of equations having $A$ as a coefficient matrix. When $A$ row-reduces to a matrix with zero rows, then $Jy$ should also have zero entries in the same rows if the system is to be consistent.

### Subsection FS
#### Four Subsets

With all the preliminaries in place we can state our main result for this section. In essence this result will allow us to say that we can find linearly independent sets to use in span constructions for all four subsets (null space, column space, row space, left null space) by analyzing only the extended echelon form of the matrix, and specifically, just the two submatrices $C$ and $L$, which will be ripe for analysis since they are already in reduced row-echelon form (Theorem PEEF 288).

**Theorem FS**

**Four Subsets**

Suppose $A$ is an $m \times n$ matrix with extended echelon form $N$. Suppose the reduced row-echelon form of $A$ has $r$ nonzero rows. Then $C$ is the submatrix of $N$ formed from the first $r$ rows and the first $n$ columns and $L$ is the submatrix of $N$ formed from the last $m$ columns and the last $m - r$ rows. Then

1. The null space of $A$ is the null space of $C$, $\mathcal{N}(A) = \mathcal{N}(C)$.
2. The row space of $A$ is the row space of $C$, $\mathcal{R}(A) = \mathcal{R}(C)$.
3. The column space of $A$ is the null space of $L$, $\mathcal{C}(A) = \mathcal{N}(L)$.
4. The left null space of $A$ is the row space of $L$, $\mathcal{L}(A) = \mathcal{R}(L)$.

**Proof** First, $\mathcal{N}(A) = \mathcal{N}(B)$ since $B$ is row-equivalent to $A$ (Theorem REMES 34). The zero rows of $B$ represent equations that are always true in the homogeneous system $\mathcal{L}S(B, 0)$, so the removal of these equations will not change the solution set. Thus, in turn, $\mathcal{N}(B) = \mathcal{N}(C)$.

Second, $\mathcal{R}(A) = \mathcal{R}(B)$ since $B$ is row-equivalent to $A$ (Theorem REMRS 269). The zero rows of $B$ contribute nothing to the span that is the row space of $B$, so the removal of these rows will not change the row space. Thus, in turn, $\mathcal{R}(B) = \mathcal{R}(C)$. 

Version 0.92
Third, we prove the set equality $C(A) = N(L)$ with Definition SE 694. Begin by showing that $C(A) \subseteq N(L)$. Choose $y \in C(A) \subseteq \mathbb{C}^m$. Then there exists a vector $x \in \mathbb{C}^n$ such that $Ax = y$ (Theorem CSCS 262). Then for $1 \leq k \leq m - r$,

$$[Ly]_k = [Jy]_{r+k} \quad L \text{ a submatrix of } J$$

$$= [Bx]_{r+k} \quad \text{Theorem PEEF 288}$$

$$= [Ox]_k \quad \text{Zero matrix a submatrix of } B$$

$$= [0]_{j_k} \quad \text{Theorem MMZM 218}$$

So, for all $1 \leq k \leq m - r$, $[Ly]_k = [0]_k$. So by Definition CVE 88 we have $Ly = 0$ and thus $y \in N(L)$.

Now, show that $N(L) \subseteq C(A)$. Choose $y \in N(L) \subseteq \mathbb{C}^m$. Form the vector $Ky \in \mathbb{C}^r$. The linear system $LS(C, Ky)$ is consistent since $C$ is in reduced row-echelon form and has no zero rows (Theorem PEEF 288). Let $x \in \mathbb{C}^n$ denote a solution to $LS(C, Ky)$.

Then for $1 \leq j \leq r$,

$$[Bx]_j = [Cx]_j \quad C \text{ a submatrix of } B$$

$$= [Ky]_j \quad x \text{ a solution to } LS(C, Ky)$$

$$= [Jy]_j \quad K \text{ a submatrix of } J$$

And for $r + 1 \leq k \leq m$,

$$[Bx]_k = [Ox]_{k-r} \quad \text{Zero matrix a submatrix of } B$$

$$= [0]_{k-r} \quad \text{Theorem MMZM 218}$$

$$= [Ly]_{k-r} \quad y \text{ in } N(L)$$

$$= [Jy]_k \quad L \text{ a submatrix of } J$$

So for all $1 \leq i \leq m$, $[Bx]_i = [Jy]_i$ and by Definition CVE 88 we have $Bx = Jy$. From Theorem PEEF 288 we know then that $Ax = y$, and therefore $y \in C(A)$ (Theorem CSCS 262). By Definition SE 694 we have $C(A) = N(L)$.

Fourth, we prove the set equality $L(A) = R(L)$ with Definition SE 694. Begin by showing that $R(L) \subseteq L(A)$. Choose $y \in R(L) \subseteq \mathbb{C}^m$. Then there exists a vector $w \in \mathbb{C}^{n-r}$ such that $y = L'w$ (Definition RSM 268, Theorem CSCS 262). Then for $1 \leq i \leq n$,

$$[A'y]_i = \sum_{k=1}^{m} [A']_{ik} [y]_k \quad \text{Theorem EMP 216}$$

$$= \sum_{k=1}^{m} [A']_{ik} [L'w]_k \quad \text{Definition of } w$$

$$= \sum_{k=1}^{m-r} [A']_{ik} \sum_{\ell=1}^{m-r} [L']_{k\ell} [w]_{\ell} \quad \text{Theorem EMP 216}$$

$$= \sum_{\ell=1}^{m-r} \left( \sum_{k=1}^{m} [A']_{ik} [L']_{k\ell} \right) [w]_{\ell} \quad \text{Commutativity, Distributivity in } \mathbb{C}$$

Version 0.92
= \sum_{\ell=1}^{m-r} \left( \sum_{k=1}^{m} [A^t]_{ik} [J^t]_{k,r+\ell} \right) [w]_\ell \quad L \text{ a submatrix of } J

= \sum_{\ell=1}^{m-r} [A^t J^t]_{i,r+\ell} [w]_\ell \quad \text{Theorem EMP 216}

= \sum_{\ell=1}^{m-r} [(J A^t)]_{i,r+\ell} [w]_\ell \quad \text{Theorem MMT 221}

= \sum_{\ell=1}^{m-r} [B^t]_{i,r+\ell} [w]_\ell \quad \text{Theorem PEEF 288}

= \sum_{\ell=1}^{m-r} 0 [w]_\ell \quad \text{Zero rows in } B

= 0 \quad \text{Zero in } \mathbb{C}

= [0], \quad \text{Definition ZCV 30}

Since \([A^t y]_i = [0]_i\) for 1 \leq i \leq n,  \text{Definition CVE 88}\) implies that \(A^t y = 0\). This means that \(y \in \mathcal{N}(A^t)\).

Now, show that \(\mathcal{L}(A) \subseteq \mathcal{R}(L)\). Choose \(y \in \mathcal{L}(A) \subseteq \mathbb{C}^m\). The matrix \(J\) is nonsingular \(\text{Theorem PEEF 288}\), so \(J^t\) is also nonsingular \(\text{Theorem MMT 240}\) and therefore the linear system \(LS(J^t, y)\) has a unique solution. Denote this solution as \(x \in \mathbb{C}^m\). We will need to work with two “halves” of \(x\), which we will denote as \(z\) and \(w\) with formal definitions given by

\[
[z]_j = [x]_i \quad 1 \leq j \leq r, \quad [w]_k = [x]_{r+k} \quad 1 \leq k \leq m - r
\]

Now, for 1 \leq j \leq r,

\[
[C^t z]_j = \sum_{k=1}^{r} [C^t]_{jk} [z]_k
\]

= \sum_{k=1}^{r} [C^t]_{jk} [z]_k + \sum_{\ell=1}^{m-r} [O]_{j\ell} [w]_\ell \quad \text{Theorem EMP 216}

= \sum_{k=1}^{r} [B^t]_{jk} [z]_k + \sum_{\ell=1}^{m-r} [B^t]_{j,r+\ell} [w]_\ell \quad \text{C, } O \text{ submatrices of } B

= \sum_{k=1}^{r} [B^t]_{jk} [x]_k + \sum_{\ell=1}^{m-r} [B^t]_{j,r+\ell} [x]_{r+\ell} \quad \text{Definitions of } z \text{ and } w

= \sum_{k=1}^{r} [B^t]_{jk} [x]_k + \sum_{k=r+1}^{m} [B^t]_{jk} [x]_k \quad \text{Re-index second sum}

= \sum_{k=1}^{m} [B^t]_{jk} [x]_k \quad \text{Combine sums}

= \sum_{k=1}^{m} [(J A^t)]_{jk} [x]_k \quad \text{Theorem PEEF 288}

= \sum_{k=1}^{m} [A^t J^t]_{jk} [x]_k \quad \text{Theorem MMT 221}
\[
= \sum_{k=1}^{m} \sum_{\ell=1}^{m} [A']_{j\ell} [J']_{\ell k} [x]_{k} \quad \text{Theorem EMP 216}
\]

\[
= \sum_{\ell=1}^{m} [A']_{j\ell} \left( \sum_{k=1}^{m} [J']_{\ell k} [x]_{k} \right) \quad \text{Commutativity, Distributivity in } \mathbb{C}
\]

\[
= \sum_{\ell=1}^{m} [A']_{j\ell} [J'x]_{\ell} \quad \text{Theorem EMP 216}
\]

\[
= [A'y]_j \quad \text{Definition of } x
\]

\[
= [0]_j \quad y \in \mathcal{L}(A)
\]

So, by Definition CVE 88, \( C'z = 0 \) and the vector \( z \) gives us a linear combination of the columns of \( C' \) that equals the zero vector. In other words, \( z \) gives a relation of linear dependence on the the rows of \( C \). However, the rows of \( C \) are a linearly independent set by Theorem BRS 271. According to Definition LICV 145 we must conclude that the entries of \( z \) are all zero, i.e. \( z = \mathbf{0} \).

Now, for \( 1 \leq i \leq m \), we have

\[
[y]_i = [J'x]_i \quad \text{Definition of } x
\]

\[
= \sum_{k=1}^{m} [J']_{ik} [x]_{k} \quad \text{Theorem EMP 216}
\]

\[
= \sum_{k=1}^{r} [J']_{ik} [x]_{k} + \sum_{k=r+1}^{m} [J']_{ik} [x]_{k} \quad \text{Break apart sum}
\]

\[
= \sum_{k=1}^{r} [J']_{ik} [z]_{k} + \sum_{k=r+1}^{m} [J']_{ik} [w]_{k-r} \quad \text{Definition of } z \text{ and } w
\]

\[
= \sum_{k=1}^{r} [J']_{ik} 0 + \sum_{\ell=1}^{m-r} [J']_{i,r+\ell} [w]_{\ell} \quad z = \mathbf{0}, \text{ re-index}
\]

\[
= 0 + \sum_{\ell=1}^{m-r} [L']_{i,\ell} [w]_{\ell} \quad L \text{ a submatrix of } J
\]

\[
= [L'w]_i \quad \text{Theorem EMP 216}
\]

So by Definition CVE 88, \( y = L'w \). The existence of \( w \) implies that \( y \in \mathcal{R}(L) \), and therefore \( \mathcal{L}(A) \subseteq \mathcal{R}(L) \). So by Definition SE 694 we have \( \mathcal{L}(A) = \mathcal{R}(L) \). ■

The first two conclusions of this theorem are nearly trivial. But they set up a pattern of results for \( C \) that is reflected in the latter two conclusions about \( L \). In total, they tell us that we can compute all four subsets just by finding null spaces and row spaces. This theorem does not tell us exactly how to compute these subsets, but instead simply expresses them as null spaces and row spaces of matrices in reduced row-echelon form without any zero rows (\( C \) and \( L \)). A linearly independent set that spans the null space of a matrix in reduced row-echelon form can be found easily with Theorem BNS 154. It
is an even easier matter to find a linearly independent set that spans the row space of a
matrix in reduced row-echelon form with \text{Theorem BRS} \[ 271 \], especially when there are
no zero rows present. So an application of \text{Theorem FS} \[ 290 \] is typically followed by two
applications each of \text{Theorem BNS} \[ 154 \] and \text{Theorem BRS} \[ 271 \].

The situation when \( r = m \) deserves comment, since now the matrix \( L \) has no rows.
What is \( C(A) \) when we try to apply \text{Theorem FS} \[ 290 \] and encounter \( N(L) \)? One
interpretation of this situation is that \( L \) is the coefficient matrix of a homogeneous system
that has no equations. How hard is it to find a solution vector to this system? Some
thought will convince you that \textit{any} proposed vector will qualify as a solution, since it
makes \textit{all} of the equations true. So every possible vector is in the null space of \( L \) and
therefore \( C(A) = N(L) = \mathbb{C}^m \). OK, perhaps this sounds like some twisted argument
from \textit{Alice in Wonderland}. Let us try another argument that might solidly convince you
of this logic.

If \( r = m \), when we row-reduce the augmented matrix of \( LS(A, b) \) the result will have
no zero rows, and all the leading 1’s will occur in first \( n \) columns, so by \text{Theorem RCLS} \[ 54 \]
the system will be consistent. By \text{Theorem CSCS} \[ 262 \], \( b \in C(A) \). Since \( b \) was arbitrary,
every possible vector is in the column space of \( A \), so we again have \( C(A) = \mathbb{C}^m \). The
situation when a matrix has \( r = m \) is known by the term \textit{full rank}, and in the case of
a square matrix coincides with nonsingularity (see \text{Exercise FS.M50} \[ 302 \]).

The properties of the matrix \( L \) described by this theorem can be explained informally
as follows. A column vector \( y \in \mathbb{C}^m \) is in the column space of \( A \) if the linear system
\( LS(A, y) \) is consistent (\text{Theorem CSCS} \[ 262 \]). By \text{Theorem RCLS} \[ 54 \], the reduced row-
echelon form of the augmented matrix \( [A | y] \) of a consistent system will have zeros in
the bottom \( m - r \) locations of the last column. By \text{Theorem PEEF} \[ 288 \] this final column
is the vector \( Jy \) and so should then have zeros in the final \( m - r \) locations. But since \( L \)
comprises the final \( m - r \) rows of \( J \), this condition is expressed by saying \( y \in N(L) \).

Additionally, the rows of \( J \) are the scalars in linear combinations of the rows of \( A 
that create the rows of \( B \). That is, the rows of \( J \) record the net effect of the sequence of
row operations that takes \( A \) to its reduced row-echelon form, \( B \). This can be seen in the
equation \( JA = B \) (\text{Theorem PEEF} \[ 288 \]). As such, the rows of \( L \) are scalars for linear
combinations of the rows of \( A \) that yield zero rows. But such linear combinations are
precisely the elements of the left null space. So any element of the row space of \( L \) is also
an element of the left null space of \( A \). We will now illustrate \text{Theorem FS} \[ 290 \] with a
few examples.

\textbf{Example FS1}

\textbf{Four subsets, \#1}

In \text{Example SEEF} \[ 288 \] we found the five relevant submatrices of the matrix
\[
A = \begin{bmatrix}
1 & -1 & -2 & 7 & 1 & 6 \\
-6 & 2 & -4 & -18 & -3 & -26 \\
4 & -1 & 4 & 10 & 2 & 17 \\
3 & -1 & 2 & 9 & 1 & 12 \\
\end{bmatrix}
\]

To apply \text{Theorem FS} \[ 290 \] we only need \( C \) and \( L \),
\[
C = \begin{bmatrix}
1 & 0 & 2 & 1 & 0 & 3 \\
0 & 1 & 4 & -6 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 2 \\
\end{bmatrix} \quad L = \begin{bmatrix}
1 & 2 & 2 & 1 \\
\end{bmatrix}
\]
Then we use Theorem FS \[290\] to obtain

\[\mathcal{N}(A) = \mathcal{N}(C) = \left\langle \begin{bmatrix} -2 \\ -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\rangle\] Theorem BNS \[154\]

\[\mathcal{R}(A) = \mathcal{R}(C) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix} \right\rangle\] Theorem BRS \[271\]

\[\mathcal{C}(A) = \mathcal{N}(L) = \left\langle \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\rangle\] Theorem BNS \[154\]

\[\mathcal{L}(A) = \mathcal{R}(L) = \left\langle \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\rangle\] Theorem BRS \[271\]

Boom! ⚡️

**Example FS2**

**Four subsets, #2**

Now let's return to the matrix \(A\) that we used to motivate this section in Example CSANS \[284\],

\[
A = \begin{bmatrix}
10 & 0 & 3 & 8 & 7 \\
-16 & -1 & -4 & -10 & -13 \\
-6 & 1 & -3 & -6 & -6 \\
0 & 2 & -2 & -3 & -2 \\
3 & 0 & 1 & 2 & 3 \\
-1 & -1 & 1 & 1 & 0
\end{bmatrix}
\]

We form the matrix \(M\) by adjoining the \(6 \times 6\) identity matrix \(I_6\),

\[
M = \begin{bmatrix}
10 & 0 & 3 & 8 & 7 & 1 & 0 & 0 & 0 & 0 & 0 \\
-16 & -1 & -4 & -10 & -13 & 0 & 1 & 0 & 0 & 0 & 0 \\
-6 & 1 & -3 & -6 & -6 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 2 & -2 & -3 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\
3 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
and row-reduce to obtain $N$

$$N = \begin{bmatrix}
1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & -1 & 2 & -1 \\
0 & 1 & 0 & 0 & -3 & 0 & 0 & -2 & 3 & -3 & 3 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 3 & 3 \\
0 & 0 & 0 & 1 & -2 & 0 & 0 & -2 & 1 & -4 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & -1 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & -1 & 1 & -1 & -1
\end{bmatrix}$$

To find the four subsets for $A$, we only need identify the $4 \times 5$ matrix $C$ and the $2 \times 6$ matrix $L$,

$$C = \begin{bmatrix}
1 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & -3 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & -2
\end{bmatrix} \quad L = \begin{bmatrix}
1 & 0 & 3 & -1 & 3 & 1 \\
0 & 1 & -2 & 1 & 1 & -1
\end{bmatrix}$$

Then we apply Theorem FS [290],

$$\mathcal{N}(A) = \mathcal{N}(C) = \langle \begin{bmatrix} -2 \\ 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} \rangle \quad \text{Theorem BNS [154]}$$

$$\mathcal{R}(A) = \mathcal{R}(C) = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 2 \\ -3 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ -3 \\ 1 \\ -2 \end{bmatrix} \rangle \quad \text{Theorem BRS [271]}$$

$$\mathcal{C}(A) = \mathcal{N}(L) = \langle \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \rangle \quad \text{Theorem BNS [154]}$$

$$\mathcal{L}(A) = \mathcal{R}(L) = \langle \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \\ -1 \end{bmatrix} \rangle \quad \text{Theorem BRS [271]}$$

The next example is just a bit different since the matrix has more rows than columns, and a trivial null space.

**Example FSAG**

**Four subsets, Archetype G**

Archetype G [748] and Archetype H [752] are both systems of $m = 5$ equations in
\( n = 2 \) variables. They have identical coefficient matrices, which we will denote here as the matrix \( G \),
\[
G = \begin{bmatrix}
2 & 3 \\
-1 & 4 \\
3 & 10 \\
3 & -1 \\
6 & 9
\end{bmatrix}.
\]

Adjoin the \( 5 \times 5 \) identity matrix, \( I_5 \), to form
\[
M = \begin{bmatrix}
2 & 3 & 1 & 0 & 0 & 0 & 0 \\
-1 & 4 & 0 & 1 & 0 & 0 & 0 \\
3 & 10 & 0 & 0 & 1 & 0 & 0 \\
3 & -1 & 0 & 0 & 0 & 1 & 0 \\
6 & 9 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

This row-reduces to
\[
N = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \frac{3}{11} & \frac{1}{11} \\
0 & 1 & 0 & 0 & 0 & -\frac{2}{11} & \frac{1}{11} \\
0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} \\
0 & 0 & 0 & 1 & 0 & 1 & -\frac{2}{3} \\
0 & 0 & 0 & 0 & 1 & 1 & -1
\end{bmatrix}
\]

The first \( n = 2 \) columns contain \( r = 2 \) leading 1’s, so we obtain \( C \) as the \( 2 \times 2 \) identity matrix and extract \( L \) from the final \( m - r = 3 \) rows in the final \( m = 5 \) columns.
\[
C = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\quad
L = \begin{bmatrix}
1 & 0 & 0 & 0 & -\frac{1}{3} \\
0 & 1 & 0 & 1 & -\frac{2}{3} \\
0 & 0 & 1 & 1 & -1
\end{bmatrix}
\]

Then we apply Theorem FS [290].
\[
\mathcal{N}(G) = \mathcal{N}(C) = \langle \rangle = \{0\} \quad \text{Theorem BNS 154}
\]
\[
\mathcal{R}(G) = \mathcal{R}(C) = \langle \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \rangle = \mathbb{C}^2 \quad \text{Theorem BRS 271}
\]
\[
\mathcal{C}(G) = \mathcal{N}(L) = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \\ 3 \end{bmatrix} \right\} \quad \text{Theorem BNS 154}
\]
\[
\mathcal{L}(G) = \mathcal{R}(L) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\} \quad \text{Theorem BRS 271}
\]
As mentioned earlier, Archetype G [748] is consistent, while Archetype H [752] is inconsistent. See if you can write the two different vectors of constants from these two archetypes as linear combinations of the two vectors in \( \mathcal{C}(G) \). How about the two columns of \( G \), can you write each individually as a linear combination of the two vectors in \( \mathcal{C}(G) \)? They must be in the column space of \( G \) also. Are your answers unique? Do you notice anything about the scalars that appear in the linear combinations you are forming?

Example COV [170] and Example CSROI [273] each describes the column space of the coefficient matrix from Archetype I [757] as the span of a set of \( r = 3 \) linearly independent vectors. It is no accident that these two different sets both have the same size. If we (you?) were to calculate the column space of this matrix using the null space of the matrix \( L \) from Theorem FS [290] then we would again find a set of 3 linearly independent vectors that span the range. More on this later.

So we have three different methods to obtain a description of the column space of a matrix as the span of a linearly independent set. Theorem BCS [264] is sometimes useful since the vectors it specifies are equal to actual columns of the matrix. Theorem BRS [271] and Theorem CSRST [273] combine to create vectors with lots of zeros, and strategically placed 1’s near the top of the vector. Theorem FS [290] and the matrix \( L \) from the extended echelon form gives us a third method, which tends to create vectors with lots of zeros, and strategically placed 1’s near the bottom of the vector. If we don’t care about linear independence we can also appeal to Definition CSM [261] and simply express the column space as the span of all the columns of the matrix, giving us a fourth description.

Although we have many ways to describe a column space, notice that one tempting strategy will usually fail. It is not possible to simply row-reduce a matrix directly and then use the columns of the row-reduced matrix as a set whose span equals the column space. In other words, row operations do not preserve column spaces (however row operations do preserve row spaces, Theorem REMRS [269]). See Exercise CRS.M21 [278].

### Subsection READ

#### Reading Questions

1. Find a nontrivial element of the left null space of \( A \).

   \[
   A = \begin{bmatrix}
   2 & 1 & -3 & 4 \\
   -1 & -1 & 2 & -1 \\
   0 & -1 & 1 & 2 
   \end{bmatrix}
   \]

2. Find the matrices \( C \) and \( L \) in the extended echelon form of \( A \).

   \[
   A = \begin{bmatrix}
   -9 & 5 & -3 \\
   2 & -1 & 1 \\
   -5 & 3 & -1 
   \end{bmatrix}
   \]
3. Why is Theorem FS 290 a great way to conclude Chapter M 197?
Subsection EXC
Exercises

C20  Example FSAG 296 concludes with several questions. Perform the analysis suggested by these questions.
Contributed by Robert Beezer

C25  Given the matrix $A$ below, use the extended echelon form of $A$ to answer each part of this problem. In each part, find a linearly independent set of vectors, $S$, so that the span of $S$, $\langle S \rangle$, equals the specified set of vectors.

$$A = \begin{bmatrix}
-5 & 3 & -1 \\
-1 & 1 & 1 \\
-8 & 5 & -1 \\
3 & -2 & 0
\end{bmatrix}$$

(a) The row space of $A$, $\mathcal{R}(A)$.
(b) The column space of $A$, $\mathcal{C}(A)$.
(c) The null space of $A$, $\mathcal{N}(A)$.
(d) The left null space of $A$, $\mathcal{L}(A)$.

Contributed by Robert Beezer  Solution 305

C26  For the matrix $D$ below use the extended echelon form to find

(a) a linearly independent set whose span is the column space of $D$.
(b) a linearly independent set whose span is the left null space of $D$.

$$D = \begin{bmatrix}
-7 & -11 & -19 & -15 \\
6 & 10 & 18 & 14 \\
3 & 5 & 9 & 7 \\
-1 & -2 & -4 & -3
\end{bmatrix}$$

Contributed by Robert Beezer  Solution 306

C41  The following archetypes are systems of equations. For each system, write the vector of constants as a linear combination of the vectors in the span construction for the column space provided by Theorem FS 290 and Theorem BNS 154 (these vectors are listed for each of these archetypes).

Archetype A 721
Archetype B 726
Archetype C 731
Archetype D 735
Archetype E 739
Archetype F 743
Archetype G 748
Archetype H 752
Archetype I 757
Archetype J 762
The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute the extended echelon form \(N\) and identify the matrices \(C\) and \(L\). Using Theorem FS, Theorem BNS and Theorem BRS express the null space, the row space, the column space and left null space of each coefficient matrix as a span of a linearly independent set.

Archetype A [721]
Archetype B [726]
Archetype C [731]
Archetype D [735]
Archetype E [739]
Archetype F [743]
Archetype G [748]
Archetype H [752]
Archetype I [757]
Archetype J [762]
Archetype K [767]
Archetype L [771]

Contributed by Robert Beezer

For the matrix \(B\) below, find sets of vectors whose span equals the column space of \(B\) \((\mathcal{C}(B))\) and which individually meet the following extra requirements.
(a) The set illustrates the definition of the column space.
(b) The set is linearly independent and the members of the set are columns of \(B\).
(c) The set is linearly independent with a “nice pattern of zeros and ones” at the top of each vector.
(d) The set is linearly independent with a “nice pattern of zeros and ones” at the bottom of each vector.

\[
B = \begin{bmatrix}
2 & 3 & 1 & 1 \\
1 & 1 & 0 & 1 \\
-1 & 2 & 3 & -4
\end{bmatrix}
\]

Let \(A\) be the matrix below, and find the indicated sets with the requested properties.

\[
A = \begin{bmatrix}
2 & -1 & 5 & -3 \\
-5 & 3 & -12 & 7 \\
1 & 1 & 4 & -3
\end{bmatrix}
\]

(a) A linearly independent set \(S\) so that \(\mathcal{C}(A) = \langle S \rangle\) and \(S\) is composed of columns of \(A\).
(b) A linearly independent set \(S\) so that \(\mathcal{C}(A) = \langle S \rangle\) and the vectors in \(S\) have a nice pattern of zeros and ones at the top of the vectors.
(c) A linearly independent set \(S\) so that \(\mathcal{C}(A) = \langle S \rangle\) and the vectors in \(S\) have a nice pattern of zeros and ones at the bottom of the vectors.
(d) A linearly independent set \(S\) so that \(\mathcal{R}(A) = \langle S \rangle\).

Suppose that \(A\) is a nonsingular matrix. Extend the four conclusions of
rem FS 290 in this special case and discuss connections with previous results (such as Theorem NME4 267).
Contributed by Robert Beezer

M51 Suppose that $A$ is a singular matrix. Extend the four conclusions of Theorem FS 290 in this special case and discuss connections with previous results (such as Theorem NME4 267).
Contributed by Robert Beezer
Subsection SOL

Solutions

C25 Contributed by Robert Beezer

Statement

Add a $4 \times 4$ identity matrix to the right of $A$ to form the matrix $M$ and then row-reduce to the matrix $N$,

$$M = \begin{bmatrix} -5 & 3 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ -8 & 5 & -1 & 0 & 0 & 1 & 0 \\ 3 & -2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & -2 & -5 \\ 0 & 1 & 3 & 0 & 0 & -3 & -8 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 \end{bmatrix} = N$$

To apply Theorem FS in each of these four parts, we need the two matrices,

$$C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

(a)

$$\mathcal{R}(A) = \mathcal{R}(C) \quad \text{Theorem FS 290}$$

$$= \left\langle \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle \quad \text{Theorem BRS 271}$$

(b)

$$\mathcal{C}(A) = \mathcal{N}(L) \quad \text{Theorem FS 290}$$

$$= \left\langle \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\rangle \quad \text{Theorem BNS 154}$$

(c)

$$\mathcal{N}(A) = \mathcal{N}(C) \quad \text{Theorem FS 290}$$

$$= \left\langle \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} \right\rangle \quad \text{Theorem BNS 154}$$

(d)

$$\mathcal{L}(A) = \mathcal{R}(L) \quad \text{Theorem FS 290}$$

$$= \left\langle \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\rangle \quad \text{Theorem BRS 271}$$
For both parts, we need the extended echelon form of the matrix.

\[
\begin{bmatrix}
-7 & -11 & -19 & -15 & 1 & 0 & 0 & 0 \\
6 & 10 & 18 & 14 & 0 & 1 & 0 & 0 \\
3 & 5 & 9 & 7 & 0 & 0 & 1 & 0 \\
-1 & -2 & -4 & -3 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\overset{\text{RREF}}{\rightarrow}
\begin{bmatrix}
-1 & 0 & 3 & 2 \\
0 & 1 & -2 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

From this matrix we extract the last two rows, in the last four columns to form the matrix \( L \),

\[
L = \begin{bmatrix}
1 & 0 & 3 & 2 \\
0 & 1 & -2 & 0
\end{bmatrix}
\]

(a) By Theorem FS and Theorem BNS we have

\[
\mathcal{C}(D) = \mathcal{N}(L) = \left\langle \begin{bmatrix}
-3 \\
2 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
-2 \\
0 \\
0 \\
1
\end{bmatrix} \right\rangle
\]

(b) By Theorem FS and Theorem BRS we have

\[
\mathcal{L}(D) = \mathcal{R}(L) = \left\langle \begin{bmatrix}
1 \\
0 \\
3 \\
2
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
-2 \\
0
\end{bmatrix} \right\rangle
\]

(c) The definition of the column space is the span of the set of columns (Definition CSM). So the desired set is just the four columns of \( B \),

\[
S = \left\{ \begin{bmatrix}
2 \\
1 \\
-1
\end{bmatrix}, \begin{bmatrix}
3 \\
1 \\
2
\end{bmatrix}, \begin{bmatrix}
1 \\
0 \\
3
\end{bmatrix}, \begin{bmatrix}
1 \\
1 \\
-4
\end{bmatrix} \right\}
\]

(b) Theorem BCS suggests row-reducing the matrix and using the columns of \( B \) that correspond to the pivot columns.

\[
B \overset{\text{RREF}}{\rightarrow}
\begin{bmatrix}
1 & 0 & -1 & 2 \\
0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

So the pivot columns are numbered by elements of \( D = \{1, 2\} \), so the requested set is

\[
S = \left\{ \begin{bmatrix}
2 \\
1 \\
-1
\end{bmatrix}, \begin{bmatrix}
3 \\
1 \\
2
\end{bmatrix} \right\}
\]

(c) We can find this set by row-reducing the transpose of \( B \), deleting the zero rows, and using the nonzero rows as column vectors in the set. This is an application of
Subsection FS.SOL Solutions 307

Theorem CSRST [273] followed by Theorem BRS [271].

\[
B^t \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 3 \\
0 & 1 & -7 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

So the requested set is

\[
S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}
\]

(d) With the column space expressed as a null space, the vectors obtained via Theorem BNS [154] will be of the desired shape. So we first proceed with Theorem FS [290] and create the extended echelon form,

\[
\begin{bmatrix} B | I_3 \end{bmatrix} = \begin{bmatrix}
2 & 3 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 \\
-1 & 2 & 3 & -4 & 0 & 0 & 1
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & -1 & 2 & 0 & \frac{2}{3} & \frac{1}{3} \\
0 & 1 & -1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\]

So, employing Theorem FS [290], we have \( C(B) = \mathcal{N}(L) \), where

\[
L = \begin{bmatrix} 1 & -\frac{7}{3} & -\frac{1}{3} \end{bmatrix}
\]

We can find the desired set of vectors from Theorem BNS [154] as

\[
S = \left\{ \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\}
\]

C61 Contributed by Robert Beezer Statement 302

(a) First find a matrix \( B \) that is row-equivalent to \( A \) and in reduced row-echelon form

\[
B = \begin{bmatrix}
1 & 0 & 3 & -2 \\
0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

By Theorem BCS [264] we can choose the columns of \( A \) that correspond to dependent variables (\( D = \{1, 2\} \)) as the elements of \( S \) and obtain the desired properties. So

\[
S = \left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\}
\]

(b) We can write the column space of \( A \) as the row space of the transpose (Theorem CSRST [273]). So we row-reduce the transpose of \( A \) to obtain the row-equivalent matrix \( C \) in reduced row-echelon form

\[
C = \begin{bmatrix}
1 & 0 & 8 \\
0 & 1 & 3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
The nonzero rows (written as columns) will be a linearly independent set that spans the row space of \( A' \), by Theorem BRS [271], and the zeros and ones will be at the top of the vectors,

\[
S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}
\]

(c) In preparation for Theorem FS [290], augment \( A \) with the \( 3 \times 3 \) identity matrix \( I_3 \) and row-reduce to obtain the extended echelon form,

\[
\begin{bmatrix}
1 & 0 & 3 & -2 & 0 & -\frac{1}{8} & \frac{3}{8} \\
0 & 1 & 1 & -1 & 0 & \frac{1}{2} & -\frac{1}{8} \\
0 & 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{8}
\end{bmatrix}
\]

Then since the first four columns of row 3 are all zeros, we extract

\[
L = \begin{bmatrix} 1 & 0 & 3 & -1 \end{bmatrix}
\]

Theorem FS [290] says that \( \mathcal{C}(A) = \mathcal{N}(L) \). We can then use Theorem BNS [154] to construct the desired set \( S \), based on the free variables with indices in \( F = \{2, 3\} \) for the homogeneous system \( LS(L, 0) \), so

\[
S = \left\{ \begin{bmatrix} -\frac{3}{8} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{8} \\ 0 \\ 1 \end{bmatrix} \right\}
\]

Notice that the zeros and ones are at the bottom of the vectors.

(d) This is a straightforward application of Theorem BRS [271]. Use the row-reduced matrix \( B \) from part (a), grab the nonzero rows, and write them as column vectors,

\[
S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}
\]
Chapter VS
Vector Spaces

We now have a computational toolkit in place and so we can begin our study of linear algebra in a more theoretical style.

Linear algebra is the study of two fundamental objects, vector spaces and linear transformations (see Chapter LT [503]). This chapter will focus on the former. The power of mathematics is often derived from generalizing many different situations into one abstract formulation, and that is exactly what we will be doing throughout this chapter.

Section VS
Vector Spaces

In this section we present a formal definition of a vector space, which will lead to an extra increment of abstraction. Once defined, we study its most basic properties.

Subsection VS
Vector Spaces

Here is one of our two most important definitions in the entire course.

Definition VS
Vector Space

Suppose that \( V \) is a set upon which we have defined two operations: (1) vector addition, which combines two elements of \( V \) and is denoted by “+”, and (2) scalar multiplication, which combines a complex number with an element of \( V \) and is denoted by juxtaposition. Then \( V \), along with the two operations, is a vector space if the following ten properties hold.

- **AC** Additive Closure
  If \( u, v \in V \), then \( u + v \in V \).

- **SC** Scalar Closure
  If \( \alpha \in \mathbb{C} \) and \( u \in V \), then \( \alpha u \in V \).
• **C** Commutativity
  If $u, v \in V$, then $u + v = v + u$.

• **AA** Additive Associativity
  If $u, v, w \in V$, then $u + (v + w) = (u + v) + w$.

• **Z** Zero Vector
  There is a vector, $0$, called the zero vector, such that $u + 0 = u$ for all $u \in V$.

• **AI** Additive Inverses
  If $u \in V$, then there exists a vector $-u \in V$ so that $u + (-u) = 0$.

• **SMA** Scalar Multiplication Associativity
  If $\alpha, \beta \in \mathbb{C}$ and $u \in V$, then $\alpha(\beta u) = (\alpha\beta)u$.

• **DVA** Distributivity across Vector Addition
  If $\alpha \in \mathbb{C}$ and $u, v \in V$, then $\alpha(u + v) = \alpha u + \alpha v$.

• **DSA** Distributivity across Scalar Addition
  If $\alpha, \beta \in \mathbb{C}$ and $u \in V$, then $(\alpha + \beta)u = \alpha u + \beta u$.

• **O** One
  If $u \in V$, then $1u = u$.

The objects in $V$ are called vectors, no matter what else they might really be, simply by virtue of being elements of a vector space.

Now, there are several important observations to make. Many of these will be easier to understand on a second or third reading, and especially after carefully studying the examples in Subsection VS.EVS [311].

An axiom is often a “self-evident” truth. Something so fundamental that we all agree it is true and accept it without proof. Typically, it would be the logical underpinning that we would begin to build theorems upon. Some might refer to the ten properties of Definition VS [309] as axioms, implying that a vector space is a very natural object and the ten properties are the essence of a vector space. We will instead emphasize that we will begin with a definition of a vector space. After studying the remainder of this chapter, you might return here and remind yourself how all our forthcoming theorems and definitions rest on this foundation.

As we will see shortly, the objects in $V$ can be anything, even though we will call them vectors. We have been working with vectors frequently, but we should stress here that these have so far just been column vectors — scalars arranged in a columnar list of fixed length. In a similar vein, you have used the symbol “+” for many years to represent the addition of numbers (scalars). We have extended its use to the addition of column vectors and to the addition of matrices, and now we are going to recycle it even further and let it denote vector addition in any possible vector space. So when describing a new vector space, we will have to define exactly what “+” is. Similar comments apply to scalar multiplication. Conversely, we can define our operations any way we like, so long as the ten properties are fulfilled (see Example CVS [314]).

A vector space is composed of three objects, a set and two operations. However, we usually use the same symbol for both the set and the vector space itself. Do not let this convenience fool you into thinking the operations are secondary!
This discussion has either convinced you that we are really embarking on a new level of abstraction, or they have seemed cryptic, mysterious or nonsensical. You might want to return to this section in a few days and give it another read then. In any case, let’s look at some concrete examples now.

Subsection EVS
Examples of Vector Spaces

Our aim in this subsection is to give you a storehouse of examples to work with, to become comfortable with the ten vector space properties and to convince you that the multitude of examples justifies (at least initially) making such a broad definition as Definition VS 309. Some of our claims will be justified by reference to previous theorems, we will prove some facts from scratch, and we will do one non-trivial example completely. In other places, our usual thoroughness will be neglected, so grab paper and pencil and play along.

Example VSCV
The vector space $\mathbb{C}^m$
Set: $\mathbb{C}^m$, all column vectors of size $m$, Definition VSCV 87.
Equality: Entry-wise, Definition CVE 88.
Vector Addition: The “usual” addition, given in Definition CVA 89.
Scalar Multiplication: The “usual” scalar multiplication, given in Definition CVSM 89.

Does this set with these operations fulfill the ten properties? Yes. And by design all we need to do is quote Theorem VSPCV 91. That was easy.

Example VSM
The vector space of matrices, $M_{mn}$
Set: $M_{mn}$, the set of all matrices of size $m \times n$ and entries from $\mathbb{C}$, Example VSM 311.
Equality: Entry-wise, Definition ME 197.
Vector Addition: The “usual” addition, given in Definition MA 198.
Scalar Multiplication: The “usual” scalar multiplication, given in Definition MSM 198.

Does this set with these operations fulfill the ten properties? Yes. And all we need to do is quote Theorem VSPM 199. Another easy one (by design).

So, the set of all matrices of a fixed size forms a vector space. That entitles us to call a matrix a vector, since a matrix is an element of a vector space. For example, if $A, B \in M_{3,4}$ then we call $A$ and $B$ “vectors,” and we even use our previous notation for column vectors to refer to $A$ and $B$. So we could legitimately write expressions like

$$u + v = A + B = B + A = v + u$$

This could lead to some confusion, but it is not too great a danger. But it is worth comment.

The previous two examples may be less than satisfying. We made all the relevant definitions long ago. And the required verifications were all handled by quoting old theorems. However, it is important to consider these two examples first. We have been studying vectors and matrices carefully (Chapter V 87, Chapter M 197), and both objects, along with their operations, have certain properties in common, as you may
have noticed in comparing [Theorem VSPCV][91] with [Theorem VSPM][199]. Indeed, it is these two theorems that motivate us to formulate the abstract definition of a vector space, [Definition VS][309]. Now, should we prove some general theorems about vector spaces (as we will shortly in Subsection VS.VSP[316]), we can instantly apply the conclusions to both $\mathbb{C}^m$ and $M_{mn}$. Notice too how we have taken six definitions and two theorems and reduced them down to two examples. With greater generalization and abstraction our old ideas get downgraded in stature.

Let us look at some more examples, now considering some new vector spaces.

**Example VSP**

**The vector space of polynomials, $P_n$**

Set: $P_n$, the set of all polynomials of degree $n$ or less in the variable $x$ with coefficients from $\mathbb{C}$.

Equality:

$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$ if and only if $a_i = b_i$ for $0 \leq i \leq n$

**Vector Addition:**

$$(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n) = (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + \cdots + (a_n + b_n) x^n$$

**Scalar Multiplication:**

$$\alpha (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = (\alpha a_0) + (\alpha a_1) x + (\alpha a_2) x^2 + \cdots + (\alpha a_n) x^n$$

This set, with these operations, will fulfill the ten properties, though we will not work all the details here. However, we will make a few comments and prove one of the properties. First, the zero vector [Property Z][310] is what you might expect, and you can check that it has the required property.

$$0 = 0 + 0 x + 0 x^2 + \cdots + 0 x^n$$

The additive inverse [Property AI][310] is also no surprise, though consider how we have chosen to write it.

$$- (a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = (-a_0) + (-a_1) x + (-a_2) x^2 + \cdots + (-a_n) x^n$$

Now let’s prove the associativity of vector addition [Property AA][310]. This is a bit tedious, though necessary. Throughout, the plus sign (“+”) does triple-duty. You might ask yourself what each plus sign represents as you work through this proof.

$$u + (v + w)$$

$$= (a_0 + a_1 x + \cdots + a_n x^n) + ((b_0 + b_1 x + \cdots + b_n x^n) + (c_0 + c_1 x + \cdots + c_n x^n))$$

$$= (a_0 + a_1 x + \cdots + a_n x^n) + ((b_0 + c_0) + (b_1 + c_1) x + \cdots + (b_n + c_n) x^n)$$

$$= (a_0 + (b_0 + c_0)) + (a_1 + (b_1 + c_1)) x + \cdots + (a_n + (b_n + c_n)) x^n$$

$$= (a_0 + b_0) + c_0 + ((a_1 + b_1) + c_1) x + \cdots + ((a_n + b_n) + c_n) x^n$$

$$= (a_0 + b_0) + (a_1 + b_1) x + \cdots + (a_n + b_n) x^n + (c_0 + c_1 x + \cdots + c_n x^n)$$

$$= ((a_0 + b_0) + b_1 x + \cdots + a_n x^n) + (b_0 + b_1 x + \cdots + b_n x^n) + (c_0 + c_1 x + \cdots + c_n x^n)$$
\[ (u + v) + w \]

Notice how it is the application of the associativity of the (old) addition of complex numbers in the middle of this chain of equalities that makes the whole proof happen. The remainder is successive applications of our (new) definition of vector (polynomial) addition. Proving the remainder of the ten properties is similar in style and tedium. You might try proving the commutativity of vector addition (Property C [310]), or one of the distributivity properties (Property DVA [310], Property DSA [310]).

Example VSIS

The vector space of infinite sequences

Set: \( \mathbb{C}^\infty = \{(c_0, c_1, c_2, c_3, \ldots) \mid c_i \in \mathbb{C}, \ i \in \mathbb{N}\} \).

Equality:

\((c_0, c_1, c_2, \ldots) = (d_0, d_1, d_2, \ldots) \) if and only if \( c_i = d_i \) for all \( i \geq 0 \)

Vector Addition:

\((c_0, c_1, c_2, \ldots) + (d_0, d_1, d_2, \ldots) = (c_0 + d_0, c_1 + d_1, c_2 + d_2, \ldots) \)

Scalar Multiplication:

\[ \alpha(c_0, c_1, c_2, \ldots) = (\alpha c_0, \alpha c_1, \alpha c_2, \alpha c_3, \ldots) \]

This should remind you of the vector space \( \mathbb{C}^m \), though now our lists of scalars are written horizontally with commas as delimiters and they are allowed to be infinite in length. What does the zero vector look like (Property Z [310])? Additive inverses (Property AI [310])? Can you prove the associativity of vector addition (Property AA [310])?

Example VSF

The vector space of functions

Set: \( F = \{f \mid f : \mathbb{C} \to \mathbb{C}\} \).

Equality: \( f = g \) if and only if \( f(x) = g(x) \) for all \( x \in \mathbb{C} \).

Vector Addition: \( f + g \) is the function with outputs defined by \( (f + g)(x) = f(x) + g(x) \).

Scalar Multiplication: \( \alpha f \) is the function with outputs defined by \( (\alpha f)(x) = \alpha f(x) \).

So this is the set of all functions of one variable that take a complex number to a complex number. You might have studied functions of one variable that take a real number to a real number, and that might be a more natural set to study. But since we are allowing our scalars to be complex numbers, we need to expand the domain and range of our functions also. Study carefully how the definitions of the operation are made, and think about the different uses of “+” and juxtaposition. As an example of what is required when verifying that this is a vector space, consider that the zero vector (Property Z [310]) is the function \( z \) whose definition is \( z(x) = 0 \) for every input \( x \).

While vector spaces of functions are very important in mathematics and physics, we will not devote them much more attention.

Here’s a unique example.
Example VSS
The singleton vector space
Set: $Z = \{ z \}$.
Equality: Huh?
Vector Addition: $z + z = z$.
Scalar Multiplication: $\alpha z = z$.

This should look pretty wild. First, just what is $z$? Column vector, matrix, polynomial, sequence, function? Mineral, plant, or animal? We aren’t saying! $z$ just is. And we have definitions of vector addition and scalar multiplication that are sufficient for an occurrence of either that may come along.

Our only concern is if this set, along with the definitions of two operations, fulfills the ten properties of Definition VS 309. Let’s check associativity of vector addition (Property AA 310). For all $u, v, w \in Z$,

$$u + (v + w) = z + (z + z) = z + z = (z + z) + z = (u + v) + w$$

What is the zero vector in this vector space (Property Z 310)? With only one element in the set, we do not have much choice. Is $z = 0$? It appears that $z$ behaves like the zero vector should, so it gets the title. Maybe now the definition of this vector space does not seem so bizarre. It is a set whose only element is the element that behaves like the zero vector, so that lone element is the zero vector.

Perhaps some of the above definitions and verifications seem obvious or like splitting hairs, but the next example should convince you that they are necessary. We will study this one carefully. Ready? Check your preconceptions at the door.

Example CVS
The crazy vector space
Set: $C = \{ (x_1, x_2) \mid x_1, x_2 \in \mathbb{C} \}$.
Vector Addition: $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)$.
Scalar Multiplication: $\alpha(x_1, x_2) = (\alpha x_1 + \alpha - 1, \alpha x_2 + \alpha - 1)$.

Now, the first thing I hear you say is “You can’t do that!” And my response is, “Oh yes, I can!” I am free to define my set and my operations any way I please. They may not look natural, or even useful, but we will now verify that they provide us with another example of a vector space. And that is enough. If you are adventurous, you might try first checking some of the properties yourself. What is the zero vector? Additive inverses? Can you prove associativity? Ready, here we go.

**Property AC 309, Property SC 309**: The result of each operation is a pair of complex numbers, so these two closure properties are fulfilled.

**Property C 310**:

$$u + v = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1) = (y_1 + x_1 + 1, y_2 + x_2 + 1) = (y_1, y_2) + (x_1, x_2) = v + u$$
**Property AA** **310**: 

\[ \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (x_1, x_2) + ((y_1, y_2) + (z_1, z_2)) \]

\[ = (x_1, x_2) + (y_1 + z_1 + 1, y_2 + z_2 + 1) \]

\[ = (x_1 + (y_1 + z_1 + 1) + 1, x_2 + (y_2 + z_2 + 1) + 1) \]

\[ = (x_1 + y_1 + z_1 + 2, x_2 + y_2 + z_2 + 2) \]

\[ = ((x_1 + y_1 + 1) + z_1 + 1, (x_2 + y_2 + 1) + z_2 + 1) \]

\[ = (x_1 + y_1 + 1, x_2 + y_2 + 1) + (z_1, z_2) \]

\[ = ((x_1, x_2) + (y_1, y_2)) + (z_1, z_2) \]

\[ = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \]

**Property Z** **310**: The zero vector is \( \ldots \mathbf{0} = (-1, -1) \). Now I hear you say, “No, no, that can’t be, it must be \((0, 0)\)”! Indulge me for a moment and let us check my proposal.

\[ \mathbf{u} + \mathbf{0} = (x_1, x_2) + (-1, -1) = (x_1 + (-1) + 1, x_2 + (-1) + 1) = (x_1, x_2) = \mathbf{u} \]

Feeling better? Or worse?

**Property AI** **310**: For each vector, \( \mathbf{u} \), we must locate an additive inverse, \(-\mathbf{u}\). Here it is, \(-(x_1, x_2) = (-x_1 - 2, -x_2 - 2)\). As odd as it may look, I hope you are withholding judgment. Check:

\[ \mathbf{u} + (-\mathbf{u}) = (x_1, x_2) + (-x_1 - 2, -x_2 - 2) = (x_1 + (-x_1 - 2) + 1, -x_2 + (x_2 - 2) + 1) = (-1, -1) = \mathbf{0} \]

**Property SMA** **310**:

\[ \alpha(\beta \mathbf{u}) = \alpha(\beta(x_1, x_2)) \]

\[ = \alpha(\beta x_1 + \beta - 1, \beta x_2 + \beta - 1) \]

\[ = (\alpha(\beta x_1 + \beta - 1) + \alpha - 1, \alpha(\beta x_2 + \beta - 1) + \alpha - 1) \]

\[ = ((\alpha \beta x_1 + \alpha \beta - \alpha) + \alpha - 1, \alpha \beta x_2 + \alpha \beta - \alpha + \alpha - 1) \]

\[ = ((\alpha \beta x_1 + \alpha \beta - 1) - \alpha x_2 + \alpha \beta - 1) \]

\[ = (\alpha \beta x_1, \alpha \beta x_2) \]

\[ = (\alpha \beta)(x_1, x_2) \]

\[ = (\alpha \beta) \mathbf{u} \]

**Property DVA** **310**: If you have hung on so far, here’s where it gets even wilder. In the next two properties we mix and mash the two operations.

\[ \alpha(\mathbf{u} + \mathbf{v}) = \alpha((x_1, x_2) + (y_1, y_2)) \]

\[ = \alpha(x_1 + y_1 + 1, x_2 + y_2 + 1) \]

\[ = (\alpha(x_1 + y_1 + 1) + \alpha - 1, \alpha(x_2 + y_2 + 1) + \alpha - 1) \]

\[ = (\alpha x_1 + \alpha y_1 + \alpha + \alpha - 1, \alpha x_2 + \alpha y_2 + \alpha + \alpha - 1) \]

\[ = ((\alpha x_1 + \alpha - 1) + (\alpha y_1 + \alpha - 1) + 1, \alpha x_2 + \alpha - 1 + \alpha y_2 + \alpha - 1 + 1) \]

\[ = (\alpha x_1 + \alpha - 1, \alpha x_2 + \alpha - 1) + (\alpha y_1 + \alpha - 1, \alpha y_2 + \alpha - 1) \]

\[ = \alpha(x_1, x_2) + \alpha(y_1, y_2) \]

\[ = \alpha \mathbf{u} + \alpha \mathbf{v} \]
Property DSA [310]:

\[(\alpha + \beta)\mathbf{u} = (\alpha + \beta)(x_1, x_2) \]
\[= ((\alpha + \beta)x_1 + (\alpha + \beta) - 1, (\alpha + \beta)x_2 + (\alpha + \beta) - 1) \]
\[= (\alpha x_1 + \beta x_1 + \alpha + \beta - 1, \alpha x_2 + \beta x_2 + \alpha + \beta - 1) \]
\[= (\alpha x_1 + \alpha - 1 + \beta x_1 + \beta - 1 + 1, \alpha x_2 + \alpha - 1 + \beta x_2 + \beta - 1 + 1) \]
\[= ((\alpha x_1 + \alpha - 1) + (\beta x_1 + \beta - 1) + 1, (\alpha x_2 + \alpha - 1) + (\beta x_2 + \beta - 1) + 1) \]
\[= (\alpha x_1 + \alpha - 1, \alpha x_2 + \alpha - 1) + (\beta x_1 + \beta - 1, \beta x_2 + \beta - 1) \]
\[= \alpha(x_1, x_2) + \beta(x_1, x_2) \]
\[= \alpha \mathbf{u} + \beta \mathbf{u} \]

Property O [310]: After all that, this one is easy, but no less pleasing.

\[1\mathbf{u} = 1(x_1, x_2) = (x_1 + 1 - 1, x_2 + 1 - 1) = (x_1, x_2) = \mathbf{u} \]

That’s it, \(C\) is a vector space, as crazy as that may seem.

Notice that in the case of the zero vector and additive inverses, we only had to propose possibilities and then verify that they were the correct choices. You might try to discover how you would arrive at these choices, though you should understand why the process of discovering them is not a necessary component of the proof itself.

Subsection VSP

Vector Space Properties

Subsection VS.EVS [311] has provided us with an abundance of examples of vector spaces, most of them containing useful and interesting mathematical objects along with natural operations. In this subsection we will prove some general properties of vector spaces. Some of these results will again seem obvious, but it is important to understand why it is necessary to state and prove them. A typical hypothesis will be “Let \(V\) be a vector space.” From this we may assume the ten properties of Definition VS [309], and nothing more. Its like starting over, as we learn about what can happen in this new algebra we are learning. But the power of this careful approach is that we can apply these theorems to any vector space we encounter — those in the previous examples, or new ones we have not yet contemplated. Or perhaps new ones that nobody has ever contemplated. We will illustrate some of these results with examples from the crazy vector space (Example CVS [314]), but mostly we are stating theorems and doing proofs. These proofs do not get too involved, but are not trivial either, so these are good theorems to try proving yourself before you study the proof given here. (See Technique P [715].)

First we show that there is just one zero vector. Notice that the properties only require there to be at least one, and say nothing about there possibly being more. That is because we can use the ten properties of a vector space (Definition VS [309]) to learn that there can never be more than one. To require that this extra condition be stated as an eleventh property would make the definition of a vector space more complicated than it needs to be.
**Theorem ZVU**  
Zero Vector is Unique  
Suppose that $V$ is a vector space. The zero vector, $0$, is unique.  
□

**Proof**  
To prove uniqueness, a standard technique is to suppose the existence of two objects (Technique U [709]). So let $0_1$ and $0_2$ be two zero vectors in $V$. Then  
\[
0_1 = 0_1 + 0_2 \quad \text{Property Z 310 for } 0_2
\]
\[
= 0_2 + 0_1 \quad \text{Property C 310}
\]
\[
= 0_2 \quad \text{Property Z 310 for } 0_1
\]

This proves the uniqueness since the two zero vectors are really the same. ■

**Theorem AIU**  
Additive Inverses are Unique  
Suppose that $V$ is a vector space. For each $u \in V$, the additive inverse, $-u$, is unique.  
□

**Proof**  
To prove uniqueness, a standard technique is to suppose the existence of two objects (Technique U [709]). So let $-u_1$ and $-u_2$ be two additive inverses for $u$. Then  
\[
-u_1 = -u_1 + 0 \quad \text{Property Z 310}
\]
\[
= -u_1 + (u + -u_2) \quad \text{Property AI 310}
\]
\[
= (-u_1 + u) + -u_2 \quad \text{Property AA 310}
\]
\[
= 0 + -u_2 \quad \text{Property AI 310}
\]
\[
= -u_2 \quad \text{Property Z 310}
\]

So the two additive inverses are really the same. ■

As obvious as the next three theorems appear, nowhere have we guaranteed that the zero scalar, scalar multiplication and the zero vector all interact this way. Until we have proved it, anyway.

**Theorem ZSSM**  
Zero Scalar in Scalar Multiplication  
Suppose that $V$ is a vector space and $u \in V$. Then $0u = 0$.  
□

**Proof**  
Notice that 0 is a scalar, $u$ is a vector, so Property SC 309 says $0u$ is again a vector. As such, $0u$ has an additive inverse, $-(0u)$ by Property AI 310.  
\[
0u = 0 + 0u \quad \text{Property Z 310}
\]
\[
= (-0u) + 0u \quad \text{Property AI 310}
\]
\[
= -(0u) + (0u + 0u) \quad \text{Property AA 310}
\]
\[
= -(0u) + (0 + 0)u \quad \text{Property DSA 310}
\]
\[
= -(0u) + 0u \quad \text{0 in } \mathbb{C}
\]
\[
= 0 \quad \text{Property AI 310}
\]

Here’s another theorem that looks like it should be obvious, but is still in need of a proof.
Theorem ZVSM
Zero Vector in Scalar Multiplication
Suppose that $V$ is a vector space and $\alpha \in \mathbb{C}$. Then $\alpha \mathbf{0} = \mathbf{0}$.

\[ \begin{align*}
\alpha \mathbf{0} &= 0 + \alpha \mathbf{0} \\
&= (-\alpha \mathbf{0}) + (\alpha \mathbf{0} + \alpha \mathbf{0}) \\
&= -\alpha \mathbf{0} + \alpha (\mathbf{0} + \mathbf{0}) \\
&= -\alpha \mathbf{0} + \alpha \mathbf{0} \\
&= 0 \\
\end{align*} \]

\[ \square \]

Here’s another one that sure looks obvious. But understand that we have chosen to use certain notation because it makes the theorem’s conclusion look so nice. The theorem is not true because the notation looks so good, it still needs a proof. If we had really wanted to make this point, we might have defined the additive inverse of $\mathbf{u}$ as $\mathbf{u}^\sharp$. Then we would have written the defining property, Property AI, as $\mathbf{u} + \mathbf{u}^\sharp = \mathbf{0}$. This theorem would become $\mathbf{u}^\sharp = (-1)\mathbf{u}$. Not really quite as pretty, is it?

Theorem AISM
Additive Inverses from Scalar Multiplication
Suppose that $V$ is a vector space and $\mathbf{u} \in V$. Then $-\mathbf{u} = (-1)\mathbf{u}$.

\[ \begin{align*}
-\mathbf{u} &= -\mathbf{u} + \mathbf{0} \\
&= -\mathbf{u} + \mathbf{0} \\
&= -\mathbf{u} + (1 + (-1))\mathbf{u} \\
&= -\mathbf{u} + (1\mathbf{u} + (-1)\mathbf{u}) \\
&= -\mathbf{u} + (\mathbf{u} + (-1)\mathbf{u}) \\
&= (\mathbf{u} + (-1)\mathbf{u}) + (-1)\mathbf{u} \\
&= 0 + (-1)\mathbf{u} \\
&= (-1)\mathbf{u} \\
\end{align*} \]

\[ \square \]

Because of this theorem, we can now write linear combinations like $6\mathbf{u}_1 + (-4)\mathbf{u}_2$ as $6\mathbf{u}_1 - 4\mathbf{u}_2$, even though we have not formally defined an operation called vector subtraction.

Example PCVS
Properties for the Crazy Vector Space
Several of the above theorems have interesting demonstrations when applied to the crazy vector space, $C$ (Example CVS). We are not proving anything new here, or learning anything we did not know already about $C$. It is just plain fun to see how these general theorems apply in a specific instance. For most of our examples, the applications are obvious or trivial, but not with $C$. 

Version 0.92
Suppose $u \in C$.
Then, as given by Theorem ZSSM [317],
$$0u = 0(x_1, x_2) = (0x_1 + 0 - 1, 0x_2 + 0 - 1) = (-1, -1) = 0$$
And as given by Theorem ZVSM [318],
$$\alpha 0 = \alpha(-1, -1) = (\alpha(-1) + \alpha - 1, \alpha(-1) + \alpha - 1)
= (-\alpha + \alpha - 1, -\alpha + \alpha - 1) = (-1, -1) = 0$$
Finally, as given by Theorem AISM [318],
$$(-1)u = (-1)(x_1, x_2) = ((-1)x_1 + (-1) - 1, (-1)x_2 + (-1) - 1)
= (-x_1 - 2, -x_2 - 2) = -u$$

Our next theorem is a bit different from several of the others in the list. Rather than making a declaration ("the zero vector is unique") it is an implication ("if... , then...") and so can be used in proofs to move from one statement to another.

**Theorem SMEZV**

**Scalar Multiplication Equals the Zero Vector**

Suppose that $V$ is a vector space and $\alpha \in \mathbb{C}$. If $\alpha u = 0$, then either $\alpha = 0$ or $u = 0$.

□

**Proof** We prove this theorem by breaking up the analysis into two cases. The first seems too trivial, and it is, but the logic of the argument is still legitimate.

Case 1. Suppose $\alpha = 0$. In this case our conclusion is true (the first part of the either/or is true) and we are done. That was easy.

Case 2. Suppose $\alpha \neq 0$.

$$u = 1u \quad \text{Property O [310]}$$
$$= \left(\frac{1}{\alpha}\right)u \quad \alpha \neq 0$$
$$= \frac{1}{\alpha}(\alpha u) \quad \text{Property SMA [310]}$$
$$= \frac{1}{\alpha}(0) \quad \text{Hypothesis}$$
$$= 0 \quad \text{Theorem ZVSM [318]}$$

So in this case, the conclusion is true (the second part of the either/or is true) and we are done since the conclusion was true in each of the two cases. □

The next three theorems give us cancellation properties. The two concerned with scalar multiplication are intimately connected with Theorem SMEZV [319]. All three are implications. So we will prove each once, here and now, and then we can apply them at will in the future, saving several steps in a proof whenever we do.

**Theorem VAC**

**Vector Addition Cancellation**

Suppose that $V$ is a vector space, and $u, v, w \in V$. If $w + u = w + v$, then $u = v$. □
Proof

\[ u = 0 + u \]
\[ = (-w + w) + u \]
\[ = -w + (w + u) \]
\[ = -w + (w + v) \]
\[ = (-w + w) + v \]
\[ = 0 + v \]
\[ = v \]

\[ \text{Property Z} \]
\[ \text{Property AI} \]
\[ \text{Property AA} \]

Theorem CSSM
Canceling Scalars in Scalar Multiplication
Suppose \( V \) is a vector space, \( u, v \in V \) and \( \alpha \) is a nonzero scalar from \( \mathbb{C} \). If \( \alpha u = \alpha v \), then \( u = v \).

Proof

\[ u = 1u \]
\[ = \left( \frac{1}{\alpha} \right) u \]
\[ = \frac{1}{\alpha} (\alpha u) \]
\[ = \frac{1}{\alpha} (\alpha v) \]
\[ = \left( \frac{1}{\alpha} \right) v \]
\[ = 1v \]
\[ = v \]

\[ \text{Property O} \]
\[ \text{Property SMA} \]

Theorem CVSM
Canceling Vectors in Scalar Multiplication
Suppose \( V \) is a vector space, \( u \neq 0 \) is a vector in \( V \) and \( \alpha, \beta \in \mathbb{C} \). If \( \alpha u = \beta u \), then \( \alpha = \beta \).

Proof

\[ 0 = \alpha u + - (\alpha u) \]
\[ = \beta u + - (\alpha u) \]
\[ = \beta u + (-1) (\alpha u) \]
\[ = \beta u + ((-1)\alpha) u \]
\[ = \beta u + (-\alpha) u \]
\[ = (\beta - \alpha) u \]

By hypothesis, \( u \neq 0 \), so \( \text{Theorem SMEZV} \) implies
\[ 0 = \beta - \alpha \]
\[ \alpha = \beta \]

So with these three theorems in hand, we can return to our practice of “slashing” out parts of an equation, so long as we are careful about not canceling a scalar that might possibly be zero, or canceling a vector in a scalar multiplication that might be the zero vector.

**Subsection RD**  
**Recycling Definitions**

When we say that \( V \) is a vector space, we then know we have a set of objects (the “vectors”), but we also know we have been provided with two operations (“vector addition” and “scalar multiplication”) and these operations behave with these objects according to the ten properties of [Definition VS](309). One combines two vectors and produces a vector, the other takes a scalar and a vector, producing a vector as the result. So if \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in V \) then an expression like

\[
5\mathbf{u}_1 + 7\mathbf{u}_2 - 13\mathbf{u}_3
\]

would be unambiguous in *any* of the vector spaces we have discussed in this section. And the resulting object would be another vector in the vector space. If you were tempted to call the above expression a linear combination, you would be right. Four of the definitions that were central to our discussions in Chapter V [87] were stated in the context of vectors being *column vectors*, but were purposely kept broad enough that they could be applied in the context of any vector space. They only rely on the presence of scalars, vectors, vector addition and scalar multiplication to make sense. We will restate them shortly, unchanged, except that their titles and acronyms no longer refer to column vectors, and the hypothesis of being in a vector space has been added. Take the time now to look forward and review each one, and begin to form some connections to what we have done earlier and what we will be doing in subsequent sections and chapters. Specifically, compare the following pairs of definitions:

- [Definition LCCV](97) and [Definition LC](331)
- [Definition SSCV](123) and [Definition SS](332)
- [Definition RLDCV](145) and [Definition RLD](345)
- [Definition LICV](145) and [Definition LI](345)

**Subsection READ**  
**Reading Questions**

1. Comment on how the vector space \( \mathbb{C}^m \) went from a theorem (Theorem VSPCV [91]) to an example (Example VSCV [311]).
2. In the crazy vector space, \( C \), (Example CVS [314]) compute the linear combination

\[
2(3, 4) + (-6)(1, 2).
\]
3. Suppose that $\alpha$ is a scalar and $\mathbf{0}$ is the zero vector. Why should we prove anything as obvious as $\alpha \mathbf{0} = \mathbf{0}$ such as we did in Theorem ZVSM 318?
T10  Prove each of the ten properties of Definition VS for each of the following examples of a vector space:
Example VSP
Example VSIS
Example VSF
Example VSS
Contributed by Robert Beezer

M10  Define a possibly new vector space by beginning with the set and vector addition from $\mathbb{C}^2$ (Example VSCV) but change the definition of scalar multiplication to

$$\alpha x = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \alpha \in \mathbb{C}, \ x \in \mathbb{C}^2$$

Prove that the first nine properties required for a vector space hold, but Property O does not hold.

This example shows us that we cannot expect to be able to derive Property O as a consequence of assuming the first nine properties. In other words, we cannot slim down our list of properties by jettisoning the last one, and still have the same collection of objects qualify as vector spaces.
Contributed by Robert Beezer
A subspace is a vector space that is contained within another vector space. So every subspace is a vector space in its own right, but it is also defined relative to some other (larger) vector space. We will discover shortly that we are already familiar with a wide variety of subspaces from previous sections. Here's the definition.

**Definition S Subspace**

Suppose that $V$ and $W$ are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that $W$ is a subset of $V$, $W \subseteq V$. Then $W$ is a subspace of $V$.

Let's look at an example of a vector space inside another vector space.

**Example SC3 A subspace of $\mathbb{C}^3$**

We know that $\mathbb{C}^3$ is a vector space (Example VSCV [311]). Consider the subset,

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \left| 2x_1 - 5x_2 + 7x_3 = 0 \right. \right\}$$

It is clear that $W \subseteq \mathbb{C}^3$, since the objects in $W$ are column vectors of size 3. But is $W$ a vector space? Does it satisfy the ten properties of Definition VS [309] when we use the same operations? That is the main question. Suppose $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are vectors from $W$. Then we know that these vectors cannot be totally arbitrary, they must have gained membership in $W$ by virtue of meeting the membership test. For example, we know that $x$ must satisfy $2x_1 - 5x_2 + 7x_3 = 0$ while $y$ must satisfy $2y_1 - 5y_2 + 7y_3 = 0$.

Our first property (Property AC [309]) asks the question, is $x + y \in W$? When our set of vectors was $\mathbb{C}^3$, this was an easy question to answer. Now it is not so obvious. Notice first that

$$x + y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

and we can test this vector for membership in $W$ as follows,

$$2(x_1 + y_1) - 5(x_2 + y_2) + 7(x_3 + y_3) = 2x_1 + 2y_1 - 5x_2 - 5y_2 + 7x_3 + 7y_3$$

$$= (2x_1 - 5x_2 + 7x_3) + (2y_1 - 5y_2 + 7y_3)$$

$$= 0 + 0$$

$$= 0$$

and by this computation we see that $x + y \in W$. One property down, nine to go.

If $\alpha$ is a scalar and $x \in W$, is it always true that $\alpha x \in W$? This is what we need to establish Property SC [309]. Again, the answer is not as obvious as it was when our set
of vectors was all of \( \mathbb{C}^3 \). Let’s see.

\[
\alpha x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}
\]

and we can test this vector for membership in \( W \) with

\[
2(\alpha x_1) - 5(\alpha x_2) + 7(\alpha x_3) = \alpha(2x_1 - 5x_2 + 7x_3) = \alpha 0 = 0
\]

\( x \in W \)

and we see that indeed \( \alpha x \in W \). Always.

If \( W \) has a zero vector, it will be unique (Theorem ZVU \[317\]). The zero vector for \( \mathbb{C}^3 \) should also perform the required duties when added to elements of \( W \). So the likely candidate for a zero vector in \( W \) is the same zero vector that we know \( \mathbb{C}^3 \) has. You can check that \( 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \) is a zero vector in \( W \) too (Property Z \[310\]).

With a zero vector, we can now ask about additive inverses (Property AI \[310\]). As you might suspect, the natural candidate for an additive inverse in \( W \) is the same as the additive inverse from \( \mathbb{C}^3 \). However, we must insure that these additive inverses actually are elements of \( W \). Given \( x \in W \), is \( -x \in W \)?

\[
-x = \begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \end{bmatrix}
\]

and we can test this vector for membership in \( W \) with

\[
2(-x_1) - 5(-x_2) + 7(-x_3) = -(2x_1 - 5x_2 + 7x_3) = 0
\]

\( x \in W \)

and we now believe that \( -x \in W \).

Is the vector addition in \( W \) commutative (Property C \[310\])? Is \( x + y = y + x \)? Of course! Nothing about restricting the scope of our set of vectors will prevent the operation from still being commutative. Indeed, the remaining five properties are unaffected by the transition to a smaller set of vectors, and so remain true. That was convenient.

So \( W \) satisfies all ten properties, is therefore a vector space, and thus earns the title of being a subspace of \( \mathbb{C}^3 \).

\[\boxplus\]

**Subsection TS**

**Testing Subspaces**

In Example SC3 \[325\] we proceeded through all ten of the vector space properties before believing that a subset was a subspace. But six of the properties were easy to prove,
and we can lean on some of the properties of the vector space (the superset) to make the
other four easier. Here is a theorem that will make it easier to test if a subset is a vector
space. A shortcut if there ever was one.

**Theorem TSS**

**Testing Subsets for Subspaces**

Suppose that $V$ is a vector space and $W$ is a subset of $V$, $W \subseteq V$. Endow $W$ with
the same operations as $V$. Then $W$ is a subspace if and only if three conditions are met

1. $W$ is non-empty, $W \neq \emptyset$.
2. If $x \in W$ and $y \in W$, then $x + y \in W$.
3. If $\alpha \in \mathbb{C}$ and $x \in W$, then $\alpha x \in W$.

□

**Proof**  $(\Rightarrow)$ We have the hypothesis that $W$ is a subspace, so by [Definition VS](#) we
know that $W$ contains a zero vector. This is enough to show that $W \neq \emptyset$. Also, since
$W$ is a vector space it satisfies the additive and scalar multiplication closure properties,
and so exactly meets the second and third conditions. If that was easy, the other
direction might require a bit more work.

$(\Leftarrow)$ We have three properties for our hypothesis, and from this we should conclude
that $W$ has the ten defining properties of a vector space. The second and third con-
ditions of our hypothesis are exactly [Property AC](#) and [Property SC](#). Our
hypothesis that $V$ is a vector space implies that [Property C](#), [Property AA](#),
[Property SMA](#), [Property DVA](#), [Property DSA](#) and [Property O](#) all
hold. They continue to be true for vectors from $W$ since passing to a subset, and keeping
the operation the same, leaves their statements unchanged. Eight down, two to go.

Suppose $x \in W$. Then by the third part of our hypothesis (scalar closure), we know
that $(-1)x \in W$. By [Theorem AISM](#) $(-1)x = -x$, so together these statements
show us that $-x \in W$. $-x$ is the additive inverse of $x$ in $V$, but will continue in this role
when viewed as element of the subset $W$. So every element of $W$ has an additive inverse
that is an element of $W$ and [Property AI](#) is completed. Just one property left.

While we have implicitly discussed the zero vector in the previous paragraph, we need
to be certain that the zero vector (of $V$) really lives in $W$. Since $W$ is non-empty, we can
choose some vector $z \in W$. Then by the argument in the previous paragraph, we know
$-z \in W$. Now by [Property AI](#) for $V$ and then by the second part of our hypothesis
(additive closure) we see that

$$0 = z + (-z) \in W$$

So $W$ contain the zero vector from $V$. Since this vector performs the required duties
of a zero vector in $V$, it will continue in that role as an element of $W$. This gives us,
[Property Z](#), the final property of the ten required. (Sarah Fellez contributed to this
proof.)

Three conditions, plus being a subset of a known vector space, gets us all ten prop-
e rties. Fabulous!

This theorem can be paraphrased by saying that a subspace is “a non-empty subset
(of a vector space) that is closed under vector addition and scalar multiplication.”

You might want to go back and rework [Example SC3](#) in light of this result,
perhaps seeing where we can now economize or where the work done in the example
mirrored the proof and where it did not. We will press on and apply this theorem in a slightly more abstract setting.

**Example SP4**

A **subspace** of \( P_4 \)

\( P_4 \) is the vector space of polynomials with degree at most 4 (Example VSP [312]). Define a subset \( W \) as

\[
W = \{ p(x) \mid p \in P_4, \ p(2) = 0 \}
\]

so \( W \) is the collection of those polynomials (with degree 4 or less) whose graphs cross the \( x \)-axis at \( x = 2 \). Whenever we encounter a new set it is a good idea to gain a better understanding of the set by finding a few elements in the set, and a few outside it. For example \( x^2 - x - 2 \in W \), while \( x^4 + x^3 - 7 \not\in W \).

Is \( W \) nonempty? Yes, \( x - 2 \in W \).

Additive closure? Suppose \( p \in W \) and \( q \in W \). Is \( p + q \in W \)? \( p \) and \( q \) are not totally arbitrary, we know that \( p(2) = 0 \) and \( q(2) = 0 \). Then we can check \( p + q \) for membership in \( W \),

\[
(p + q)(2) = p(2) + q(2) = 0 + 0 = 0
\]

so we see that \( p + q \) qualifies for membership in \( W \).

Scalar multiplication closure? Suppose that \( \alpha \in \mathbb{C} \) and \( p \in W \). Then we know that \( p(2) = 0 \). Testing \( \alpha p \) for membership,

\[
(\alpha p)(2) = \alpha p(2) = \alpha 0 = 0
\]

so \( \alpha p \in W \).

We have shown that \( W \) meets the three conditions of Theorem TSS [327] and so qualifies as a subspace of \( P_4 \). Notice that by Definition S [325] we now know that \( W \) is also a vector space. So all the properties of a vector space (Definition VS [309]) and the theorems of Section VS [309] apply in full.

 Much of the power of Theorem TSS [327] is that we can easily establish new vector spaces if we can locate them as subsets of other vector spaces, such as the ones presented in Subsection VS.EVS [311].

It can be as instructive to consider some subsets that are *not* subspaces. Since Theorem TSS [327] is an equivalence (see Technique E [704]) we can be assured that a subset is not a subspace if it violates one of the three conditions, and in any example of interest this will not be the “non-empty” condition. However, since a subspace has to be a vector space in its own right, we can also search for a violation of any one of the ten defining properties in Definition VS [309] or any inherent property of a vector space, such as those given by the basic theorems of Subsection VS.VSP [316]. Notice also that a violation need only be for a specific vector or pair of vectors.

**Example NSC2Z**

A non-subspace in \( \mathbb{C}^2 \), zero vector
Consider the subset \( W \) below as a candidate for being a subspace of \( \mathbb{C}^2 \)

\[
W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid 3x_1 - 5x_2 = 12 \right\}
\]

The zero vector of \( \mathbb{C}^2 \), \( \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) will need to be the zero vector in \( W \) also. However, \( \mathbf{0} \notin W \) since \( 3(0) - 5(0) = 0 \neq 12 \). So \( W \) has no zero vector and fails Property Z of Definition VS. This subspace also fails to be closed under addition and scalar multiplication. Can you find examples of this? ☒

**Example NSC2A**

**A non-subspace in \( \mathbb{C}^2 \), additive closure**

Consider the subset \( X \) below as a candidate for being a subspace of \( \mathbb{C}^2 \)

\[
X = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1x_2 = 0 \right\}
\]

You can check that \( \mathbf{0} \in X \), so the approach of the last example will not get us anywhere. However, notice that \( \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in X \) and \( \mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in X \). Yet

\[
\mathbf{x + y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin X
\]

So \( X \) fails the additive closure requirement of either Property AC or Theorem TSS, and is therefore not a subspace.

**Example NSC2S**

**A non-subspace in \( \mathbb{C}^2 \), scalar multiplication closure**

Consider the subset \( Y \) below as a candidate for being a subspace of \( \mathbb{C}^2 \)

\[
Y = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1 \in \mathbb{Z}, \ x_2 \in \mathbb{Z} \right\}
\]

\( \mathbb{Z} \) is the set of integers, so we are only allowing “whole numbers” as the constituents of our vectors. Now, \( \mathbf{0} \in Y \), and additive closure also holds (can you prove these claims?). So we will have to try something different. Note that \( \alpha = \frac{1}{2} \in \mathbb{C} \) and \( \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in Y \), but

\[
\alpha \mathbf{x} = \frac{1}{2} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} \notin Y
\]

So \( Y \) fails the scalar multiplication closure requirement of either Property SC or Theorem TSS, and is therefore not a subspace.

There are two examples of subspaces that are trivial. Suppose that \( V \) is any vector space. Then \( V \) is a subset of itself and is a vector space. By Definition S, \( V \) qualifies as a subspace of itself. The set containing just the zero vector \( Z = \{ \mathbf{0} \} \) is also a subspace as can be seen by applying Theorem TSS or by simple modifications of the techniques hinted at in Example VSS. Since these subspaces are so obvious (and therefore not too interesting) we will refer to them as being trivial.
Definition TS

Trivial Subspaces

Given the vector space $V$, the subspaces $V$ and $\{0\}$ are each called a trivial subspace.

We can also use Theorem TSS to prove more general statements about subspaces, as illustrated in the next theorem.

Theorem NSMS

Null Space of a Matrix is a Subspace

Suppose that $A$ is an $m \times n$ matrix. Then the null space of $A$, $\mathcal{N}(A)$, is a subspace of $\mathbb{C}^n$. □

Proof

We will examine the three requirements of Theorem TSS. Recall that $\mathcal{N}(A) = \{ x \in \mathbb{C}^n \mid Ax = 0 \}$.

First, $0 \in \mathcal{N}(A)$, which can be inferred as a consequence of Theorem HSC. So $\mathcal{N}(A) \neq \emptyset$.

Second, check additive closure by supposing that $x \in \mathcal{N}(A)$ and $y \in \mathcal{N}(A)$. So we know a little something about $x$ and $y$: $Ax = 0$ and $Ay = 0$, and that is all we know. Question: Is $x + y \in \mathcal{N}(A)$? Let’s check.

$$A(x + y) = Ax + Ay$$

Theorem MMDAA

$$= 0 + 0$$

$x \in \mathcal{N}(A)$, $y \in \mathcal{N}(A)$

Theorem VSPCV

$$= 0$$

So, yes, $x + y$ qualifies for membership in $\mathcal{N}(A)$.

Third, check scalar multiplication closure by supposing that $\alpha \in \mathbb{C}$ and $x \in \mathcal{N}(A)$. So we know a little something about $x$: $Ax = 0$, and that is all we know. Question: Is $\alpha x \in \mathcal{N}(A)$? Let’s check.

$$A(\alpha x) = \alpha(Ax)$$

Theorem MMSMM

$$= \alpha 0$$

$x \in \mathcal{N}(A)$

Theorem ZVSM

$$= 0$$

So, yes, $\alpha x$ qualifies for membership in $\mathcal{N}(A)$.

Having met the three conditions in Theorem TSS we can now say that the null space of a matrix is a subspace (and hence a vector space in its own right!). □

Here is an example where we can exercise Theorem NSMS.

Example RSNS

Recasting a subspace as a null space

Consider the subset of $\mathbb{C}^5$ defined as

$$W = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \begin{array}{l} 3x_1 + x_2 - 5x_3 + 7x_4 + x_5 = 0, \\ 4x_1 + 6x_2 + 3x_3 - 6x_4 - 5x_5 = 0, \\ -2x_1 + 4x_2 + 7x_4 + x_5 = 0 \end{array}$$

It is possible to show that $W$ is a subspace of $\mathbb{C}^5$ by checking the three conditions of Theorem TSS directly, but it will get tedious rather quickly. Instead, give $W$ a
Subsection S.TSS  The Span of a Set 331

fresh look and notice that it is a set of solutions to a homogeneous system of equations. Define the matrix

\[ A = \begin{bmatrix}
3 & 1 & -5 & 7 & 1 \\
4 & 6 & 3 & -6 & -5 \\
-2 & 4 & 0 & 7 & 1
\end{bmatrix} \]

and then recognize that \( W = \mathcal{N}(A) \). By Theorem NSMS \(^{330}\) we can immediately see that \( W \) is a subspace. Boom!

Subsection TSS  The Span of a Set

The span of a set of column vectors got a heavy workout in Chapter V \(^{87}\) and Chapter M \(^{197}\). The definition of the span depended only on being able to formulate linear combinations. In any of our more general vector spaces we always have a definition of vector addition and of scalar multiplication. So we can build linear combinations and manufacture spans. This subsection contains two definitions that are just mild variants of definitions we have seen earlier for column vectors. If you haven’t already, compare them with Definition LCCV \(^{97}\) and Definition SSCV \(^{123}\).

Definition LC  Linear Combination

Suppose that \( V \) is a vector space. Given \( n \) vectors \( u_1, u_2, u_3, \ldots, u_n \) and \( n \) scalars \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \), their linear combination is the vector

\[ \alpha_1u_1 + \alpha_2u_2 + \alpha_3u_3 + \cdots + \alpha_nu_n. \]

Example LCM  A linear combination of matrices

In the vector space \( M_{23} \) of \( 2 \times 3 \) matrices, we have the vectors

\[ x = \begin{bmatrix}
1 & 3 & -2 \\
2 & 0 & 7
\end{bmatrix} \quad y = \begin{bmatrix}
3 & -1 & 2 \\
5 & 5 & 1
\end{bmatrix} \quad z = \begin{bmatrix}
4 & 2 & -4 \\
1 & 1 & 1
\end{bmatrix} \]

and we can form linear combinations such as

\[ 2x + 4y + (-1)z = 2 \begin{bmatrix}
1 & 3 & -2 \\
2 & 0 & 7
\end{bmatrix} + 4 \begin{bmatrix}
3 & -1 & 2 \\
5 & 5 & 1
\end{bmatrix} + (-1) \begin{bmatrix}
4 & 2 & -4 \\
1 & 1 & 1
\end{bmatrix} \]

\[ = \begin{bmatrix}
2 & 6 & -4 \\
4 & 0 & 14
\end{bmatrix} + \begin{bmatrix}
12 & -4 & 8 \\
20 & 20 & 4
\end{bmatrix} + \begin{bmatrix}
-4 & -2 & 4 \\
-1 & -1 & -1
\end{bmatrix} \]

\[ = \begin{bmatrix}
10 & 0 & 8 \\
23 & 19 & 17
\end{bmatrix} \]

or,

\[ 4x - 2y + 3z = 4 \begin{bmatrix}
1 & 3 & -2 \\
2 & 0 & 7
\end{bmatrix} - 2 \begin{bmatrix}
3 & -1 & 2 \\
5 & 5 & 1
\end{bmatrix} + 3 \begin{bmatrix}
4 & 2 & -4 \\
1 & 1 & 1
\end{bmatrix} \]

Version 0.92
When we realize that we can form linear combinations in any vector space, then it is natural to revisit our definition of the span of a set, since it is the set of all possible linear combinations of a set of vectors.

**Definition SS**

**Span of a Set**

Suppose that $V$ is a vector space. Given a set of vectors $S = \{ u_1, u_2, u_3, \ldots, u_t \}$, their span, $\langle S \rangle$, is the set of all possible linear combinations of $u_1, u_2, u_3, \ldots, u_t$. Symbolically,

$$\langle S \rangle = \left\{ \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_t u_t \mid \alpha_i \in \mathbb{C}, 1 \leq i \leq t \right\}$$

**Theorem SSS**

**Span of a Set is a Subspace**

Suppose $V$ is a vector space. Given a set of vectors $S = \{ u_1, u_2, u_3, \ldots, u_t \} \subseteq V$, their span, $\langle S \rangle$, is a subspace. □

**Proof** We will verify the three conditions of [Theorem TSS](#). First,

$$0 = 0 + 0 + 0 + \cdots + 0$$

$$= 0 u_1 + 0 u_2 + 0 u_3 + \cdots + 0 u_t$$

Property [Z](#) for $V$

Theorem [ZSSM](#)

So we have written $0$ as a linear combination of the vectors in $S$ and by [Definition SS](#), $0 \in \langle S \rangle$ and therefore $S \neq \emptyset$.

Second, suppose $x \in \langle S \rangle$ and $y \in \langle S \rangle$. Can we conclude that $x + y \in \langle S \rangle$? What do we know about $x$ and $y$ by virtue of their membership in $\langle S \rangle$? There must be scalars from $\mathbb{C}$, $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_t$ and $\beta_1, \beta_2, \beta_3, \ldots, \beta_t$ so that

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_t u_t$$

$$y = \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \cdots + \beta_t u_t$$

Then

$$x + y = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_t u_t$$

$$+ \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \cdots + \beta_t u_t$$

$$= (\alpha_1 + \beta_1) u_1 + (\alpha_2 + \beta_2) u_2$$

$$+ (\alpha_3 + \beta_3) u_3 + \cdots + (\alpha_t + \beta_t) u_t$$

Property [AA](#), Property [C](#)
Since each $\alpha_i + \beta_i$ is again a scalar from $\mathbb{C}$ we have expressed the vector sum $x + y$ as a linear combination of the vectors from $S$, and therefore by Definition SS [332] we can say that $x + y \in \langle S \rangle$.

Third, suppose $\alpha \in \mathbb{C}$ and $x \in \langle S \rangle$. Can we conclude that $\alpha x \in \langle S \rangle$? What do we know about $x$ by virtue of its membership in $\langle S \rangle$? There must be scalars from $\mathbb{C}$, $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_t$ so that

$$x = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_t u_t$$

Then

$$\alpha x = \alpha (\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_t u_t)$$

$$= \alpha (\alpha_1 u_1) + \alpha (\alpha_2 u_2) + \alpha (\alpha_3 u_3) + \cdots + \alpha (\alpha_t u_t) \quad \text{(Property DVA 310)}$$

$$= (\alpha \alpha_1) u_1 + (\alpha \alpha_2) u_2 + (\alpha \alpha_3) u_3 + \cdots + (\alpha \alpha_t) u_t \quad \text{(Property SMA 310)}$$

Since each $\alpha \alpha_i$ is again a scalar from $\mathbb{C}$ we have expressed the scalar multiple $\alpha x$ as a linear combination of the vectors from $S$, and therefore by Definition SS [332] we can say that $\alpha x \in \langle S \rangle$.

With the three conditions of Theorem TSS [327] met, we can say that $\langle S \rangle$ is a subspace (and so is also vector space, Definition VS [309]). (See Exercise SS.T20 [137], Exercise SS.T21 [137], Exercise SS.T22 [137].)

**Example SSP**

**Span of a set of polynomials**

In Example SP4 [328] we proved that

$$W = \{ p(x) \mid p \in P_4, \ p(2) = 0 \}$$

is a subspace of $P_4$, the vector space of polynomials of degree at most 4. Since $W$ is a vector space itself, let’s construct a span within $W$. First let

$$S = \{ x^4 - 4x^3 + 5x^2 - x - 2, \ 2x^4 - 3x^3 - 6x^2 + 6x + 4 \}$$

and verify that $S$ is a subset of $W$ by checking that each of these two polynomials has $x = 2$ as a root. Now, if we define $U = \langle S \rangle$, then Theorem SSS [332] tells us that $U$ is a subspace of $W$. So quite quickly we have built a chain of subspaces, $U$ inside $W$, and $W$ inside $P_4$.

Rather than dwell on how quickly we can build subspaces, let’s try to gain a better understanding of just how the span construction creates subspaces, in the context of this example. We can quickly build representative elements of $U$,

$$3(x^4 - 4x^3 + 5x^2 - x - 2) + 5(2x^4 - 3x^3 - 6x^2 + 6x + 4) = 13x^4 - 27x^3 - 15x^2 + 27x + 14$$

and

$$(-2)(x^4 - 4x^3 + 5x^2 - x - 2) + 8(2x^4 - 3x^3 - 6x^2 + 6x + 4) = 14x^4 - 16x^3 - 58x^2 + 50x + 36$$

and each of these polynomials must be in $W$ since it is closed under addition and scalar multiplication. But you might check for yourself that both of these polynomials have $x = 2$ as a root.
I can tell you that \( y = 3x^4 - 7x^3 - x^2 + 7x - 2 \) is not in \( U \), but would you believe me? A first check shows that \( y \) does have \( x = 2 \) as a root, but that only shows that \( y \in W \). What does \( y \) have to do to gain membership in \( U = \langle S \rangle \)? It must be a linear combination of the vectors in \( S \), \( x^4 - 4x^3 + 5x^2 - x - 2 \) and \( 2x^4 - 3x^3 - 6x^2 + 6x + 4 \). So let’s suppose that \( y \) is such a linear combination,

\[
y = 3x^4 - 7x^3 - x^2 + 7x - 2 = \alpha_1(x^4 - 4x^3 + 5x^2 - x - 2) + \alpha_2(2x^4 - 3x^3 - 6x^2 + 6x + 4)
\]

\( y \) has to do to gain membership in \( U = \langle S \rangle \)? It must be a linear combination of the vectors in \( S \), \( x^4 - 4x^3 + 5x^2 - x - 2 \) and \( 2x^4 - 3x^3 - 6x^2 + 6x + 4 \). So let’s suppose that \( y \) is such a linear combination,

\[
y = \alpha_1(x^4 - 4x^3 + 5x^2 - x - 2) + \alpha_2(2x^4 - 3x^3 - 6x^2 + 6x + 4)
\]

Notice that operations above are done in accordance with the definition of the vector space of polynomials (Example VSP [312]). Now, if we equate coefficients, which is the definition of equality for polynomials, then we obtain the system of five linear equations in two variables

\[
\begin{align*}
\alpha_1 + 2\alpha_2 &= 3 \\
-4\alpha_1 - 3\alpha_2 &= -7 \\
5\alpha_1 - 6\alpha_2 &= -1 \\
-\alpha_1 + 6\alpha_2 &= 7 \\
-2\alpha_1 + 4\alpha_2 &= -2
\end{align*}
\]

Build an augmented matrix from the system and row-reduce,

\[
\begin{bmatrix}
1 & 2 & 3 \\
-4 & -3 & -7 \\
5 & -6 & -1 \\
-1 & 6 & 7 \\
-2 & 4 & -2
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

With a leading 1 in the final column of the row-reduced augmented matrix, Theorem RCLS [54] tells us the system of equations is inconsistent. Therefore, there are no scalars, \( \alpha_1 \) and \( \alpha_2 \), to establish \( y \) as a linear combination of the elements in \( U \). So \( y \notin U \).

Let’s again examine membership in a span.

**Example SM32**

**A subspace of \( M_{32} \)**

The set of all \( 3 \times 2 \) matrices forms a vector space when we use the operations of matrix addition (Definition MA [198]) and scalar matrix multiplication (Definition MSM [198]), as was shown in Example VSM [311]. Consider the subset

\[
S = \left\{ \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 14 & -2 \\ -19 & 11 \end{bmatrix} \right\}
\]

and define a new subset of vectors \( W \) in \( M_{32} \) using the span (Definition SS [332]), \( W = \langle S \rangle \). So by Theorem SSS [332] we know that \( W \) is a subspace of \( M_{32} \). While \( W \) is an infinite set, and this is a precise description, it would still be worthwhile to investigate whether or not \( W \) contains certain elements.
First, is

\[ y = \begin{bmatrix} 9 & 3 \\ 7 & 3 \\ 10 & -11 \end{bmatrix} \]

in \( W \)? To answer this, we want to determine if \( y \) can be written as a linear combination of the five matrices in \( S \). Can we find scalars, \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \) so that

\[
\begin{bmatrix} 9 & 3 \\ 7 & 3 \\ 10 & -11 \end{bmatrix} = \alpha_1 \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & -5 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 14 & -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 & -1 \\ 1 & -2 \\ 14 & -2 \end{bmatrix} + \alpha_4 \begin{bmatrix} 4 & 2 \\ -19 & -11 \end{bmatrix} + \alpha_5 \begin{bmatrix} 3 & 1 \\ -17 & 7 \end{bmatrix}
\]

Using our definition of matrix equality (Definition ME [197]) we can translate this statement into six equations in the five unknowns,

\[
\begin{align*}
3\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 &= 9 \\
\alpha_1 + \alpha_2 - \alpha_3 + 2\alpha_4 + \alpha_5 &= 3 \\
4\alpha_1 + 2\alpha_2 - \alpha_3 + \alpha_4 - 4\alpha_5 &= 7 \\
2\alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4 &= 3 \\
5\alpha_1 + 14\alpha_2 - 19\alpha_3 + 14\alpha_4 - 17\alpha_5 &= 10 \\
5\alpha_1 - \alpha_2 - 11\alpha_3 - 2\alpha_4 + 7\alpha_5 &= -11
\end{align*}
\]

This is a linear system of equations, which we can represent with an augmented matrix and row-reduce in search of solutions. The matrix that is row-equivalent to the augmented matrix is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & \frac{5}{2} \\
0 & 1 & 0 & 0 & -\frac{7}{2} \\
0 & 0 & 1 & 0 & -\frac{7}{2} \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

So we recognize that the system is consistent since there is no leading 1 in the final column (Theorem RCLS [54]), and compute \( n-r = 5-4 = 1 \) free variables (Theorem FVCS [56]). While there are infinitely many solutions, we are only in pursuit of a single solution, so let’s choose the free variable \( \alpha_5 = 0 \) for simplicity’s sake. Then we easily see that \( \alpha_1 = 2, \alpha_2 = -1, \alpha_3 = 0, \alpha_4 = 1 \). So the scalars \( \alpha_1 = 2, \alpha_2 = -1, \alpha_3 = 0, \alpha_4 = 1, \alpha_5 = 0 \) will provide a linear combination of the elements of \( S \) that equals \( y \), as we can verify by checking,

\[
\begin{bmatrix} 9 & 3 \\ 7 & 3 \\ 10 & -11 \end{bmatrix} = 2 \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & -5 \end{bmatrix} + (-1) \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 14 & -1 \end{bmatrix} + (1) \begin{bmatrix} 4 & 2 \\ 1 & -2 \\ 14 & -2 \end{bmatrix}
\]

So with one particular linear combination in hand, we are convinced that \( y \) deserves to be a member of \( W = \langle S \rangle \). Second, is

\[
x = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 4 & -2 \end{bmatrix}
\]
in \( W \)? To answer this, we want to determine if \( x \) can be written as a linear combination of the five matrices in \( S \). Can we find scalars, \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \) so that

\[
\begin{bmatrix}
2 & 1 \\
3 & 1 \\
4 & -2
\end{bmatrix} = \alpha_1 \begin{bmatrix}
3 & 1 \\
4 & 2 \\
5 & -5
\end{bmatrix} + \alpha_2 \begin{bmatrix}
1 & 1 \\
2 & -1 \\
14 & -1
\end{bmatrix} + \alpha_3 \begin{bmatrix}
3 & -1 \\
-1 & 2 \\
-19 & -11
\end{bmatrix} + \alpha_4 \begin{bmatrix}
4 & 2 \\
1 & -2 \\
14 & -2
\end{bmatrix} + \alpha_5 \begin{bmatrix}
3 & 1 \\
-4 & 0 \\
-17 & 7
\end{bmatrix}
\]

Using our definition of matrix equality (Definition ME [197]), we can translate this statement into six equations in the five unknowns,

\[
\begin{align*}
3\alpha_1 + \alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 &= 2 \\
\alpha_1 + \alpha_2 - \alpha_3 + 2\alpha_4 + \alpha_5 &= 1 \\
4\alpha_1 + 2\alpha_2 - \alpha_3 + \alpha_4 - 4\alpha_5 &= 3 \\
2\alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4 &= 1 \\
5\alpha_1 + 14\alpha_2 - 19\alpha_3 + 14\alpha_4 - 17\alpha_5 &= 4 \\
-5\alpha_1 - \alpha_2 - 11\alpha_3 - 2\alpha_4 + 7\alpha_5 &= -2
\end{align*}
\]

This is a linear system of equations, which we can represent with an augmented matrix and row-reduce in search of solutions. The matrix that is row-equivalent to the augmented matrix is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 5 \\
0 & 1 & 0 & 0 & -\frac{38}{5} & 0 \\
0 & 0 & 1 & 0 & -\frac{17}{5} & 0 \\
0 & 0 & 0 & 1 & -\frac{7}{5} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

With a leading 1 in the last column, Theorem RCLS [54] tells us that the system is inconsistent. Therefore, there are no values for the scalars that will place \( x \) in \( W \), and so we conclude that \( x \not\in W \). ❇️

Notice how Example SSP [333] and Example SM32 [334] contained questions about membership in a span, but these questions quickly became questions about solutions to a system of linear equations. This will be a common theme going forward.

### Subsection SC

#### Subspace Constructions

Several of the subsets of vectors spaces that we worked with in Chapter M [197] are also subspaces — they are closed under vector addition and scalar multiplication in \( \mathbb{C}^m \).

**Theorem CSMS**

**Column Space of a Matrix is a Subspace**

Suppose that \( A \) is an \( m \times n \) matrix. Then \( \mathcal{C}(A) \) is a subspace of \( \mathbb{C}^m \). □
Proof. Definition CSM [261] shows us that $\mathcal{C}(A)$ is a subset of $\mathbb{C}^m$, and that it is defined as the span of a set of vectors from $\mathbb{C}^m$ (the columns of the matrix). Since $\mathcal{C}(A)$ is a span, Theorem SSS [332] says it is a subspace.

That was easy! Notice that we could have used this same approach to prove that the null space is a subspace, since Theorem SSNS [129] provided a description of the null space of a matrix as the span of a set of vectors. However, I much prefer the current proof of Theorem NSMS [330]. Speaking of easy, here is a very easy theorem that exposes another of our constructions as creating subspaces.

Theorem RSMS
Row Space of a Matrix is a Subspace
Suppose that $A$ is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of $\mathbb{C}^n$. □

Proof. Definition RSM [268] says $\mathcal{R}(A) = \mathcal{C}(A^t)$, so the row space of a matrix is a column space, and every column space is a subspace by Theorem CSMS [336]. That’s enough. ■

One more.

Theorem LNSMS
Left Null Space of a Matrix is a Subspace
Suppose that $A$ is an $m \times n$ matrix. Then $\mathcal{L}(A)$ is a subspace of $\mathbb{C}^m$. □

Proof. Definition LNS [283] says $\mathcal{L}(A) = \mathcal{N}(A^t)$, so the left null space is a null space, and every null space is a subspace by Theorem NSMS [330]. Done. ■

So the span of a set of vectors, and the null space, column space, row space and left null space of a matrix are all subspaces, and hence are all vector spaces, meaning they have all the properties detailed in Definition VS [309] and in the basic theorems presented in Section VS [309]. We have worked with these objects as just sets in Chapter V [87] and Chapter M [197], but now we understand that they have much more structure. In particular, being closed under vector addition and scalar multiplication means a subspace is also closed under linear combinations.

Subsection READ
Reading Questions

1. Summarize the three conditions that allow us to quickly test if a set is a subspace.

2. Consider the set of vectors

$$\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid 3a - 2b + c = 5 \right\}$$

Is this set a subspace of $\mathbb{C}^3$? Explain your answer.

3. Name five general constructions of sets of column vectors (subsets of $\mathbb{C}^m$) that we now know as subspaces.
Subsection EXC
Exercises

C20  Working within the vector space $P_3$ of polynomials of degree 3 or less, determine if $p(x) = x^3 + 6x + 4$ is in the subspace $W$ below.

$$W = \langle \{ x^3 + x^2 + x, x^3 + 2x - 6, x^2 - 5 \} \rangle$$

Contributed by Robert Beezer  Solution 341

C21  Consider the subspace

$$W = \langle \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} \rangle$$

of the vector space of $2 \times 2$ matrices, $M_{22}$. Is $C = \begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix}$ an element of $W$?

Contributed by Robert Beezer  Solution 341

C25  Show that the set $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid 3x_1 - 5x_2 = 12 \right\}$ from Example NSC2Z 329 fails Property AC 309 and Property SC 309.

Contributed by Robert Beezer

C26  Show that the set $Y = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1 \in \mathbb{Z}, x_2 \in \mathbb{Z} \right\}$ from Example NSC2S 329 has Property AC 309.

Contributed by Robert Beezer

M20  In $\mathbb{C}^3$, the vector space of column vectors of size 3, prove that the set $Z$ is a subspace.

$$Z = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mid 4x_1 - x_2 + 5x_3 = 0 \right\}$$

Contributed by Robert Beezer  Solution 342

T20  A square matrix $A$ of size $n$ is upper-triangular if $[A]_{ij} = 0$ whenever $i > j$. Let $UT_n$ be the set of all upper-triangular matrices of size $n$. Prove that $UT_n$ is a subspace of the vector space of all square matrices of size $n$, $M_{nn}$.

Contributed by Robert Beezer  Solution 343
The question is if \( p \) can be written as a linear combination of the vectors in \( W \). To check this, we set \( p \) equal to a linear combination and massage with the definitions of vector addition and scalar multiplication that we get with \( P_3 \) (Example VSP [312]).

\[
p(x) = a_1(x^3 + x^2 + x) + a_2(x^3 + 2x - 6) + a_3(x^2 - 5)
\]

\[
x^3 + 6x + 4 = (a_1 + a_2)x^3 + (a_1 + a_3)x^2 + (a_1 + 2a_2)x + (-6a_2 - 5a_3)
\]

Equating coefficients of equal powers of \( x \), we get the system of equations,

\[
\begin{align*}
   a_1 + a_2 &= 1 \\
   a_1 + a_3 &= 0 \\
   a_1 + 2a_2 &= 6 \\
   -6a_2 - 5a_3 &= 4
\end{align*}
\]

The augmented matrix of this system of equations row-reduces to

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

There is a leading 1 in the last column, so Theorem RCLS [54] implies that the system is inconsistent. So there is no way for \( p \) to gain membership in \( W \), so \( p \notin W \).

In order to belong to \( W \), we must be able to express \( C \) as a linear combination of the elements in the spanning set of \( W \). So we begin with such an expression, using the unknowns \( a, b, c \) for the scalars in the linear combination.

\[
C = \begin{bmatrix}
-3 & 3 \\
6 & -4
\end{bmatrix}
= a \begin{bmatrix}
2 & 1 \\
3 & -1
\end{bmatrix}
+ b \begin{bmatrix}
4 & 0 \\
2 & 3
\end{bmatrix}
+ c \begin{bmatrix}
-3 & 1 \\
2 & 1
\end{bmatrix}
\]

Massaging the right-hand side, according to the definition of the vector space operations in \( M_{22} \) (Example VSM [311]), we find the matrix equality,

\[
\begin{bmatrix}
-3 & 3 \\
6 & -4
\end{bmatrix}
= \begin{bmatrix}
2a + 4b - 3c & a + c \\
3a + 2b + 2c & -a + 3b + c
\end{bmatrix}
\]

Matrix equality allows us to form a system of four equations in three variables, whose augmented matrix row-reduces as follows,

\[
\begin{bmatrix}
2 & 4 & -3 & -3 \\
1 & 0 & 1 & 3 \\
3 & 2 & 2 & 6 \\
-1 & 3 & 1 & -4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
Since this system of equations is consistent (Theorem RCLS 54), a solution will provide values for \( a, b \) and \( c \) that allow us to recognize \( C \) as an element of \( W \).

M20 Contributed by Robert Beezer Statement 339

The membership criteria for \( Z \) is a single linear equation, which comprises a homogeneous system of equations. As such, we can recognize \( Z \) as the solutions to this system, and therefore \( Z \) is a null space. Specifically, \( Z = N(\begin{bmatrix} 4 & -1 & 5 \end{bmatrix}) \). Every null space is a subspace by Theorem NSMS 330.

A less direct solution appeals to Theorem TSS 327.

First, we want to be certain \( Z \) is non-empty. The zero vector of \( C^3, 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \), is a good candidate, since if it fails to be in \( Z \), we will know that \( Z \) is not a vector space. Check that 
\[
4(0) - (0) + 5(0) = 0
\]
so that \( 0 \in Z \).

Suppose \( \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \) and \( \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \) are vectors from \( Z \). Then we know that these vectors cannot be totally arbitrary, they must have gained membership in \( Z \) by virtue of meeting the membership test. For example, we know that \( \mathbf{x} \) must satisfy \( 4x_1 - x_2 + 5x_3 = 0 \) while \( \mathbf{y} \) must satisfy \( 4y_1 - y_2 + 5y_3 = 0 \). Our second criteria asks the question, is \( \mathbf{x} + \mathbf{y} \in Z \)? Notice first that
\[
\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}
\]
and we can test this vector for membership in \( Z \) as follows,
\[
4(x_1 + y_1) - 1(x_2 + y_2) + 5(x_3 + y_3) \\
= 4x_1 + 4y_1 - x_2 - y_2 + 5x_3 + 5y_3 \\
= (4x_1 - x_2 + 5x_3) + (4y_1 - y_2 + 5y_3) \\
= 0 + 0 \\
= 0
\]
and by this computation we see that \( \mathbf{x} + \mathbf{y} \in Z \).

If \( \alpha \) is a scalar and \( \mathbf{x} \in Z \), is it always true that \( \alpha \mathbf{x} \in Z \)? To check our third criteria, we examine
\[
\alpha \mathbf{x} = \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}
\]
and we can test this vector for membership in \( Z \) with
\[
4(\alpha x_1) - (\alpha x_2) + 5(\alpha x_3) \\
= \alpha(4x_1 - x_2 + 5x_3) \\
= \alpha 0 \\
= 0
\]
and we see that indeed \( \alpha \mathbf{x} \in Z \). With the three conditions of Theorem TSS 327 fulfilled, we can conclude that \( Z \) is a subspace of \( C^3 \).
First, the zero vector of $M_{nn}$ is the zero matrix, $\mathcal{O}$, whose entries are all zero (Definition ZM [200]). This matrix then meets the condition that $[\mathcal{O}]_{ij} = 0$ for $i > j$ and so is an element of $UT_n$.

Suppose $A, B \in UT_n$. Is $A + B \in UT_n$? We examine the entries of $A + B$ “below” the diagonal. That is, in the following, assume that $i > j$.

\[
[A + B]_{ij} = [A]_{ij} + [B]_{ij} \quad \text{Definition MA [198]}
\]
\[
= 0 + 0 \quad A, B \in UT_n
\]
\[
= 0
\]

which qualifies $A + B$ for membership in $UT_n$.

Suppose $\alpha \in \mathbb{C}$ and $A \in UT_n$. Is $\alpha A \in UT_n$? We examine the entries of $\alpha A$ “below” the diagonal. That is, in the following, assume that $i > j$.

\[
[\alpha A]_{ij} = \alpha [A]_{ij} \quad \text{Definition MSM [198]}
\]
\[
= \alpha 0 \quad A \in UT_n
\]
\[
= 0
\]

which qualifies $\alpha A$ for membership in $UT_n$.

Having fulfilled the three conditions of Theorem TSS [327] we see that $UT_n$ is a subspace of $M_{nn}$. 
A vector space is defined as a set with two operations, meeting ten properties (Definition VS [309]). Just as the definition of span of a set of vectors only required knowing how to add vectors and how to multiply vectors by scalars, so it is with linear independence. A definition of a linear independent set of vectors in an arbitrary vector space only requires knowing how to form linear combinations and equating these with the zero vector. Since every vector space must have a zero vector (Property Z [310]), we always have a zero vector at our disposal.

In this section we will also put a twist on the notion of the span of a set of vectors. Rather than beginning with a set of vectors and creating a subspace that is the span, we will instead begin with a subspace and look for a set of vectors whose span equals the subspace.

The combination of linear independence and spanning will be very important going forward.

Subsection LI
Linear independence

Our previous definition of linear independence (Definition LI [345]) employed a relation of linear dependence that was a linear combination on one side of an equality and a zero vector on the other side. As a linear combination in a vector space (Definition LC [331]) depends only on vector addition and scalar multiplication, and every vector space must have a zero vector (Property Z [310]), we can extend our definition of linear independence from the setting of \( \mathbb{C}^m \) to the setting of a general vector space \( V \) with almost no changes. Compare these next two definitions with Definition RLDCV [145] and Definition LICV [145].

Definition RLD
Relation of Linear Dependence

Suppose that \( V \) is a vector space. Given a set of vectors \( S = \{u_1, u_2, u_3, \ldots, u_n\} \), an equation of the form

\[
\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_n u_n = 0
\]

is a relation of linear dependence on \( S \). If this equation is formed in a trivial fashion, i.e. \( \alpha_i = 0, 1 \leq i \leq n \), then we say it is a trivial relation of linear dependence on \( S \). \( \triangle \)

Definition LI
Linear Independence

Suppose that \( V \) is a vector space. The set of vectors \( S = \{u_1, u_2, u_3, \ldots, u_n\} \) from \( V \) is linearly dependent if there is a relation of linear dependence on \( S \) that is not
trivial. In the case where the only relation of linear dependence on \( S \) is the trivial one, then \( S \) is a \textbf{linearly independent} set of vectors.

Notice the emphasis on the word "only." This might remind you of the definition of a nonsingular matrix, where if the matrix is employed as the coefficient matrix of a homogeneous system then the only solution is the \textit{trivial} one.

\textbf{Example LIP4}

\textbf{Linear independence in} \( P_4 \)

In the vector space of polynomials with degree 4 or less, \( P_4 \) (Example VSP 312) consider the set

\[ S = \{2x^4 + 3x^3 + 2x^2 - x + 10, -x^4 - 2x^3 + x^2 + 5x - 8, 2x^4 + x^3 + 10x^2 + 17x - 2\}. \]

Is this set of vectors linearly independent or dependent? Consider that

\[
3 (2x^4 + 3x^3 + 2x^2 - x + 10) + 4 (-x^4 - 2x^3 + x^2 + 5x - 8) \\
+ (-1) (2x^4 + x^3 + 10x^2 + 17x - 2) = 0x^4 + 0x^3 + 0x^2 + 0x + 0 = 0
\]

This is a nontrivial relation of linear dependence (Definition RLD 345) on the set \( S \) and so convinces us that \( S \) is linearly dependent (Definition LI 345).

Now, I hear you say, "Where did those scalars come from?" Do not worry about that right now, just be sure you understand why the above explanation is sufficient to prove that \( S \) is linearly dependent. The remainder of the example will demonstrate how we might find these scalars if they had not been provided so readily. Let’s look at another set of vectors (polynomials) from \( P_4 \). Let

\[ T = \{3x^4 - 2x^3 + 4x^2 + 6x - 1, -3x^4 + 1x^3 + 0x^2 + 4x + 2, \\
4x^4 + 5x^3 - 2x^2 + 3x + 1, 2x^4 - 7x^3 + 4x^2 + 2x + 1\} \]

Suppose we have a relation of linear dependence on this set,

\[
0 = 0x^4 + 0x^3 + 0x^2 + 0x + 0 \\
= \alpha_1 (3x^4 - 2x^3 + 4x^2 + 6x - 1) + \alpha_2 (-3x^4 + 1x^3 + 0x^2 + 4x + 2) \\
+ \alpha_3 (4x^4 + 5x^3 - 2x^2 + 3x + 1) + \alpha_4 (2x^4 - 7x^3 + 4x^2 + 2x + 1)
\]

Using our definitions of vector addition and scalar multiplication in \( P_4 \) (Example VSP 312), we arrive at,

\[
0x^4 + 0x^3 + 0x^2 + 0x + 0 = (3\alpha_1 - 3\alpha_2 + 4\alpha_3 + 2\alpha_4) x^4 + (-2\alpha_1 + \alpha_2 + 5\alpha_3 - 7\alpha_4) x^3 \\
+ (4\alpha_1 + -2\alpha_3 + 4\alpha_4) x^2 + (6\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4) x \\
+ (-\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4).
\]

Equating coefficients, we arrive at the homogeneous system of equations,

\[
3\alpha_1 - 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = 0 \\
-2\alpha_1 + \alpha_2 + 5\alpha_3 - 7\alpha_4 = 0 \\
4\alpha_1 + -2\alpha_3 + 4\alpha_4 = 0 \\
6\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 = 0 \\
-\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 0
\]
We form the coefficient matrix of this homogeneous system of equations and row-reduce to find
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

We expected the system to be consistent (Theorem HSC [65]) and so can compute \( n - r = 4 - 4 = 0 \) and Theorem CSRN [56] tells us that the solution is unique. Since this is a homogeneous system, this unique solution is the trivial solution (Definition TSHSE [66]), \( \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \alpha_4 = 0 \). So by Definition LI [345] the set \( T \) is linearly independent.

A few observations. If we had discovered infinitely many solutions, then we could have used one of the non-trivial ones to provide a linear combination in the manner we used to show that \( S \) was linearly dependent. It is important to realize that it is not interesting that we can create a relation of linear dependence with zero scalars — we can always do that — but that for \( T \), this is the only way to create a relation of linear dependence. It was no accident that we arrived at a homogeneous system of equations in this example, it is related to our use of the zero vector in defining a relation of linear dependence. It is easy to present a convincing statement that a set is linearly dependent (just exhibit a nontrivial relation of linear dependence) but a convincing statement of linear independence requires demonstrating that there is no relation of linear dependence other than the trivial one. Notice how we relied on theorems from Chapter SLE [3] to provide this demonstration. Whew! There’s a lot going on in this example. Spend some time with it, we’ll be waiting patiently right here when you get back.

**Example LIM32**

**Linear Independence in \( M_{32} \)**

Consider the two sets of vectors \( R \) and \( S \) from the vector space of all \( 3 \times 2 \) matrices, \( M_{32} \) (Example VSM [311])

\[
R = \left\{ \begin{bmatrix} 3 & -1 \\ 1 & 4 \\ 6 & -6 \end{bmatrix}, \begin{bmatrix} -2 & 3 \\ 1 & -3 \\ -2 & -6 \end{bmatrix}, \begin{bmatrix} 6 & -6 \\ -1 & 0 \\ 7 & -9 \end{bmatrix}, \begin{bmatrix} 7 & 9 \\ -4 & -5 \\ 2 & 5 \end{bmatrix} \right\}
\]

\[
S = \left\{ \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} -4 & 0 \\ -2 & 2 \\ -2 & -6 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} -5 & 3 \\ -10 & 7 \\ -5 \end{bmatrix} \right\}
\]

One set is linearly independent, the other is not. Which is which? Let’s examine \( R \) first. Build a generic relation of linear dependence (Definition RLD [345]),

\[
\alpha_1 \begin{bmatrix} 3 & -1 \\ 1 & 4 \\ 6 & -6 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 & 3 \\ 1 & -3 \\ -2 & -6 \end{bmatrix} + \alpha_3 \begin{bmatrix} 6 & -6 \\ -1 & 0 \\ 7 & -9 \end{bmatrix} + \alpha_4 \begin{bmatrix} 7 & 9 \\ -4 & -5 \\ 2 & 5 \end{bmatrix} = 0
\]

Massaging the left-hand side with our definitions of vector addition and scalar multiplication in \( M_{32} \) (Example VSM [311]) we obtain,

\[
\begin{bmatrix} 3a_1 - 2a_2 + 6a_3 + 7a_4 & -1a_1 + 3a_2 - 6a_3 + 9a_4 \\ 1a_1 + a_2 - a_3 - 4a_4 & 4a_1 - 3a_2 + 5a_4 \\ 6a_1 - 2a_2 + 7a_3 + 2a_4 & -6a_1 - 6a_2 - 9a_3 + 5a_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
Using our definition of matrix equality (Definition ME [197]) and equating corresponding entries we get the homogeneous system of six equations in four variables,

\[ \begin{align*}
3\alpha_1 - 2\alpha_2 + 6\alpha_3 + 7\alpha_4 &= 0 \\
-1\alpha_1 + 3\alpha_2 - 6\alpha_3 + 9\alpha_4 &= 0 \\
1\alpha_1 + 1\alpha_2 - \alpha_3 - 4\alpha_4 &= 0 \\
4\alpha_1 - 3\alpha_2 + -5\alpha_4 &= 0 \\
6\alpha_1 - 2\alpha_2 + 7\alpha_3 + 2\alpha_4 &= 0 \\
-6\alpha_1 - 6\alpha_2 - 9\alpha_3 + 5\alpha_4 &= 0
\end{align*} \]

Form the coefficient matrix of this homogeneous system and row-reduce to obtain

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Analyzing this matrix we are led to conclude that \( \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \alpha_4 = 0 \). This means there is only a trivial relation of linear dependence on the vectors of \( R \) and so we call \( R \) a linearly independent set (Definition LI [345]).

So it must be that \( S \) is linearly dependent. Let’s see if we can find a non-trivial relation of linear dependence on \( S \). We will begin as with \( R \), by constructing a relation of linear dependence (Definition RLD [345]) with unknown scalars,

\[
\begin{align*}
\alpha_1 \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -4 & 0 \\ -2 & 2 \\ -2 & -6 \end{bmatrix} &+ \alpha_3 \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 2 & 4 \end{bmatrix} + \alpha_4 \begin{bmatrix} -5 & 3 \\ -10 & 7 \\ 2 & 0 \end{bmatrix} = 0
\end{align*}
\]

Massaging the left-hand side with our definitions of vector addition and scalar multiplication in \( M_{32} \) (Example VSM [311]) we obtain,

\[
\begin{bmatrix}
2\alpha_1 - 4\alpha_2 + \alpha_3 - 5\alpha_4 \\
\alpha_1 - 2\alpha_2 - 2\alpha_3 - 10\alpha_4 \\
\alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\alpha_4
\end{bmatrix}
\begin{bmatrix}
\alpha_3 + 3\alpha_4 \\
-\alpha_1 + 2\alpha_2 + \alpha_3 + 7\alpha_4 \\
3\alpha_1 - 6\alpha_2 + 4\alpha_3
\end{bmatrix}
= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

Using our definition of matrix equality (Definition ME [197]) and equating corresponding entries we get the homogeneous system of six equations in four variables,

\[ \begin{align*}
2\alpha_1 - 4\alpha_2 + \alpha_3 - 5\alpha_4 &= 0 \\
+\alpha_3 + 3\alpha_4 &= 0 \\
\alpha_1 - 2\alpha_2 - 2\alpha_3 - 10\alpha_4 &= 0 \\
-\alpha_1 + 2\alpha_2 + \alpha_3 + 7\alpha_4 &= 0 \\
\alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\alpha_4 &= 0 \\
3\alpha_1 - 6\alpha_2 + 4\alpha_3 &= 0
\end{align*} \]
Form the coefficient matrix of this homogeneous system and row-reduce to obtain

\[
\begin{bmatrix}
1 & -2 & 0 & -4 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Analyzing this we see that the system is consistent (we expected this since the system is homogeneous, Theorem HSC [65]) and has \( n - r = 4 - 2 = 2 \) free variables, namely \( \alpha_2 \) and \( \alpha_4 \). This means there are infinitely many solutions, and in particular, we can find a non-trivial solution, so long as we do not pick all of our free variables to be zero. The mere presence of a nontrivial solution for these scalars is enough to conclude that \( S \) is a linearly dependent set (Definition LI [345]). But let’s go ahead and explicitly construct a non-trivial relation of linear dependence.

Choose \( \alpha_2 = 1 \) and \( \alpha_4 = -1 \). There is nothing special about this choice, there are infinitely many possibilities, some “easier” than this one, just avoid picking both variables to be zero. Then we find the corresponding dependent variables to be \( \alpha_1 = -2 \) and \( \alpha_3 = 3 \). So the relation of linear dependence,

\[
\begin{bmatrix}
2 & 0 \\
1 & -1 \\
1 & 3 \\
\end{bmatrix}
\begin{bmatrix}
-2 \\
2 \\
-6 \\
\end{bmatrix}
+ \begin{bmatrix}
-4 & 0 \\
-2 & 2 \\
-2 & -6 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
-1 & 1 \\
2 & 4 \\
\end{bmatrix}
+ (-1)
\begin{bmatrix}
-5 & 3 \\
-10 & 7 \\
2 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

is an iron-clad demonstration that \( S \) is linearly dependent. Can you construct another such demonstration?

Example LIC
Linearly independent set in the crazy vector space

Is the set \( R = \{(1, 0), (6, 3)\} \) linearly independent in the crazy vector space \( C \) (Example CVS [314])? We begin with a relation of linear independence and massage it to a point where we can apply the definition of equality in \( C \). Recall the definitions of vector addition and scalar multiplication in \( C \).

\[
0 = a_1(1, 0) + a_2(6, 3)
\]

\[
\begin{align*}
(-1, -1) &= (1a_1 + a_1 - 1, 0a_1 + a_1 - 1) + (6a_2 + a_2 - 1, 3a_2 + a_2 - 1) \\
&= (2a_1 - 1, a_1 - 1) + (7a_2 - 1, 4a_2 - 1) \\
&= (2a_1 - 1 + 7a_2 - 1 + 1, a_1 - 1 + 4a_2 - 1 + 1) \\
&= (2a_1 + 7a_2 - 1, a_1 + 4a_2 - 1) \\
\end{align*}
\]

Equality in \( C \) then yields the two equations,

\[
\begin{align*}
2a_1 + 7a_2 - 1 &= -1 \\
a_1 + 4a_2 - 1 &= -1 \\
\end{align*}
\]
which becomes the homogeneous system

\[
\begin{align*}
2a_1 + 7a_2 &= 0 \\
a_1 + 4a_2 &= 0
\end{align*}
\]

Since the coefficient matrix of this system is nonsingular (check this!) the system has only the trivial solution \(a_1 = a_2 = 0\). By Definition LI [345] the set \(R\) is linearly independent. Notice that even though the zero vector of \(C\) is not what we might first suspected, a question about linear independence still concludes with a question about a homogeneous system of equations.

Subsection SS
Spanning Sets

In a vector space \(V\), suppose we are given a set of vectors \(S \subseteq V\). Then we can immediately construct a subspace, \(\langle S \rangle\), using Definition SS [332] and then be assured by Theorem SSS [332] that the construction does provide a subspace. We now turn the situation upside-down. Suppose we are first given a subspace \(W \subseteq V\). Can we find a set \(S\) so that \(\langle S \rangle = W\)? Typically \(W\) is infinite and we are searching for a finite set of vectors \(S\) that we can combine in linear combinations and “build” all of \(W\).

I like to think of \(S\) as the raw materials that are sufficient for the construction of \(W\). If you have nails, lumber, wire, copper pipe, drywall, plywood, carpet, shingles, paint (and a few other things), then you can combine them in many different ways to create a house (or infinitely many different houses for that matter). A fast-food restaurant may have beef, chicken, beans, cheese, tortillas, taco shells and hot sauce and from this small list of ingredients build a wide variety of items for sale. Or maybe a better analogy comes from Ben Cordes — the additive primary colors (red, green and blue) can be combined to create many different colors by varying the intensity of each. The intensity is like a scalar multiple, and the combination of the three intensities is like vector addition. The three individual colors, red, yellow and blue, are the elements of the spanning set.

Because we will use terms like “spanned by” and “spanning set,” there is the potential for confusion with “the span.” Come back and reread the first paragraph of this subsection whenever you are uncertain about the difference. Here’s the working definition.

Definition TSVS
To Span a Vector Space
Suppose \(V\) is a vector space. A subset \(S\) of \(V\) is a spanning set for \(V\) if \(\langle S \rangle = V\). In this case, we also say \(S\) spans \(V\).

The definition of a spanning set requires that two sets (subspaces actually) be equal. If \(S\) is a subset of \(V\), then \(\langle S \rangle \subseteq V\), always. Thus it is usually only necessary to prove that \(V \subseteq \langle S \rangle\). Now would be a good time to review Definition SE [694].

Example SSP4
Spanning set in \(P_4\)
In Example SP4 [328] we showed that

\[ W = \{ p(x) \mid p \in P_4, p(2) = 0 \} \]
is a subspace of $P_4$, the vector space of polynomials with degree at most 4 (Example VSP [312]). In this example, we will show that the set

$$S = \{ x - 2, x^2 - 4x + 4, x^3 - 6x^2 + 12x - 8, x^4 - 8x^3 + 24x^2 - 32x + 16 \}$$

is a spanning set for $W$. To do this, we require that $W = \langle S \rangle$. This is an equality of sets. We can check that every polynomial in $S$ has $x = 2$ as a root and therefore $S \subseteq W$. Since $W$ is closed under addition and scalar multiplication, $\langle S \rangle \subseteq W$ also.

So it remains to show that $W \subseteq \langle S \rangle$ (Definition SE [694]). To do this, begin by choosing an arbitrary polynomial in $W$, say $r(x) = ax^4 + bx^3 + cx^2 + dx + e \in W$. This polynomial is not as arbitrary as it would appear, since we also know it must have $x = 2$ as a root. This translates to

$$0 = a(2)^4 + b(2)^3 + c(2)^2 + d(2) + e = 16a + 8b + 4c + 2d + e$$
as a condition on $r$.

We wish to show that $r$ is a polynomial in $\langle S \rangle$, that is, we want to show that $r$ can be written as a linear combination of the vectors (polynomials) in $S$. So let’s try.

$$r(x) = ax^4 + bx^3 + cx^2 + dx + e = a_1 (x - 2) + a_2 (x^2 - 4x + 4) + a_3 (x^3 - 6x^2 + 12x - 8) + a_4 (x^4 - 8x^3 + 24x^2 - 32x + 16) = a_4 x^4 + (a_3 - 8a_4) x^3 + (a_2 - 6a_3 + 24a_4) x^2 + (a_1 - 4a_2 + 12a_3 - 32a_4) x + (-2a_1 + 4a_2 - 8a_3 + 16a_4)$$

Equating coefficients (vector equality in $P_4$) gives the system of five equations in four variables,

- $a_4 = a$
- $a_3 - 8a_4 = b$
- $a_2 - 6a_3 + 24a_4 = c$
- $a_1 - 4a_2 + 12a_3 - 32a_4 = d$
- $-2a_1 + 4a_2 - 8a_3 + 16a_4 = e$

Any solution to this system of equations will provide the linear combination we need to determine if $r \in \langle S \rangle$, but we need to be convinced there is a solution for any values of $a, b, c, d, e$ that qualify $r$ to be a member of $W$. So the question is: is this system of equations consistent? We will form the augmented matrix, and row-reduce. (We probably need to do this by hand, since the matrix is symbolic — reversing the order of the first four rows is the best way to start). We obtain a matrix in reduced row-echelon form

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 32a + 12b + 4c + d \\
0 & 1 & 0 & 0 & 24a + 6b + c \\
0 & 0 & 1 & 0 & 8a + b \\
0 & 0 & 0 & 1 & a \\
0 & 0 & 0 & 0 & 16a + 8b + 4c + 2d + e
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 32a + 12b + 4c + d \\
0 & 1 & 0 & 0 & 24a + 6b + c \\
0 & 0 & 1 & 0 & 8a + b \\
0 & 0 & 0 & 1 & a \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$
For your results to match our first matrix, you may find it necessary to multiply the final row of your row-reduced matrix by the appropriate scalar, and/or add multiples of this row to some of the other rows. To obtain the second version of the matrix, the last entry of the last column has been simplified to zero according to the one condition we were able to impose on an arbitrary polynomial from $W$. So with no leading 1’s in the last column, Theorem RCLS tells us this system is consistent. Therefore, any polynomial from $W$ can be written as a linear combination of the polynomials in $S$, so $W \subseteq \langle S \rangle$. Therefore, $W = \langle S \rangle$ and $S$ is a spanning set for $W$ by Definition TSVS.

Notice that an alternative to row-reducing the augmented matrix by hand would be to appeal to Theorem FS by expressing the column space of the coefficient matrix as a null space, and then verifying that the condition on $r$ guarantees that $r$ is in the column space, thus implying that the system is always consistent. Give it a try, we’ll wait. This has been a complicated example, but worth studying carefully.

Given a subspace and a set of vectors, as in Example SSP4 it can take some work to determine that the set actually is a spanning set. An even harder problem is to be confronted with a subspace and required to construct a spanning set with no guidance. We will now work an example of this flavor, but some of the steps will be unmotivated. Fortunately, we will have some better tools for this type of problem later on.

Example SSM22

Spanning set in $M_{22}$

In the space of all $2 \times 2$ matrices, $M_{22}$ consider the subspace

$$Z = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \bigg| a + 3b - c - 5d = 0, -2a - 6b + 3c + 14d = 0 \right\}$$

and find a spanning set for $Z$.

We need to construct a limited number of matrices in $Z$ so that every matrix in $Z$ can be expressed as a linear combination of this limited number of matrices. Suppose that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a matrix in $Z$. Then we can form a column vector with the entries of $B$ and write

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in N\left( \begin{bmatrix} 1 & 3 & -1 & -5 \\ -2 & -6 & 3 & 14 \end{bmatrix} \right)$$

Row-reducing this matrix and applying Theorem REMES we obtain the equivalent statement,

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in N\left( \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix} \right)$$

We can then express the subspace $Z$ in the following equal forms,

$$Z = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \bigg| a + 3b - c - 5d = 0, -2a - 6b + 3c + 14d = 0 \right\} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \bigg| a + 3b - d = 0, c + 4d = 0 \right\}$$
\[
\begin{align*}
&= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \bigg| a = -3b + d, \ c = -4d \right\} \\
&= \left\{ \begin{bmatrix} -3b + d & b \\ -4d & d \end{bmatrix} \bigg| b, \ d \in \mathbb{C} \right\} \\
&= \left\{ \begin{bmatrix} -3 & b \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \bigg| b, \ d \in \mathbb{C} \right\} \\
&= \left\{ \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \right\}
\end{align*}
\]

So the set
\[
Q = \left\{ \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \right\}
\]

spans \( \mathbb{Z} \) by Definition TSVS \([350]\).

**Example SSC**

**Spanning set in the crazy vector space**

In Example LIC \([349]\) we determined that the set \( R = \{(1, 0), (6, 3)\} \) is linearly independent in the crazy vector space \( C \) (Example CVS \([314]\)). We now show that \( R \) is a spanning set for \( C \).

Given an arbitrary vector \((x, y) \in C\) we desire to show that it can be written as a linear combination of the elements of \( R \). In other words, are there scalars \( a_1 \) and \( a_2 \) so that
\[
(x, y) = a_1(1, 0) + a_2(6, 3)
\]

We will act as if this equation is true and try to determine just what \( a_1 \) and \( a_2 \) would be (as functions of \( x \) and \( y \)).

\[
(x, y) = a_1(1, 0) + a_2(6, 3)
\]
\[
= (a_1 + a_1 - 1, 0a_1 + a_1 - 1) + (6a_2 + a_2 - 1, 3a_2 + a_2 - 1) \quad \text{Scalar mult in } C
\]
\[
= (2a_1 - 1, a_1 - 1) + (7a_2 - 1, 4a_2 - 1) \quad \text{Addition in } C
\]
\[
= (2a_1 + 7a_2 - 1 + 1, a_1 - 1 + 4a_2 - 1 + 1)
\]
\[
= (2a_1 + 7a_2 - 1, a_1 + 4a_2 - 1)
\]

Equality in \( C \) then yields the two equations,
\[
\begin{align*}
2a_1 + 7a_2 - 1 &= x \\
a_1 + 4a_2 - 1 &= y
\end{align*}
\]

which becomes the linear system with a matrix representation
\[
\begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} x + 1 \\ y + 1 \end{bmatrix}
\]

The coefficient matrix of this system is nonsingular, hence invertible (Theorem NI \([251]\)), and we can employ its inverse to find a solution (Theorem TTMI \([234]\), Theorem SNCM \([252]\)),
\[
\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} x + 1 \\ y + 1 \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x + 1 \\ y + 1 \end{bmatrix} = \begin{bmatrix} 4x - 7y - 3 \\ -x + 2y + 1 \end{bmatrix}
\]
We could chase through the above implications backwards and take the existence of these solutions as sufficient evidence for \( R \) being a spanning set for \( C \). Instead, let us view the above as simply scratchwork and now get serious with a simple direct proof that \( R \) is a spanning set. Ready? Suppose \((x, y)\) is any vector from \( C \), then compute the following linear combination using the definitions of the operations in \( C \),

\[
(4x - 7y - 3)(1, 0) + (-x + 2y + 1)(6, 3)
\]

\[
= (1(4x - 7y - 3) + (4x - 7y - 3) - 1, 0(4x - 7y - 3) + (4x - 7y - 3) - 1) +
\]

\[
= (6(-x + 2y + 1) + (-x + 2y + 1) - 1, 3(-x + 2y + 1) + (-x + 2y + 1) - 1) +
\]

\[
= (8x - 14y - 7, 4x - 7y - 4) + (-7x + 14y + 6, -4x + 8y + 3)
\]

\[
= ((8x - 14y - 7) + (-7x + 14y + 6) + 1, (4x - 7y - 4) + (-4x + 8y + 3) + 1)
\]

\[
= (x, y)
\]

This final sequence of computations in \( C \) is sufficient to demonstrate that any element of \( C \) can be written (or expressed) as a linear combination of the two vectors in \( R \), so \( C \subseteq \langle R \rangle \). Since the reverse inclusion \( \langle R \rangle \subseteq C \) is trivially true, \( C = \langle R \rangle \) and we say \( R \) spans \( C \) (Definition TSVS [350]). Notice that this demonstration is no more or less valid if we hide from the reader our scratchwork that suggested \( a_1 = 4x - 7y - 3 \) and \( a_2 = -x + 2y + 1 \).

### Subsection VR

#### Vector Representation

In Chapter R [587] we will take up the matter of representations fully, where Theorem VRBB [355] will be critical for Definition VR [587]. We will now motivate and prove a critical theorem that tells us how to “represent” a vector. This theorem could wait, but working with it now will provide some extra insight into the nature of linearly independent spanning sets. First an example, then the theorem.

**Example AVR**

**A vector representation**

Consider the set

\[
S = \left\{ \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix} \right\}
\]

from the vector space \( \mathbb{C}^3 \). Let \( A \) be the matrix whose columns are the set \( S \), and verify that \( A \) is nonsingular. By Theorem NMLIC [151] the elements of \( S \) form a linearly independent set. Suppose that \( \mathbf{b} \in \mathbb{C}^3 \). Then \( L S(A, \mathbf{b}) \) has a (unique) solution (Theorem NMUS [79]) and hence is consistent. By Theorem SLSLC [100], \( \mathbf{b} \in \langle S \rangle \). Since \( \mathbf{b} \) is arbitrary, this is enough to show that \( \langle S \rangle = \mathbb{C}^3 \), and therefore \( S \) is a spanning set for \( \mathbb{C}^3 \) (Definition TSVS [350]). (This set comes from the columns of the coefficient matrix of Archetype B [726].)

Now examine the situation for a particular choice of \( \mathbf{b} \), say \( \mathbf{b} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix} \). Because \( S \) is a spanning set for \( \mathbb{C}^3 \), we know we can write \( \mathbf{b} \) as a linear combination of the vectors
in $S$, 
\[
\begin{bmatrix}
-33 \\
24 \\
5
\end{bmatrix} = (-3) \begin{bmatrix}
-7 \\
5 \\
1
\end{bmatrix} + (5) \begin{bmatrix}
-6 \\
5 \\
0
\end{bmatrix} + (2) \begin{bmatrix}
-12 \\
7 \\
4
\end{bmatrix}.
\]

The nonsingularity of the matrix $A$ tells that the scalars in this linear combination are unique. More precisely, it is the linear independence of $S$ that provides the uniqueness. We will refer to the scalars $a_1 = -3$, $a_2 = 5$, $a_3 = 2$ as a “representation of $b$ relative to $S$.” In other words, once we settle on $S$ as a linearly independent set that spans $\mathbb{C}^3$, the vector $b$ is recoverable just by knowing the scalars $a_1 = -3$, $a_2 = 5$, $a_3 = 2$ (use these scalars in a linear combination of the vectors in $S$). This is all an illustration of the following important theorem, which we prove in the setting of a general vector space. □

**Theorem VRRRB**

**Vector Representation Relative to a Basis**

Suppose that $V$ is a vector space and $B = \{v_1, v_2, v_3, \ldots, v_m\}$ is a linearly independent set that spans $V$. Let $w$ be any vector in $V$. Then there exist unique scalars $a_1, a_2, a_3, \ldots, a_m$ such that

\[w = a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_m v_m.\]

\[\square\]

**Proof** That $w$ can be written as a linear combination of the vectors in $B$ follows from the spanning property of the set [Definition TSVS 350]. This is good, but not the meat of this theorem. We now know that for any choice of the vector $w$ there exist some scalars that will create $w$ as a linear combination of the basis vectors. The real question is: Is there more than one way to write $w$ as a linear combination of $\{v_1, v_2, v_3, \ldots, v_m\}$? Are the scalars $a_1, a_2, a_3, \ldots, a_m$ unique? [Technique U 709]

Assume there are two ways to express $w$ as a linear combination of $\{v_1, v_2, v_3, \ldots, v_m\}$. In other words there exist scalars $a_1, a_2, a_3, \ldots, a_m$ and $b_1, b_2, b_3, \ldots, b_m$ so that

\[w = a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_m v_m\]

\[w = b_1 v_1 + b_2 v_2 + b_3 v_3 + \cdots + b_m v_m.\]

Then notice that

\[0 = w + (-w) = w + (-1)w = (a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_m v_m) + (-1)(b_1 v_1 + b_2 v_2 + b_3 v_3 + \cdots + b_m v_m) = (a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_m v_m) + (-b_1 v_1 - b_2 v_2 - b_3 v_3 - \cdots - b_m v_m) = (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + (a_3 - b_3)v_3 + \cdots + (a_m - b_m)v_m.\]

But this is a relation of linear dependence on a linearly independent set of vectors [Definition RLD 345]! Now we are using the other assumption about $B$, that $\{v_1, v_2, v_3, \ldots, v_m\}$
is a linearly independent set. So by Definition LI \[345\] it must happen that the scalars are all zero. That is,

\[
\begin{align*}
(a_1 - b_1) &= 0 \\
(a_2 - b_2) &= 0 \\
(a_3 - b_3) &= 0 \\
&\quad \vdots \\
(a_m - b_m) &= 0
\end{align*}
\]

\[
\begin{align*}
a_1 &= b_1 \\
a_2 &= b_2 \\
a_3 &= b_3 \\
&\quad \vdots \\
a_m &= b_m.
\end{align*}
\]

And so we find that the scalars are unique.

This is a very typical use of the hypothesis that a set is linearly independent — obtain a relation of linear dependence and then conclude that the scalars must all be zero. The result of this theorem tells us that we can write any vector in a vector space as a linear combination of the vectors in a linearly independent spanning set, but only just. There is only enough raw material in the spanning set to write each vector one way as a linear combination. So in this sense, we could call a linearly independent spanning set a “minimal spanning set.” These sets are so important that we will give them a simpler name (“basis”) and explore their properties further in the next section.

**Subsection READ**

**Reading Questions**

1. Is the set of matrices below linearly independent or linearly dependent in the vector space \(M_{22}\)? Why or why not?

\[
\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}, \begin{bmatrix} -2 & 3 \\ 3 & -5 \end{bmatrix}, \begin{bmatrix} 0 & 9 \\ -1 & 3 \end{bmatrix}
\]

2. Explain the difference between the following two uses of the term “span”:

(a) \(S\) is a subset of the vector space \(V\) and the span of \(S\) is a subspace of \(V\).

(b) \(W\) is subspace of the vector space \(Y\) and \(T\) spans \(W\).

3. The set

\[
S = \left\{ \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix} \right\}
\]

is linearly independent and spans \(\mathbb{C}^3\). Write the vector \(\mathbf{x} = \begin{bmatrix} -6 \\ 2 \\ 2 \end{bmatrix}\) a linear combination of the elements of \(S\). How many ways are there to answer this question?
Subsection EXC
Exercises

C20  In the vector space of $2 \times 2$ matrices, $M_{22}$, determine if the set $S$ below is linearly independent.

$$ S = \left\{ \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \right\} $$

Contributed by Robert Beezer  Solution 359

C21  In the crazy vector space $C$ (Example CVS 314), is the set $S = \{(0, 2), (2, 8)\}$ linearly independent?

Contributed by Robert Beezer  Solution 359

C22  In the vector space of polynomials $P_3$, determine if the set $S$ is linearly independ-ent or linearly dependent.

$$ S = \{2 + x - 3x^2 - 8x^3, 1 + x + x^2 + 5x^3, 3 - 4x^2 - 7x^3\} $$

Contributed by Robert Beezer  Solution 360

C23  Determine if the set $S = \{(3, 1), (7, 3)\}$ is linearly independent in the crazy vector space $C$ (Example CVS 314).

Contributed by Robert Beezer  Solution 360

C30  In Example LIM32 347, find another nontrivial relation of linear dependence on the linearly dependent set of $3 \times 2$ matrices, $S$.

Contributed by Robert Beezer

C40  Determine if the set $T = \{x^2 - x + 5, 4x^3 - x^2 + 5x, 3x + 2\}$ spans the vector space of polynomials with degree 4 or less, $P_4$.

Contributed by Robert Beezer  Solution 360

C41  The set $W$ is a subspace of $M_{22}$, the vector space of all $2 \times 2$ matrices. Prove that $S$ is a spanning set for $W$.

$$ W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid 2a - 3b + 4c - d = 0 \right\} \quad S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix} \right\} $$

Contributed by Robert Beezer  Solution 360

C42  Determine if the set $S = \{(3, 1), (7, 3)\}$ spans the crazy vector space $C$ (Example CVS 314).

Contributed by Robert Beezer  Solution 361

M10  Halfway through Example SSP4 350, we need to show that the system of equations

$$ LS = \left( \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -8 \\ 0 & 1 & -6 & 24 \\ 1 & -4 & 12 & -32 \\ -2 & 4 & -8 & 16 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} \right) $$

Version 0.92
is consistent for every choice of the vector of constants for which $16a + 8b + 4c + 2d + e = 0$.

Express the column space of the coefficient matrix of this system as a null space, using Theorem FS [290]. From this use Theorem CSCI [262] to establish that the system is always consistent. Notice that this approach removes from Example SSP4 [350] the need to row-reduce a symbolic matrix.

Contributed by Robert Beezer   Solution [361]
Subsection SOL

Solutions

C20  Contributed by Robert Beezer  Statement 357

Begin with a relation of linear dependence on the vectors in $S$ and massage it according to the definitions of vector addition and scalar multiplication in $M_{22},$

$$\mathbf{0} = a_1 \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix} + a_3 \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

By our definition of matrix equality (Definition ME 197) we arrive at a homogeneous system of linear equations,

$$2a_1 + 4a_3 = 0$$
$$-a_1 + 4a_2 + 2a_3 = 0$$
$$a_1 - a_2 + a_3 = 0$$
$$3a_1 + 2a_2 + 3a_3 = 0$$

The coefficient matrix of this system row-reduces to the matrix,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and from this we conclude that the only solution is $a_1 = a_2 = a_3 = 0$. Since the relation of linear dependence (Definition RLD 345) is trivial, the set $S$ is linearly independent (Definition LI 345).

C21  Contributed by Robert Beezer  Statement 357

We begin with a relation of linear dependence using unknown scalars $a$ and $b$. We wish to know if these scalars must both be zero. Recall that the zero vector in $C$ is $(-1, -1)$ and that the definitions of vector addition and scalar multiplication are not what we might expect.

$0 = (-1, -1)$

$$= a(0, 2) + b(2, 8)$$

$$= (0a + a - 1, 2a + a - 1) + (2b + b - 1, 8b + b - 1)$$

Scalar mult., Example CVS 314

$$= (a - 1, 3a - 1) + (3b - 1, 9b - 1)$$

Vector addition, Example CVS 314

$$= (a - 1 + 3b - 1 + 1, 3a - 1 + 9b - 1 + 1)$$

$$= (a + 3b - 1, 3a + 9b - 1)$$

From this we obtain two equalities, which can be converted to a homogeneous system of equations,

$$-1 = a + 3b - 1$$
$$a + 3b = 0$$
This homogeneous system has a singular coefficient matrix (Theorem SMZD 432), and so has more than just the trivial solution (Definition NM 75). Any nontrivial solution will give us a nontrivial relation of linear dependence on \(S\). So \(S\) is linearly dependent (Definition LI 345).

\[
\begin{align*}
-1 &= 3a + 9b - 1 \\
3a + 9b &= 0
\end{align*}
\]

Row-reducing the coefficient matrix of this homogeneous system leads to the unique solution \(a_1 = a_2 = a_3 = 0\). So the only relation of linear dependence on \(S\) is the trivial one, and this is linear independence for \(S\) (Definition LI 345).

\[C22\] Contributed by Robert Beezer Statement 357

Begin with a relation of linear dependence (Definition RLD 345),

\[a_1 \left(2 + x - 3x^2 - 8x^3\right) + a_2 \left(1 + x + x^2 + 5x^3\right) + a_3 \left(3 - 4x^2 - 7x^3\right) = 0\]

Massage according to the definitions of scalar multiplication and vector addition in the definition of \(P_3\) (Example VSP 312) and use the zero vector dro this vector space,

\[(2a_1 + a_2 + 3a_3) + (a_1 + a_2) x + (-3a_1 + a_2 - 4a_3) x^2 + (-8a_1 + 5a_2 - 7a_3) x^3 = 0 + 0x + 0x^2 + 0x^3\]

The definition of the equality of polynomials allows us to deduce the following four equations,

\[
\begin{align*}
2a_1 + a_2 + 3a_3 &= 0 \\
-3a_1 + a_2 - 4a_3 &= 0 \\
-8a_1 + 5a_2 - 7a_3 &= 0
\end{align*}
\]

\[C40\] Contributed by Robert Beezer Statement 357

The vector space \(P_4\) has dimension 5 by Theorem DP 383. Since \(T\) contains only 3 vectors, and \(3 < 5\), Theorem G 398 tells us that \(T\) does not span \(P_5\).

\[C41\] Contributed by Robert Beezer Statement 357

We want to show that \(W = \langle S \rangle\) (Definition TSVS 350), which is an equality of sets (Definition SE 694).

First, show that \(\langle S \rangle \subseteq W\). Begin by checking that each of the three matrices in \(S\) is a member of the set \(W\). Then, since \(W\) is a vector space, the closure properties (Property AC 309, Property SC 309) guarantee that every linear combination of elements of \(S\) remains in \(W\).

Second, show that \(W \subseteq \langle S \rangle\). We want to convince ourselves that an arbitrary element of \(W\) is a linear combination of elements of \(S\). Choose

\[x = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in W\]
The values of $a, b, c, d$ are not totally arbitrary, since membership in $W$ requires that $2a - 3b + 4c - d = 0$. Now, rewrite as follows,

\[
\begin{align*}
x &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ c & 2a - 3b + 4c \end{bmatrix} \\
&= \begin{bmatrix} a & 0 \\ 0 & 2a \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & -3b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 4c \end{bmatrix} \\
&= a \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix}
\end{align*}
\]

Equality in $C$ leads to the system

\[
\begin{align*}
4a_1 + 8a_2 &= x + 1 \\
2a_1 + 4a_2 &= y + 1
\end{align*}
\]

This system has a singular coefficient matrix whose column space is simply $\langle \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rangle$. So any choice of $x$ and $y$ that causes the column vector $\begin{bmatrix} x + 1 \\ y + 1 \end{bmatrix}$ to lie outside the column space will lead to an inconsistent system, and hence create an element $(x, y)$ that is not in the span of $S$. So $S$ does not span $C$.

For example, choose $x = 0$ and $y = 5$, and then we can see that $\begin{bmatrix} 1 \\ 6 \end{bmatrix} \notin \langle \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rangle$ and we know that $(0, 5)$ cannot be written as a linear combination of the vectors in $S$. A shorter solution might begin by asserting that $(0, 5)$ is not in $\langle S \rangle$ and then establishing this claim alone.

**M10** Contributed by Robert Beezer  Statement 357

Theorem FS provides the matrix

\[
L = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \end{bmatrix}
\]

and so if $A$ denotes the coefficient matrix of the system, then $C(A) = N(L)$. The single homogeneous equation in $LS(L, 0)$ is equivalent to the condition on the vector of constants (use $a, b, c, d, e$ as variables and then multiply by 16).
A basis of a vector space is one of the most useful concepts in linear algebra. It often provides a finite description of an infinite vector space.

We now have all the tools in place to define a basis of a vector space.

**Definition B**

**Basis**

Suppose \( V \) is a vector space. Then a subset \( S \subseteq V \) is a **basis** of \( V \) if it is linearly independent and spans \( V \).

\( \triangle \)

So, a basis is a linearly independent spanning set for a vector space. The requirement that the set spans \( V \) insures that \( S \) has enough raw material to build \( V \), while the linear independence requirement insures that we do not have any more raw material than we need. As we shall see soon in Section D [379], a basis is a minimal spanning set.

You may have noticed that we used the term basis for some of the titles of previous theorems (e.g. Theorem BNS [154], Theorem BCS [264], Theorem BRS [271]) and if you review each of these theorems you will see that their conclusions provide linearly independent spanning sets for sets that we now recognize as subspaces of \( \mathbb{C}^m \). Examples associated with these theorems include Example NSLIL [155], Example CSOCD [265] and Example IAS [272]. As we will see, these three theorems will continue to be powerful tools, even in the setting of more general vector spaces.

Furthermore, the archetypes contain an abundance of bases. For each coefficient matrix of a system of equations, and for each archetype defined simply as a matrix, there is a basis for the null space, three bases for the column space, and a basis for the row space. For this reason, our subsequent examples will concentrate on bases for vector spaces other than \( \mathbb{C}^m \). Notice that Definition B [363] does not preclude a vector space from having many bases, and this is the case, as hinted above by the statement that the archetypes contain three bases for the column space of a matrix. More generally, we can grab any basis for a vector space, multiply any one basis vector by a non-zero scalar and create a slightly different set that is still a basis. For “important” vector spaces, it will be convenient to have a collection of “nice” bases. When a vector space has a single particularly nice basis, it is sometimes called the **standard basis** though there is nothing precise enough about this term to allow us to define it formally — it is a question of style. Here are some nice bases for important vector spaces.

**Theorem SUVB**

**Standard Unit Vectors are a Basis**

The set of standard unit vectors for \( \mathbb{C}^m \) (Definition SUV [234]), \( B = \{ e_1, e_2, e_3, \ldots, e_m \} = \{ e_i \mid 1 \leq i \leq m \} \) is a basis for the vector space \( \mathbb{C}^m \).
Proof We must show that the set $B$ is both linearly independent and a spanning set for $C^m$. First, the vectors in $B$ are, by Definition SUV [234], the columns of the identity matrix, which we know is nonsingular (since it row-reduces to the identity matrix, Theorem NMRRI [77]). And the columns of a nonsingular matrix are linearly independent by Theorem NMLIC [151].

Suppose we grab an arbitrary vector from $C^m$, say $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix}$.

Can we write $v$ as a linear combination of the vectors in $B$? Yes, and quite simply.

$$v = v_1 e_1 + v_2 e_2 + v_3 e_3 + \cdots + v_m e_m$$

this shows that $C^m \subseteq \langle B \rangle$, which is sufficient to show that $B$ is a spanning set for $C^m$.

Example BP
Bases for $P_n$

The vector space of polynomials with degree at most $n$, $P_n$, has the basis $B = \{1, x, x^2, x^3, \ldots, x^n\}$.

Another nice basis for $P_n$ is

$$C = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3, \ldots, 1 + x + x^2 + x^3 + \cdots + x^n\}.$$ 

Checking that each of $B$ and $C$ is a linearly independent spanning set are good exercises.

Example BM
A basis for the vector space of matrices

In the vector space $M_{mn}$ of matrices (Example VSM [311]) define the matrices $B_{k\ell}$, $1 \leq k \leq m$, $1 \leq \ell \leq n$ by

$$[B_{k\ell}]_{ij} = \begin{cases} 1 & \text{if } k = i, \ell = j \\ 0 & \text{otherwise} \end{cases}$$

So these matrices have entries that are all zeros, with the exception of a lone entry that is one. The set of all $mn$ of them,

$$B = \{B_{k\ell} \mid 1 \leq k \leq m, 1 \leq \ell \leq n\}$$

forms a basis for $M_{mn}$.
The bases described above will often be convenient ones to work with. However a basis doesn’t have to obviously look like a basis.

**Example BSP4**

A basis for a subspace of $P_4$

In Example SSP4 we showed that

$$S = \{ x - 2, x^2 - 4x + 4, x^3 - 6x^2 + 12x - 8, x^4 - 8x^3 + 24x^2 - 32x + 16 \}$$

is a spanning set for $W = \{ p(x) \mid p \in P_4, p(2) = 0 \}$. We will now show that $S$ is also linearly independent in $W$. Begin with a relation of linear dependence,

$$0 + 0x + 0x^2 + 0x^3 + 0x^4 = \alpha_1 (x - 2) + \alpha_2 (x^2 - 4x + 4) + \alpha_3 (x^3 - 6x^2 + 12x - 8) + \alpha_4 (x^4 - 8x^3 + 24x^2 - 32x + 16)$$

$$= \alpha_4 x^4 + (\alpha_3 - 8\alpha_4) x^3 + (\alpha_2 - 6\alpha_3 + 24\alpha_4) x^2 + (\alpha_1 - 4\alpha_2 + 12\alpha_3 - 32\alpha_4) x + (-2\alpha_1 + 4\alpha_2 - 8\alpha_3 + 16\alpha_4)$$

Equating coefficients (vector equality in $P_4$) gives the homogeneous system of five equations in four variables,

$$\begin{align*}
\alpha_4 &= 0 \\
\alpha_3 - 8\alpha_4 &= 0 \\
\alpha_2 - 6\alpha_3 + 24\alpha_4 &= 0 \\
\alpha_1 - 4\alpha_2 + 12\alpha_3 - 32\alpha_4 &= 0 \\
-2\alpha_1 + 4\alpha_2 - 8\alpha_3 + 16\alpha_4 &= 0
\end{align*}$$

We form the coefficient matrix, and row-reduce to obtain a matrix in reduced row-echelon form

$$\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

With only the trivial solution to this homogeneous system, we conclude that only scalars that will form a relation of linear dependence are the trivial ones, and therefore the set $S$ is linearly independent (Definition LI). Finally, $S$ has earned the right to be called a basis for $W$ (Definition B).

**Example BSM22**

A basis for a subspace of $M_{22}$

In Example SSM22 we discovered that

$$Q = \left\{ \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \right\}$$

is a spanning set for the subspace

$$Z = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \; \mid \; a + 3b - c - 5d = 0, \; -2a - 6b + 3c + 14d = 0 \right\}$$
of the vector space of all $2 \times 2$ matrices, $M_{22}$. If we can also determine that $Q$ is linearly independent in $Z$ (or in $M_{22}$), then it will qualify as a basis for $Z$. Let’s begin with a relation of linear dependence.

$$
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} = \alpha_1 \begin{bmatrix}
-3 & 1 \\
0 & 0
\end{bmatrix} + \alpha_2 \begin{bmatrix}
1 & 0 \\
-4 & 1
\end{bmatrix} = \begin{bmatrix}
-3\alpha_1 + \alpha_2 & \alpha_1 \\
-4\alpha_2 & \alpha_2
\end{bmatrix}
$$

Using our definition of matrix equality (Definition ME [197]) we equate corresponding entries and get a homogeneous system of four equations in two variables,

$$
-3\alpha_1 + \alpha_2 = 0 \\
\alpha_1 = 0 \\
-4\alpha_2 = 0 \\
\alpha_2 = 0
$$

We could row-reduce the coefficient matrix of this homogeneous system, but it is not necessary. The second and fourth equations tell us that $\alpha_1 = 0, \alpha_2 = 0$ is the only solution to this homogeneous system. This qualifies the set $Q$ as being linearly independent, since the only relation of linear dependence is trivial (Definition LI [345]). Therefore $Q$ is a basis for $Z$ (Definition B [363]).

**Example BC**

**Basis for the crazy vector space**

In Example LIC [349] and Example SSC [353] we determined that the set $R = \{(1, 0), (6, 3)\}$ from the crazy vector space, $C$ (Example CVS [314]), is linearly independent and is a spanning set for $C$. By Definition B [363] we see that $R$ is a basis for $C$. ☒

We have seen that several of the sets associated with a matrix are subspaces of vector spaces of column vectors. Specifically these are the null space (Theorem NSMS [330]), column space (Theorem CSMS [336]), row space (Theorem RSMS [337]) and left null space (Theorem LNSMS [337]). As subspaces they are vector spaces (Definition S [325]) and it is natural to ask about bases for these vector spaces. Theorem BNS [154], Theorem BCS [264], Theorem BRS [271] each have conclusions that provide linearly independent spanning sets for (respectively) the null space, column space, and row space. Notice that each of these theorems contains the word “basis” in its title, even though we did not know the precise meaning of the word at the time. To find a basis for a left null space we can use the definition of this subspace as a null space (Definition LNS [283]) and apply Theorem BNS [154]. Or Theorem FS [290] tells us that the left null space can be expressed as a row space and we can then use Theorem BRS [271].

Theorem BS [172] is another early result that provides a linearly independent spanning set (i.e. a basis) as its conclusion. If a vector space of column vectors can be expressed as a span of a set of column vectors, then Theorem BS [172] can be employed in a straightforward manner to quickly yield a basis.
We have seen several examples of bases in different vector spaces. In this subsection, and the next (Subsection B.BNM), we will consider building bases for \( \mathbb{C}^m \) and its subspaces.

Suppose we have a subspace of \( \mathbb{C}^m \) that is expressed as the span of a set of vectors, \( S \), and \( S \) is not necessarily linearly independent, or perhaps not very attractive. Theorem REMRS says that row-equivalent matrices have identical row spaces, while Theorem BRS says the nonzero rows of a matrix in reduced row-echelon form are a basis for the row space. These theorems together give us a great computational tool for quickly finding a basis for a subspace that is expressed originally as a span.

Example RSB
Row space basis

When we first defined the span of a set of column vectors, in Example SCAD, we looked at the set

\[
W = \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \\ 1 \\ 4 \\ 1 \\ 7 \\ -5 \\ 4 \\ -7 \\ -6 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -5 \\ 4 \\ 11 \end{bmatrix}, \begin{bmatrix} 7 \\ -5 \\ 4 \\ -6 \\ -5 \end{bmatrix} \right\}
\]

with an eye towards realizing \( W \) as the span of a smaller set. By building relations of linear dependence (though we did not know them by that name then) we were able to remove two vectors and write \( W \) as the span of the other two vectors. These two remaining vectors formed a linearly independent set, even though we did not know that at the time.

Now we know that \( W \) is a subspace and must have a basis. Consider the matrix, \( C \), whose rows are the vectors in the spanning set for \( W \),

\[
C = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 4 & 1 \\ 7 & -5 & 4 \\ -7 & -6 & -5 \end{bmatrix}
\]

Then, by Definition RSM, the row space of \( C \) will be \( W \), \( \mathcal{R}(C) = W \). Theorem BRS tells us that if we row-reduce \( C \), the nonzero rows of the row-equivalent matrix in reduced row-echelon form will be a basis for \( \mathcal{R}(C) \), and hence a basis for \( W \). Let’s do it — \( C \) row-reduces to

\[
\begin{bmatrix} 1 & 0 & \frac{7}{11} \\ 0 & 1 & \frac{1}{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

If we convert the two nonzero rows to column vectors then we have a basis,

\[
B = \left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{7}{11} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{1}{11} \end{bmatrix} \right\}
\]
and

\[ W = \left\{ \begin{bmatrix} 1 \\ 0 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 11 \\ 1 \end{bmatrix} \right\} \]

For aesthetic reasons, we might wish to multiply each vector in \( B \) by 11, which will not change the spanning or linear independence properties of \( B \) as a basis. Then we can also write

\[ W = \left\{ \begin{bmatrix} 11 \\ 0 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 11 \\ 1 \\ 7 \end{bmatrix} \right\} \]

Example IAS \[272\] provides another example of this flavor, though now we can notice that \( X \) is a subspace, and that the resulting set of three vectors is a basis. This is such a powerful technique that we should do one more example.

Example RS
Reducing a span
In Example RSC5 \[168\] we began with a set of \( n = 4 \) vectors from \( \mathbb{C}^5 \),

\[ R = \{ v_1, v_2, v_3, v_4 \} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -7 \\ 6 \\ -11 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \\ 6 \end{bmatrix} \right\} \]

and defined \( V = \langle R \rangle \). Our goal in that problem was to find a relation of linear dependence on the vectors in \( R \), solve the resulting equation for one of the vectors, and re-express \( V \) as the span of a set of three vectors.

Here is another way to accomplish something similar. The row space of the matrix

\[ A = \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 2 & 1 & 3 & 1 & 2 \\ 0 & -7 & 6 & -11 & -2 \\ 4 & 1 & 2 & 1 & 6 \end{bmatrix} \]

is equal to \( \langle R \rangle \). By Theorem BRS \[271\] we can row-reduce this matrix, ignore any zero rows, and use the non-zero rows as column vectors that are a basis for the row space of \( A \). Row-reducing \( A \) creates the matrix

\[
\begin{bmatrix}
1 & 0 & 0 & -\frac{1}{17} & -\frac{30}{17} \\
0 & 1 & 0 & -\frac{25}{17} & -\frac{17}{2} \\
0 & 0 & 1 & -\frac{2}{17} & -\frac{8}{17} \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

So

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{17} \\ \frac{25}{17} \\ -\frac{2}{17} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{30}{17} \\ \frac{17}{2} \\ -\frac{8}{17} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{17}{2} \\ -\frac{2}{17} \\ \frac{17}{2} \end{bmatrix} \right\}
\]
is a basis for $V$. Our theorem tells us this is a basis, there is no need to verify that the subspace spanned by three vectors (rather than four) is the identical subspace, and there is no need to verify that we have reached the limit in reducing the set, since the set of three vectors is guaranteed to be linearly independent.

**Subsection BNM**  
**Bases and Nonsingular Matrices**

A quick source of diverse bases for $\mathbb{C}^m$ is the set of columns of a nonsingular matrix.

**Theorem CNMB**  
**Columns of Nonsingular Matrix are a Basis**

Suppose that $A$ is a square matrix of size $m$. Then the columns of $A$ are a basis of $\mathbb{C}^m$ if and only if $A$ is nonsingular. \[\square\]

**Proof**  
($\Rightarrow$) Suppose that the columns of $A$ are a basis for $\mathbb{C}^m$. Then Definition B \[363\] says the set of columns is linearly independent. Theorem NMLIC \[151\] then says that $A$ is nonsingular.

($\Leftarrow$) Suppose that $A$ is nonsingular. Then by Theorem NMLIC \[151\] this set of columns is linearly independent. Theorem CSNM \[267\] says that for a nonsingular matrix, $\mathcal{C}(A) = \mathbb{C}^m$. This is equivalent to saying that the columns of $A$ are a spanning set for the vector space $\mathbb{C}^m$. As a linearly independent spanning set, the columns of $A$ qualify as a basis for $\mathbb{C}^m$ (Definition B \[363\]). \[\blacksquare\]

**Example CABAK**  
**Columns as Basis, Archetype K**

Archetype K \[767\] is the $5 \times 5$ matrix

$$K = \begin{bmatrix} 10 & 18 & 24 & 24 & -12 \\ 12 & -2 & -6 & 0 & -18 \\ -30 & -21 & -23 & -30 & 39 \\ 27 & 30 & 36 & 37 & -30 \\ 18 & 24 & 30 & 30 & -20 \end{bmatrix}$$

which is row-equivalent to the $5 \times 5$ identity matrix $I_5$. So by Theorem NMRRI \[77\], $K$ is nonsingular. Then Theorem CNMB \[369\] says the set

$$\begin{Bmatrix} \begin{bmatrix} 10 \\ 12 \\ -30 \\ 27 \\ 18 \end{bmatrix} , \begin{bmatrix} 18 \\ -2 \\ -21 \\ 30 \\ 24 \end{bmatrix} , \begin{bmatrix} 24 \\ -6 \\ -23 \\ 36 \\ 30 \end{bmatrix} , \begin{bmatrix} 24 \\ 0 \\ -30 \\ 37 \\ 30 \end{bmatrix} , \begin{bmatrix} -12 \\ -18 \\ 39 \\ -30 \\ -20 \end{bmatrix} \end{Bmatrix}$$

is a (novel) basis of $\mathbb{C}^5$. \[\blacktriangle\]

Perhaps we should view the fact that the standard unit vectors are a basis (Theorem SUVB \[363\]) as just a simple corollary of Theorem CNMB \[369\]? (See Technique LC \[716\].)
With a new equivalence for a nonsingular matrix, we can update our list of equivalences.

**Theorem NME5**

**Nonsingular Matrix Equivalences, Round 5**

Suppose that $A$ is a square matrix of size $n$. The following are equivalent.

1. $A$ is nonsingular.
2. $A$ row-reduces to the identity matrix.
3. The null space of $A$ contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
4. The linear system $\mathcal{L}S(A, b)$ has a unique solution for every possible choice of $b$.
5. The columns of $A$ are a linearly independent set.
6. $A$ is invertible.
7. The column space of $A$ is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.
8. The columns of $A$ are a basis for $\mathbb{C}^n$.

\[\square\]

**Proof** With a new equivalence for a nonsingular matrix in Theorem CNMB [369] we can expand Theorem NME4 [267].

---

**Subsection OBC**

**Orthonormal Bases and Coordinates**

We learned about orthogonal sets of vectors in $\mathbb{C}^m$ back in Section O [183], and we also learned that orthogonal sets are automatically linearly independent (Theorem OSLI [190]). When an orthogonal set also spans a subspace of $\mathbb{C}^m$, then the set is a basis. And when the set is orthonormal, then the set is an incredibly nice basis. We will back up this claim with a theorem, but first consider how you might manufacture such a set.

Suppose that $W$ is a subspace of $\mathbb{C}^m$ with basis $B$. Then $B$ spans $W$ and is a linearly independent set of nonzero vectors. We can apply the Gram-Schmidt Procedure (Theorem GSPCV [191]) and obtain a linearly independent set $T$ such that $\langle T \rangle = \langle B \rangle = W$ and $T$ is orthogonal. In other words, $T$ is a basis for $W$, and is an orthogonal set. By scaling each vector of $T$ to norm $1$, we can convert $T$ into an orthonormal set, without destroying the properties that make it a basis of $W$. In short, we can convert any basis into an orthonormal basis. Example GSTV [192], followed by Example ONTV [193], illustrates this process.

Unitary matrices (Definition UM [252]) are another good source of orthonormal bases (and vice versa). Suppose that $Q$ is a unitary matrix of size $n$. Then the $n$ columns of $Q$ form an orthonormal set (Theorem CUMOS [253]) that is therefore linearly independent (Theorem OSLI [190]). Since $Q$ is invertible (Theorem OMI [??]), we know $Q$ is nonsingular (Theorem NI [251]), and then the columns of $Q$ span $\mathbb{C}^n$ (Theorem CSNM [267]). So the columns of a unitary matrix of size $n$ are an orthonormal basis for $\mathbb{C}^n$. 

Version 0.92
Why all the fuss about orthonormal bases? Theorem VRRB told us that any vector in a vector space could be written, uniquely, as a linear combination of basis vectors. For an orthonormal basis, finding the scalars for this linear combination is extremely easy, and this is the content of the next theorem. Furthermore, with vectors written this way (as linear combinations of the elements of an orthonormal set) certain computations and analysis become much easier. Here’s the promised theorem.

**Theorem COB**

**Coordinates and Orthonormal Bases**

Suppose that \( B = \{ v_1, v_2, v_3, \ldots, v_p \} \) is an orthonormal basis of the subspace \( W \) of \( \mathbb{C}^m \). For any \( w \in W \),

\[
  w = \langle w, v_1 \rangle v_1 + \langle w, v_2 \rangle v_2 + \langle w, v_3 \rangle v_3 + \cdots + \langle w, v_p \rangle v_p
\]

\[\square\]

**Proof** Because \( B \) is a basis of \( W \), Theorem VRRB tells us that we can write \( w \) uniquely as a linear combination of the vectors in \( B \). So it is not this aspect of the conclusion that makes this theorem interesting. What is interesting is that the particular scalars are so easy to compute. No need to solve big systems of equations — just do an inner product of \( w \) with \( v_i \) to arrive at the coefficient of \( v_i \) in the linear combination.

So begin the proof by writing \( w \) as a linear combination of the vectors in \( B \), using unknown scalars,

\[
  w = a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_p v_p
\]

and compute,

\[
  \langle w, v_i \rangle = \left\langle \sum_{k=1}^{p} a_k v_k, v_i \right\rangle
  = \sum_{k=1}^{p} \langle a_k v_k, v_i \rangle \quad \text{Theorem IPVA}
  = \sum_{k=1}^{p} a_k \langle v_k, v_i \rangle \quad \text{Theorem IPSM}
  = a_i \langle v_i, v_i \rangle + \sum_{k \neq i} a_k \langle v_k, v_i \rangle \quad \text{Isolate term with } k = i
  = a_i (1) + \sum_{k \neq i} a_k (0) \quad T \text{ orthonormal}
  = a_i
\]

So the (unique) scalars for the linear combination are indeed the inner products advertised in the conclusion of the theorem’s statement. \[\blacksquare\]

**Example CROB4**

**Coordinatization relative to an orthonormal basis, \( \mathbb{C}^4 \)**

The set

\[
  \{ x_1, x_2, x_3, x_4 \} = \begin{bmatrix}
  1 + i & 1 + 5i & -7 + 34i & -2 - 4i \\
  1 & 6 + 5i & -8 - 23i & 6 + i \\
  1 - i & -7 - i & -10 + 22i & 4 + 3i \\
  i & 1 - 6i & 30 + 13i & 6 - i
\end{bmatrix}
\]

Version 0.92
was proposed, and partially verified, as an orthogonal set in Example AOS [189]. Let’s scale each vector to norm 1, so as to form an orthonormal basis of $\mathbb{C}^4$. (Notice that by Theorem OSLI [190] the set is linearly independent. Since we know the dimension of $\mathbb{C}^4$ is 4, Theorem G [398] tells us the set is just the right size to be a basis of $\mathbb{C}^4$.) The norms of these vectors are, 
\[
\|x_1\| = \sqrt{6} \quad \|x_2\| = \sqrt{174} \quad \|x_3\| = \sqrt{3451} \quad \|x_4\| = \sqrt{119}
\]
So an orthonormal basis is 
\[
B = \{v_1, v_2, v_3, v_4\} = \{ \begin{pmatrix} 1 + i \\ 1 - i \end{pmatrix}, \begin{pmatrix} 1 + 5i \\ 6 + 5i \end{pmatrix}, \begin{pmatrix} -7 + 34i \\ -8 - 23i \end{pmatrix}, \begin{pmatrix} -2 - 4i \\ 6 + i \end{pmatrix} \}
\]
Now, choose any vector from $\mathbb{C}^4$, say 
\[
w = \begin{pmatrix} 2 \\ -3 \\ 1 \\ 4 \end{pmatrix},
\]
and compute 
\[
\langle w, v_1 \rangle = -\frac{5i}{\sqrt{6}}, \quad \langle w, v_2 \rangle = -\frac{19 + 30i}{\sqrt{174}}, \quad \langle w, v_3 \rangle = \frac{120 - 211i}{\sqrt{3451}}, \quad \langle w, v_4 \rangle = \frac{6 + 12i}{\sqrt{119}}
\]
then Theorem COB [371] guarantees that 
\[
\begin{pmatrix} 2 \\ -3 \\ 1 \\ 4 \end{pmatrix} = -\frac{5i}{\sqrt{6}} \begin{pmatrix} 1 + i \\ 1 - i \end{pmatrix} + \frac{19 + 30i}{\sqrt{174}} \begin{pmatrix} 1 + 5i \\ 6 + 5i \end{pmatrix} + \frac{120 - 211i}{\sqrt{3451}} \begin{pmatrix} -7 + 34i \\ -8 - 23i \end{pmatrix} + \frac{6 + 12i}{\sqrt{119}} \begin{pmatrix} -2 - 4i \\ 6 + i \end{pmatrix}
\]
as you might want to check (if you have unlimited patience).

A slightly less intimidating example follows, in three dimensions and with just real numbers.

**Example CROB3**

**Coordinitization relative to an orthonormal basis, $\mathbb{C}^3$**

The set 
\[
\{x_1, x_2, x_3\} = \{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \}
\]
is a linearly independent set, which the Gram-Schmidt Process (Theorem GSPCV [191]) converts to an orthogonal set, and which can then be converted to the orthonormal set, 
\[
B = \{v_1, v_2, v_3\} = \{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 + i \\ 1 - i \end{pmatrix}, \begin{pmatrix} 1 + 5i \\ 6 + 5i \end{pmatrix} \}
\]
which is therefore an orthonormal basis of $\mathbb{C}^3$. With three vectors in $\mathbb{C}^3$, all with real number entries, the inner product (Definition IP 184) reduces to the usual “dot product” (or scalar product) and the orthogonal pairs of vectors can be interpreted as perpendicular pairs of directions. So the vectors in $B$ serve as replacements for our usual 3-D axes, or the usual 3-D unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$. We would like to decompose arbitrary vectors into “components” in the directions of each of these basis vectors. It is Theorem COB 371 that tells us how to do this.

Suppose that we choose $w = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$. Compute

$$\langle w, v_1 \rangle = \frac{5}{\sqrt{6}}, \quad \langle w, v_2 \rangle = \frac{3}{\sqrt{2}}, \quad \langle w, v_3 \rangle = \frac{8}{\sqrt{3}}$$

then Theorem COB 371 guarantees that

$$\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \frac{5}{\sqrt{6}} \begin{bmatrix} 1 \\ \frac{1}{\sqrt{6}} \\ 2 \end{bmatrix} + \frac{3}{\sqrt{2}} \begin{bmatrix} -1 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + \frac{8}{\sqrt{3}} \begin{bmatrix} 1 \\ \frac{1}{\sqrt{3}} \\ -1 \end{bmatrix}$$

which you should be able to check easily, even if you do not have much patience.

Subsection B.READ
Reading Questions

1. The matrix below is nonsingular. What can you now say about its columns?

$$A = \begin{bmatrix} -3 & 0 & 1 \\ 1 & 2 & 1 \\ 5 & 1 & 6 \end{bmatrix}$$

2. Write the vector $w = \begin{bmatrix} 6 \\ 6 \\ 15 \end{bmatrix}$ as a linear combination of the columns of the matrix $A$ above. How many ways are there to answer this question?

3. Why is an orthonormal basis desirable?
Subsection EXC
Exercises

C40 From Example RSB, form an arbitrary (and nontrivial) linear combination of the four vectors in the original spanning set for $W$. So the result of this computation is of course an element of $W$. As such, this vector should be a linear combination of the basis vectors in $B$. Find the (unique) scalars that provide this linear combination. Repeat with another linear combination of the original four vectors.
Contributed by Robert Beezer Solution

C80 Prove that \{(1, 2), (2, 3)\} is a basis for the crazy vector space $C$. Example CVS.
Contributed by Robert Beezer

M20 In Example BM provide the verifications (linear independence and spanning) to show that $B$ is a basis of $M_{mn}$.
Contributed by Robert Beezer Solution
We need to establish the linear independence and spanning properties of the set

\[ B = \{ B_{k\ell} \mid 1 \leq k \leq m, \ 1 \leq \ell \leq n \} \]

relative to the vector space \( M_{mn} \).

This proof is more transparent if you write out individual matrices in the basis with lots of zeros and dots and a lone one. But we don’t have room for that here, so we will use summation notation. Think carefully about each step, especially when the double summations seem to “disappear.” Begin with a relation of linear dependence, using double subscripts on the scalars to align with the basis elements.

\[
\mathcal{O} = \sum_{k=1}^{m} \sum_{\ell=1}^{n} \alpha_{k\ell} B_{k\ell}
\]

Now consider the entry in row \( i \) and column \( j \) for these equal matrices,

\[
0 = [\mathcal{O}]_{ij} = \left[ \sum_{k=1}^{m} \sum_{\ell=1}^{n} \alpha_{k\ell} B_{k\ell} \right]_{ij} = \sum_{k=1}^{m} \sum_{\ell=1}^{n} [\alpha_{k\ell} B_{k\ell}]_{ij} = \sum_{k=1}^{m} \sum_{\ell=1}^{n} \alpha_{k\ell} [B_{k\ell}]_{ij} = \alpha_{ij} [B_{ij}]_{ij} = \alpha_{ij} (1) [B_{ij}]_{ij} = \alpha_{ij}
\]

Since \( i \) and \( j \) were arbitrary, we find that each scalar is zero and so \( B \) is linearly independent (Definition LI [345]).

To establish the spanning property of \( B \) we need only show that an arbitrary matrix \( A \) can be written as a linear combination of the elements of \( B \). So suppose that \( A \) is an arbitrary \( m \times n \) matrix and consider the matrix \( C \) defined as a linear combination of the elements of \( B \) by

\[
C = \sum_{k=1}^{m} \sum_{\ell=1}^{n} [A]_{k\ell} B_{k\ell}
\]

Then,

\[
[C]_{ij} = \left[ \sum_{k=1}^{m} \sum_{\ell=1}^{n} [A]_{k\ell} B_{k\ell} \right]_{ij}
\]
\[
= \sum_{k=1}^{m} \sum_{\ell=1}^{n} [A]_{k\ell} [B_{k\ell}]_{ij} \quad \text{Definition MA 198}
\]

\[
= \sum_{k=1}^{m} \sum_{\ell=1}^{n} [A]_{k\ell} [B_{k\ell}]_{ij} \quad \text{Definition MSM 198}
\]

\[
= [A]_{ij} [B_{ij}]_{ij} \quad [B_{k\ell}]_{ij} = 0 \text{ when } (k, \ell) \neq (i, j)
\]

\[
= [A]_{ij} (1) \quad [B_{ij}]_{ij} = 1
\]

So by Definition ME 197, \(A = C\), and therefore \(A \in \langle B \rangle\). By Definition B 363, the set \(B\) is a basis of the vector space \(M_{mn}\).

\textbf{C40} Contributed by Robert Beezer Statement 375

An arbitrary linear combination is

\[
y = 3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix} + (-2) \begin{bmatrix} -7 \\ -6 \\ -5 \end{bmatrix} = \begin{bmatrix} 25 \\ -10 \\ 15 \end{bmatrix}
\]

(You probably used a different collection of scalars.) We want to write \(y\) as a linear combination of

\[
B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 7/11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}
\]

We could set this up as vector equation with variables as scalars in a linear combination of the vectors in \(B\), but since the first two slots of \(B\) have such a nice pattern of zeros and ones, we can determine the necessary scalars easily and then double-check our answer with a computation in the third slot,

\[
25 \begin{bmatrix} 1 \\ 0 \\ 7/11 \end{bmatrix} + (-10) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 25 \\ -10 \\ (25)7/11 + (-10)1/11 \end{bmatrix} = \begin{bmatrix} 25 \\ -10 \\ 15 \end{bmatrix} = y
\]

Notice how the uniqueness of these scalars arises. They are forced to be 25 and \(-10\).
Section D
Dimension

Almost every vector space we have encountered has been infinite in size (an exception is [Example VSS 314]). But some are bigger and richer than others. Dimension, once suitably defined, will be a measure of the size of a vector space, and a useful tool for studying its properties. You probably already have a rough notion of what a mathematical definition of dimension might be — try to forget these imprecise ideas and go with the new ones given here.

Subsection D
Dimension

Definition D
Dimension
Suppose that \( V \) is a vector space and \( \{v_1, v_2, v_3, \ldots, v_t\} \) is a basis of \( V \). Then the \textbf{dimension} of \( V \) is defined by \( \dim(V) = t \). If \( V \) has no finite bases, we say \( V \) has infinite dimension.
(This definition contains Notation D.) △

This is a very simple definition, which belies its power. Grab a basis, any basis, and count up the number of vectors it contains. That’s the dimension. However, this simplicity causes a problem. Given a vector space, you and I could each construct different bases — remember that a vector space might have \textit{many} bases. And what if your basis and my basis had different sizes? Applying \textbf{Definition D 379} we would arrive at different numbers! With our current knowledge about vector spaces, we would have to say that dimension is not “well-defined.” Fortunately, there is a theorem that will correct this problem.

In a strictly logical progression, the next two theorems would \textit{precede} the definition of dimension. Many subsequent theorems will trace their lineage back to the following fundamental result.

Theorem SSLLD
Spanning Sets and Linear Dependence
Suppose that \( S = \{v_1, v_2, v_3, \ldots, v_t\} \) is a finite set of vectors which spans the vector space \( V \). Then any set of \( t + 1 \) or more vectors from \( V \) is linearly dependent. □

Proof We want to prove that any set of \( t + 1 \) or more vectors from \( V \) is linearly dependent. So we will begin with a totally arbitrary set of vectors from \( V \), \( R = \{u_1, u_2, u_3, \ldots, u_m\} \), where \( m > t \). We will now construct a nontrivial relation of linear dependence on \( R \).

Each vector \( u_1, u_2, u_3, \ldots, u_m \) can be written as a linear combination of \( v_1, v_2, v_3, \ldots, v_t \) since \( S \) is a spanning set of \( V \). This means there exist scalars \( a_{ij}, 1 \leq i \leq t, 1 \leq j \leq m \), so that
\[
u_1 = a_{11}v_1 + a_{21}v_2 + a_{31}v_3 + \cdots + a_{t1}v_t\]
As a collection of nontrivial scalars, of linear dependence we desire,

\[
\begin{align*}
\mathbf{u}_2 &= a_{12} \mathbf{v}_1 + a_{22} \mathbf{v}_2 + a_{32} \mathbf{v}_3 + \cdots + a_{t2} \mathbf{v}_t \\
\mathbf{u}_3 &= a_{13} \mathbf{v}_1 + a_{23} \mathbf{v}_2 + a_{33} \mathbf{v}_3 + \cdots + a_{t3} \mathbf{v}_t \\
&\vdots \\
\mathbf{u}_m &= a_{1m} \mathbf{v}_1 + a_{2m} \mathbf{v}_2 + a_{3m} \mathbf{v}_3 + \cdots + a_{tm} \mathbf{v}_t
\end{align*}
\]

Now we form, unmotivated, the homogeneous system of \( t \) equations in the \( m \) variables, \( x_1, x_2, x_3, \ldots, x_m \), where the coefficients are the just-discovered scalars \( a_{ij} \),

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1m}x_m &= 0 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2m}x_m &= 0 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3m}x_m &= 0 \\
&\vdots \\
a_{t1}x_1 + a_{t2}x_2 + a_{t3}x_3 + \cdots + a_{tm}x_m &= 0
\end{align*}
\]

This is a homogeneous system with more variables than equations (our hypothesis is expressed as \( m > t \)), so by Theorem HMVEI \[67\] there are infinitely many solutions. Choose a nontrivial solution and denote it by \( x_1 = c_1, x_2 = c_2, x_3 = c_3, \ldots, x_m = c_m \).

As a solution to the homogeneous system, we then have

\[
\begin{align*}
a_{11}c_1 + a_{12}c_2 + a_{13}c_3 + \cdots + a_{1m}c_m &= 0 \\
a_{21}c_1 + a_{22}c_2 + a_{23}c_3 + \cdots + a_{2m}c_m &= 0 \\
a_{31}c_1 + a_{32}c_2 + a_{33}c_3 + \cdots + a_{3m}c_m &= 0 \\
&\vdots \\
a_{t1}c_1 + a_{t2}c_2 + a_{t3}c_3 + \cdots + a_{tm}c_m &= 0
\end{align*}
\]

As a collection of nontrivial scalars, \( c_1, c_2, c_3, \ldots, c_m \) will provide the nontrivial relation of linear dependence we desire,

\[
\begin{align*}
c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3 + \cdots + c_m \mathbf{u}_m \\
&= c_1 \left( a_{11} \mathbf{v}_1 + a_{21} \mathbf{v}_2 + a_{31} \mathbf{v}_3 + \cdots + a_{t1} \mathbf{v}_t \right) \\
&\quad + c_2 \left( a_{12} \mathbf{v}_1 + a_{22} \mathbf{v}_2 + a_{32} \mathbf{v}_3 + \cdots + a_{t2} \mathbf{v}_t \right) \\
&\quad + c_3 \left( a_{13} \mathbf{v}_1 + a_{23} \mathbf{v}_2 + a_{33} \mathbf{v}_3 + \cdots + a_{t3} \mathbf{v}_t \right) \\
&\quad \vdots \\
&\quad + c_m \left( a_{1m} \mathbf{v}_1 + a_{2m} \mathbf{v}_2 + a_{3m} \mathbf{v}_3 + \cdots + a_{tm} \mathbf{v}_t \right) \\
&= c_1 a_{11} \mathbf{v}_1 + c_1 a_{21} \mathbf{v}_2 + c_1 a_{31} \mathbf{v}_3 + \cdots + c_1 a_{t1} \mathbf{v}_t \\
&\quad + c_2 a_{12} \mathbf{v}_1 + c_2 a_{22} \mathbf{v}_2 + c_2 a_{32} \mathbf{v}_3 + \cdots + c_2 a_{t2} \mathbf{v}_t \\
&\quad + c_3 a_{13} \mathbf{v}_1 + c_3 a_{23} \mathbf{v}_2 + c_3 a_{33} \mathbf{v}_3 + \cdots + c_3 a_{t3} \mathbf{v}_t \\
&\quad \vdots \\
&\quad + c_m a_{1m} \mathbf{v}_1 + c_m a_{2m} \mathbf{v}_2 + c_m a_{3m} \mathbf{v}_3 + \cdots + c_m a_{tm} \mathbf{v}_t \\
&= \left( c_1 a_{11} + c_2 a_{12} + c_3 a_{13} + \cdots + c_m a_{1m} \right) \mathbf{v}_1 \\
&\quad + \left( c_1 a_{21} + c_2 a_{22} + c_3 a_{23} + \cdots + c_m a_{2m} \right) \mathbf{v}_2 \text{ Property DSA 310}
\end{align*}
\]
\[ + (c_1 a_{31} + c_2 a_{32} + c_3 a_{33} + \cdots + c_m a_{3m}) v_3 \]
\[ + (c_1 a_{41} + c_2 a_{42} + c_3 a_{43} + \cdots + c_m a_{4m}) v_4 \]
\[ \vdots \]
\[ + (c_1 a_{t1} + c_2 a_{t2} + c_3 a_{t3} + \cdots + c_m a_{tm}) v_t \]
\[ = (a_{11} c_1 + a_{12} c_2 + a_{13} c_3 + \cdots + a_{1m} c_m) v_1 \]
\[ + (a_{21} c_1 + a_{22} c_2 + a_{23} c_3 + \cdots + a_{2m} c_m) v_2 \]
\[ + (a_{31} c_1 + a_{32} c_2 + a_{33} c_3 + \cdots + a_{3m} c_m) v_3 \]
\[ \vdots \]
\[ + (a_{t1} c_1 + a_{t2} c_2 + a_{t3} c_3 + \cdots + a_{tm} c_m) v_t \]
\[ = 0 v_1 + 0 v_2 + 0 v_3 + \cdots + 0 v_t \]
\[ = 0 + 0 + 0 + \cdots + 0 \]
\[ = 0 \]

That does it. \( R \) has been undeniably shown to be a linearly dependent set. \( \blacksquare \)

The proof just given has some rather monstrous expressions in it, mostly owing to the double subscripts present. Now is a great opportunity to show the value of a more compact notation. We will rewrite the key steps of the previous proof using summation notation, resulting in a more economical presentation, and even greater insight into the key aspects of the proof. So here is an alternate proof — study it carefully.

**Proof (Alternate Proof of Theorem SSLD)** We want to prove that any set of \( t + 1 \) or more vectors from \( V \) is linearly dependent. So we will begin with a totally arbitrary set of vectors from \( V, R = \{ u_j | 1 \leq j \leq m \}, \) where \( m > t \). We will now construct a nontrivial relation of linear dependence on \( R \).

Each vector \( u_j, 1 \leq j \leq m \) can be written as a linear combination of \( v_i, 1 \leq i \leq t \) since \( S \) is a spanning set of \( V \). This means there are scalars \( a_{ij}, 1 \leq i \leq t, 1 \leq j \leq m \), so that

\[ u_j = \sum_{i=1}^{t} a_{ij} v_i \]

Now we form, unmotivated, the homogeneous system of \( t \) equations in the \( m \) variables, \( x_j, 1 \leq j \leq m \), where the coefficients are the just-discovered scalars \( a_{ij}, \)

\[ \sum_{j=1}^{m} a_{ij} x_j = 0 \quad 1 \leq i \leq t \]

This is a homogeneous system with more variables than equations (our hypothesis is expressed as \( m > t \)), so by [Theorem HMVEI 67] there are infinitely many solutions. Choose one of these solutions that is not trivial and denote it by \( x_j = c_j, 1 \leq j \leq m \).

As a solution to the homogeneous system, we then have \( \sum_{j=1}^{m} a_{ij} c_j = 0 \) for \( 1 \leq i \leq t \). As a collection of nontrivial scalars, \( c_j, 1 \leq j \leq m \), will provide the nontrivial relation of linear dependence we desire,

\[ \sum_{j=1}^{m} c_j u_j = \sum_{j=1}^{m} c_j \left( \sum_{i=1}^{t} a_{ij} v_i \right) \]
\[ = \sum_{j=1}^{m} \sum_{i=1}^{t} c_j a_{ij} v_i \]

\( S \) spans \( V \)

\[ \text{Property DVA 310} \]
\[ = \sum_{i=1}^{t} \sum_{j=1}^{m} c_{ij} v_i \quad \text{Commutativity in } \mathbb{C} \]

\[ = \sum_{i=1}^{t} \sum_{j=1}^{m} a_{ij} c_j v_i \quad \text{Commutativity in } \mathbb{C} \]

\[ = \sum_{i=1}^{t} \left( \sum_{j=1}^{m} a_{ij} c_j \right) v_i \quad \text{Property DSA 310} \]

\[ = \sum_{i=1}^{t} 0 v_i \quad c_j \text{ as solution} \]

\[ = \sum_{i=1}^{t} 0 \quad \text{Theorem ZSSM 317} \]

\[ = 0 \quad \text{Property Z 310} \]

That does it. \( R \) has been undeniably shown to be a linearly dependent set. ■

Notice how the swap of the two summations is so much easier in the third step above, as opposed to all the rearranging and regrouping that takes place in the previous proof. In about half the space. And there are no ellipses (\ldots).

**Theorem SSLD 379** can be viewed as a generalization of **Theorem MVSLD 150**. We know that \( \mathbb{C}^m \) has a basis with \( m \) vectors in it (Theorem SUVB 363), so it is a set of \( m \) vectors that spans \( \mathbb{C}^m \). By **Theorem SSLD 379**, any set of more than \( m \) vectors from \( \mathbb{C}^m \) will be linearly dependent. But this is exactly the conclusion we have in **Theorem MVSLD 150**. Maybe this is not a total shock, as the proofs of both theorems rely heavily on **Theorem HMVEI 67**. The beauty of **Theorem SSLD 379** is that it applies in any vector space. We illustrate the generality of this theorem, and hint at its power, in the next example.

**Example LDP4**

**Linearly dependent set in \( P_4 \)**

In **Example SSP4 350** we showed that

\[ S = \{ x - 2, x^2 - 4x + 4, x^3 - 6x^2 + 12x - 8, x^4 - 8x^3 + 24x^2 - 32x + 16 \} \]

is a spanning set for \( W = \{ p(x) \mid p \in P_4, p(2) = 0 \} \). So we can apply **Theorem SSLD 379** to \( W \) with \( t = 4 \). Here is a set of five vectors from \( W \), as you may check by verifying that each is a polynomial of degree 4 or less and has \( x = 2 \) as a root,

\[ T = \{ p_1, p_2, p_3, p_4, p_5 \} \subseteq W \]

\[ p_1 = x^4 - 2x^3 + 2x^2 - 8x + 8 \]
\[ p_2 = -x^3 + 6x^2 - 5x - 6 \]
\[ p_3 = 2x^4 - 5x^3 + 5x^2 - 7x + 2 \]
\[ p_4 = -x^4 + 4x^3 - 7x^2 + 6x \]
\[ p_5 = 4x^3 - 9x^2 + 5x - 6 \]

By **Theorem SSLD 379** we conclude that \( T \) is linearly dependent, with no further computations. ■
Theorem SSLD\[379\] is indeed powerful, but our main purpose in proving it right now was to make sure that our definition of dimension (Definition D\[379\]) is well-defined. Here’s the theorem.

**Theorem BIS**

**Bases have Identical Sizes**

Suppose that \(V\) is a vector space with a finite basis \(B\) and a second basis \(C\). Then \(B\) and \(C\) have the same size.

**Proof** Suppose that \(C\) has more vectors than \(B\). (Allowing for the possibility that \(C\) is infinite, we can replace \(C\) by a subset that has more vectors than \(B\).) As a basis, \(B\) is a spanning set for \(V\) (Definition B\[363\]), so Theorem SSLD\[379\] says that \(C\) is linearly dependent. However, this contradicts the fact that as a basis \(C\) is linearly independent (Definition B\[363\]). So \(C\) must also be a finite set, with size less than, or equal to, that of \(B\).

Suppose that \(B\) has more vectors than \(C\). As a basis, \(C\) is a spanning set for \(V\) (Definition B\[363\]), so Theorem SSLD\[379\] says that \(B\) is linearly dependent. However, this contradicts the fact that as a basis \(B\) is linearly independent (Definition B\[363\]). So \(C\) cannot be strictly smaller than \(B\).

The only possibility left for the sizes of \(B\) and \(C\) is for them to be equal. □

Theorem BIS\[383\] tells us that if we find one finite basis in a vector space, then they all have the same size. This (finally) makes Definition D\[379\] unambiguous.

**Subsection DVS**

**Dimension of Vector Spaces**

We can now collect the dimension of some common, and not so common, vector spaces.

**Theorem DCM**

**Dimension of \(\mathbb{C}^m\)**

The dimension of \(\mathbb{C}^m\) (Example VSCV\[311\]) is \(m\). □

**Proof** Theorem SUVB\[363\] provides a basis with \(m\) vectors. ■

**Theorem DP**

**Dimension of \(P_n\)**

The dimension of \(P_n\) (Example VSP\[312\]) is \(n + 1\). □

**Proof** Example BP\[364\] provides two bases with \(n + 1\) vectors. Take your pick. ■

**Theorem DM**

**Dimension of \(M_{mn}\)**

The dimension of \(M_{mn}\) (Example VSM\[311\]) is \(mn\). □

**Proof** Example BM\[364\] provides a basis with \(mn\) vectors. ■

**Example DSM22**

**Dimension of a subspace of \(M_{22}\)**

It should now be plausible that

\[
Z = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid 2a + b + 3c + 4d = 0, -a + 3b - 5c - d = 0 \right\}
\]
is a subspace of the vector space $M_{22}$ (Example VSM [311]). (It is.) To find the dimension of $Z$ we must first find a basis, though any old basis will do.

First concentrate on the conditions relating $a$, $b$, $c$ and $d$. They form a homogeneous system of two equations in four variables with coefficient matrix

$$
\begin{bmatrix}
2 & 1 & 3 & 4 \\
-1 & 3 & -5 & -1
\end{bmatrix}
$$

We can row-reduce this matrix to obtain

$$
\begin{bmatrix}
1 & 0 & 2 & 2 \\
0 & 1 & -1 & 0
\end{bmatrix}
$$

Rewrite the two equations represented by each row of this matrix, expressing the dependent variables ($a$ and $b$) in terms of the free variables ($c$ and $d$), and we obtain,

$$a = -2c - 2d$$
$$b = c$$

We can now write a typical entry of $Z$ strictly in terms of $c$ and $d$, and we can decompose the result,

$$\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} = \begin{bmatrix}
-2c - 2d & c \\
c & d
\end{bmatrix} = \begin{bmatrix}
-2c & c \\
c & 0
\end{bmatrix} + \begin{bmatrix}
-2d & 0 \\
0 & d
\end{bmatrix} = c \begin{bmatrix}
-2 & 1 \\
1 & 0
\end{bmatrix} + d \begin{bmatrix}
-2 & 0 \\
0 & 1
\end{bmatrix}$$

This equation says that an arbitrary matrix in $Z$ can be written as a linear combination of the two vectors in

$$S = \left\{ \begin{bmatrix}
-2 & 1 \\
1 & 0
\end{bmatrix}, \begin{bmatrix}
-2 & 0 \\
0 & 1
\end{bmatrix} \right\}$$

so we know that

$$Z = \langle S \rangle = \left\langle \left\{ \begin{bmatrix}
-2 & 1 \\
1 & 0
\end{bmatrix}, \begin{bmatrix}
-2 & 0 \\
0 & 1
\end{bmatrix} \right\} \right\}$$

Are these two matrices (vectors) also linearly independent? Begin with a relation of linear dependence on $S$,

$$a_1 \begin{bmatrix}
-2 & 1 \\
1 & 0
\end{bmatrix} + a_2 \begin{bmatrix}
-2 & 0 \\
0 & 1
\end{bmatrix} = \mathbf{0}$$

From the equality of the two entries in the last row, we conclude that $a_1 = 0$, $a_2 = 0$. Thus the only possible relation of linear dependence is the trivial one, and therefore $S$ is linearly independent (Definition LI [345]). So $S$ is a basis for $V$ (Definition B [363]). Finally, we can conclude that dim $(Z) = 2$ (Definition D [379]) since $S$ has two elements.

\*

**Example DSP4**

**Dimension of a subspace of $P_4$**

In Example BSP4 [365] we showed that

$$S = \{ x - 2, x^2 - 4x + 4, x^3 - 6x^2 + 12x - 8, x^4 - 8x^3 + 24x^2 - 32x + 16 \}$$
is a basis for \( W = \{ p(x) \mid p \in P_4, \ p(2) = 0 \} \). Thus, the dimension of \( W \) is four, \( \dim(W) = 4 \).

Example DC

Dimension of the crazy vector space

In Example BC [366] we determined that the set \( R = \{ (1,0), (6,3) \} \) from the crazy vector space, \( C \) (Example CVS [314]), is a basis for \( C \). By Definition D [379] we see that \( C \) has dimension 2, \( \dim(C) = 2 \).

It is possible for a vector space to have no finite bases, in which case we say it has infinite dimension. Many of the best examples of this are vector spaces of functions, which lead to constructions like Hilbert spaces. We will focus exclusively on finite-dimensional vector spaces. OK, one infinite-dimensional example, and then we will focus exclusively on finite-dimensional vector spaces.

Example VSPUD

Vector space of polynomials with unbounded degree

Define the set \( P \) by

\[
P = \{ p \mid p(x) \text{ is a polynomial in } x \}
\]

Our operations will be the same as those defined for \( P_n \) (Example VSP [312]).

With no restrictions on the possible degrees of our polynomials, any finite set that is a candidate for spanning \( P \) will come up short. We will give a proof by contradiction (Technique CD [708]). To this end, suppose that the dimension of \( P \) is finite, say \( \dim(P) = n \).

The set \( T = \{ 1, x, x^2, \ldots, x^n \} \) is a linearly independent set (check this!) containing \( n + 1 \) polynomials from \( P \). However, a basis of \( P \) will be a spanning set of \( P \) containing \( n \) vectors. This situation is a contradiction of Theorem SSLD [379], so our assumption that \( P \) has finite dimension is false. Thus, we say \( \dim(P) = \infty \).

Subsection RNM

Rank and Nullity of a Matrix

For any matrix, we have seen that we can associate several subspaces — the null space (Theorem NSMS [330]), the column space (Theorem CSMS [336]), row space (Theorem RSMS [337]) and the left null space (Theorem LNSMS [337]). As vector spaces, each of these has a dimension, and for the null space and column space, they are important enough to warrant names.

Definition NOM

Nullity Of a Matrix

Suppose that \( A \) is an \( m \times n \) matrix. Then the nullity of \( A \) is the dimension of the null space of \( A \), \( n(A) = \dim(\mathcal{N}(A)) \).

(This definition contains Notation NOM.)

Definition ROM

Rank Of a Matrix

Suppose that \( A \) is an \( m \times n \) matrix. Then the rank of \( A \) is the dimension of the column space of \( A \), \( r(A) = \dim(\mathcal{C}(A)) \).
Example RNM

Rank and nullity of a matrix

Let’s compute the rank and nullity of

\[
A = \begin{bmatrix}
2 & -4 & -1 & 3 & 2 & 1 & -4 \\
1 & -2 & 0 & 0 & 4 & 0 & 1 \\
-2 & 4 & 1 & 0 & -5 & -4 & -8 \\
1 & -2 & 1 & 1 & 6 & 1 & -3 \\
2 & -4 & -1 & 1 & 4 & -2 & -1 \\
-1 & 2 & 3 & -1 & 6 & 3 & -1 \\
\end{bmatrix}
\]

To do this, we will first row-reduce the matrix since that will help us determine bases for the null space and column space.

\[
\begin{bmatrix}
1 & -2 & 0 & 0 & 4 & 0 & 1 \\
0 & 0 & 1 & 0 & 3 & 0 & -2 \\
0 & 0 & 0 & 1 & -1 & 0 & -3 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

From this row-equivalent matrix in reduced row-echelon form we record

\[D = \{1, 3, 4, 6\}\]

and \[F = \{2, 5, 7\}\.\]

For each index in \(D\), \text{Theorem BCS 264\} creates a single basis vector. In total the basis will have 4 vectors, so the column space of \(A\) will have dimension 4 and we write \(r(A) = 4\).

For each index in \(F\), \text{Theorem BNS 154\} creates a single basis vector. In total the basis will have 3 vectors, so the null space of \(A\) will have dimension 3 and we write \(n(A) = 3\).

There were no accidents or coincidences in the previous example — with the row-reduced version of a matrix in hand, the rank and nullity are easy to compute.

\textbf{Theorem CRN}

\textbf{Computing Rank and Nullity}

Suppose that \(A\) is an \(m \times n\) matrix and \(B\) is a row-equivalent matrix in reduced row-echelon form with \(r\) nonzero rows. Then \(r(A) = r\) and \(n(A) = n - r\).

\textbf{Proof} \text{Theorem BCS 264\} provides a basis for the column space by choosing columns of \(A\) that correspond to the dependent variables in a description of the solutions to \(LS(A, 0)\.\) In the analysis of \(B\), there is one dependent variable for each leading 1, one per nonzero row, or one per pivot column. So there are \(r\) column vectors in a basis for \(C(A)\.\)

\text{Theorem BNS 154\} provide a basis for the null space by creating basis vectors of the null space of \(A\) from entries of \(B\), one for each independent variable, one per column without a leading 1. So there are \(n - r\) column vectors in a basis for \(n(A)\.\)

Every archetype (Appendix A 177\) that involves a matrix lists its rank and nullity. You may have noticed as you studied the archetypes that the larger the column space
is the smaller the null space is. A simple corollary states this trade-off succinctly. (See Technique LC 716.)

**Theorem RPNC**

**Rank Plus Nullity is Columns**

Suppose that $A$ is an $m \times n$ matrix. Then $r (A) + n (A) = n$. □

**Proof** Let $r$ be the number of nonzero rows in a row-equivalent matrix in reduced row-echelon form. By Theorem CRN 386,

$$r (A) + n (A) = r + (n - r) = n$$

When we first introduced $r$ as our standard notation for the number of nonzero rows in a matrix in reduced row-echelon form you might have thought $r$ stood for “rows.” Not really — it stands for “rank”!

**Subsection RNNM**

**Rank and Nullity of a Nonsingular Matrix**

Let’s take a look at the rank and nullity of a square matrix.

**Example RNSM**

**Rank and nullity of a square matrix**

The matrix

$$E = \begin{bmatrix}
0 & 4 & -1 & 2 & 2 & 3 & 1 \\
2 & -2 & 1 & -1 & 0 & -4 & -3 \\
-2 & -3 & 9 & -3 & 9 & -1 & 9 \\
-3 & -4 & 9 & 4 & -1 & 6 & -2 \\
-3 & -4 & 6 & -2 & 5 & 9 & -4 \\
9 & -3 & 8 & -2 & -4 & 2 & 4 \\
8 & 2 & 2 & 9 & 3 & 0 & 9
\end{bmatrix}$$

is row-equivalent to the matrix in reduced row-echelon form,

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

With $n = 7$ columns and $r = 7$ nonzero rows Theorem CRN 386 tells us the rank is $r (E) = 7$ and the nullity is $n (E) = 7 - 7 = 0$. □

The value of either the nullity or the rank are enough to characterize a nonsingular matrix.

**Theorem RNNM**

**Rank and Nullity of a Nonsingular Matrix**

Suppose that $A$ is a square matrix of size $n$. The following are equivalent.
1. A is nonsingular.

2. The rank of $A$ is $n$, $r(A) = n$.

3. The nullity of $A$ is zero, $n(A) = 0$.

Proof \hspace{1cm} (1 $\Rightarrow$ 2) Theorem CSNM \[267\] says that if $A$ is nonsingular then $\mathcal{C}(A) = \mathbb{C}^n$. If $\mathcal{C}(A) = \mathbb{C}^n$, then the column space has dimension $n$ by Theorem DCM \[383\], so the rank of $A$ is $n$.

(2 $\Rightarrow$ 3) Suppose $r(A) = n$. Then Theorem RPNC \[387\] gives

$$
\begin{align*}
n(A) &= n - r(A) \\
&= n - n \\
&= 0
\end{align*}
$$

(3 $\Rightarrow$ 1) Suppose $n(A) = 0$, so a basis for the null space of $A$ is the empty set. This implies that $\mathcal{N}(A) = \{0\}$ and Theorem NMTNS \[78\] says $A$ is nonsingular.

With a new equivalence for a nonsingular matrix, we can update our list of equivalences (Theorem NME5 \[370\]) which now becomes a list requiring double digits to number.

Theorem NME6
Nonsingular Matrix Equivalences, Round 6
Suppose that $A$ is a square matrix of size $n$. The following are equivalent.

1. $A$ is nonsingular.

2. $A$ row-reduces to the identity matrix.

3. The null space of $A$ contains only the zero vector, $\mathcal{N}(A) = \{0\}$.

4. The linear system $\mathcal{L}S(A, b)$ has a unique solution for every possible choice of $b$.

5. The columns of $A$ are a linearly independent set.

6. $A$ is invertible.

7. The column space of $A$ is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of $A$ are a basis for $\mathbb{C}^n$.

9. The rank of $A$ is $n$, $r(A) = n$.

10. The nullity of $A$ is zero, $n(A) = 0$.

Proof Building on Theorem NME5 \[370\] we can add two of the statements from Theorem RNNM \[387\].

Version 0.92
1. What is the dimension of the vector space $P_6$, the set of all polynomials of degree 6 or less?

2. How are the rank and nullity of a matrix related?

3. Explain why we might say that a nonsingular matrix has “full rank.”
Subsection EXC
Exercises

C20  The archetypes listed below are matrices, or systems of equations with coefficient matrices. For each, compute the nullity and rank of the matrix. This information is listed for each archetype (along with the number of columns in the matrix, so as to illustrate Theorem RPNC [387]), and notice how it could have been computed immediately after the determination of the sets $D$ and $F$ associated with the reduced row-echelon form of the matrix.

Archetype A  [721]
Archetype B  [726]
Archetype C  [731]
Archetype D  [735] Archetype E  [739]
Archetype F  [743]
Archetype G  [748] Archetype H  [752]
Archetype I  [757]
Archetype J  [762]
Archetype K  [767]
Archetype L  [771]
Contributed by Robert Beezer

C30  For the matrix $A$ below, compute the dimension of the null space of $A$, $\dim \langle N(A) \rangle$.

$$A = \begin{bmatrix} 2 & -1 & -3 & 11 & 9 \\ 1 & 2 & 1 & -7 & -3 \\ 3 & 1 & -3 & 6 & 8 \\ 2 & 1 & 2 & -5 & -3 \end{bmatrix}$$

Contributed by Robert Beezer  Solution 393

C31  The set $W$ below is a subspace of $\mathbb{C}^4$. Find the dimension of $W$.

$$W = \left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 2 \\ 5 \end{bmatrix} \right\}$$

Contributed by Robert Beezer  Solution 393

C40  In Example LDP4 [382] we determined that the set of five polynomials, $T$, is linearly dependent by a simple invocation of Theorem SSLD [379]. Prove that $T$ is linearly dependent from scratch, beginning with Definition LI [345].

Contributed by Robert Beezer

M20  $M_{22}$ is the vector space of $2 \times 2$ matrices. Let $S_{22}$ denote the set of all $2 \times 2$ symmetric matrices. That is

$$S_{22} = \{ A \in M_{22} \mid A^t = A \}$$
(a) Show that $S_{22}$ is a subspace of $M_{22}$.
(b) Exhibit a basis for $S_{22}$ and prove that it has the required properties.
(c) What is the dimension of $S_{22}$?
Contributed by Robert Beezer  Solution 393

**M21** A $2 \times 2$ matrix $B$ is upper-triangular if $[B]_{21} = 0$. Let $UT_2$ be the set of all $2 \times 2$ upper-triangular matrices. Then $UT_2$ is a subspace of the vector space of all $2 \times 2$ matrices, $M_{22}$ (you may assume this). Determine the dimension of $UT_2$ providing all of the necessary justifications for your answer.
Contributed by Robert Beezer  Solution 394
Subsection SOL
Solutions

C30 Contributed by Robert Beezer Statement 391

Row reduce $A$,

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & -3 & -1 \\
0 & 0 & 1 & -2 & -2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

So $r = 3$ for this matrix. Then

$$
\dim (\mathcal{N}(A)) = n(A) = (n(A) + r(A)) - r(A) = 5 - r(A) = 5 - 3 = 2
$$

We could also use Theorem BNS and create a basis for $\mathcal{N}(A)$ with $n - r = 5 - 3 = 2$ vectors (because the solutions are described with 2 free variables) and arrive at the dimension as the size of this basis.

C31 Contributed by Robert Beezer Statement 391

We will appeal to Theorem BS (or you could consider this an appeal to Theorem BCS). Put the three column vectors of this spanning set into a matrix as columns and row-reduce.

$$
\begin{bmatrix}
2 & 3 & -4 \\
-3 & 0 & -3 \\
4 & 1 & 2 \\
1 & -2 & 5
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

The pivot columns are $D = \{1, 2\}$ so we can “keep” the vectors corresponding to the pivot columns and set

$$
T = \left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}
$$

and conclude that $W = \langle T \rangle$ and $T$ is linearly independent. In other words, $T$ is a basis with two vectors, so $W$ has dimension 2.

M20 Contributed by Robert Beezer Statement 391

(a) We will use the three criteria of Theorem TSS. The zero matrix of $M_{22}$ is the zero matrix, $0$ (Definition ZM), which is a symmetric matrix. So $S_{22}$ is not empty, since $0 \in S_{22}$.

Suppose that $A$ and $B$ are two matrices in $S_{22}$. Then we know that $A^t = A$ and $B^t = B$. We want to know if $A + B \in S_{22}$, so test $A + B$ for membership,

$$
(A + B)^t = A^t + B^t
$$
\[ A + B \quad A, B \in S_{22} \]

So \( A + B \) is symmetric and qualifies for membership in \( S_{22} \).

Suppose that \( A \in S_{22} \) and \( \alpha \in \mathbb{C} \). Is \( \alpha A \in S_{22} \)? We know that \( A^t = A \). Now check that,

\[
\alpha A^t = \alpha A^t \\
= \alpha A \\
A \in S_{22}
\]

So \( \alpha A \) is also symmetric and qualifies for membership in \( S_{22} \).

With the three criteria of Theorem TSS fulfilled, we see that \( S_{22} \) is a subspace of \( M_{22} \).

(b) An arbitrary matrix from \( S_{22} \) can be written as \( \begin{bmatrix} a & b \\ b & d \end{bmatrix} \). We can express this matrix as

\[
\begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}
\]

\[
= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

this equation says that the set

\[
T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]

spans \( S_{22} \). Is it also linearly independent?

Write a relation of linear dependence on \( S \),

\[
O = a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}
\]

The equality of these two matrices (Definition ME) tells us that \( a_1 = a_2 = a_3 = 0 \), and the only relation of linear dependence on \( T \) is trivial. So \( T \) is linearly independent, and hence is a basis of \( S_{22} \).

(c) The basis \( T \) found in part (b) has size 3. So by Definition D, \( \dim (S_{22}) = 3 \).

M21 Contributed by Robert Beezer Statement

A typical matrix from \( UT_2 \) looks like

\[
\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}
\]

where \( a, b, c \in \mathbb{C} \) are arbitrary scalars. Observing this we can then write

\[
\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

which says that

\[
R = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]
is a spanning set for $UT_2$ (Definition TSVS [350]). Is $R$ is linearly independent? If so, it is a basis for $UT_2$. So consider a relation of linear dependence on $R$,

$$
\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

From this equation, one rapidly arrives at the conclusion that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. So $R$ is a linearly independent set (Definition LI [345]), and hence is a basis (Definition B [363]) for $UT_2$. Now, we simply count up the size of the set $R$ to see that the dimension of $UT_2$ is $\dim(UT_2) = 3$. 
Once the dimension of a vector space is known, then the determination of whether or not a set of vectors is linearly independent, or if it spans the vector space, can often be much easier. In this section we will state a workhorse theorem and then apply it to the column space and row space of a matrix. It will also help us describe a super-basis for $\mathbb{C}^m$.

Subsection GT
Goldilocks’ Theorem

We begin with a useful theorem that we will need later, and in the proof of the main theorem in this subsection. This theorem says that we can extend linearly independent sets, one vector at a time, by adding vectors from outside the span of the linearly independent set, all the while preserving the linear independence of the set.

**Theorem ELIS**
Extending Linearly Independent Sets

Suppose $V$ is vector space and $S$ is a linearly independent set of vectors from $V$. Suppose $w$ is a vector such that $w \notin \langle S \rangle$. Then the set $S' = S \cup \{w\}$ is linearly independent.

**Proof**
Suppose $S = \{v_1, v_2, v_3, \ldots, v_m\}$ and begin with a relation of linear dependence on $S'$,

$$a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_mv_m + a_{m+1}w = 0.$$

There are two cases to consider. First suppose that $a_{m+1} = 0$. Then the relation of linear dependence on $S'$ becomes

$$a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_mv_m = 0.$$

and by the linear independence of the set $S$, we conclude that $a_1 = a_2 = a_3 = \cdots = a_m = 0$. So all of the scalars in the relation of linear dependence on $S'$ are zero.

In the second case, suppose that $a_{m+1} \neq 0$. Then the relation of linear dependence on $S'$ becomes

$$a_{m+1}w = -a_1v_1 - a_2v_2 - a_3v_3 - \cdots - a_mv_m$$

$$w = -\frac{a_1}{a_{m+1}}v_1 - \frac{a_2}{a_{m+1}}v_2 - \frac{a_3}{a_{m+1}}v_3 - \cdots - \frac{a_m}{a_{m+1}}v_m$$

This equation expresses $w$ as a linear combination of the vectors in $S$, contrary to the assumption that $w \notin \langle S \rangle$, so this case leads to a contradiction.

The first case yielded only a trivial relation of linear dependence on $S'$ and the second case led to a contradiction. So $S'$ is a linearly independent set since any relation of linear dependence is trivial. ■
In the story *Goldilocks and the Three Bears*, the young girl Goldilocks visits the empty house of the three bears while out walking in the woods. One bowl of porridge is too hot, the other too cold, the third is just right. One chair is too hard, one too soft, the third is just right. So it is with sets of vectors — some are too big (linearly dependent), some are too small (they don’t span), and some are just right (bases). Here’s Goldilocks’ Theorem.

**Theorem G**

**Goldilocks**

Suppose that $V$ is a vector space of dimension $t$. Let $S = \{v_1, v_2, v_3, \ldots, v_m\}$ be a set of vectors from $V$. Then

1. If $m > t$, then $S$ is linearly dependent.
2. If $m < t$, then $S$ does not span $V$.
3. If $m = t$ and $S$ is linearly independent, then $S$ spans $V$.
4. If $m = t$ and $S$ spans $V$, then $S$ is linearly independent.

**Proof**

Let $B$ be a basis of $V$. Since $\dim(V) = t$, Definition B [363] and Theorem BIS [383] imply that $B$ is a linearly independent set of $t$ vectors that spans $V$.

1. Suppose to the contrary that $S$ is linearly independent. Then $B$ is a smaller set of vectors that spans $V$. This contradicts Theorem SSLD [379].

2. Suppose to the contrary that $S$ does span $V$. Then $B$ is a larger set of vectors that is linearly independent. This contradicts Theorem SSLD [379].

3. Suppose to the contrary that $S$ does not span $V$. Then we can choose a vector $w$ such that $w \in V$ and $w \notin \langle S \rangle$. By Theorem ELIS [397], the set $S' = S \cup \{w\}$ is again linearly independent. Then $S'$ is a set of $m + 1 = t + 1$ vectors that are linearly independent, while $B$ is a set of $t$ vectors that span $V$. This contradicts Theorem SSLD [379].

4. Suppose to the contrary that $S$ is linearly dependent. Then by Theorem DLDS [167] (which can be upgraded, with no changes in the proof, to the setting of a general vector space), there is a vector in $S$, say $v_k$, that is equal to a linear combination of the other vectors in $S$. Let $S' = S \setminus \{v_k\}$, the set of “other” vectors in $S$. Then it is easy to show that $V = \langle S \rangle = \langle S' \rangle$. So $S'$ is a set of $m - 1 = t - 1$ vectors that spans $V$, while $B$ is a set of $t$ linearly independent vectors in $V$. This contradicts Theorem SSLD [379].

There is a tension in the construction of basis. Make a set too big and you will end up with relations of linear dependence among the vectors. Make a set too small and you will not have enough raw material to span the entire vector space. Make a set just the right size (the dimension) and you only need to have linear independence or spanning, and you get the other property for free. These roughly-stated ideas are made precise by Theorem G [398].
The structure and proof of this theorem also deserve comment. The hypotheses seem innocuous. We presume we know the dimension of the vector space in hand, then we mostly just look at the size of the set \( S \). From this we get big conclusions about spanning and linear independence. Each of the four proofs relies on ultimately contradicting Theorem SSLD [379], so in a way we could think of this entire theorem as a corollary of Theorem SSLD [379]. (See Technique LC [716].) The proofs of the third and fourth parts parallel each other in style (add \( w \), toss \( v_k \)) and then turn on Theorem ELIS [397] before contradicting Theorem SSLD [379]. 

Theorem G [398] is useful in both concrete examples and as a tool in other proofs. We will use it often to bypass verifying linear independence or spanning.

**Example BPR**

**Bases for \( P_n \), reprised**

In Example BP [364] we claimed that

\[
B = \{1, x, x^2, x^3, \ldots, x^n\}
\]

\[
C = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3, \ldots, 1 + x + x^2 + x^3 + \cdots + x^n\}
\]

were both bases for \( P_n \) (Example VSP [312]). Suppose we had first verified that \( B \) was a basis, so we would then know that \( \dim(P_n) = n + 1 \). The size of \( C \) is \( n + 1 \), the right size to be a basis. We could then verify that \( C \) is linearly independent. We would not have to make any special efforts to prove that \( C \) spans \( P_n \), since Theorem G [398] would allow us to conclude this property of \( C \) directly. Then we would be able to say that \( C \) is a basis of \( P_n \) also.

**Example BDM22**

**Basis by dimension in \( M_{22} \)**

In Example DSM22 [383] we showed that

\[
B = \begin{\{\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\end{\} }
\]

is a basis for the subspace \( Z \) of \( M_{22} \) (Example VSM [311]) given by

\[
Z = \begin{\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid 2a + b + 3c + 4d = 0, -a + 3b - 5c - d = 0\end{\} }
\]

This tells us that \( \dim(Z) = 2 \). In this example we will find another basis. We can construct two new matrices in \( Z \) by forming linear combinations of the matrices in \( B \).

\[
2 \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} + (-3) \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -3 \end{bmatrix}
\]

\[
3 \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 3 \\ 3 & 1 \end{bmatrix}
\]

Then the set

\[
C = \begin{\{\begin{bmatrix} 2 & 2 \\ 2 & -3 \end{bmatrix}, \begin{bmatrix} -8 & 3 \\ 3 & 1 \end{bmatrix}\end{\} }
\]

has the right size to be a basis of \( Z \). Let’s see if it is a linearly independent set. The relation of linear dependence

\[
a_1 \begin{bmatrix} 2 & 2 \\ 2 & -3 \end{bmatrix} + a_2 \begin{bmatrix} -8 & 3 \\ 3 & 1 \end{bmatrix} = O
\]
leads to the homogeneous system of equations whose coefficient matrix

$$\begin{bmatrix} 2 & -8 \\ 2 & 3 \\ 2 & 3 \\ -3 & 1 \end{bmatrix}$$

row-reduces to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So with $a_1 = a_2 = 0$ as the only solution, the set is linearly independent. Now we can apply Theorem G 398 to see that $C$ also spans $Z$ and therefore is a second basis for $Z$.

**Example SVP4**

**Sets of vectors in $P_4$**

In Example BSP4 365 we showed that

$$B = \{ x - 2, x^2 - 4x + 4, x^3 - 6x^2 + 12x - 8, x^4 - 8x^3 + 24x^2 - 32x + 16 \}$$

is a basis for $W = \{ p(x) \mid p \in P_4, \ p(2) = 0 \}$. So $\dim (W) = 4$.

The set

$$\{ 3x^2 - 5x - 2, 2x^2 - 7x + 6, x^3 - 2x^2 + x - 2 \}$$

is a subset of $W$ (check this) and it happens to be linearly independent (check this, too). However, by Theorem G 398 it cannot span $W$.

The set

$$\{ 3x^2 - 5x - 2, 2x^2 - 7x + 6, x^3 - 2x^2 + x - 2, -x^4 + 2x^3 + 5x^2 - 10x, x^4 - 16 \}$$

is another subset of $W$ (check this) and Theorem G 398 tells us that it must be linearly dependent.

The set

$$\{ x - 2, x^2 - 2x, x^3 - 2x^2, x^4 - 2x^3 \}$$

is a third subset of $W$ (check this) and is linearly independent (check this). Since it has the right size to be a basis, and is linearly independent, Theorem G 398 tells us that it also spans $W$, and therefore is a basis of $W$.

A simple consequence of Theorem G 398 is the observation that proper subspaces have strictly smaller dimensions. Hopefully this may seem intuitively obvious, but it still requires proof, and we will cite this result later.

**Theorem PSSD**

**Proper Subspaces have Smaller Dimension**

Suppose that $U$ and $V$ are subspaces of the vector space $W$, such that $U \subset V$. Then $\dim (U) < \dim (V)$. □
Proof Suppose that \( \dim(U) = m \) and \( \dim(V) = t \). Then \( U \) has a basis \( B \) of size \( m \). If \( m > t \), then by Theorem G [398], \( B \) is linearly dependent, which is a contradiction. If \( m = t \), then by Theorem G [398], \( B \) spans \( V \). Then \( U = \langle B \rangle = V \), also a contradiction. All that remains is that \( m < t \), which is the desired conclusion.

The final theorem of this subsection is an extremely powerful tool for establishing the equality of two sets that are subspaces. Notice that the hypotheses include the equality of two integers (dimensions) while the conclusion is the equality of two sets (subspaces). It is the extra “structure” of a vector space and its dimension that makes possible this huge leap from an integer equality to a set equality.

Theorem EDYES
Equal Dimensions Yields Equal Subspaces

Suppose that \( U \) and \( V \) are subspaces of the vector space \( W \), such that \( U \subseteq V \) and \( \dim(U) = \dim(V) \). Then \( U = V \).

Proof We give a proof by contradiction (Technique CD [708]). Suppose to the contrary that \( U \neq V \). Since \( U \subseteq V \), there must be a vector \( v \) such that \( v \in V \) and \( v \notin U \). Let \( B = \{u_1, u_2, u_3, \ldots, u_t\} \) be a basis for \( U \). Then, by Theorem ELIS [397], the set \( C = B \cup \{v\} = \{u_1, u_2, u_3, \ldots, u_t, v\} \) is a linearly independent set of \( t + 1 \) vectors in \( V \). However, by hypothesis, \( V \) has the same dimension as \( U \) (namely \( t \)) and therefore Theorem G [398] says that \( C \) is too big to be linearly independent. This contradiction shows that \( U = V \).

Subsection RT
Ranks and Transposes

We now prove one of the most surprising theorems about matrices. Notice the paucity of hypotheses compared to the precision of the conclusion.

Theorem RMRT
Rank of a Matrix is the Rank of the Transpose

Suppose \( A \) is an \( m \times n \) matrix. Then \( r(A) = r(A^t) \).

Proof Suppose we row-reduce \( A \) to the matrix \( B \) in reduced row-echelon form, and \( B \) has \( r \) non-zero rows. The quantity \( r \) tells us three things about \( B \): the number of leading 1’s, the number of non-zero rows and the number of pivot columns. For this proof we will be interested in the latter two. Theorem BRS [271] and Theorem BCS [264] each has a conclusion that provides a basis, for the row space and the column space, respectively. In each case, these bases contain \( r \) vectors. This observation makes the following go.

\[
\begin{align*}
  r(A) &= \dim(C(A)) \\
        &= \dim(R(A)) \\
        &= \dim(C(A^t)) \\
        &= r(A^t)
\end{align*}
\]

Jacob Linenthal helped with this proof.

This says that the row space and the column space of a matrix have the same dimension, which should be very surprising. It does not say that column space and the row space are identical. Indeed, if the matrix is not square, then the sizes (number of slots) of the vectors in each space are different, so the sets are not even comparable.

It is not hard to construct by yourself examples of matrices that illustrate Theorem RMRT, since it applies equally well to any matrix. Grab a matrix, row-reduce it, count the nonzero rows or the leading 1’s. That’s the rank. Transpose the matrix, row-reduce that, count the nonzero rows or the leading 1’s. That’s the rank of the transpose. The theorem says the two will be equal. Here’s an example anyway.

Example RRTI
Rank, rank of transpose, Archetype I
Archetype I has a 4 × 7 coefficient matrix which row-reduces to

\[
\begin{bmatrix}
1 & 4 & 0 & 0 & 2 & 1 & -3 \\
0 & 0 & 1 & 0 & 1 & -3 & 5 \\
0 & 0 & 0 & 1 & 2 & -6 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

so the rank is 3. Row-reducing the transpose yields

\[
\begin{bmatrix}
1 & 0 & 0 & -\frac{31}{7} \\
0 & 1 & 0 & \frac{12}{7} \\
0 & 0 & 1 & \frac{13}{7} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

demonstrating that the rank of the transpose is also 3.

Subsection DFS
Dimension of Four Subspaces

That the rank of a matrix equals the rank of its transpose is a fundamental and surprising result. However, applying Theorem FS we can easily determine the dimension of all four fundamental subspaces associated with a matrix.

Theorem DFS
Dimensions of Four Subspaces

Suppose that A is an m × n matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

1. \( \dim(\mathcal{N}(A)) = n - r \)
2. \( \dim(\mathcal{C}(A)) = r \)
3. \( \dim(\mathcal{R}(A)) = r \)
4. \( \dim(\mathcal{L}(A)) = m - r \) □

**Proof**  
If \( A \) row-reduces to a matrix in reduced row-echelon form with \( r \) nonzero rows, then the matrix \( C \) of extended echelon form (Definition EEF [287]) will be an \( r \times n \) matrix in reduced row-echelon form with no zero rows and \( r \) pivot columns (Theorem PEEF [288]). Similarly, the matrix \( L \) of extended echelon form (Definition EEF [287]) will be an \( m - r \times m \) matrix in reduced row-echelon form with no zero rows and \( m - r \) pivot columns (Theorem PEEF [288]).

\[
\begin{align*}
\dim(\mathcal{N}(A)) &= \dim(\mathcal{N}(C)) \\
&= n - r \\
&= n - r \\
\end{align*}
\]

**Theorem FS [290]**  
**Theorem BNS [154]**

\[
\begin{align*}
\dim(\mathcal{C}(A)) &= \dim(\mathcal{N}(L)) \\
&= m - (m - r) \\
&= r \\
\end{align*}
\]

**Theorem FS [290]**  
**Theorem BNS [154]**

\[
\begin{align*}
\dim(\mathcal{R}(A)) &= \dim(\mathcal{R}(C)) \\
&= r \\
\end{align*}
\]

**Theorem FS [290]**  
**Theorem BRS [271]**

\[
\begin{align*}
\dim(\mathcal{L}(A)) &= \dim(\mathcal{R}(L)) \\
&= m - r \\
\end{align*}
\]

**Theorem FS [290]**  
**Theorem BRS [271]**

There are many different ways to state and prove this result, and indeed, the equality of the dimensions of the column space and row space is just a slight expansion of Theorem RMRT [401]. However, we have restricted our techniques to applying Theorem FS [290] and then determining dimensions with bases provided by Theorem BNS [154] and Theorem BRS [271]. This provides an appealing symmetry to the results and the proof.

**Subsection READ**  
**Reading Questions**

1. Why does Theorem G [398] have the title it does?
2. What is so surprising about Theorem RMRT [401]?
3. Row-reduce the matrix \( A \) to reduced row-echelon form. Without any further computations, compute the dimensions of the four subspaces, \( \mathcal{N}(A) \), \( \mathcal{C}(A) \), \( \mathcal{R}(A) \) and \( \mathcal{L}(A) \).

\[
A = \begin{bmatrix}
1 & -1 & 2 & 8 & 5 \\
1 & 1 & 1 & 4 & -1 \\
0 & 2 & -3 & -8 & -6 \\
2 & 0 & 1 & 8 & 4
\end{bmatrix}
\]
Subsection EXC
Exercises

C10 Example SVP4 [400] leaves several details for the reader to check. Verify these five claims.
Contributed by Robert Beezer

T05 Trivially, if $U$ and $V$ are two subspaces of $W$, then $\dim(U) = \dim(V)$. Combine this fact, Theorem PSSD [400], and Theorem EDYES [401] all into one grand combined theorem. You might look to Theorem PIP [188] stylistic inspiration. (Notice this problem does not ask you to prove anything. It just asks you to roll up three theorems into one compact, logically equivalent statement.)
Contributed by Robert Beezer

T10 Prove the following theorem, which could be viewed as a reformulation of parts (3) and (4) of Theorem G [398], or more appropriately as a corollary of Theorem G [398] (Technique LC [716]).
Suppose $V$ is a vector space and $S$ is a subset of $V$ such that the number of vectors in $S$ equals the dimension of $V$. Then $S$ is linearly independent if and only if $S$ spans $V$.
Contributed by Robert Beezer

T15 Suppose that $A$ is an $m \times n$ matrix and let $\min(m, n)$ denote the minimum of $m$ and $n$. Prove that $r(A) \leq \min(m, n)$.
Contributed by Robert Beezer

T20 Suppose that $A$ is an $m \times n$ matrix and $b \in \mathbb{C}^m$. Prove that the linear system $L(A, b)$ is consistent if and only if $r(A) = r([A | b])$.
Contributed by Robert Beezer

T25 Suppose that $V$ is a vector space with finite dimension. Let $W$ be any subspace of $V$. Prove that $W$ has finite dimension.
Contributed by Robert Beezer

T60 Suppose that $W$ is a vector space with dimension 5, and $U$ and $V$ are subspaces of $W$, each of dimension 3. Prove that $U \cap V$ contains a non-zero vector. State a more general result.
Contributed by Joe Riegsecker

Solution [407]
Subsection SOL
Solutions

T20  Contributed by Robert Beezer  Statement [405]

(⇒) Suppose first that \( L \mathcal{S}(A, b) \) is consistent. Then by Theorem CS CS [262], \( b \in \mathcal{C}(A) \).
This means that \( \mathcal{C}(A) = \mathcal{C}(\begin{bmatrix} A & b \end{bmatrix}) \) and so it follows that \( r(A) = r(\begin{bmatrix} A & b \end{bmatrix}) \).

(⇐) Adding a column to a matrix will only increase the size of its column space, so in all cases, \( \mathcal{C}(A) \subseteq \mathcal{C}(\begin{bmatrix} A & b \end{bmatrix}) \). However, if we assume that \( r(A) = r(\begin{bmatrix} A & b \end{bmatrix}) \), then by Theorem ED YES [401] we conclude that \( \mathcal{C}(A) = \mathcal{C}(\begin{bmatrix} A & b \end{bmatrix}) \). Then \( b \in \mathcal{C}(\begin{bmatrix} A & b \end{bmatrix}) = \mathcal{C}(A) \) so by Theorem CS CS [262], \( L \mathcal{S}(A, b) \) is consistent.

T60  Contributed by Robert Beezer  Statement [405]

Let \( \{u_1, u_2, u_3\} \) and \( \{v_1, v_2, v_3\} \) be bases for \( U \) and \( V \) (respectively). Then, the set \( \{u_1, u_2, u_3, v_1, v_2, v_3\} \) is linearly dependent, since Theorem G [398] says we cannot have 6 linearly independent vectors in a vector space of dimension 6. So we can assert that there is a non-trivial relation of linear dependence,

\[
a_1 u_1 + a_2 u_2 + a_3 u_3 + b_1 v_1 + b_2 v_2 + b_3 v_3 = 0
\]

where \( a_1, a_2, a_3 \) and \( b_1, b_2, b_3 \) are not all zero.

We can rearrange this equation as

\[
a_1 u_1 + a_2 u_2 + a_3 u_3 = -b_1 v_1 - b_2 v_2 - b_3 v_3
\]

This is an equality of two vectors, so we can give this common vector a name, say \( w \),

\[
w = a_1 u_1 + a_2 u_2 + a_3 u_3 = -b_1 v_1 - b_2 v_2 - b_3 v_3
\]

This is the desired non-zero vector, as we will now show.

First, since \( w = a_1 u_1 + a_2 u_2 + a_3 u_3 \), we can see that \( w \in U \). Similarly, \( w = -b_1 v_1 - b_2 v_2 - b_3 v_3 \), so \( w \in V \). This establishes that \( w \in U \cap V \) (Definition SI [695]).

Is \( w \neq 0 \)? Suppose not, in other words, suppose \( w = 0 \). Then

\[
0 = w = a_1 u_1 + a_2 u_2 + a_3 u_3
\]

Because \( \{u_1, u_2, u_3\} \) is a basis for \( U \), it is a linearly independent set and the relation of linear dependence above means we must conclude that \( a_1 = a_2 = a_3 = 0 \). By a similar process, we would conclude that \( b_1 = b_2 = b_3 = 0 \). But this is a contradiction since \( a_1, a_2, a_3, b_1, b_2, b_3 \) were chosen so that some were nonzero. So \( w \neq 0 \).

How does this generalize? All we really needed was the original relation of linear dependence that resulted because we had “too many” vectors in \( W \). A more general statement would be: Suppose that \( W \) is a vector space with dimension \( n \), \( U \) is a subspace of dimension \( p \) and \( V \) is a subspace of dimension \( q \). If \( p + q > n \), then \( U \cap V \) contains a non-zero vector.
Chapter D
Determinants

The determinant is a function that takes a square matrix as an input and produces a scalar as an output. So unlike a vector space, it is not an algebraic structure. However, it has many beneficial properties for studying vector spaces, matrices and systems of equations, so it is hard to ignore (though some have tried). While the properties of a determinant can be very useful, they are also complicated to prove.

Section DM
Determinant of a Matrix

First, a slight detour, as we introduce elementary matrices, which will bring us back to the beginning of the course and our old friend, row operations.

Subsection EM
Elementary Matrices

Elementary matrices are very simple, as you might have suspected from their name. Their purpose is to effect row operations (Definition RO [33]) on a matrix through matrix multiplication (Definition MM [215]). Their definitions look more complicated than they really are, so be sure to read ahead after you read the definition for some explanations and an example.

Definition ELEM
Elementary Matrices
1. $E_{i,j}$ is the square matrix of size $n$ with

$$
[E_{i,j}]_{k\ell} = \begin{cases} 
0 & k \neq i, k \neq j, \ell \neq k \\
1 & k \neq i, k \neq j, \ell = k \\
0 & k = i, \ell \neq j \\
1 & k = i, \ell = j \\
0 & k = j, \ell \neq i \\
1 & k = j, \ell = i 
\end{cases}
$$

2. $E_i(\alpha)$, for $\alpha \neq 0$, is the square matrix of size $n$ with

$$
[E_i(\alpha)]_{k\ell} = \begin{cases} 
0 & k \neq i, \ell \neq k \\
1 & k \neq i, \ell = k \\
\alpha & k = i, \ell = i 
\end{cases}
$$

3. $E_{i,j}(\alpha)$ is the square matrix of size $n$ with

$$
[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 
0 & k \neq j, \ell \neq k \\
1 & k \neq j, \ell = k \\
0 & k = j, \ell \neq i, \ell \neq j \\
1 & k = j, \ell = j \\
\alpha & k = j, \ell = i 
\end{cases}
$$

(This definition contains Notation ELEM.)

Again, these matrices are not as complicated as they appear, since they are mostly perturbations of the $n \times n$ identity matrix (Definition IM [76]). $E_{i,j}$ is the identity matrix with rows (or columns) $i$ and $j$ trading places, $E_i(\alpha)$ is the identity matrix where the diagonal entry in row $i$ and column $i$ has been replaced by $\alpha$, and $E_{i,j}(\alpha)$ is the identity matrix where the entry in row $j$ and column $i$ has been replaced by $\alpha$. (Yes, those subscripts look backwards in the description of $E_{i,j}(\alpha)$). Notice that our notation makes no reference to the size of the elementary matrix, since this will always be apparent from the context, or unimportant.

The raison d’être for elementary matrices is to “do” row operations on matrices with matrix multiplication. So here is an example where we will both see some elementary matrices and see how they can accomplish row operations.

**Example EMRO**

**Elementary matrices and row operations**

We will perform a sequence of row operations (Definition RO [33]) on the $3 \times 4$ matrix $A$, while also multiplying the matrix on the left by the appropriate $3 \times 3$ elementary matrix.

$$
A = \begin{bmatrix} 
2 & 1 & 3 & 1 \\
1 & 3 & 2 & 4 \\
5 & 0 & 3 & 1 
\end{bmatrix}
$$
The next three theorems establish that each elementary matrix effects a row operation via matrix multiplication.

**Theorem EMDRO**

**Elementary Matrices Do Row Operations**

Suppose that $A$ is a matrix, and $B$ is a matrix of the same size that is obtained from $A$ by a single row operation (Definition RO [33]).

1. If the row operation swaps rows $i$ and $j$, then $B = E_{i,j} A$.

2. If the row operation multiplies row $i$ by $\alpha$, then $B = E_i (\alpha) A$.

3. If the row operation multiplies row $i$ by $\alpha$ and adds the result to row $j$, then $B = E_{i,j} (\alpha) A$.

\[\square\]

**Proof** In each of the three conclusions, performing the row operation on $A$ will create the matrix $B$ where only one or two rows will have changed. So we will establish the equality of the matrix entries row by row, first for the unchanged rows, then for the changed rows, showing in each case that the result of the matrix product is the same as the result of the row operation. Here we go.

Row $k$ of the product $E_{i,j} A$, where $k \neq i, k \neq j$, is unchanged from $A$,

\[
[E_{i,j} A]_{k\ell} = \sum_{p=1}^{n} [E_{i,j}]_{kp} [A]_{p\ell} \quad \text{(Theorem EMP 216)}
\]

\[= [E_{i,j}]_{kk} [A]_{k\ell} + \sum_{p=1, p \neq k}^{n} [E_{i,j}]_{kp} [A]_{p\ell} \]

\[= 1 [A]_{k\ell} + \sum_{p=1, p \neq k}^{n} 0 [A]_{p\ell} \quad \text{(Definition ELEM 409)}
\]

\[= [A]_{k\ell}
\]

Row $i$ of the product $E_{i,j} A$ is row $j$ of $A$,

\[
[E_{i,j} A]_{i\ell} = \sum_{p=1}^{n} [E_{i,j}]_{ip} [A]_{p\ell} \quad \text{(Theorem EMP 216)}
\]
\[ [E_{i,j}]_{ji} A_{j\ell} + \sum_{p=1, p \neq j}^{n} [E_{i,j}]_{jp} A_{p\ell} \]
\[ = 1 [A]_{j\ell} + \sum_{p=1, p \neq j}^{n} 0 [A]_{p\ell} \quad \text{Definition ELEM 409} \]
\[ = [A]_{j\ell} \]

Row \( j \) of the product \( E_{i,j} A \) is row \( i \) of \( A \),

\[ [E_{i,j} A]_{ji} = \sum_{p=1}^{n} [E_{i,j}]_{jp} [A]_{p\ell} \quad \text{Theorem EMP 216} \]
\[ = [E_{i,j}]_{ji} [A]_{i\ell} + \sum_{p=1, p \neq i}^{n} [E_{i,j}]_{jp} [A]_{p\ell} \]
\[ = 1 [A]_{i\ell} + \sum_{p=1, p \neq i}^{n} 0 [A]_{p\ell} \quad \text{Definition ELEM 409} \]
\[ = [A]_{i\ell} \]

So the matrix product \( E_{i,j} A \) is the same as the row operation that swaps rows \( i \) and \( j \).

Row \( k \) of the product \( E_{i,j} (\alpha) A \), where \( k \neq i \), is unchanged from \( A \),

\[ [E_{i,j} (\alpha) A]_{kk} = \sum_{p=1}^{n} [E_{i,j} (\alpha)]_{kp} [A]_{p\ell} \quad \text{Theorem EMP 216} \]
\[ = [E_{i,j}]_{ki} [A]_{k\ell} + \sum_{p=1, p \neq k}^{n} [E_{i,j} (\alpha)]_{kp} [A]_{p\ell} \]
\[ = 1 [A]_{k\ell} + \sum_{p=1, p \neq k}^{n} 0 [A]_{p\ell} \quad \text{Definition ELEM 409} \]
\[ = [A]_{k\ell} \]

Row \( i \) of the product \( E_{i} (\alpha) A \) is \( \alpha \) times row \( i \) of \( A \),

\[ [E_{i} (\alpha) A]_{i\ell} = \sum_{p=1}^{n} [E_{i} (\alpha)]_{ip} [A]_{p\ell} \quad \text{Theorem EMP 216} \]
\[ = [E_{i}]_{ii} [A]_{i\ell} + \sum_{p=1, p \neq i}^{n} [E_{i} (\alpha)]_{ip} [A]_{p\ell} \]
\[ = \alpha [A]_{i\ell} + \sum_{p=1, p \neq i}^{n} 0 [A]_{p\ell} \quad \text{Definition ELEM 409} \]
\[ = \alpha [A]_{i\ell} \]

So the matrix product \( E_{i} (\alpha) A \) is the same as the row operation that swaps multiplies row \( i \) by \( \alpha \).

Row \( k \) of the product \( E_{i,j} (\alpha) A \), where \( k \neq j \), is unchanged from \( A \),

\[ [E_{i,j} (\alpha) A]_{k\ell} = \sum_{p=1}^{n} [E_{i,j} (\alpha)]_{kp} [A]_{p\ell} \quad \text{Theorem EMP 216} \]
Subsection DM.EM  Elementary Matrices  413

\[
= [E_{i,j} (\alpha)]_{kk} [A]_{k\ell} + \sum_{p=1, \ p \neq k}^{n} [E_{i,j} (\alpha)]_{kp} [A]_{p\ell}
\]

\[
= 1 [A]_{k\ell} + \sum_{p=1, \ p \neq k}^{n} 0 [A]_{p\ell} \quad \text{Definition ELEM 409}
\]

\[
= [A]_{k\ell}
\]

Row \( j \) of the product \( E_{i,j} (\alpha) A \), is \( \alpha \) times row \( i \) of \( A \) and then added to row \( j \) of \( A \),

\[
[E_{i,j} (\alpha) A]_{j\ell} = \sum_{p=1}^{n} [E_{i,j} (\alpha)]_{jp} [A]_{p\ell} \quad \text{Theorem EMP 216}
\]

\[
= [E_{i,j} (\alpha)]_{jj} [A]_{j\ell} + \sum_{p=1, \ p \neq j,i}^{n} [E_{i,j} (\alpha)]_{jp} [A]_{p\ell}
\]

\[
= 1 [A]_{j\ell} + \alpha [A]_{i\ell} + \sum_{p=1, \ p \neq j,i}^{n} 0 [A]_{p\ell} \quad \text{Definition ELEM 409}
\]

\[
= [A]_{j\ell} + \alpha [A]_{i\ell}
\]

So the matrix product \( E_{i,j} (\alpha) A \) is the same as the row operation that multiplies row \( i \) by \( \alpha \) and adds the result to row \( j \). ■

Later in this section we will need two facts about elementary matrices.

**Theorem EMN**

**Elementary Matrices are Nonsingular**

If \( E \) is an elementary matrix, then \( E \) is nonsingular. □

**Proof** We can row-reduce each elementary matrix to the identity matrix. Given an elementary matrix of the form \( E_{i,j} \), perform the row operation that swaps row \( j \) with row \( i \). Given an elementary matrix of the form \( E_{i} (\alpha) \), with \( \alpha \neq 0 \), perform the row operation that multiplies row \( i \) by \( 1/\alpha \). Given an elementary matrix of the form \( E_{i,j} (\alpha) \), with \( \alpha \neq 0 \), perform the row operation that multiplies row \( i \) by \( -\alpha \) and adds it to row \( j \). In each case, the result of the single row operation is the identity matrix. So each elementary matrix is row-equivalent to the identity matrix, and by **Theorem NMRRI 77** is nonsingular. ■

Notice that we have now made use of the nonzero restriction on \( \alpha \) in the definition of \( E_{i} (\alpha) \). One more key property of elementary matrices.

**Theorem NMPEM**

**Nonsingular Matrices are Products of Elementary Matrices**

Suppose that \( A \) is a nonsingular matrix. Then there exists elementary matrices \( E_{1}, E_{2}, E_{3}, \ldots, E_{t} \) so that \( A = E_{1} E_{2} E_{3} \ldots E_{t} \). □

**Proof** Since \( A \) is nonsingular, it is row-equivalent to the identity matrix by **Theorem NMRRI 77**, so there is a sequence of \( t \) row operations that converts \( I \) to \( A \). For each of these row operations, form the associated elementary matrix from **Theorem EM-DRO 111** and denote these matrices by \( E_{1}, E_{2}, E_{3}, \ldots, E_{t} \). Applying the first row
operation to \( I \) yields the matrix \( E_1I \). The second row operation yields \( E_2(E_1I) \), and the third row operation creates \( E_3E_2E_1I \). The result of the full sequence of \( t \) row operations will yield \( A \), so

\[
A = E_t \ldots E_3E_2E_1I = E_t \ldots E_3E_2E_1
\]

Other than the cosmetic matter of re-indexing these elementary matrices in the opposite order, this is the desired result.

\[\square\]

**Subsection DD**

**Definition of the Determinant**

We’ll now turn to the definition of a determinant and do some sample computations. The definition of the determinant function is **recursive**, that is, the determinant of a large matrix is defined in terms of the determinant of smaller matrices. To this end, we will make a few definitions.

**Definition SM**

**SubMatrix**

Suppose that \( A \) is an \( m \times n \) matrix. Then the **submatrix** \( A (i|j) \) is the \((m-1) \times (n-1)\) matrix obtained from \( A \) by removing row \( i \) and column \( j \).

(This definition contains Notation SM.)

\[\triangle\]

**Example SS**

**Some submatrices**

For the matrix

\[
A = \begin{bmatrix}
1 & -2 & 3 & 9 \\
4 & -2 & 0 & 1 \\
3 & 5 & 2 & 1
\end{bmatrix}
\]

we have the submatrices

\[
A(2|3) = \begin{bmatrix}
1 & -2 & 9 \\
3 & 5 & 1
\end{bmatrix} \quad A(3|1) = \begin{bmatrix}
-2 & 3 & 9 \\
-2 & 0 & 1
\end{bmatrix}
\]

\[\boxdot\]

**Definition DM**

**Determinant of a Matrix**

Suppose \( A \) is a square matrix. Then its **determinant**, \( \det (A) = |A| \), is an element of \( \mathbb{C} \) defined recursively by:

If \( A \) is a 1 \times 1 matrix, then \( \det (A) = [A]_{11} \).

If \( A \) is a matrix of size \( n \) with \( n \geq 2 \), then

\[
\det (A) = [A]_{11} \det (A (1|1)) - [A]_{12} \det (A (1|2)) + [A]_{13} \det (A (1|3)) - [A]_{14} \det (A (1|4)) + \cdots + (-1)^{n+1} [A]_{1n} \det (A (1|n))
\]

Version 0.92
Subsection DM-DD Definition of the Determinant

So to compute the determinant of a $5 \times 5$ matrix we must build 5 submatrices, each of size 4. To compute the determinants of each the $4 \times 4$ matrices we need to create 4 submatrices each, these now of size 3 and so on. To compute the determinant of a $10 \times 10$ matrix would require computing the determinant of $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 3,628,800$ $1 \times 1$ matrices. Fortunately there are better ways. However this does suggest an excellent computer programming exercise to write a recursive procedure to compute a determinant.

Let’s compute the determinant of a reasonable sized matrix by hand.

**Example D33M**

**Determinant of a $3 \times 3$ matrix**

Suppose that we have the $3 \times 3$ matrix

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{bmatrix}$$

Then

$$\text{det}(A) = |A| = \begin{vmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{vmatrix} = 3 \begin{vmatrix} 1 & 6 \\ -1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ -3 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 4 & 1 \\ -3 & -1 \end{vmatrix}$$

$$= 3 (1 \cdot 2 - 6 \cdot -1) - 2 (4 \cdot 2 - 6 \cdot -3) - (-1) (4 \cdot -1 - 1 \cdot -3)$$

$$= 3 (2) - 2 (4) - (-3) = 24 - 52 + 1 = -27$$

In practice it is a bit silly to decompose a $2 \times 2$ matrix down into a couple of $1 \times 1$ matrices and then compute the exceedingly easy determinant of these puny matrices. So here is a simple theorem.

**Theorem DMST**

**Determinant of Matrices of Size Two**

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\text{det}(A) = ad - bc$

**Proof** Applying Definition DM [414],

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot d - b \cdot c = ad - bc$$

Do you recall seeing the expression $ad - bc$ before? (Hint: Theorem TTMI [234])
Subsection CD
Computing Determinants

There are a variety of ways to compute the determinant. We will establish first that we can choose to mimic our definition of the determinant, but by using matrix entries and submatrices based on a row other than the first one.

**Theorem DER**

**Determinant Expansion about Rows**

Suppose that $A$ is a square matrix of size $n$. Then

$$
\det (A) = (-1)^{i+1} [A]_{i1} \det (A (i|1)) + (-1)^{i+2} [A]_{i2} \det (A (i|2)) + \cdots + (-1)^{i+n} [A]_{in} \det (A (i|n))
$$

where $1 \leq i \leq n$

which is known as **expansion** about row $i$.

**Proof**

Given the recursive definition of the determinant, it should be no surprise that we will use induction for this proof (Technique I [713]). When $n = 1$, there is nothing to prove since there is but one row. When $n = 2$, we just examine expansion about the second row,

$$
(-1)^{2+1} [A]_{21} \det (A (2|1)) + (-1)^{2+2} [A]_{22} \det (A (2|2)) = - [A]_{21} [A]_{12} + [A]_{22} [A]_{11} = [A]_{11} [A]_{22} - [A]_{12} [A]_{21} = \det (A)
$$

So the theorem is true for matrices of size $n = 1$ and $n = 2$. Now assume the result is true for all matrices of size $n - 1$ as we derive an expression for expansion about row $i$ for a matrix of size $n$. We will abuse our notation for a submatrix slightly, so $A (i_1, i_2|j_1, j_2)$ will denote the matrix formed by removing rows $i_1$ and $i_2$, along with removing columns $j_1$ and $j_2$. Also, as we take a determinant of a submatrix, we will need to “jump up” the index of summation partway through as we “skip over” a missing column. To do this smoothly we will set

$$
\epsilon_{\ell j} = \begin{cases} 
0 & \ell < j \\
1 & \ell > j 
\end{cases}
$$

Now,

$$
\det (A) = \sum_{j=1}^{n} (-1)^{1+j} [A]_{1j} \det (A (1|j))
$$

$$
= \sum_{j=1}^{n} (-1)^{1+j} [A]_{1j} \sum_{1 \leq \ell \leq n, \ell \neq j} (-1)^{i-1+\ell-\epsilon_{\ell j}} [A]_{i\ell} \det (A (1,i|j,\ell))
$$

Induction, row $i$

$$
= \sum_{1 \leq i < n, 1 \leq j < n, i \neq j} (-1)^{j+i+\ell-\epsilon_{\ell j}} [A]_{1j} [A]_{i\ell} \det (A (1,i|j,\ell))
$$

Version 0.92
\[
\begin{align*}
&= \sum_{\ell=1}^{n} (-1)^{i+\ell} [A]_{i\ell} \sum_{1 \leq j \leq n} (-1)^{j-\epsilon_{ij}} [A]_{ij} \det (A (1, i|j, \ell)) \\
&= \sum_{\ell=1}^{n} (-1)^{i+\ell} [A]_{i\ell} \sum_{1 \leq j \leq n} (-1)^{i+j+\epsilon_{ij}} [A]_{ij} \det (A (i, 1|j, \ell)) \\
&= \sum_{\ell=1}^{n} (-1)^{i+\ell} [A]_{i\ell} \det (A (i|\ell))
\end{align*}
\]

Subsection DM.CD Computing Determinants

We can also obtain a formula that computes a determinant by expansion about a column, but this will be simpler if we first prove a result about the interplay of determinants and transposes. Notice how the following proof makes use of the ability to compute a determinant by expanding about any row.

Theorem DT
Determinant of the Transpose

Suppose that \( A \) is a square matrix. Then \( \det (A^t) = \det (A) \).

Proof
As before, with a recursive definition, a proof by induction will be natural (Technique I [713]). For the base case, a square matrix of size 1 is symmetric, so \( A = A^t \), and the determinants will be equal. Now assume the theorem is true for all square matrices of size \( n-1 \) and consider the determinant of a matrix of size \( n \).

\[
\det (A^t) = \frac{1}{n} \sum_{i=1}^{n} \det (A^t)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} [A^t]_{ij} \det (A^t (i|j))
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} [A]_{ji} \det (A (j|i))
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{n} (-1)^{i+j} [A]_{ji} \det (A (j|i))
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \det (A)
\]

\[
= \det (A)
\]

Now we can easily get the result that a determinant can be computed by expansion about any column as well.

Theorem DEC
Determinant Expansion about Columns

Suppose that \( A \) is a square matrix of size \( n \). Then

\[
\det (A) = (-1)^{1+j} [A]_{1j} \det (A (1|j)) + (-1)^{2+j} [A]_{2j} \det (A (2|j))
\]

\[
+ (-1)^{3+j} [A]_{3j} \det (A (3|j)) + \cdots + (-1)^{n+j} [A]_{nj} \det (A (n|j)) \quad 1 \leq j \leq n
\]
which is known as **expansion** about column \( j \).

**Proof**

\[
det(A) = det(A^t) \quad \text{Theorem DT [417]}
\]

\[
= \sum_{j=1}^{n} [A^t]_{ji} \det(A^t(j|i)) \quad \text{Theorem DER [416]}
\]

\[
= \sum_{j=1}^{n} [A]_{ij} \det(A(i|j)) \quad \text{Definition TM [200]}
\]

That the determinant of an \( n \times n \) matrix can be computed in \( 2n \) different (albeit similar) ways is nothing short of remarkable. For the doubters among us, we will do an example, computing a \( 4 \times 4 \) matrix in two different ways.

**Example TCSD**

Two computations, same determinant

Let

\[
A = \begin{bmatrix}
-2 & 3 & 0 & 1 \\
9 & -2 & 0 & 1 \\
1 & 3 & -2 & -1 \\
4 & 1 & 2 & 6
\end{bmatrix}
\]

Then expanding about the fourth row (Theorem DER [416] with \( i = 4 \)) yields,

\[
|A| = (4)(-1)^{4+1} \begin{vmatrix}
3 & 0 & 1 \\
-2 & 0 & 1 \\
3 & -2 & -1
\end{vmatrix} + (1)(-1)^{4+2} \begin{vmatrix}
-2 & 0 & 1 \\
9 & 0 & 1 \\
1 & -2 & -1
\end{vmatrix} + (2)(-1)^{4+3} \begin{vmatrix}
-2 & 3 & 1 \\
9 & -2 & 1 \\
1 & 3 & -1
\end{vmatrix} + (6)(-1)^{4+4} \begin{vmatrix}
-2 & 3 & 0 \\
9 & -2 & 0 \\
1 & 3 & -2
\end{vmatrix} = (-4)(10) + (1)(-22) + (-2)(61) + 6(46) = 92
\]

while expanding about column 3 (Theorem DEC [417] with \( j = 3 \)) gives

\[
|A| = (0)(-1)^{1+3} \begin{vmatrix}
9 & -2 & 2 \\
1 & 3 & -1 \\
4 & 1 & 6
\end{vmatrix} + (0)(-1)^{2+3} \begin{vmatrix}
-2 & 3 & 1 \\
1 & 3 & -1 \\
4 & 1 & 6
\end{vmatrix} + (-2)(-1)^{3+3} \begin{vmatrix}
-2 & 3 & 1 \\
9 & -2 & 1 \\
4 & 1 & 6
\end{vmatrix} + (2)(-1)^{4+3} \begin{vmatrix}
-2 & 3 & 1 \\
9 & -2 & 1 \\
1 & 3 & -1
\end{vmatrix} = 0 + 0 + (-2)(-107) + (-2)(61) = 92
\]

Notice how much easier the second computation was. By choosing to expand about the third column, we have two entries that are zero, so two \( 3 \times 3 \) determinants need not be computed at all!
When a matrix has all zeros above (or below) the diagonal, exploiting the zeros by expanding about the proper row or column makes computing a determinant insanely easy.

**Example DUTM**

**Determinant of an upper-triangular matrix**

Suppose that

\[
T = \begin{bmatrix}
2 & 3 & -1 & 3 & 3 \\
0 & -1 & 5 & 2 & -1 \\
0 & 0 & 3 & 9 & 2 \\
0 & 0 & 0 & -1 & 3 \\
0 & 0 & 0 & 0 & 5
\end{bmatrix}
\]

We will compute the determinant of this $5 \times 5$ matrix by consistently expanding about the first column for each submatrix that arises and does not have a zero entry multiplying it.

\[
det(T) = \begin{vmatrix}
2 & 3 & -1 & 3 & 3 \\
0 & -1 & 5 & 2 & -1 \\
0 & 0 & 3 & 9 & 2 \\
0 & 0 & 0 & -1 & 3 \\
0 & 0 & 0 & 0 & 5
\end{vmatrix}
\]

\[
= 2(-1)^{1+1} \begin{vmatrix}
-1 & 5 & 2 & -1 \\
0 & 3 & 9 & 2 \\
0 & 0 & -1 & 3 \\
0 & 0 & 0 & 5
\end{vmatrix}
\]

\[
= 2(-1)(-1)^{1+1} \begin{vmatrix}
3 & 9 & 2 \\
0 & -1 & 3 \\
0 & 0 & 5
\end{vmatrix}
\]

\[
= 2(-1)(3)(-1)^{1+1} \begin{vmatrix}
-1 & 3 \\
0 & 5
\end{vmatrix}
\]

\[
= 2(-1)(3)(-1)(-1)^{1+1} 5
\]

\[
= 2(-1)(3)(-1)(5) = 30
\]

If you consult other texts in your study of determinants, you may run into the terms "minor" and "cofactor," especially in a discussion centered on expansion about rows and columns. We’ve chosen not to make these definitions formally since we’ve been able to get along without them. However, informally, a minor is a determinant of a submatrix, specifically $det(A(i|j))$ and is usually referenced as the minor of $[A]_{ij}$. A cofactor is a signed minor, specifically the cofactor of $[A]_{ij}$ is $(-1)^{i+j} det(A(i|j))$.

**Subsection READ**

**Reading Questions**

1. Construct the elementary matrix that will effect the row operation $-6R_2 + R_3$ on a $4 \times 7$ matrix.
2. Compute the determinant of the matrix
\[
\begin{bmatrix}
2 & 3 & -1 \\
3 & 8 & 2 \\
4 & -1 & -3 \\
\end{bmatrix}
\]

3. Compute the determinant of the matrix
\[
\begin{bmatrix}
3 & 9 & -2 & 4 & 2 \\
0 & 1 & 4 & -2 & 7 \\
0 & 0 & -2 & 5 & 2 \\
0 & 0 & 0 & -1 & 6 \\
0 & 0 & 0 & 0 & 4 \\
\end{bmatrix}
\]
Subsection EXC
Exercises

C24  Doing the computations by hand, find the determinant of the matrix below.

\[
\begin{bmatrix}
-2 & 3 & -2 \\
-4 & -2 & 1 \\
2 & 4 & 2 \\
\end{bmatrix}
\]

Contributed by Robert Beezer  Solution 423

C25  Doing the computations by hand, find the determinant of the matrix below.

\[
\begin{bmatrix}
3 & -1 & 4 \\
2 & 5 & 1 \\
2 & 0 & 6 \\
\end{bmatrix}
\]

Contributed by Robert Beezer  Solution 423

C26  Doing the computations by hand, find the determinant of the matrix \(A\).

\[
A = \begin{bmatrix}
2 & 0 & 3 & 2 \\
5 & 1 & 2 & 4 \\
3 & 0 & 1 & 2 \\
5 & 3 & 2 & 1 \\
\end{bmatrix}
\]

Contributed by Robert Beezer  Solution 423
Subsection SOL
Solutions

C24 Contributed by Robert Beezer Statement [421]
We’ll expand about the first row since there are no zeros to exploit,
\[
\begin{vmatrix}
-2 & 3 & -2 \\
-4 & -2 & 1 \\
2 & 4 & 2 \\
\end{vmatrix}
= (-2) \begin{vmatrix}
-2 & 1 \\
4 & 2 \\
\end{vmatrix}
+ (-1)(3) \begin{vmatrix}
-4 & 1 \\
2 & 2 \\
\end{vmatrix}
+ (-2) \begin{vmatrix}
-4 & -2 \\
2 & 4 \\
\end{vmatrix}
\]
\[
= (-2)((-2)(2) - 1(4)) + (-3)((-4)(2) - 1(2)) + (-2)((-4)(4) - (-2)(2))
= (-2)(-8) + (-3)(-10) + (-2)(-12) = 70
\]

C25 Contributed by Robert Beezer Statement [421]
We can expand about any row or column, so the zero entry in the middle of the last row is attractive. Let’s expand about column 2. By Theorem DER [416] and Theorem DEC [417] you will get the same result by expanding about a different row or column. We will use Theorem DMST [415] twice.
\[
\begin{vmatrix}
3 & -1 & 4 \\
2 & 5 & 1 \\
2 & 0 & 6 \\
\end{vmatrix}
= (-1)(-1)^{1+2} \begin{vmatrix}
2 & 1 \\
2 & 6 \\
\end{vmatrix}
+ (5)(-1)^{2+2} \begin{vmatrix}
3 & 4 \\
2 & 6 \\
\end{vmatrix}
+ (0)(-1)^{3+2} \begin{vmatrix}
3 & 4 \\
2 & 1 \\
\end{vmatrix}
\]
\[
= (1)(10) + (5)(10) + 0 = 60
\]

C26 Contributed by Robert Beezer Statement [421]
With two zeros in column 2, we choose to expand about that column (Theorem DEC [417]),
\[
\begin{vmatrix}
2 & 0 & 3 & 2 \\
5 & 1 & 2 & 4 \\
3 & 0 & 1 & 2 \\
5 & 3 & 2 & 1 \\
\end{vmatrix}
\]
\[
\det(A) = 0(-1) \begin{vmatrix}
5 & 2 & 4 \\
3 & 1 & 2 \\
5 & 2 & 1 \\
\end{vmatrix}
+ 1(1) \begin{vmatrix}
2 & 3 & 2 \\
3 & 1 & 2 \\
5 & 2 & 1 \\
\end{vmatrix}
+ 0(-1) \begin{vmatrix}
2 & 3 & 2 \\
3 & 1 & 2 \\
5 & 2 & 1 \\
\end{vmatrix}
+ 3(1) \begin{vmatrix}
2 & 3 & 2 \\
3 & 1 & 2 \\
5 & 2 & 1 \\
\end{vmatrix}
\]
\[
= (1)(2(1)(1) - 2(2)) - 3(3(1) - 5(2)) + 2(3(2) - 5(1)) +
(3)(2(2)(2) - 4(1)) - 3(5(2) - 4(3)) + 2(5(1) - 3(2))
= (-6 + 21 + 2) + (3)(0 + 6 - 2) = 29
\]
Section PDM
Properties of Determinants of Matrices

We have seen how to compute the determinant of a matrix, and the incredible fact that we can perform expansion about any row or column to make this computation. In this largely theoretical section, we will state and prove several more intriguing properties about determinants. Our main goal will be the two results in Theorem SMZD [432] and Theorem DRMM [434], but more specifically, we will see how the value of a determinant will allow us to gain insight into the various properties of a square matrix.

Subsection DRO
Determinants and Row Operations

We start easy with a straightforward theorem whose proof presages the style of subsequent proofs in this subsection.

**Theorem DZRC**
Determinant with Zero Row or Column

Suppose that $A$ is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then $\det(A) = 0$.

**Proof**
Suppose that $A$ is a square matrix of size $n$ and row $i$ has every entry equal to zero. We compute $\det(A)$ via expansion about row $i$.

\[
\begin{align*}
\det(A) &= \sum_{j=1}^{n} (-1)^{i+j} [A]_{ij} \det(A(i|j)) \\
&= \sum_{j=1}^{n} (-1)^{i+j} 0 \det(A(i|j)) \\
&= \sum_{j=1}^{n} 0 = 0
\end{align*}
\]

The proof for the case of a zero column is entirely similar, or could be derived from an application of Theorem DT [417] employing the transpose of the matrix.

**Theorem DRCS**
Determinant for Row or Column Swap

Suppose that $A$ is a square matrix. Let $B$ be the square matrix obtained from $A$ by interchanging the location of two rows, or interchanging the location of two columns. Then $\det(B) = -\det(A)$.

**Proof** Begin with the special case where $A$ is a square matrix of size $n$ and we form $B$ by swapping adjacent rows $i$ and $i+1$ for some $1 \leq i \leq n-1$. Notice that the assumption about swapping adjacent rows means that $B(i+1|j) = A(i|j)$ for all $1 \leq j \leq n$, and
\[ [B]_{i+1,j} = [A]_{ij} \text{ for all } 1 \leq j \leq n. \] We compute det \((B)\) via expansion about row \(i + 1\).

\[
\det (B) = \sum_{j=1}^{n} (-1)^{(i+1)+j} [B]_{i+1,j} \det (B (i+1|j)) \quad \text{Theorem DER 416}
\]

\[
= \sum_{j=1}^{n} (-1)^{(i+1)+j} [A]_{ij} \det (A (i|j)) \quad \text{Hypothesis}
\]

\[
= \sum_{j=1}^{n} (-1)^{1}(-1)^{i+j} [A]_{ij} \det (A (i|j))
\]

\[
= (-1) \sum_{j=1}^{n}(-1)^{i+j} [A]_{ij} \det (A (i|j))
\]

\[
= - \det (A) \quad \text{Theorem DER 416}
\]

So the result holds for the special case where we swap adjacent rows of the matrix. As any computer scientist knows, we can accomplish any rearrangement of an ordered list by swapping adjacent elements. This principle can be demonstrated by naïve sorting algorithms such as “bubble sort.” In any event, we don’t need to discuss every possible reordering, we just need to consider a swap of two rows, say rows \(s\) and \(t\) with \(1 \leq s < t \leq n\).

Begin with row \(s\), and repeatedly swap it with each row just below it, including row \(t\) and stopping there. This will total \(t - s\) swaps. Now swap the former row \(t\), which currently lives in row \(t - 1\), with each row above it, stopping when it becomes row \(s\). This will total another \(t - s - 1\) swaps. In this way, we create \(B\) through a sequence of \(2(t - s) - 1\) swaps of adjacent rows, each of which adjusts det \((A)\) by a multiplicative factor of \(-1\). So

\[
\det (B) = (-1)^{2(t-s)-1} \det (A) = \left((-1)^{2}\right)^{t-s} (-1)^{-1} \det (A) = - \det (A)
\]

as desired.

The proof for the case of swapping two columns is entirely similar, or could be derived from an application of Theorem DT 417 employing the transpose of the matrix. □

So Theorem DRCS 425 tells us the effect of the first row operation (Definition RO 33) on the determinant of a matrix. Here’s the effect of the second row operation.

**Theorem DRCM**

**Determinant for Row or Column Multiples**

Suppose that \(A\) is a square matrix. Let \(B\) be the square matrix obtained from \(A\) by multiplying a single row by the scalar \(\alpha\), or by multiplying a single column by the scalar \(\alpha\). Then det \((B) = \alpha \det (A)\).

**Proof** Suppose that \(A\) is a square matrix of size \(n\) and we form the square matrix \(B\) by multiplying each entry of row \(i\) of \(A\) by \(\alpha\). Notice that the other rows of \(A\) and \(B\) are equal, so \(A(i|j) = B(i|j)\), for all \(1 \leq j \leq n\). We compute det \((B)\) via expansion about row \(i\).

\[
\det (B) = \sum_{j=1}^{n} (-1)^{i+j} [B]_{ij} \det (B (i|j)) \quad \text{Theorem DER 416}
\]
\[
= \sum_{j=1}^{n} (-1)^{i+j} [B]_{ij} \det (A (i|j)) \text{ Hypothesis}
\]

\[
= \sum_{j=1}^{n} (-1)^{i+j} \alpha [A]_{ij} \det (A (i|j)) \text{ Hypothesis}
\]

\[
= \alpha \sum_{j=1}^{n} (-1)^{i+j} [A]_{ij} \det (A (i|j))
\]

\[
= \alpha \det (A) \text{ Theorem DER } 416
\]

The proof for the case of a multiple of a column is entirely similar, or could be derived from an application of Theorem DT [417] employing the transpose of the matrix. □

Let’s go for understanding the effect of all three row operations. But first we need an intermediate result, but it is an easy one.

**Theorem DERC**

**Determinant with Equal Rows or Columns**

Suppose that \( A \) is a square matrix with two equal rows, or two equal columns. Then \( \det (A) = 0. \)

**Proof**  Suppose that \( A \) is a square matrix of size \( n \) where the two rows \( s \) and \( t \) are equal. Form the matrix \( B \) by swapping rows \( r \) and \( s \). Notice that as a consequence of our hypothesis, \( A = B \). Then

\[
\det (A) = \frac{1}{2} (\det (A) + \det (A)) = \frac{1}{2} (\det (A) - \det (B)) \text{ Theorem DRCS } 425
\]

\[
= \frac{1}{2} (\det (A) - \det (A)) \text{ Hypothesis, } A = B
\]

\[
= \frac{1}{2} (0) = 0
\]

The proof for the case of two equal columns is entirely similar, or could be derived from an application of Theorem DT [417] employing the transpose of the matrix. □

Now explain the third row operation. Here we go.

**Theorem DRCMA**

**Determinant for Row or Column Multiples and Addition**

Suppose that \( A \) is a square matrix. Let \( B \) be the square matrix obtained from \( A \) by multiplying a row by the scalar \( \alpha \) and then adding it to another row, or by multiplying a column by the scalar \( \alpha \) and then adding it to another column. Then \( \det (B) = \det (A). \)

**Proof**  Suppose that \( A \) is a square matrix of size \( n \). Form the matrix \( B \) by multiplying row \( s \) by \( \alpha \) and adding it to row \( t \). Let \( C \) be the auxiliary matrix where we replace row \( t \) of \( A \) by row \( s \) of \( A \). Notice that \( A (t|j) = B (t|j) = C (t|j) \) for all \( 1 \leq j \leq n \). We compute the determinant of \( B \) by expansion about row \( t \).

\[
\det (B) = \sum_{j=1}^{n} (-1)^{t+j} [B]_{tj} \det (B (t|j)) \text{ Theorem DER } 416
\]
\[
\begin{align*}
\sum_{j=1}^{n} (-1)^{t+j} \left( \alpha [A]_{s+j} + [A]_{t+j} \right) \det (B (s|j)) & \quad \text{Hypothesis} \\
= \sum_{j=1}^{n} (-1)^{t+j} \alpha [A]_{s+j} \det (B (t|j)) & \\
+ \sum_{j=1}^{n} (-1)^{t+j} [A]_{t+j} \det (B (t|j)) & \\
= \alpha \sum_{j=1}^{n} (-1)^{t+j} [A]_{s+j} \det (B (t|j)) & \\
+ \sum_{j=1}^{n} (-1)^{t+j} [A]_{t+j} \det (B (t|j)) & \\
= \alpha \sum_{j=1}^{n} (-1)^{t+j} [C]_{t+j} \det (C (t|j)) & \\
+ \sum_{j=1}^{n} (-1)^{t+j} [A]_{t+j} \det (A (t|j)) & \\
= \alpha \det (C) + \det (A) & \quad \text{Theorem DER 416} \\
= \alpha 0 + \det (A) = \det (A) & \quad \text{Theorem DERC 427}
\end{align*}
\]

The proof for the case of adding a multiple of a column is entirely similar, or could be derived from an application of Theorem DT 417 employing the transpose of the matrix.

Is this what you expected? We could argue that the third row operation is the most popular, and yet it has no effect whatsoever on the determinant of a matrix! We can exploit this, along with our understanding of the other two row operations, to provide another approach to computing a determinant. We’ll explain this in the context of an example.

**Example DRO**

**Determinant by row operations**

Suppose we desire the determinant of the \(4 \times 4\) matrix

\[
A = \begin{bmatrix}
2 & 0 & 2 & 3 \\
1 & 3 & -1 & 1 \\
-1 & 1 & -1 & 2 \\
3 & 5 & 4 & 0
\end{bmatrix}
\]

We will perform a sequence of row operations on this matrix, shooting for an upper-triangular matrix, whose determinant will be simply the product of its diagonal entries. For each row operation, we will track the effect on the determinant via Theorem DRCS 425, Theorem DRCM 426, Theorem DRCMA 427.

\[
R_{1} \leftrightarrow R_{2} \quad A_{1} = \begin{bmatrix}
1 & 3 & -1 & 1 \\
2 & 0 & 2 & 3 \\
-1 & 1 & -1 & 2 \\
3 & 5 & 4 & 0
\end{bmatrix} \quad \det (A) = - \det (A_{1}) \quad \text{Theorem DRCS 425}
\]
\[ -2R_1 + R_2 \quad A_2 = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & -6 & 4 & 1 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{bmatrix} = -\det(A_2) \quad \text{Theorem DRCMA} [427] \]

\[ R_1 + R_3 \quad A_3 = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & -6 & 4 & 1 \\ 0 & 4 & -2 & 3 \\ 3 & 5 & 4 & 0 \end{bmatrix} = -\det(A_3) \quad \text{Theorem DRCMA} [427] \]

\[ -3R_1 + R_4 \quad A_4 = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & -6 & 4 & 1 \\ 0 & 4 & -2 & 3 \\ 0 & -4 & 7 & -3 \end{bmatrix} = -\det(A_4) \quad \text{Theorem DRCMA} [427] \]

\[ R_3 + R_2 \quad A_5 = \begin{bmatrix} 0 & -2 & 2 & 4 \\ 1 & 3 & -1 & 1 \\ 0 & 4 & -2 & 3 \\ 0 & -4 & 7 & -3 \end{bmatrix} = -\det(A_5) \quad \text{Theorem DRCMA} [427] \]

\[ -\frac{1}{2}R_2 \quad A_6 = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 4 & -2 & 3 \\ 0 & -4 & 7 & -3 \end{bmatrix} = 2\det(A_6) \quad \text{Theorem DRCM} [426] \]

\[ -4R_2 + R_3 \quad A_7 = \begin{bmatrix} 0 & 1 & -1 & -2 \\ 1 & 3 & -1 & 1 \\ 0 & 0 & 2 & 11 \\ 0 & -4 & 7 & -3 \end{bmatrix} = 2\det(A_7) \quad \text{Theorem DRCMA} [427] \]

\[ 4R_2 + R_4 \quad A_8 = \begin{bmatrix} 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 11 \\ 0 & 0 & 3 & -11 \end{bmatrix} = 2\det(A_8) \quad \text{Theorem DRCMA} [427] \]

\[ -R_3 + R_4 \quad A_9 = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 11 \\ 0 & 0 & 1 & -22 \end{bmatrix} = 2\det(A_9) \quad \text{Theorem DRCMA} [427] \]

\[ -2R_4 + R_3 \quad A_{10} = \begin{bmatrix} 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 55 \\ 0 & 0 & 1 & -22 \end{bmatrix} = 2\det(A_{10}) \quad \text{Theorem DRCMA} [427] \]

\[ R_3 - R_4 \quad A_{11} = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 55 \\ 0 & 0 & 1 & -22 \end{bmatrix} = -2\det(A_{11}) \quad \text{Theorem DRCS} [425] \]

\[ \frac{1}{2}R_4 \quad A_{12} = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -110\det(A_{12}) \quad \text{Theorem DRCM} [426] \]
The matrix $A_{12}$ is upper-triangular, so expansion about the first column (repeatedly) will result in $\det(A_{12}) = (1)(1)(1)(1) = 1$ (see Example DUTM [419]) and thus, $\det(A) = -110(1) = -110$.

Notice that our sequence of row operations was somewhat ad hoc, such as the transformation to $A_5$. We could have been even more methodical, and strictly followed the process that converts a matrix to reduced row-echelon form (Theorem REMEF [36]), eventually achieving the same numerical result with a final matrix that equaled the $4 \times 4$ identity matrix. Notice too that we could have stopped with $A_8$, since at this point we could compute $\det(A_8)$ by two expansions about first columns, followed by a simple determinant of a $2 \times 2$ matrix (Theorem DMST [415]).

The beauty of this approach is that computationally we should already have written a procedure to convert matrices to reduced-row echelon form, so all we need to do is track the multiplicative changes to the determinant as the algorithm proceeds. Further, for a square matrix of size $n$ this approach requires on the order of $n^3$ multiplications, while a recursive application of expansion about a row or column (Theorem DER [416], Theorem DEC [417]) will require in the vicinity of $(n-1)(n!)$ multiplications. So even for very small matrices, a computational approach utilizing row operations will have superior run-time. Tracking, and controlling, the effects of round-off errors is another story, best saved for a numerical linear algebra course.

Subsection DROEM
Determinants, Row Operations, Elementary Matrices

As a final preparation for our two most important theorems about determinants, we prove a handful of facts about the interplay of row operations and matrix multiplication with elementary matrices with regard to the determinant. But first, a simple, but crucial, fact about the identity matrix.

**Theorem DIM**

**Determinant of the Identity Matrix**

For every $n \geq 1$, $\det(I_n) = 1$. □

**Proof** It may be overkill, but this is a good situation to run through a proof by induction on $n$ (Technique I [713]). Is the result true when $n = 1$? Yes,

$$\det(I_1) = [I_1]_{11} = 1$$

Definition DM [414] Definition IM [76]

Now assume the theorem is true for the identity matrix of size $n - 1$ and investigate the determinant of the identity matrix of size $n$ with expansion about row 1,

$$\det(I_n) = \sum_{j=1}^{n} (-1)^{1+j} [I_n]_{1j} \det(I_n(1|j))$$

Definition DM [414]

$$= (-1)^{1+1} [I_n]_{11} \det(I_n(1|1))$$

$$+ \sum_{j=2}^{n} (-1)^{1+j} [I_n]_{1j} \det(I_n(1|j))$$

Version 0.92
\[ 1 \det (I_{n-1}) + \sum_{j=2}^{n} (-1)^{1+j} 0 \det (I_n (1|j)) \quad \text{Definition IM 76} \]

\[ = 1(1) + \sum_{j=2}^{n} 0 = 1 \quad \text{Induction Hypothesis} \]

Theorem DEM
Determinants of Elementary Matrices
For the three possible versions of an elementary matrix (Definition ELEM 409) we have the determinants,

1. \( \det (E_{i,j}) = -1 \)
2. \( \det (E_i (\alpha)) = \alpha \)
3. \( \det (E_{i,j} (\alpha)) = 1 \)

Proof
Swapping rows \( i \) and \( j \) of the identity matrix will create \( E_{i,j} \) (Definition ELEM 409), so

\[ \det (E_{i,j}) = - \det (I_n) \quad \text{Theorem DRCS 425} \]
\[ = -1 \quad \text{Theorem DIM 430} \]

Multiplying row \( i \) of the identity matrix by \( \alpha \) will create \( E_i (\alpha) \) (Definition ELEM 409), so

\[ \det (E_i (\alpha)) = \alpha \det (I_n) \quad \text{Theorem DRCM 426} \]
\[ = \alpha(1) = \alpha \quad \text{Theorem DIM 430} \]

Multiplying row \( i \) of the identity matrix by \( \alpha \) and adding to row \( j \) will create \( E_i (\alpha) j \) (Definition ELEM 409), so

\[ \det (E_i (\alpha) j) = \det (I_n) \quad \text{Theorem DRCMA 427} \]
\[ = 1 \quad \text{Theorem DIM 430} \]

Theorem DEMMM
Determinants, Elementary Matrices, Matrix Multiplication
Suppose that \( A \) is a square matrix of size \( n \) and \( E \) is any elementary matrix of size \( n \). Then

\[ \det (EA) = \det (E) \det (A) \]
Proof  The proof proceeds in three parts, one for each type of elementary matrix, with each part very similar to the other two. First, let $B$ be the matrix obtained from $A$ by swapping rows $i$ and $j$,

\[
\begin{align*}
\det (E_{i,j}A) &= \det (B) \\
&= -\det (A) \\
&= \det (E_{i,j}) \det (A)
\end{align*}
\]

Second, let $B$ be the matrix obtained from $A$ by multiplying row $i$ by $\alpha$,

\[
\begin{align*}
\det (E_i(\alpha)A) &= \det (B) \\
&= \alpha \det (A) \\
&= \det (E_i(\alpha)) \det (A)
\end{align*}
\]

Third, let $B$ be the matrix obtained from $A$ by multiplying row $i$ by $\alpha$ and adding to row $j$,

\[
\begin{align*}
\det (E_{i,j}(\alpha)A) &= \det (B) \\
&= \det (A) \\
&= \det (E_{i,j}(\alpha)) \det (A)
\end{align*}
\]

Since the desired result holds for each variety of elementary matrix individually, we are done.

Subsection DNMMNM
Determinants, Nonsingular Matrices, Matrix Multiplication

If you asked someone with substantial experience working with matrices about the value of the determinant, they’d be likely to quote the following theorem as the first thing to come to mind.

**Theorem SMZD**

**Singular Matrices have Zero Determinants**

Let $A$ be a square matrix. Then $A$ is singular if and only if $\det (A) = 0$.

Proof  $\Rightarrow$  Suppose that $A$ is a singular matrix of size $n$. Then $A$ is row-equivalent to a square matrix $B$ in reduced row-echelon form (Theorem REMEF [36]). Since $A$ is singular, the matrix $B$ is not the identity matrix (Theorem NMRRR [77]). Therefore, the number of pivot columns is strictly less than $n$, i.e. $r < n$, and so $B$ has at least one row of all zeros.

There is a sequence of row operations $R_1$, $R_2$, $R_3$, $\ldots$, $R_s$ that will convert $B$ into $A$. For each of these row operations, there is an elementary matrix $E_i$ which effects the row operation by matrix multiplication (Theorem EMDRO [411]). Repeated applications of Theorem EMDRO [411] allow us to write

\[A = E_sE_{s-1}\ldots E_2E_1B\]
Then
\[
\det(A) = \det(E_sE_{s-1} \ldots E_2E_1B)
= \det(E_s)\det(E_{s-1}) \ldots \det(E_2)\det(E_1)\det(B) \quad \text{Theorem DEMMM} 431
= \det(E_s)\det(E_{s-1}) \ldots \det(E_2)\det(E_1) 0 \quad \text{Theorem DZRC} 425
= 0
\]

\((\Leftarrow)\) We will establish the contrapositive of this implication. So begin by assuming that
\(A\) is nonsingular. Then \(A\) is row-equivalent to the identity matrix by Theorem NM-RRI 77. As above, there is a sequence of row operations that will convert \(I_n\) to
\(A\), which can be effected by matrix multiplication by elementary matrices and Theorem DEMMM 431 allows us to “distribute” the determinant through this product. Mimicking the first half of the proof, we would arrive at

\[
\det(A) = \det(E_s)\det(E_{s-1}) \ldots \det(E_2)\det(E_1)\det(I_n)
\]

We know that \(\det(I_n) = 1 \neq 0\). From Theorem DEM 431 we can infer that the
determinant of an elementary matrix is never zero (note the ban on \(\alpha = 0\) for \(E_i(\alpha)\) in
Definition ELEM 409). So the product on the right is composed of nonzero scalars, and
so is also nonzero. This is the result we needed. 

For the case of \(2 \times 2\) matrices you might compare the application of Theorem SMZD 432
with the combination of the results stated in Theorem DMST 415 and Theorem TTMI 234.

**Example ZNDAB**

**Zero and nonzero determinant, Archetypes A and B**

The coefficient matrix in Archetype A 721 has a zero determinant (check this!) while
the coefficient matrix Archetype B 726 has a nonzero determinant (check this, too).
These matrices are singular and nonsingular, respectively. This is exactly what Theorem SMZD 432 says, and continues our list of contrasts between these two archetypes. 

Since Theorem SMZD 432 is an equivalence (Technique E 704) we can expand
on our growing list of equivalences about nonsingular matrices. The addition of the
condition \(\det(A) \neq 0\) is one of the best motivations for learning about determinants.

**Theorem NME7**

**Nonsingular Matrix Equivalences, Round 7**

Suppose that \(A\) is a square matrix of size \(n\). The following are equivalent.

1. \(A\) is nonsingular.
2. \(A\) row-reduces to the identity matrix.
3. The null space of \(A\) contains only the zero vector, \(N(A) = \{0\}\).
4. The linear system \(LS(A, b)\) has a unique solution for every possible choice of \(b\).
5. The columns of \(A\) are a linearly independent set.
6. \(A\) is invertible.
7. The column space of $A$ is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.

8. The columns of $A$ are a basis for $\mathbb{C}^n$.

9. The rank of $A$ is $n$, $r(A) = n$.

10. The nullity of $A$ is zero, $n(A) = 0$.

11. The determinant of $A$ is nonzero, $\det(A) \neq 0$.

**Proof** [Theorem SMZD](#) says $A$ is singular if and only if $\det(A) = 0$. If we negate each of these statements, we arrive at two contrapositives that we can combine as the equivalence, $A$ is nonsingular if and only if $\det(A) \neq 0$. This allows us to add a new statement to the list found in [Theorem NME6](#).

Computationally, row-reducing a matrix is the most efficient way to determine if a matrix is nonsingular, though the effect of using division in a computer can lead to round-off errors that confuse small quantities with critical zero quantities. Conceptually, the determinant may seem the most efficient way to determine if a matrix is nonsingular. The definition of a determinant uses just addition, subtraction and multiplication, so division is never a problem. And the final test is easy: is the determinant zero or not? However, the number of operations involved in computing a determinant by the definition very quickly becomes so excessive as to be impractical.

Now for the *coup de grâce*. We will generalize [Theorem DEMMM](#) to the case of any two square matrices. You may recall thinking that matrix multiplication was defined in a needlessly complicated manner. For sure, the definition of a determinant seems even stranger. (Though [Theorem SMZD](#) might be forcing you to reconsider.) Read the statement of the next theorem and contemplate how nicely matrix multiplication and determinants play with each other.

**Theorem DRMM**

**Determinant Respects Matrix Multiplication**

Suppose that $A$ and $B$ are square matrices of the same size. Then $\det(AB) = \det(A)\det(B)$.

**Proof** Suppose that $A$ or $B$ is singular. Then either $\det(A) = 0$ or $\det(B) = 0$ by [Theorem SMZD](#). In either case, $\det(A)\det(B) = 0$. By the contrapositive of [Theorem NPNT](#), we know $AB$ is singular as well. So by [Theorem SMZD](#), $\det(AB) = 0$. So in this case, we have the desired equality.

Now assume that $A$ and $B$ are both nonsingular. By [Theorem NMPEM](#) there are elementary matrices $E_1, E_2, E_3, \ldots, E_s$ and $E_{s+1}, E_{s+2}, E_{s+3}, \ldots, E_{s+t}$ such that

$$A = E_1E_2E_3\ldots E_s \quad B = E_{s+1}E_{s+2}E_{s+3}\ldots E_{s+t}$$

Then

$$\det(AB) = \det(E_1E_2\ldots E_sE_{s+1}E_{s+2}\ldots E_{s+t})$$

$$= \det(E_1)\det(E_2)\ldots\det(E_s)\det(E_{s+1}E_{s+2}\ldots E_{s+t}) \quad \text{[Theorem DEMMM]}$$

$$= \det(E_1E_2\ldots E_s)\det(E_{s+1}E_{s+2}\ldots E_{s+t}) \quad \text{[Theorem DEMMM]}$$

$$= \det(A)\det(B)$$
It’s an amazing thing that matrix multiplication and the determinant interact this way. Might it also be true that \( \det (A + B) = \det (A) + \det (B) \)? (See Exercise PDM.M30.)

Subsection READ
Reading Questions

1. Consider the two matrices below, and suppose you already have computed \( \det (A) = -120 \). What is \( \det (B) \)? Why?

\[
A = \begin{bmatrix}
0 & 8 & 3 & -4 \\
-1 & 2 & -2 & 5 \\
-2 & 8 & 4 & 3 \\
0 & -4 & 2 & -3 \\
\end{bmatrix} \quad B = \begin{bmatrix}
0 & 8 & 3 & -4 \\
0 & -4 & 2 & -3 \\
-2 & 8 & 4 & 3 \\
-1 & 2 & -2 & 5 \\
\end{bmatrix}
\]

2. State the theorem that allows us to make yet another extension to our NMEx series of theorems.

3. What is amazing about the interaction between matrix multiplication and the determinant?
Subsection EXC
Exercises

C30 Each of the archetypes below is a system of equations with a square coefficient matrix, or is a square matrix itself. Compute the determinant of each matrix, noting how Theorem SMZD indicates when the matrix is singular or nonsingular.

Archetype A 721
Archetype B 726
Archetype F 743
Archetype K 767
Archetype L 771

Contributed by Robert Beezer

M20 Construct a $3 \times 3$ nonsingular matrix and call it $A$. Then, for each entry of the matrix, compute the corresponding cofactor, and create a new $3 \times 3$ matrix full of these cofactors by placing the cofactor of an entry in the same location as the entry it was based on. Once complete, call this matrix $C$. Compute $AC^t$. Any observations? Repeat with a new matrix, or perhaps with a $4 \times 4$ matrix.

Contributed by Robert Beezer Solution 439

M30 Construct an example to show that the following statement is not true for all square matrices $A$ and $B$ of the same size: $\det (A + B) = \det (A) + \det (B)$.

Contributed by Robert Beezer

T10 Theorem NPNT says that if the product of square matrices $AB$ is nonsingular, then the individual matrices $A$ and $B$ are nonsingular also. Construct a new proof of this result making use of theorems about determinants of matrices.

Contributed by Robert Beezer

T15 Use Theorem DRCM to prove Theorem DZRC as a corollary. (See Technique LC.)

Contributed by Robert Beezer

T20 Suppose that $A$ is a square matrix of size $n$ and $\alpha \in \mathbb{C}$ is a scalar. Prove that $\det (\alpha A) = \alpha^n \det (A)$.

Contributed by Robert Beezer

T25 Employ Theorem DT to construct the second half of the proof of Theorem DRCM (the portion about a multiple of a column).

Contributed by Robert Beezer
M20  Contributed by Robert Beezer  Statement 437

The result of these computations should be a matrix with the value of det \( A \) in the diagonal entries and zeros elsewhere. The suggestion of using a nonsingular matrix was partially so that it was obvious that the value of the determinant appears on the diagonal.

This result (which is true in general) provides a method for computing the inverse of a nonsingular matrix. Since \( AC^t = \det(A) I_n \), we can multiply by the reciprocal of the determinant (which is nonzero!) and the inverse of \( A \) (it exists!) to arrive at an expression for the matrix inverse:

\[
A^{-1} = \frac{1}{\det(A)} C^t
\]
Chapter E
Eigenvalues

When we have a square matrix of size \( n \), \( A \), and we multiply it by a vector \( x \) from \( \mathbb{C}^n \) to form the matrix-vector product (Definition MVP [211]), the result is another vector in \( \mathbb{C}^n \). So we can adopt a functional view of this computation — the act of multiplying by a square matrix is a function that converts one vector \( (x) \) into another one \( (Ax) \) of the same size. For some vectors, this seemingly complicated computation is really no more complicated than scalar multiplication. The vectors vary according to the choice of \( A \), so the question is to determine, for an individual choice of \( A \), if there are any such vectors, and if so, which ones. It happens in a variety of situations that these vectors (and the scalars that go along with them) are of special interest.

We will be solving polynomial equations in this chapter, which raises the specter of roots that are complex numbers. This distinct possibility is our main reason for entertaining the complex numbers throughout the course. You might be moved to revisit Section CNO [687] and Section O [183].

Section EE
Eigenvalues and Eigenvectors

We start with the principal definition for this chapter.

Subsection EEM
Eigenvalues and Eigenvectors of a Matrix

Definition EEM
Eigenvalues and Eigenvectors of a Matrix
Suppose that \( A \) is a square matrix of size \( n \), \( x \neq 0 \) is a vector in \( \mathbb{C}^n \), and \( \lambda \) is a scalar in \( \mathbb{C} \). Then we say \( x \) is an eigenvector of \( A \) with eigenvalue \( \lambda \) if

\[
Ax = \lambda x
\]
Before going any further, perhaps we should convince you that such things ever happen at all. Understand the next example, but do not concern yourself with where the pieces come from. We will have methods soon enough to be able to discover these eigenvectors ourselves.

**Example SEE**

**Some eigenvalues and eigenvectors**

Consider the matrix

\[
A = \begin{bmatrix}
204 & 98 & -26 & -10 \\
-280 & -134 & 36 & 14 \\
716 & 348 & -90 & -36 \\
-472 & -232 & 60 & 28
\end{bmatrix}
\]

and the vectors

\[
x = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix}, \quad y = \begin{bmatrix} -3 \\ 4 \\ -10 \\ 4 \end{bmatrix}, \quad z = \begin{bmatrix} -3 \\ 7 \\ 0 \\ 8 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 0 \end{bmatrix}
\]

Then

\[
Ax = \begin{bmatrix} 204 & 98 & -26 & -10 \\
-280 & -134 & 36 & 14 \\
716 & 348 & -90 & -36 \\
-472 & -232 & 60 & 28 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix} = 4x
\]

so \(x\) is an eigenvector of \(A\) with eigenvalue \(\lambda = 4\). Also,

\[
Ay = \begin{bmatrix} 204 & 98 & -26 & -10 \\
-280 & -134 & 36 & 14 \\
716 & 348 & -90 & -36 \\
-472 & -232 & 60 & 28 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ -10 \\ 4 \end{bmatrix} = 0 \begin{bmatrix} -3 \\ 4 \\ -10 \\ 4 \end{bmatrix} = 0y
\]

so \(y\) is an eigenvector of \(A\) with eigenvalue \(\lambda = 0\). Also,

\[
Az = \begin{bmatrix} 204 & 98 & -26 & -10 \\
-280 & -134 & 36 & 14 \\
716 & 348 & -90 & -36 \\
-472 & -232 & 60 & 28 \end{bmatrix} \begin{bmatrix} -3 \\ 7 \\ 0 \\ 8 \end{bmatrix} = 2 \begin{bmatrix} -3 \\ 7 \\ 0 \\ 8 \end{bmatrix} = 2z
\]

so \(z\) is an eigenvector of \(A\) with eigenvalue \(\lambda = 2\). Also,

\[
Aw = \begin{bmatrix} 204 & 98 & -26 & -10 \\
-280 & -134 & 36 & 14 \\
716 & 348 & -90 & -36 \\
-472 & -232 & 60 & 28 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 4 \\ 0 \end{bmatrix} = 2w
\]

so \(w\) is an eigenvector of \(A\) with eigenvalue \(\lambda = 2\).

So we have demonstrated four eigenvectors of \(A\). Are there more? Yes, any nonzero scalar multiple of an eigenvector is again an eigenvector. In this example, set \(u = 30x\). Then

\[
Au = A(30x)
\]
Theorem MMSMM [219]

Property SMAM [199]

Property DVAC [91]

Theorem MMDAA [219]

Subsection EE.PM Polynomials and Matrices

A polynomial is a combination of powers, multiplication by scalar coefficients, and addition (with subtraction just being the inverse of addition). We never have occasion to divide when computing the value of a polynomial. So it is with matrices. We can add and subtract matrices, we can multiply matrices by scalars, and we can form powers of square matrices by repeated applications of matrix multiplication. We do not normally divide matrices (though sometimes we can multiply by an inverse). If a matrix is square, all the operations constituting a polynomial will preserve the size of the matrix. So it is natural to consider evaluating a polynomial with a matrix, effectively replacing the variable of the polynomial by a matrix. We’ll demonstrate with an example.

Example PM

Polynomial of a matrix

Let

\[ p(x) = 14 + 19x - 3x^2 - 7x^3 + x^4 \]

\[ D = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} \]
and we will compute \( p(D) \). First, the necessary powers of \( D \). Notice that \( D^0 \) is defined to be the multiplicative identity, \( I_3 \), as will be the case in general.

\[
D^0 = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
D^1 = D = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix}
\]

\[
D^2 = DD^1 = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 & -6 \\ 5 & 1 & 0 \\ 1 & -8 & -7 \end{bmatrix}
\]

\[
D^3 = DD^2 = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & -6 \\ 5 & 1 & 0 \\ 1 & -8 & -7 \end{bmatrix} = \begin{bmatrix} 19 & -12 & -8 \\ -4 & 15 & 8 \\ 12 & -4 & 11 \end{bmatrix}
\]

\[
D^4 = DD^3 = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 19 & -12 & -8 \\ -4 & 15 & 8 \\ 12 & -4 & 11 \end{bmatrix} = \begin{bmatrix} -7 & 49 & 54 \\ -5 & -4 & -30 \\ -49 & 47 & 43 \end{bmatrix}
\]

Then

\[
p(D) = 14 + 19D - 3D^2 - 7D^3 + D^4
\]

\[
= 14 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 19 \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} - 3 \begin{bmatrix} -2 & -1 & -6 \\ 5 & 1 & 0 \\ 1 & -8 & -7 \end{bmatrix} - 7 \begin{bmatrix} 19 & -12 & -8 \\ -4 & 15 & 8 \\ 12 & -4 & 11 \end{bmatrix} + \begin{bmatrix} -7 & 49 & 54 \\ -5 & -4 & -30 \\ -49 & 47 & 43 \end{bmatrix}
\]

\[
= \begin{bmatrix} -139 & 193 & 166 \\ 27 & -98 & -124 \\ -193 & 118 & 20 \end{bmatrix}
\]

Notice that \( p(x) \) factors as

\[
p(x) = 14 + 19x - 3x^2 - 7x^3 + x^4 = (x - 2)(x - 7)(x + 1)^2
\]

Because \( D \) commutes with itself (\( DD = DD \)), we can use distributivity of matrix multiplication across matrix addition (Theorem MMDAA [219]) without being careful with any of the matrix products, and just as easily evaluate \( p(D) \) using the factored form of \( p(x) \),

\[
p(D) = 14 + 19D - 3D^2 - 7D^3 + D^4 = (D - 2I_3)(D - 7I_3)(D + I_3)^2
\]

\[
= \begin{bmatrix} -3 & 3 & 2 \\ 1 & -2 & -2 \\ -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} -8 & 3 & 2 \\ 1 & -7 & -2 \\ -3 & 1 & -6 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 \\ 1 & 1 & -2 \\ -3 & 1 & 2 \end{bmatrix}^2
\]

\[
= \begin{bmatrix} -139 & 193 & 166 \\ 27 & -98 & -124 \\ -193 & 118 & 20 \end{bmatrix}
\]
This example is not meant to be too profound. It is meant to show you that it is natural to evaluate a polynomial with a matrix, and that the factored form of the polynomial is as good as (or maybe better than) the expanded form. And do not forget that constant terms in polynomials are really multiples of the identity matrix when we are evaluating the polynomial with a matrix.

Subsection EE.EEE
Existence of Eigenvalues and Eigenvectors

Before we embark on computing eigenvalues and eigenvectors, we will prove that every matrix has at least one eigenvalue (and an eigenvector to go with it). Later, in Theorem MNEM, we will determine the maximum number of eigenvalues a matrix may have.

The determinant will be a powerful tool in Subsection EE.CEE when it comes time to compute eigenvalues. However, it is possible, with some more advanced machinery, to compute eigenvalues without ever making use of the determinant. Sheldon Axler does just that in his book, Linear Algebra Done Right. Here and now, we give Axler’s “determinant-free” proof that every matrix has an eigenvalue. The result is not too startling, but the proof is most enjoyable.

Theorem EMHE
Every Matrix Has an Eigenvalue

Suppose $A$ is a square matrix. Then $A$ has at least one eigenvalue.

Proof Suppose that $A$ has size $n$, and choose $x$ as any nonzero vector from $\mathbb{C}^n$. (Notice how much latitude we have in our choice of $x$. Only the zero vector is off-limits.) Consider the set

$$S = \{x, Ax, A^2x, A^3x, \ldots, A^n x\}$$

This is a set of $n + 1$ vectors from $\mathbb{C}^n$, so by Theorem MVSLD, $S$ is linearly dependent. Let $a_0, a_1, a_2, \ldots, a_n$ be a collection of $n + 1$ scalars from $\mathbb{C}$, not all zero, that provide a relation of linear dependence on $S$. In other words,

$$a_0x + a_1Ax + a_2A^2x + a_3A^3x + \cdots + a_n A^n x = 0$$

Some of the $a_i$ are nonzero. Suppose that just $a_0 \neq 0$, and $a_1 = a_2 = a_3 = \cdots = a_n = 0$. Then $a_0x = 0$ and by Theorem SMEZV, either $a_0 = 0$ or $x = 0$, which are both contradictions. So $a_i \neq 0$ for some $i \geq 1$. Let $m$ be the largest integer such that $a_m \neq 0$. From this discussion we know that $m \geq 1$. We can also assume that $a_m = 1$, for if not, replace each $a_i$ by $a_i/a_m$ to obtain scalars that serve equally well in providing a relation of linear dependence on $S$.

Define the polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_m x^m$$

Because we have consistently used $\mathbb{C}$ as our set of scalars (rather than $\mathbb{R}$), we know that we can factor $p(x)$ into linear factors of the form $(x - b_i)$, where $b_i \in \mathbb{C}$. So there are scalars, $b_1, b_2, b_3, \ldots, b_m$, from $\mathbb{C}$ so that,

$$p(x) = (x - b_m)(x - b_{m-1}) \cdots (x - b_3)(x - b_2)(x - b_1)$$

Version 0.92
Put it all together and
\[ 0 = a_0 \mathbf{x} + a_1 A \mathbf{x} + a_2 A^2 \mathbf{x} + a_3 A^3 \mathbf{x} + \cdots + a_m A^m \mathbf{x} \]
\[ = (a_0 I_n + a_1 A + a_2 A^2 + a_3 A^3 + \cdots + a_m A^m) \mathbf{x} \]
\[ = p(A) \mathbf{x} \]
\[ = (A - b_m I_n)(A - b_{m-1} I_n) \cdots (A - b_3 I_n)(A - b_2 I_n)(A - b_1 I_n) \mathbf{x} \]

Let \( k \) be the smallest integer such that
\[ (A - b_k I_n)(A - b_{k-1} I_n) \cdots (A - b_3 I_n)(A - b_2 I_n)(A - b_1 I_n) \mathbf{x} = 0. \]

From the preceding equation, we know that \( k \leq m \). Define the vector \( \mathbf{z} \) by
\[ \mathbf{z} = (A - b_{k-1} I_n) \cdots (A - b_3 I_n)(A - b_2 I_n)(A - b_1 I_n) \mathbf{x} \]

Notice that by the definition of \( k \), the vector \( \mathbf{z} \) must be nonzero. In the case where \( k = 1 \), we understand that \( \mathbf{z} \) is defined by \( \mathbf{z} = \mathbf{x} \), and \( \mathbf{z} \) is still nonzero. Now
\[ (A - b_k I_n)\mathbf{z} = (A - b_k I_n)(A - b_{k-1} I_n) \cdots (A - b_3 I_n)(A - b_2 I_n)(A - b_1 I_n) \mathbf{x} = 0 \]
which allows us to write
\[ A\mathbf{z} = (A + O)\mathbf{z} \]
\[ = (A - b_k I_n + b_k I_n)\mathbf{z} \]
\[ = (A - b_k I_n)\mathbf{z} + b_k I_n \mathbf{z} \]
\[ = 0 + b_k I_n \mathbf{z} \]
\[ = b_k I_n \mathbf{z} \]
\[ = b_k \mathbf{z} \]

Since \( \mathbf{z} \neq \mathbf{0} \), this equation says that \( \mathbf{z} \) is an eigenvector of \( A \) for the eigenvalue \( \lambda = b_k \) (**Definition EEM** [441]), so we have shown that any square matrix \( A \) does have at least one eigenvalue. \( \blacksquare \)

The proof of **Theorem EMHE** [445] is constructive (it contains an unambiguous procedure that leads to an eigenvalue), but it is not meant to be practical. We will illustrate the theorem with an example, the purpose being to provide a companion for studying the proof and not to suggest this is the best procedure for computing an eigenvalue.

**Example CAEHW**

**Computing an eigenvalue the hard way**

This example illustrates the proof of **Theorem EMHE** [445], so will employ the same notation as the proof — look there for full explanations. It is *not* meant to be an example of a reasonable computational approach to finding eigenvalues and eigenvectors.

OK, warnings in place, here we go.

Let
\[ A = \begin{bmatrix} -7 & -1 & 11 & 0 & -4 \\ 4 & 1 & 0 & 2 & 0 \\ -10 & -1 & 14 & 0 & -4 \\ 8 & 2 & -15 & -1 & 5 \\ -10 & -1 & 16 & 0 & -6 \end{bmatrix} \]

Version 0.92
and choose

\[
x = \begin{bmatrix}
3 \\
0 \\
3 \\
-5 \\
4
\end{bmatrix}
\]

It is important to notice that the choice of \( x \) could be \textit{anything}, so long as it is \textit{not} the zero vector. We have not chosen \( x \) totally at random, but so as to make our illustration of the theorem as general as possible. You could replicate this example with your own choice and the computations are guaranteed to be reasonable, provided you have a computational tool that will factor a fifth degree polynomial for you.

The set

\[
S = \{ x, Ax, A^2x, A^3x, A^4x, A^5x \}
\]

is guaranteed to be linearly dependent, as it has six vectors from \( \mathbb{C}^5 \) (Theorem MVSLD [150]).

We will search for a non-trivial relation of linear dependence by solving a homogeneous system of equations whose coefficient matrix has the vectors of \( S \) as columns through row operations,

\[
\begin{bmatrix}
3 & -4 & 6 & -10 & 18 & -34 \\
0 & 2 & -6 & 14 & -30 & 62 \\
3 & -4 & 6 & -10 & 18 & -34 \\
-5 & 4 & -2 & -2 & 10 & -26 \\
4 & -6 & 10 & -18 & 34 & -66
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & -2 & 6 & -14 & 30 \\
0 & 1 & 3 & 7 & -15 & 31 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

There are four free variables for describing solutions to this homogeneous system, so we have our pick of solutions. The most expedient choice would be to set \( x_3 = 1 \) and \( x_4 = x_5 = x_6 = 0 \). However, we will again opt to maximize the generality of our illustration of Theorem EMHE [445] and choose \( x_3 = -8, x_4 = -3, x_5 = 1 \) and \( x_6 = 0 \). The leads to a solution with \( x_1 = 16 \) and \( x_2 = 12 \).

This relation of linear dependence then says that

\[
0 = 16x + 12Ax - 8A^2x - 3A^3x + A^4x + 0A^5x
\]

So we define \( p(x) = 16 + 12x - 8x^2 - 3x^3 + x^4, \) and as advertised in the proof of Theorem EMHE [445], we have a polynomial of degree \( m = 4 > 1 \) such that \( p(A)x = 0 \).

Now we need to factor \( p(x) \) over \( \mathbb{C} \). If you made your own choice of \( x \) at the start, this is where you might have a fifth degree polynomial, and where you might need to use a computational tool to find roots and factors. We have

\[
p(x) = 16 + 12x - 8x^2 - 3x^3 + x^4 = (x - 4)(x + 2)(x - 2)(x + 1)
\]
So we know that

\[ 0 = p(A)x = (A - 4I_5)(A + 2I_5)(A - 2I_5)(A + 1I_5)x \]

We apply one factor at a time, until we get the zero vector, so as to determine the value of \( k \) described in the proof of Theorem EMHE 445,

\[
(A + 1I_5)x = \begin{bmatrix}
-6 & -1 & 11 & 0 & -4 \\
4 & 2 & 0 & 2 & 0 \\
-10 & -1 & 15 & 0 & -4 \\
8 & 2 & -15 & 0 & 5 \\
-10 & -1 & 16 & 0 & -5
\end{bmatrix} \begin{bmatrix}
3 \\
0 \\
3 \\
-5 \\
4
\end{bmatrix} = \begin{bmatrix}
-1 \\
2 \\
-1 \\
-1 \\
-2
\end{bmatrix}
\]

\[
(A - 2I_5)(A + 1I_5)x = \begin{bmatrix}
-9 & -1 & 11 & 0 & -4 \\
4 & -1 & 0 & 2 & 0 \\
-10 & -1 & 12 & 0 & -4 \\
8 & 2 & -15 & 0 & 5 \\
-10 & -1 & 16 & 0 & -8
\end{bmatrix} \begin{bmatrix}
-1 \\
2 \\
-1 \\
-1 \\
-2
\end{bmatrix} = \begin{bmatrix}
4 \\
4 \\
-1 \\
4 \\
-8
\end{bmatrix}
\]

\[
(A + 2I_5)(A - 2I_5)(A + 1I_5)x = \begin{bmatrix}
-5 & -1 & 11 & 0 & -4 \\
4 & 3 & 0 & 2 & 0 \\
-10 & -1 & 16 & 0 & -4 \\
8 & 2 & -15 & 1 & 5 \\
-10 & -1 & 16 & 0 & -4
\end{bmatrix} \begin{bmatrix}
4 \\
4 \\
4 \\
8 \\
8
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

So \( k = 3 \) and

\[
z = (A - 2I_5)(A + 1I_5)x = \begin{bmatrix}
4 \\
-8 \\
4 \\
4 \\
8
\end{bmatrix}
\]

is an eigenvector of \( A \) for the eigenvalue \( \lambda = -2 \), as you can check by doing the computation \( Az \). If you work through this example with your own choice of the vector \( x \) (strongly recommended) then the eigenvalue you will find may be different, but will be in the set \{3, 0, 1, -1, -2\}. See Exercise EE.M60 461 for a suggested starting vector.

\[ \Box \]

**Subsection CEE**

**Computing Eigenvalues and Eigenvectors**

Fortunately, we need not rely on the procedure of Theorem EMHE 445 each time we need an eigenvalue. It is the determinant, and specifically Theorem SMZD 432, that provides the main tool for computing eigenvalues. Here is an informal sequence of equivalences that is the key to determining the eigenvalues and eigenvectors of a matrix,

\[
A\mathbf{x} = \lambda \mathbf{x} \iff A\mathbf{x} - \lambda I_n \mathbf{x} = \mathbf{0} \iff (A - \lambda I_n) \mathbf{x} = \mathbf{0}
\]
So, for an eigenvalue $\lambda$ and associated eigenvector $x \neq 0$, the vector $x$ will be a nonzero element of the null space of $A - \lambda I_n$, while the matrix $A - \lambda I_n$ will be singular and therefore have zero determinant. These ideas are made precise in Theorem EMRCP 449 and Theorem EMNS 451, but for now this brief discussion should suffice as motivation for the following definition and example.

**Definition CP**

**Characteristic Polynomial**

Suppose that $A$ is a square matrix of size $n$. Then the characteristic polynomial of $A$ is the polynomial $p_A(x)$ defined by

$$p_A(x) = \det (A - xI_n)$$

\[\square\]

**Example CPMS3**

**Characteristic polynomial of a matrix, size 3**

Consider

$$F = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix}$$

Then

$$p_F(x) = \det (F - xI_3)$$

\[\begin{aligned} & = \begin{vmatrix} -13 - x & -8 & -4 \\ 12 & 7 - x & 4 \\ 24 & 16 & 7 - x \end{vmatrix} \\ & = (-13 - x) \begin{vmatrix} 7 - x & 4 \\ 16 & 7 - x \end{vmatrix} + (-8)(-1) \begin{vmatrix} 12 & 4 \\ 24 & 7 - x \end{vmatrix} \\ & \quad + (-4) \begin{vmatrix} 12 & 7 - x \\ 24 & 16 \end{vmatrix} \\ & = (-13 - x)((7 - x)(7 - x) - 4(16)) \\ & \quad + (-8)(-1)(12(7 - x) - 4(24)) \\ & \quad + (-4)(12(16) - (7 - x)(24)) \\ & = 3 + 5x + x^2 - x^3 \\ & = -(x - 3)(x + 1)^2 \end{aligned}\]

The characteristic polynomial is our main computational tool for finding eigenvalues, and will sometimes be used to aid us in determining the properties of eigenvalues.

**Theorem EMRCP**

**Eigenvalues of a Matrix are Roots of Characteristic Polynomials**

Suppose $A$ is a square matrix. Then $\lambda$ is an eigenvalue of $A$ if and only if $p_A(\lambda) = 0$.

\[\square\]

**Proof** Suppose $A$ has size $n$.

$\lambda$ is an eigenvalue of $A$
\[\text{Definition EEM [441]}\]
\[\text{Theorem MMIM [218]}\]
\[\text{Theorem MMDAA [219]}\]
\[\text{Theorem SMZD [432]}\]
\[\text{Definition CP [449]}\]

Example EMS3

Eigenvalues of a matrix, size 3

In Example CPMS3 [449] we found the characteristic polynomial of
\[F = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix}\]
to be \(p_F(x) = -(x - 3)(x + 1)^2\). Factored, we can find all of its roots easily, they are \(x = 3\) and \(x = -1\). By Theorem EMRCP [449], \(\lambda = 3\) and \(\lambda = -1\) are both eigenvalues of \(F\), and these are the only eigenvalues of \(F\). We’ve found them all.

Let us now turn our attention to the computation of eigenvectors.

Definition EM

Eigenspace of a Matrix

Suppose that \(A\) is a square matrix and \(\lambda\) is an eigenvalue of \(A\). Then the eigenspace of \(A\) for \(\lambda\), \(\mathcal{E}_A(\lambda)\), is the set of all the eigenvectors of \(A\) for \(\lambda\), together with the inclusion of the zero vector.

Example SEE [442] hinted that the set of eigenvectors for a single eigenvalue might have some closure properties, and with the addition of the non-eigenvector, 0, we indeed get a whole subspace.

Theorem EMS

Eigenspace for a Matrix is a Subspace

Suppose \(A\) is a square matrix of size \(n\) and \(\lambda\) is an eigenvalue of \(A\). Then the eigenspace \(\mathcal{E}_A(\lambda)\) is a subspace of the vector space \(\mathbb{C}^n\).

Proof

We will check the three conditions of Theorem TSS [327]. First, Definition EM [450] explicitly includes the zero vector in \(\mathcal{E}_A(\lambda)\), so the set is non-empty.

Suppose that \(x, y \in \mathcal{E}_A(\lambda)\), that is, \(x\) and \(y\) are two eigenvectors of \(A\) for \(\lambda\). Then
\[A(x + y) = Ax + Ay = \lambda x + \lambda y = \lambda(x + y)\]
So either \(x + y = 0\), or \(x + y\) is an eigenvector of \(A\) for \(\lambda\) (Definition EEM [441]). So, in either event, \(x + y \in \mathcal{E}_A(\lambda)\), and we have additive closure.

Suppose that \(\alpha \in \mathbb{C}\), and that \(x \in \mathcal{E}_A(\lambda)\), that is, \(x\) is an eigenvector of \(A\) for \(\lambda\). Then
\[A(\alpha x) = \alpha(Ax)\]

\[\text{Theorem MMSMM [219]}\]
So either $\alpha \mathbf{x} = 0$, or $\alpha \mathbf{x}$ is an eigenvector of $A$ for $\lambda$ (Definition EEM 441). So, in either event, $\alpha \mathbf{x} \in E_A(\lambda)$, and we have scalar closure.

With the three conditions of Theorem TSS 327 met, we know $E_A(\lambda)$ is a subspace.

Theorem EMS 450 tells us that an eigenspace is a subspace (and hence a vector space in its own right). Our next theorem tells us how to quickly construct this subspace.

**Theorem EMNS**

**Eigenspace of a Matrix is a Null Space**

Suppose $A$ is a square matrix of size $n$ and $\lambda$ is an eigenvalue of $A$. Then

$$E_A(\lambda) = \mathcal{N}(A - \lambda I_n)$$

**Proof** The conclusion of this theorem is an equality of sets, so normally we would follow the advice of Definition SE 694. However, in this case we can construct a sequence of equivalences which will together provide the two subset inclusions we need. First, notice that $0 \in E_A(\lambda)$ by Definition EM 450 and $0 \in \mathcal{N}(A - \lambda I_n)$ by Theorem HSC 65. Now consider any nonzero vector $\mathbf{x} \in \mathbb{C}^n$,

\[
\begin{align*}
\mathbf{x} \in E_A(\lambda) & \iff A\mathbf{x} = \lambda \mathbf{x} & \text{Definition EM 450} \\
& \iff A\mathbf{x} - \lambda \mathbf{x} = 0 \\
& \iff (A - \lambda I_n) \mathbf{x} = 0 & \text{Theorem MMIM 218} \\
& \iff \mathbf{x} \in \mathcal{N}(A - \lambda I_n) & \text{Definition NSM 68}
\end{align*}
\]

You might notice the close parallels (and differences) between the proofs of Theorem EMRCP 449 and Theorem EMNS 451. Since Theorem EMNS 451 describes the set of all the eigenvectors of $A$ as a null space we can use techniques such as Theorem BNS 154 to provide concise descriptions of eigenspaces.

**Example ESMS3**

**Eigenspaces of a matrix, size 3**

Example CPMS3 449 and Example EMS3 450 describe the characteristic polynomial and eigenvalues of the $3 \times 3$ matrix

$$F = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix}$$

We will now take the each eigenvalue in turn and compute its eigenspace. To do this, we row-reduce the matrix $F - \lambda I_3$ in order to determine solutions to the homogeneous system $\mathcal{L}S(F - \lambda I_3, \mathbf{0})$ and then express the eigenspace as the null space of $F - \lambda I_3$
Theorem EMNS \[451\]. Theorem BNS \[154\] then tells us how to write the null space as the span of a basis.

\[
\lambda = 3 \quad F - 3I_3 = \begin{bmatrix}
-16 & -8 & -4 \\
12 & 4 & 4 \\
24 & 16 & 4
\end{bmatrix}
\xrightarrow{RREF} \begin{bmatrix}
1 & 0 & \frac{1}{2} \\
0 & 1 & -\frac{1}{2} \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\mathcal{E}_F(3) = \mathcal{N}(F - 3I_3) = \left\langle \left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\} \right\rangle = \left\langle \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} \right\rangle
\]

\[
\lambda = -1 \quad F + I_3 = \begin{bmatrix}
-12 & -8 & -4 \\
12 & 8 & 4 \\
24 & 16 & 8
\end{bmatrix}
\xrightarrow{RREF} \begin{bmatrix}
1 & \frac{2}{3} & \frac{1}{3} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\mathcal{E}_F(-1) = \mathcal{N}(F + I_3) = \left\langle \left\{ \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle = \left\langle \left\{ \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\} \right\rangle
\]

Eigenspaces in hand, we can easily compute eigenvectors by forming nontrivial linear combinations of the basis vectors describing each eigenspace. In particular, notice that we can “pretty up” our basis vectors by using scalar multiples to clear out fractions.

Subsection ECEE
Examples of Computing Eigenvalues and Eigenvectors

No theorems in this section, just a selection of examples meant to illustrate the range of possibilities for the eigenvalues and eigenvectors of a matrix. These examples can all be done by hand, though the computation of the characteristic polynomial would be very time-consuming and error-prone. It can also be difficult to factor an arbitrary polynomial, though if we were to suggest that most of our eigenvalues are going to be integers, then it can be easier to hunt for roots. These examples are meant to look similar to a concatenation of Example CPMS3 \[449\], Example EMS3 \[450\] and Example ESMS3 \[451\].

First, we will sneak in a pair of definitions so we can illustrate them throughout this sequence of examples.

Definition AME
Algebraic Multiplicity of an Eigenvalue
Suppose that \( A \) is a square matrix and \( \lambda \) is an eigenvalue of \( A \). Then the algebraic multiplicity of \( \lambda \), \( \alpha_A(\lambda) \), is the highest power of \( (x - \lambda) \) that divides the characteristic polynomial, \( p_A(x) \).

Since an eigenvalue \( \lambda \) is a root of the characteristic polynomial, there is always a factor of \( (x - \lambda) \), and the algebraic multiplicity is just the power of this factor in a factorization of \( p_A(x) \). So in particular, \( \alpha_A(\lambda) \geq 1 \). Compare the definition of algebraic multiplicity with the next definition.

Definition GME
Geometric Multiplicity of an Eigenvalue
Suppose that \( A \) is a square matrix and \( \lambda \) is an eigenvalue of \( A \). Then the geometric multiplicity of \( \lambda \), \( \gamma_A(\lambda) \), is the dimension of the eigenspace \( \mathcal{E}_A(\lambda) \).
Since every eigenvalue must have at least one eigenvector, the associated eigenspace cannot be trivial, and so $\gamma_A(\lambda) \geq 1$.

**Example EMMS4**  
**Eigenvalue multiplicities, matrix of size 4**

Consider the matrix

$$B = \begin{bmatrix} -2 & 1 & -2 & -4 \\ 12 & 1 & 4 & 9 \\ 6 & 5 & -2 & -4 \\ 3 & -4 & 5 & 10 \end{bmatrix}$$

then

$$p_B(x) = 8 - 20x + 18x^2 - 7x^3 + x^4 = (x - 1)(x - 2)^3$$

So the eigenvalues are $\lambda = 1, 2$ with algebraic multiplicities $\alpha_B(1) = 1$ and $\alpha_B(2) = 3$.

Computing eigenvectors,

$$\lambda = 1 \quad B - 1I_4 = \begin{bmatrix} -3 & 1 & -2 & -4 \\ 12 & 0 & 4 & 9 \\ 6 & 5 & -3 & -4 \\ 3 & -4 & 5 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$E_B(1) = N(B - 1I_4) = \langle \left\{ \begin{bmatrix} \frac{1}{3} \\ 1 \\ 1 \end{bmatrix} \right\} \rangle$

$$\lambda = 2 \quad B - 2I_4 = \begin{bmatrix} -4 & 1 & -2 & -4 \\ 12 & -1 & 4 & 9 \\ 6 & 5 & -4 & -4 \\ 3 & -4 & 5 & 8 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$E_B(2) = N(B - 2I_4) = \langle \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ 1 \end{bmatrix} \right\} \rangle$

So each eigenspace has dimension 1 and so $\gamma_B(1) = 1$ and $\gamma_B(2) = 1$. This example is of interest because of the discrepancy between the two multiplicities for $\lambda = 2$. In many of our examples the algebraic and geometric multiplicities will be equal for all of the eigenvalues (as it was for $\lambda = 1$ in this example), so keep this example in mind. We will have some explanations for this phenomenon later (see Example NDMS4 [491]).

**Example ESMS4**  
**Eigenvalues, symmetric matrix of size 4**

Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

then

$$p_C(x) = -3 + 4x + 2x^2 - 4x^3 + x^4 = (x - 3)(x - 1)^2(x + 1)$$
So the eigenvalues are $\lambda = 3, 1, -1$ with algebraic multiplicities $\alpha_C(3) = 1$, $\alpha_C(1) = 2$ and $\alpha_C(-1) = 1$.

Computing eigenvectors,

$$\lambda = 3 \quad C - 3I_4 = \begin{bmatrix} -2 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_C(3) = \mathcal{N}(C - 3I_4) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\lambda = 1 \quad C - 1I_4 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_C(1) = \mathcal{N}(C - 1I_4) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\lambda = -1 \quad C + 1I_4 = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_C(-1) = \mathcal{N}(C + 1I_4) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

So the eigenspace dimensions yield geometric multiplicities $\gamma_C(3) = 1$, $\gamma_C(1) = 2$ and $\gamma_C(-1) = 1$, the same as for the algebraic multiplicities. This example is of interest because $A$ is a symmetric matrix, and will be the subject of Theorem HMRE 476.  

**Example HMEM5**

**High multiplicity eigenvalues, matrix of size 5**

Consider the matrix

$$E = \begin{bmatrix} 29 & 14 & 2 & 6 & -9 \\ -47 & -22 & -1 & -11 & 13 \\ 19 & 10 & 5 & 4 & -8 \\ -19 & -10 & -3 & -2 & 8 \\ 7 & 4 & 3 & 1 & -3 \end{bmatrix}$$

then

$$p_E(x) = -16 + 16x + 8x^2 - 16x^3 + 7x^4 - x^5 = -(x - 2)^4(x + 1)$$

So the eigenvalues are $\lambda = 2, -1$ with algebraic multiplicities $\alpha_E(2) = 4$ and $\alpha_E(-1) = 1$.  

Version 0.92
Computing eigenvectors,\
\[ \lambda = 2 \quad E - 2I_5 = \begin{bmatrix} 27 & 14 & 2 & 6 & -9 \\ -47 & -24 & -1 & -11 & 13 \\ 19 & 10 & 3 & 4 & -8 \\ -19 & -10 & -3 & -4 & 8 \\ 7 & 4 & 3 & 1 & -5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
\[ \mathcal{E}_E(2) = \mathcal{N}(E - 2I_5) = \langle \begin{bmatrix} -1 \\ \frac{3}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \\ 0 \\ 1 \end{bmatrix} \rangle = \langle \begin{bmatrix} -2 \\ 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \rangle \]
\[ \lambda = -1 \quad E + 1I_5 = \begin{bmatrix} 30 & 14 & 2 & 6 & -9 \\ -47 & -21 & -1 & -11 & 13 \\ 19 & 10 & 6 & 4 & -8 \\ -19 & -10 & -3 & -1 & 8 \\ 7 & 4 & 3 & 1 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
\[ \mathcal{E}_E(-1) = \mathcal{N}(E + 1I_5) = \langle \begin{bmatrix} -2 \\ 4 \\ -1 \\ 1 \\ 0 \end{bmatrix} \rangle \]

So the eigenspace dimensions yield geometric multiplicities \( \gamma_E(2) = 2 \) and \( \gamma_E(-1) = 1 \).
This example is of interest because \( \lambda = 2 \) has such a large algebraic multiplicity, which is also not equal to its geometric multiplicity.

**Example CEMS6**

**Complex eigenvalues, matrix of size 6**

Consider the matrix
then
\[ p_F(x) = -50 + 55x + 13x^2 - 50x^3 + 32x^4 - 9x^5 + x^6 \]
\[ = (x - 2)(x + 1)(x^2 - 4x + 5)^2 \]
\[ = (x - 2)(x + 1)((x - (2 + i))(x - (2 - i)))^2 \]
\[ = (x - 2)(x + 1)(x - (2 + i))^2(x - (2 - i))^2 \]

So the eigenvalues are \( \lambda = 2, -1, 2 + i, 2 - i \) with algebraic multiplicities \( \alpha_F(2) = 1, \alpha_F(-1) = 1, \alpha_F(2 + i) = 2 \) and \( \alpha_F(2 - i) = 2 \).
Computing eigenvectors,

\[ \lambda = 2 \]

\[
F - 2I_6 = 
\begin{bmatrix}
-61 & -34 & 41 & 12 & 25 & 30 \\
1 & 5 & -46 & -36 & -11 & -29 \\
-233 & -119 & 56 & -35 & 75 & 54 \\
157 & 81 & -43 & 19 & -51 & -39 \\
-91 & -48 & 32 & -5 & 30 & 26 \\
209 & 107 & -55 & 28 & -69 & -52
\end{bmatrix}
\]

\[
RREF \rightarrow 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[ \mathcal{E}_F(2) = \mathcal{N}(F - 2I_6) = \left\langle \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ -4 \\ 5 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} -\frac{3}{5} \\ \frac{4}{5} \\ 1 \\ -\frac{3}{5} \\ \frac{4}{5} \\ 1 \end{bmatrix} \right\rangle \]

\[ \lambda = -1 \]

\[ F + I_6 = 
\begin{bmatrix}
-58 & -34 & 41 & 12 & 25 & 30 \\
1 & 8 & -46 & -36 & -11 & -29 \\
-233 & -119 & 59 & -35 & 75 & 54 \\
157 & 81 & -43 & 22 & -51 & -39 \\
-91 & -48 & 32 & 5 & 33 & 26 \\
209 & 107 & -55 & 28 & -69 & -49
\end{bmatrix}
\]

\[
RREF \rightarrow 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[ \mathcal{E}_F(-1) = \mathcal{N}(F + I_6) = \left\langle \begin{bmatrix} 1 \\ 1 \\ -9 - 2i \\ 0 \\ 1 \\ -\frac{1}{2} \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} -\frac{3}{2} \\ -\frac{3}{2} \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\rangle \]

\[ \lambda = 2 + i \]

\[ F - (2 + i)I_6 = 
\begin{bmatrix}
-61 - i & -34 & 41 & 12 & 25 & 30 \\
1 - 5i & -46 & -36 & -35 & -11 & -29 \\
-233 - 119i & 56 - i & -35 & 75 & 54 \\
157 + 81i & 81 - 43i & 19 - i & -51 & -39 \\
-91 & -48 + 32i & -5 & 30 - i & 26 \\
209 & 107 & -55 & 28 & -69 & -52 - i
\end{bmatrix}
\]

\[
RREF \rightarrow 
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \frac{1}{5}(7 + i) \\
0 & 1 & 0 & 0 & 0 & \frac{1}{5}(-9 - 2i) \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
\mathcal{E}_F (2 + i) = \mathcal{N}(F - (2 + i)I_6) = \begin{pmatrix}
-\frac{1}{5}(7 + i) \\
\frac{1}{5}(9 + 2i) \\
-1 \\
1 \\
-1 \\
1
\end{pmatrix}
\begin{pmatrix}
-7 - i \\
9 + 2i \\
-5 \\
5 \\
-5 \\
5
\end{pmatrix}
\]
So the eigenvalues are $\lambda = 2, 1, 0, -1, -3$ with algebraic multiplicities $\alpha_H(2) = 1$, $\alpha_H(1) = 1$, $\alpha_H(0) = 1$, $\alpha_H(-1) = 1$ and $\alpha_H(-3) = 1$.

Computing eigenvectors,

\[
\lambda = 2 \quad H - 2I_5 = \begin{bmatrix}
13 & 18 & -8 & 6 & -5 \\
5 & 1 & 1 & -1 & -3 \\
0 & -4 & 3 & -4 & -2 \\
-43 & -46 & 17 & -16 & 15 \\
26 & 30 & -12 & 8 & -12
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\mathcal{E}_H(2) = \mathcal{N}(H - 2I_5) = \left\{ \begin{bmatrix}
1 \\
-1 \\
2 \\
-1 \\
1
\end{bmatrix} \right\}
\]

\[
\lambda = 1 \quad H - 1I_5 = \begin{bmatrix}
14 & 18 & -8 & 6 & -5 \\
5 & 2 & 1 & -1 & -3 \\
0 & -4 & 4 & -4 & -2 \\
-43 & -46 & 17 & -15 & 15 \\
26 & 30 & -12 & 8 & -11
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\mathcal{E}_H(1) = \mathcal{N}(H - 1I_5) = \left\{ \begin{bmatrix}
1 \\
-\frac{3}{2} \\
0 \\
-1 \\
2
\end{bmatrix} \right\}
\]

\[
\lambda = 0 \quad H - 0I_5 = \begin{bmatrix}
15 & 18 & -8 & 6 & -5 \\
5 & 3 & 1 & -1 & -3 \\
0 & -4 & 5 & -4 & -2 \\
-43 & -46 & 17 & -14 & 15 \\
26 & 30 & -12 & 8 & -10
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & -2 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\mathcal{E}_H(0) = \mathcal{N}(H - 0I_5) = \left\{ \begin{bmatrix}
1 \\
2 \\
0 \\
1
\end{bmatrix} \right\}
\]

\[
\lambda = -1 \quad H + 1I_5 = \begin{bmatrix}
16 & 18 & -8 & 6 & -5 \\
5 & 4 & 1 & -1 & -3 \\
0 & -4 & 6 & -4 & -2 \\
-43 & -46 & 17 & -13 & 15 \\
26 & 30 & -12 & 8 & -9
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 0 & -1/2 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1/2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\mathcal{E}_H(-1) = \mathcal{N}(H + 1I_5) = \left\{ \begin{bmatrix}
1 \\
0 \\
-\frac{3}{2} \\
-1 \\
2
\end{bmatrix} \right\}
\]
\[ \lambda = -3 \quad H + 3I_5 = \begin{bmatrix} 18 & 18 & -8 & 6 & -5 \\ 5 & 6 & 1 & -1 & -3 \\ 0 & -4 & 8 & -4 & -2 \\ -43 & -46 & 17 & -11 & 15 \\ 26 & 30 & -12 & 8 & -7 \end{bmatrix} \]

\[
\begin{array}{c}
\mathcal{E}_H(-3) = \mathcal{N}(H + 3I_5) = \left\{ \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -1 \\ -2 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 2 \\ 4 \\ -2 \end{bmatrix} \right\}
\end{array}
\]

So the eigenspace dimensions yield geometric multiplicities \( \gamma_H(2) = 1 \), \( \gamma_H(1) = 1 \), \( \gamma_H(0) = 1 \), \( \gamma_H(-1) = 1 \) and \( \gamma_H(-3) = 1 \), identical to the algebraic multiplicities. This example is of interest for two reasons. First, \( \lambda = 0 \) is an eigenvalue, illustrating the upcoming Theorem SMZE [468]. Second, all the eigenvalues are distinct, yielding algebraic and geometric multiplicities of 1 for each eigenvalue, illustrating Theorem DED [492]. ☑

Subsection EE.READ  
Reading Questions

Suppose \( A \) is the 2 \times 2 matrix

\[ A = \begin{bmatrix} -5 & 8 \\ -4 & 7 \end{bmatrix} \]

1. Find the eigenvalues of \( A \).

2. Find the eigenspaces of \( A \).

3. For the polynomial \( p(x) = 3x^2 - x + 2 \), compute \( p(A) \).
Subsection EXC
Exercises

C19 Find the eigenvalues, eigenspaces, algebraic multiplicities and geometric multiplicities for the matrix below. It is possible to do all these computations by hand, and it would be instructive to do so.

\[ C = \begin{bmatrix} -1 & 2 \\ -6 & 6 \end{bmatrix} \]

Contributed by Robert Beezer Solution 463

C20 Find the eigenvalues, eigenspaces, algebraic multiplicities and geometric multiplicities for the matrix below. It is possible to do all these computations by hand, and it would be instructive to do so.

\[ B = \begin{bmatrix} -12 & 30 \\ -5 & 13 \end{bmatrix} \]

Contributed by Robert Beezer Solution 463

C21 The matrix \( A \) below has \( \lambda = 2 \) as an eigenvalue. Find the geometric multiplicity of \( \lambda = 2 \) using your calculator only for row-reducing matrices.

\[ A = \begin{bmatrix} 18 & -15 & 33 & -15 \\ -4 & 8 & -6 & 6 \\ -9 & 9 & -16 & 9 \\ 5 & -6 & 9 & -4 \end{bmatrix} \]

Contributed by Robert Beezer Solution 464

C22 Without using a calculator, find the eigenvalues of the matrix \( B \).

\[ B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \]

Contributed by Robert Beezer Solution 464

M60 Repeat Example CAEHW 446 by choosing \( x = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} \) and then arrive at an eigenvalue and eigenvector of the matrix \( A \). The hard way.

Contributed by Robert Beezer Solution 464

T10 A matrix \( A \) is idempotent if \( A^2 = A \). Show that the only possible eigenvalues of an idempotent matrix are \( \lambda = 0 \) and \( \lambda = 1 \). Then give an example of a matrix that is idempotent and has both of these two values as eigenvalues.

Contributed by Robert Beezer Solution 465

T20 Suppose that \( \lambda \) and \( \rho \) are two different eigenvalues of the square matrix \( A \). Prove that the intersection of the eigenspaces for these two eigenvalues is trivial. That is,
\[ \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho) = \{0\}. \]

Contributed by Robert Beezer Solution 466
Subsection SOL
Solutions

C19 Contributed by Robert Beezer Statement 461
First compute the characteristic polynomial,

\[ p_C(x) = \det(C - xI_2) \]

\[ = \begin{vmatrix} -1 - x & 2 \\ -6 & 6 - x \end{vmatrix} \]

\[ = (-1 - x)(6 - x) - (2)(-6) \]

\[ = x^2 - 5x - 6 \]

\[ = (x - 3)(x - 2) \]

So the eigenvalues of \( C \) are the solutions to \( p_C(x) = 0 \), namely, \( \lambda = 2 \) and \( \lambda = 3 \).

To obtain the eigenspaces, construct the appropriate singular matrices and find expressions for the null spaces of these matrices.

\[ \lambda = 2 \]

\[ C - (2)I_2 = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix} \text{ RREF } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ E_C(2) = \mathcal{N}(C - (2)I_2) = \langle \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} \rangle = \langle \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\} \rangle \]

\[ \lambda = 3 \]

\[ C - (3)I_2 = \begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix} \text{ RREF } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ E_C(3) = \mathcal{N}(C - (3)I_2) = \langle \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \rangle = \langle \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \rangle \]

C20 Contributed by Robert Beezer Statement 461
The characteristic polynomial of \( B \) is

\[ p_B(x) = \det(B - xI_2) \]

\[ = \begin{vmatrix} -12 - x & 30 \\ -5 & 13 - x \end{vmatrix} \]

\[ = (-12 - x)(13 - x) - (30)(-5) \]

\[ = x^2 - x - 6 \]

\[ = (x - 3)(x + 2) \]

From this we find eigenvalues \( \lambda = 3, -2 \) with algebraic multiplicities \( \alpha_B(3) = 1 \) and \( \alpha_B(-2) = 1 \).

For eigenvectors and geometric multiplicities, we study the null spaces of \( B - \lambda I_2 \) (Theorem EMNS 451).

\[ \lambda = 3 \]

\[ B - 3I_2 = \begin{bmatrix} -15 & 30 \\ -5 & 10 \end{bmatrix} \text{ RREF } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]
\[ \mathcal{E}_B(3) = \mathcal{N}(B - 3I_2) = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \]

\[ \lambda = -2 \]

\[ B + 2I_2 = \begin{bmatrix} -10 & 30 \\ -5 & 15 \end{bmatrix} \overset{\text{RREF,}}{\longrightarrow} \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \]

\[ \mathcal{E}_B(-2) = \mathcal{N}(B + 2I_2) = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \]

Each eigenspace has dimension one, so we have geometric multiplicities \( \gamma_B(3) = 1 \) and \( \gamma_B(-2) = 1 \).

C21 Contributed by Robert Beezer Statement [461]

If \( \lambda = 2 \) is an eigenvalue of \( A \), the matrix \( A - 2I_4 \) will be singular, and its null space will be the eigenspace of \( A \). So we form this matrix and row-reduce,

\[ A - 2I_4 = \begin{bmatrix} 16 & -15 & 33 & -15 \\ -4 & 6 & -6 & 6 \\ -9 & 9 & -18 & 9 \\ 5 & -6 & 9 & -6 \end{bmatrix} \overset{\text{RREF,}}{\longrightarrow} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -3 \\ -9 \\ -30 \end{bmatrix} \]

With two free variables, we know a basis of the null space (Theorem BNS 154) will contain two vectors. Thus the null space of \( A - 2I_4 \) has dimension two, and so the eigenspace of \( \lambda = 2 \) has dimension two also (Theorem EMNS 451), \( \gamma_A(2) = 2 \).

C22 Contributed by Robert Beezer Statement [461]

The characteristic polynomial (Definition CP 449) is

\[ p_B(x) = \text{det}(B - xI_2) = \begin{vmatrix} 2 - x & -1 \\ 1 & 1 - x \end{vmatrix} = (2-x)(1-x) - (1)(-1) = x^2 - 3x + 3 \]

\[ = \left( x - \frac{3 + 3i}{2} \right) \left( x - \frac{3 - 3i}{2} \right) \]

where the factorization can be obtained by finding the roots of \( p_B(x) = 0 \) with the quadratic equation. By Theorem EMRCP 449 the eigenvalues of \( B \) are the complex numbers \( \lambda_1 = \frac{3 + 3i}{2} \) and \( \lambda_2 = \frac{3 - 3i}{2} \).

M60 Contributed by Robert Beezer Statement [461]

Form the matrix \( C \) whose columns are \( x, Ax, A^2x, A^3x, A^4x, A^5x \) and row-reduce the matrix,

\[ \begin{bmatrix} 0 & 6 & 32 & 102 & 320 & 966 \\ 8 & 10 & 24 & 58 & 168 & 490 \\ 2 & 12 & 50 & 156 & 482 & 1452 \\ 1 & -5 & -47 & -149 & -479 & -1445 \\ 2 & 12 & 50 & 156 & 482 & 1452 \end{bmatrix} \overset{\text{RREF,}}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 & -3 & -9 & -30 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 10 & 30 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

The simplest possible relation of linear dependence on the columns of \( C \) comes from using scalars \( \alpha_4 = 1 \) and \( \alpha_5 = \alpha_6 = 0 \) for the free variables in a solution to \( \mathcal{L}S(C, 0) \). The
remainder of this solution is \( \alpha_1 = 3, \alpha_2 = -1, \alpha_3 = -3 \). This solution gives rise to the polynomial
\[
p(x) = 3 - x - 3x^2 + x^3 = (x - 3)(x - 1)(x + 1)
\]
which then has the property that \( p(A)x = 0 \).

No matter how you choose to order the factors of \( p(x) \), the value of \( k \) (in the language of Theorem EMHE and Example CAEH) is \( k = 2 \). For each of the three possibilities, we list the resulting eigenvector and the associated eigenvalue:

\[
(C - 3I_5)(C - I_5)z = \begin{bmatrix} 8 \\ 8 \\ -24 \\ 8 \end{bmatrix}, \quad \lambda = -1
\]

\[
(C - 3I_5)(C + I_5)z = \begin{bmatrix} 20 \\ -20 \\ 20 \\ -40 \\ 20 \end{bmatrix}, \quad \lambda = 1
\]

\[
(C + I_5)(C - I_5)z = \begin{bmatrix} 32 \\ 16 \\ 48 \\ -48 \\ 48 \end{bmatrix}, \quad \lambda = 3
\]

Note that each of these eigenvectors can be simplified by an appropriate scalar multiple, but we have shown here the actual vector obtained by the product specified in the theorem.

**T10** Contributed by Robert Beezer Statement 461

Suppose that \( \lambda \) is an eigenvalue of \( A \). Then there is an eigenvector \( x \), such that \( Ax = \lambda x \). We have,

\[
\lambda x = Ax \\
= A^2x \\
= A(Ax) \\
= A(\lambda x) \\
= \lambda(Ax) \\
= \lambda(\lambda x) \\
= \lambda^2x
\]

From this we get
\[
0 = \lambda^2 x - \lambda x \\
= (\lambda^2 - \lambda)x \quad \text{Property DSAC 91}
\]

Since \( x \) is an eigenvector, it is nonzero, and Theorem SMEZV leaves us with the conclusion that \( \lambda^2 - \lambda = 0 \), and the solutions to this quadratic polynomial equation in \( \lambda \) are \( \lambda = 0 \) and \( \lambda = 1 \).
The matrix
\[
\begin{bmatrix}
1 & 0 \\
0 & 0 
\end{bmatrix}
\]
is idempotent (check this!) and since it is a diagonal matrix, its eigenvalues are the diagonal entries, \(\lambda = 0\) and \(\lambda = 1\), so each of these possible values for an eigenvalue of an idempotent matrix actually occurs as an eigenvalue of some idempotent matrix.

**T20** Contributed by Robert Beezer

This problem asks you to prove that two sets are equal, so use Definition SE [694].

First show that \(\{0\} \subseteq \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho)\). Choose \(x \in \{0\}\). Then \(x = 0\). Eigenspaces are subspaces (Theorem EMS [450]), so both \(\mathcal{E}_A(\lambda)\) and \(\mathcal{E}_A(\rho)\) contain the zero vector, and therefore \(x \in \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho)\) [Definition SI [695]].

To show that \(\mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho) \subseteq \{0\}\), suppose that \(x \in \mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho)\). Then \(x\) is an eigenvector of \(A\) for both \(\lambda\) and \(\rho\) [Definition SI [695]] and so

\[
x = 1x = \frac{1}{\lambda - \rho} (\lambda - \rho)x \quad \text{Property O [310]}
\]

\[
= \frac{1}{\lambda - \rho} (\lambda x - \rho x) \quad \lambda \neq \rho, \lambda - \rho \neq 0
\]

\[
= \frac{1}{\lambda - \rho} (Ax - Ax) \quad \text{Property DSAC [91]}
\]

\[
= \frac{1}{\lambda - \rho} (0) \quad x \text{ eigenvector of } A \text{ for } \lambda, \rho
\]

\[
= 0 \quad \text{Theorem ZVSM [318]}
\]

So \(x = 0\), and trivially, \(x \in \{0\}\).
Section PEE
Properties of Eigenvalues and Eigenvectors

The previous section introduced eigenvalues and eigenvectors, and concentrated on their existence and determination. This section will be more about theorems, and the various properties eigenvalues and eigenvectors enjoy. Like a good 4 × 100 meter relay, we will lead-off with one of our better theorems and save the very best for the anchor leg.

**Theorem EDELI**

**Eigenvectors with Distinct Eigenvalues are Linearly Independent**

Suppose that $A$ is an $n \times n$ square matrix and $S = \{x_1, x_2, x_3, \ldots, x_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then $S$ is a linearly independent set.

**Proof** If $p = 1$, then the set $S = \{x_1\}$ is linearly independent since eigenvectors are nonzero (Definition EEM [141]), so assume for the remainder that $p \geq 2$.

We will prove this result by contradiction (Technique CD [708]). Suppose to the contrary that $S$ is a linearly dependent set. Define $k$ to be an integer such that $\{x_1, x_2, x_3, \ldots, x_{k-1}\}$ is linearly independent and $\{x_1, x_2, x_3, \ldots, x_k\}$ is linearly dependent. We have to ask if there is even such an integer? Since eigenvectors are nonzero, the set $\{x_1\}$ is linearly independent. Think of adding in vectors to this set, one at a time, $x_2, x_3, x_4, \ldots$. Since we are assuming that $S$ is linearly dependent, eventually this set will convert from being linearly independent to being linearly dependent. In other words, it is the addition of the vector $x_k$ that converts the set from linear independence to linear dependence. So there is such a $k$, and furthermore $2 \leq k \leq p$.

Since $\{x_1, x_2, x_3, \ldots, x_k\}$ is linearly dependent there are scalars, $a_1, a_2, a_3, \ldots, a_k$, some non-zero, so that

$$0 = a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_kx_k$$

Then,

$$0 = (A - \lambda_k I_n) 0 = (A - \lambda_k I_n) (a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_kx_k)$$

$$(A - \lambda_k I_n) a_1x_1 + (A - \lambda_k I_n) a_2x_2 + \cdots + (A - \lambda_k I_n) a_kx_k = (A - \lambda_k I_n) x_1 + (A - \lambda_k I_n) x_2 + \cdots + (A - \lambda_k I_n) x_k$$

$$(A - \lambda_k I_n) x_1 + (A - \lambda_k I_n) x_2 + \cdots + (A - \lambda_k I_n) x_k = 0$$

$$(A - \lambda_k I_n) x_1 = (A - \lambda_k I_n) x_2 = \cdots = (A - \lambda_k I_n) x_k = 0$$

$$(A - \lambda_k I_n) x_1 = (A - \lambda_k I_n) x_2 = \cdots = (A - \lambda_k I_n) x_k = 0$$

$$(A - \lambda_k I_n) x_1 = (A - \lambda_k I_n) x_2 = \cdots = (A - \lambda_k I_n) x_k = 0$$

$$a_1 (A - \lambda_k I_n) x_1 = a_2 (A - \lambda_k I_n) x_2 = \cdots = a_k (A - \lambda_k I_n) x_k = 0$$

$$a_1 (A - \lambda_k I_n) x_1 = a_2 (A - \lambda_k I_n) x_2 = \cdots = a_k (A - \lambda_k I_n) x_k = 0$$

$$a_1 (A - \lambda_k I_n) x_1 = a_2 (A - \lambda_k I_n) x_2 = \cdots = a_k (A - \lambda_k I_n) x_k = 0$$

$$a_1 (A - \lambda_k I_n) x_1 = a_2 (A - \lambda_k I_n) x_2 = \cdots = a_k (A - \lambda_k I_n) x_k = 0$$

$$a_1 (A - \lambda_k I_n) x_1 = a_2 (A - \lambda_k I_n) x_2 = \cdots = a_k (A - \lambda_k I_n) x_k = 0$$

$$a_1 (A - \lambda_k I_n) x_1 = a_2 (A - \lambda_k I_n) x_2 = \cdots = a_k (A - \lambda_k I_n) x_k = 0$$

$$a_1 (A - \lambda_k I_n) x_1 = a_2 (A - \lambda_k I_n) x_2 = \cdots = a_k (A - \lambda_k I_n) x_k = 0$$

$$a_1 (A - \lambda_k I_n) x_1 = a_2 (A - \lambda_k I_n) x_2 = \cdots = a_k (A - \lambda_k I_n) x_k = 0$$

$$a_1 (A - \lambda_k I_n) x_1 = a_2 (A - \lambda_k I_n) x_2 = \cdots = a_k (A - \lambda_k I_n) x_k = 0$$

This is a relation of linear dependence on the linearly independent set $\{x_1, x_2, x_3, \ldots, x_{k-1}\}$, so the scalars must all be zero. That is, $a_i (\lambda_i - \lambda_k) = 0$ for $1 \leq i \leq k - 1$. However, we
have the hypothesis that the eigenvalues are distinct, so \( \lambda_i \neq \lambda_k \) for \( 1 \leq i \leq k - 1 \). Thus \( a_i = 0 \) for \( 1 \leq i \leq k - 1 \).

This reduces the original relation of linear dependence on \( \{x_1, x_2, x_3, \ldots, x_k\} \) to the simpler equation \( a_k x_k = 0 \). By Theorem SMEZV we conclude that \( a_k = 0 \) or \( x_k = 0 \). Eigenvectors are never the zero vector (Definition EEM), so \( a_k = 0 \). So all of the scalars \( a_i, 1 \leq i \leq k \) are zero, contradicting their introduction as the scalars creating a nontrivial relation of linear dependence on the set \( \{x_1, x_2, x_3, \ldots, x_k\} \). With a contradiction in hand, we conclude that \( S \) must be linearly independent.

There is a simple connection between the eigenvalues of a matrix and whether or not the matrix is nonsingular.

**Theorem SMZE**

**Singular Matrices have Zero Eigenvalues**

Suppose \( A \) is a square matrix. Then \( A \) is singular if and only if \( \lambda = 0 \) is an eigenvalue of \( A \).

**Proof**

We have the following equivalences:

\[
A \text{ is singular } \iff \text{ there exists } x \neq 0, Ax = 0 \quad \text{[Definition NSM 68]}
\]

\[
\implies \text{ there exists } x \neq 0, Ax = 0x \quad \text{[Theorem ZSSM 317]}
\]

\[
\implies \lambda = 0 \text{ is an eigenvalue of } A \quad \text{[Definition EEM 441]}
\]

With an equivalence about singular matrices we can update our list of equivalences about nonsingular matrices.

**Theorem NME8**

**Nonsingular Matrix Equivalences, Round 8**

Suppose that \( A \) is a square matrix of size \( n \). The following are equivalent.

1. \( A \) is nonsingular.
2. \( A \) row-reduces to the identity matrix.
3. The null space of \( A \) contains only the zero vector, \( \mathcal{N}(A) = \{0\} \).
4. The linear system \( LS(A, b) \) has a unique solution for every possible choice of \( b \).
5. The columns of \( A \) are a linearly independent set.
6. \( A \) is invertible.
7. The column space of \( A \) is \( \mathbb{C}^n \), \( \mathcal{C}(A) = \mathbb{C}^n \).
8. The columns of \( A \) are a basis for \( \mathbb{C}^n \).
9. The rank of \( A \) is \( n \), \( \text{r}(A) = n \).
10. The nullity of \( A \) is zero, \( n(A) = 0 \).
11. The determinant of \( A \) is nonzero, \( \det(A) \neq 0 \).
12. \( \lambda = 0 \) is not an eigenvalue of \( A \).
**Proof** The equivalence of the first and last statements is the contrapositive of Theorem SMZE 468, so we are able to improve on Theorem NME7 433.

Certain changes to a matrix change its eigenvalues in a predictable way.

**Theorem ESMM**

**Eigenvalues of a Scalar Multiple of a Matrix**

Suppose $A$ is a square matrix and $\lambda$ is an eigenvalue of $A$. Then $\alpha \lambda$ is an eigenvalue of $\alpha A$.

**Proof** Let $x \neq 0$ be one eigenvector of $A$ for $\lambda$. Then

\[
(\alpha A)x = \alpha (Ax) = \alpha (\lambda x) = (\alpha \lambda) x
\]

So $x \neq 0$ is an eigenvector of $\alpha A$ for the eigenvalue $\alpha \lambda$.

Unfortunately, there are not parallel theorems about the sum or product of arbitrary matrices. But we can prove a similar result for powers of a matrix.

**Theorem EOMP**

**Eigenvalues Of Matrix Powers**

Suppose $A$ is a square matrix, $\lambda$ is an eigenvalue of $A$, and $s \geq 0$ is an integer. Then $\lambda^s$ is an eigenvalue of $A^s$.

**Proof** Let $x \neq 0$ be one eigenvector of $A$ for $\lambda$. Suppose $A$ has size $n$. Then we proceed by induction on $s$ (Technique I 713). First, for $s = 0$,

\[
A^s x = A^0 x = I_n x = x = \lambda^0 x = \lambda^s x
\]

so $\lambda^s$ is an eigenvalue of $A^s$ in this special case. If we assume the theorem is true for $s$, then we find

\[
A^{s+1} x = A^s Ax = A^s (\lambda x) = \lambda (A^s x) = \lambda (\lambda^s x) = (\lambda \lambda^s) x = \lambda^{s+1} x
\]

So $x \neq 0$ is an eigenvector of $A^{s+1}$ for $\lambda^{s+1}$, and induction tells us the theorem is true for all $s \geq 0$. 

\[\square\]
While we cannot prove that the sum of two arbitrary matrices behaves in any reasonable way with regard to eigenvalues, we can work with the sum of dissimilar powers of the same matrix. We have already seen two connections between eigenvalues and polynomials, in the proof of Theorem EMHE and the characteristic polynomial (Definition CP). Our next theorem strengthens this connection.

**Theorem EPM**

**Eigenvalues of the Polynomial of a Matrix**

Suppose $A$ is a square matrix and $\lambda$ is an eigenvalue of $A$. Let $q(x)$ be a polynomial in the variable $x$. Then $q(\lambda)$ is an eigenvalue of the matrix $q(A)$.

**Proof** Let $x \neq 0$ be one eigenvector of $A$ for $\lambda$, and write $q(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$. Then

$$q(A)x = (a_0 A^0 + a_1 A^1 + a_2 A^2 + \cdots + a_m A^m) x$$

$$= (a_0 A^0 x) + (a_1 A^1 x) + (a_2 A^2 x) + \cdots + (a_m A^m x)$$

$$= a_0 (A^0 x) + a_1 (A^1 x) + a_2 (A^2 x) + \cdots + a_m (A^m x)$$

$$= a_0 (\lambda_0 x) + a_1 (\lambda_1 x) + a_2 (\lambda_2 x) + \cdots + a_m (\lambda^m x)$$

$$= (a_0 \lambda^0 x) + (a_1 \lambda^1 x) + (a_2 \lambda^2 x) + \cdots + (a_m \lambda^m x)$$

$$= (a_0 \lambda^0 + a_1 \lambda^1 + a_2 \lambda^2 + \cdots + a_m \lambda^m) x$$

$$= q(\lambda)x$$

So $x \neq 0$ is an eigenvector of $q(A)$ for the eigenvalue $q(\lambda)$. □

**Example BDE**

**Building desired eigenvalues**

In Example ESMS4 the $4 \times 4$ symmetric matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

is shown to have the three eigenvalues $\lambda = 3, 1, -1$. Suppose we wanted a $4 \times 4$ matrix that has the three eigenvalues $\lambda = 4, 0, -2$. We can employ Theorem EPM by finding a polynomial that converts 3 to 4, 1 to 0, and $-1$ to $-2$. Such a polynomial is called an interpolating polynomial, and in this example we can use

$$r(x) = \frac{1}{4} x^2 + x - \frac{5}{4}$$

We will not discuss how to concoct this polynomial, but a text on numerical analysis should provide the details. In our case, simply verify that $r(3) = 4, r(1) = 0$ and $r(-1) = -2$.

Now compute

$$r(C) = \frac{1}{4} C^2 + C - \frac{5}{4} I_4$$

$$= \frac{1}{4} \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} - \frac{5}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
Theorem EPM \[470\] tells us that if \(r(x)\) transforms the eigenvalues in the desired manner, then \(r(C)\) will have the desired eigenvalues. You can check this by computing the eigenvalues of \(r(C)\) directly. Furthermore, notice that the multiplicities are the same, and the eigenspaces of \(C\) and \(r(C)\) are identical.

Inverses and transposes also behave predictably with regard to their eigenvalues.

**Theorem EIM**

**Eigenvalues of the Inverse of a Matrix**

Suppose \(A\) is a square nonsingular matrix and \(\lambda\) is an eigenvalue of \(A\). Then \(\frac{1}{\lambda}\) is an eigenvalue of the matrix \(A^{-1}\).

**Proof**

Notice that since \(A\) is assumed nonsingular, \(A^{-1}\) exists by Theorem NI \[251\], but more importantly, \(\frac{1}{\lambda}\) does not involve division by zero since Theorem SMZE \[468\] prohibits this possibility.

Let \(x \neq 0\) be one eigenvector of \(A\) for \(\lambda\). Suppose \(A\) has size \(n\). Then

\[
A^{-1}x = A^{-1}(1x) = A^{-1} \left( \frac{1}{\lambda} \lambda x \right) = \frac{1}{\lambda} A^{-1}(\lambda x) = \frac{1}{\lambda} A^{-1}(Ax) = \frac{1}{\lambda} (A^{-1}A)x = \frac{1}{\lambda} I_n x = \frac{1}{\lambda} x
\]

So \(x \neq 0\) is an eigenvector of \(A^{-1}\) for the eigenvalue \(\frac{1}{\lambda}\).

The theorems above have a similar style to them, a style you should consider using when confronted with a need to prove a theorem about eigenvalues and eigenvectors. So far we have been able to reserve the characteristic polynomial for strictly computational purposes. However, the next theorem, whose statement resembles the preceding theorems, has an easier proof if we employ the characteristic polynomial and results about determinants.

**Theorem ETM**

**Eigenvalues of the Transpose of a Matrix**

Suppose \(A\) is a square matrix and \(\lambda\) is an eigenvalue of \(A\). Then \(\lambda\) is an eigenvalue of the matrix \(A^t\).

**Proof**

Let \(x \neq 0\) be one eigenvector of \(A\) for \(\lambda\). Suppose \(A\) has size \(n\). Then

\[
p_A(x) = \det(A - xI_n) = \det(\lambda - xI_n) = \frac{1}{\lambda} \det(AI_n - xI_n) = \frac{1}{\lambda} \det(\lambda I_n - xI_n) = \frac{1}{\lambda} \det(\lambda I_n - xI_n)
\]

So \(x \neq 0\) is an eigenvector of \(A^{-1}\) for the eigenvalue \(\frac{1}{\lambda}\).
$= \det \left( (A - xI_n)^t \right) \quad \text{Theorem DT} \ [417]$

$= \det \left( A^t - (xI_n)^t \right) \quad \text{Theorem TMA} \ [202]$

$= \det \left( A^t - xI_n^t \right) \quad \text{Theorem TMSM} \ [202]$

$= \det \left( A^t - xI_n \right) \quad \text{Definition IM} \ [76]$  

$= p_A (x) \quad \text{Definition CP} \ [449]$

So $A$ and $A^t$ have the same characteristic polynomial, and by Theorem EMRCP [449], their eigenvalues are identical and have equal algebraic multiplicities. Notice that what we have proved here is a bit stronger than the stated conclusion in the theorem. ■

If a matrix has only real entries, then the computation of the characteristic polynomial (Definition CP [449]) will result in a polynomial with coefficients that are real numbers. Complex numbers could result as roots of this polynomial, but they are roots of quadratic factors with real coefficients, and as such, come in conjugate pairs. The next theorem proves this, and a bit more, without mentioning the characteristic polynomial.

**Theorem ERMCP**

**Eigenvalues of Real Matrices come in Conjugate Pairs**

Suppose $A$ is a square matrix with real entries and $x$ is an eigenvector of $A$ for the eigenvalue $\lambda$. Then $\bar{x}$ is an eigenvector of $A$ for the eigenvalue $\bar{\lambda}$. □

**Proof**

$$A\bar{x} = \bar{A}\bar{x} \quad A \text{ has real entries}$$

$$= \bar{A}\bar{x} \quad \text{Theorem MMCC} \ [221]$$

$$= \bar{\lambda}\bar{x} \quad \text{x eigenvector of A}$$

$$= \bar{\lambda}\bar{x} \quad \text{Theorem CRSM} \ [184]$$

So $\bar{x}$ is an eigenvector of $A$ for the eigenvalue $\bar{\lambda}$. ■

This phenomenon is amply illustrated in Example CEMS6 [455], where the four complex eigenvalues come in two pairs, and the two basis vectors of the eigenspaces are complex conjugates of each other. Theorem ERMCP [472] can be a time-saver for computing eigenvalues and eigenvectors of real matrices with complex eigenvalues, since the conjugate eigenvalue and eigenspace can be inferred from the theorem rather than computed.

**Subsection ME**

**Multiplicities of Eigenvalues**

A polynomial of degree $n$ will have exactly $n$ roots. From this fact about polynomial equations we can say more about the algebraic multiplicities of eigenvalues.

**Theorem DCP**

**Degree of the Characteristic Polynomial**

Suppose that $A$ is a square matrix of size $n$. Then the characteristic polynomial of $A$, $p_A (x)$, has degree $n$. □
Proof We will prove a more general result by induction (Technique I [713]). Then the theorem will be true as a special case. We will carefully state this result as a proposition indexed by \( m, m \geq 1 \).

\( P(m) \): Suppose that \( A \) is an \( m \times m \) matrix whose entries are complex numbers or linear polynomials in the variable \( x \) of the form \( c - x \), where \( c \) is a complex number. Suppose further that there are exactly \( k \) entries that contain \( x \) and that no row or column contains more than one such entry. Then, when \( k = m \), det \((A)\) is a polynomial in \( x \) of degree \( m \), with leading coefficient \( \pm 1 \), and when \( k < m \), det \((A)\) is a polynomial in \( x \) of degree \( k \) or less.

Base Case: Suppose \( A \) is a \( 1 \times 1 \) matrix. Then its determinant is equal to the lone entry (Definition DM [414]). When \( k = m = 1 \), the entry is of the form \( c - x \), a polynomial in \( x \) of degree \( m = 1 \) with leading coefficient \(-1\). When \( k < m \), then \( k = 0 \) and the entry is simply a complex number, a polynomial of degree \( 0 \leq k \). So \( P(1) \) is true.

Induction Step: Assume \( P(m) \) is true, and that \( A \) is an \((m+1) \times (m+1)\) matrix with \( k \) entries of the form \( c - x \). There are two cases to consider.

Suppose \( k = m+1 \). Then every row and every column will contain an entry of the form \( c - x \). Suppose that for the first row, this entry is in column \( t \). Compute the determinant of \( A \) by an expansion about this first row (Definition DM [414]). The term associated with entry \( t \) of this row will be of the form

\[ (c - x)(-1)^{t+1} \text{det}(A(1|t)) \]

The submatrix \( A(1|t) \) is an \( m \times m \) matrix with \( k = m \) terms of the form \( c - x \), no more than one per row or column. By the induction hypothesis, det \((A(1|t))\) will be a polynomial in \( x \) of degree \( m \) with coefficient \( \pm 1 \). So this entire term is then a polynomial of degree \( m + 1 \) with leading coefficient \( \pm 1 \).

The remaining terms (which constitute the sum that is the determinant of \( A \)) are products of complex numbers from the first row with cofactors built from submatrices that lack the first row of \( A \) and lack some column of \( A \), other than column \( t \). As such, these submatrices are \( m \times m \) matrices with \( k = m - 1 < m \) entries of the form \( c - x \), no more than one per row or column. Applying the induction hypothesis, we see that these terms are polynomials in \( x \) of degree \( m - 1 \) or less. Adding the single term from the entry in column \( t \) with all these others, we see that det \((A)\) is a polynomial in \( x \) of degree \( m + 1 \) and leading coefficient \( \pm 1 \).

The second case occurs when \( k < m + 1 \). Now there is a row of \( A \) that does not contain an entry of the form \( c - x \). We consider the determinant of \( A \) by expanding about this row (Theorem DER [416]), whose entries are all complex numbers. The cofactors employed are built from submatrices that are \( m \times m \) matrices with either \( k \) or \( k - 1 \) entries of the form \( c - x \), no more than one per row or column. In either case, \( k \leq m \), and we can apply the induction hypothesis to see that the determinants computed for the cofactors are all polynomials of degree \( k \) or less. Summing these contributions to the determinant of \( A \) yields a polynomial in \( x \) of degree \( k \) or less, as desired.

**Theorem NEM**

**Number of Eigenvalues of a Matrix**

Suppose that \( A \) is a square matrix of size \( n \) with distinct eigenvalues \( \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k \).
Then
\[
\sum_{i=1}^{k} \alpha_A(\lambda_i) = n
\]

**Proof**  By the definition of the algebraic multiplicity (Definition AME 452), we can factor the characteristic polynomial as
\[
p_A(x) = c(x - \lambda_1)^{\alpha_A(\lambda_1)}(x - \lambda_2)^{\alpha_A(\lambda_2)}(x - \lambda_3)^{\alpha_A(\lambda_3)} \cdots (x - \lambda_k)^{\alpha_A(\lambda_k)}
\]
where \(c\) is a nonzero constant. (We could prove that \(c = (-1)^n\), but we do not need that specificity right now. See Exercise PEE.T30 479) The left-hand side is a polynomial of degree \(n\) by Theorem DCP 472 and the right-hand side is a polynomial of degree \(\sum_{i=1}^{k} \alpha_A(\lambda_i)\). So the equality of the polynomials’ degrees gives the equality \(\sum_{i=1}^{k} \alpha_A(\lambda_i) = n\).

**Theorem ME**

**Multiplicities of an Eigenvalue**

Suppose that \(A\) is a square matrix of size \(n\) and \(\lambda\) is an eigenvalue. Then
\[
1 \leq \gamma_A(\lambda) \leq \alpha_A(\lambda) \leq n
\]

**Proof**  Since \(\lambda\) is an eigenvalue of \(A\), there is an eigenvector of \(A\) for \(\lambda, x\). Then \(x \in \mathcal{E}_A(\lambda)\), so \(\gamma_A(\lambda) \geq 1\), since we can extend \(\{x\}\) into a basis of \(\mathcal{E}_A(\lambda)\) (Theorem ELIS 397).

To show that \(\gamma_A(\lambda) \leq \alpha_A(\lambda)\) is the most involved portion of this proof. To this end, let \(g = \gamma_A(\lambda)\) and let \(x_1, x_2, x_3, \ldots, x_g\) be a basis for the eigenspace of \(\lambda, \mathcal{E}_A(\lambda)\).

Construct another \(n - g\) vectors, \(y_1, y_2, y_3, \ldots, y_{n-g}\), so that
\[
\{x_1, x_2, x_3, \ldots, x_g, y_1, y_2, y_3, \ldots, y_{n-g}\}
\]
is a basis of \(\mathbb{C}^n\). This can be done by repeated applications of Theorem ELIS 397. Finally, define a matrix \(S\) by
\[
S = [x_1 | x_2 | x_3 | \ldots | x_g | y_1 | y_2 | y_3 | \ldots | y_{n-g}] = [x_1 | x_2 | x_3 | \ldots | x_g | R]
\]
where \(R\) is an \(n \times (n - g)\) matrix whose columns are \(y_1, y_2, y_3, \ldots, y_{n-g}\). The columns of \(S\) are linearly independent by design, so \(S\) is nonsingular (Theorem NMLIC 151) and therefore invertible (Theorem NI 251). Then,
\[
[e_1 | e_2 | e_3 | \ldots | e_n] = I_n = S^{-1}S = S^{-1}[x_1 | x_2 | x_3 | \ldots | x_g | R] = [S^{-1}x_1 | S^{-1}x_2 | S^{-1}x_3 | \ldots | S^{-1}x_g | S^{-1}R]
\]
So
\[
S^{-1}x_i = e_i \quad 1 \leq i \leq g
\]

Preparations in place, we compute the characteristic polynomial of \(A\),
\[
p_A(x) = \det (A - xI_n)
\]

Definition CP 449
What can we learn then about the matrix $S^{-1}AS$?

\[
S^{-1}AS = S^{-1}A[x_1|x_2|x_3|\ldots|x_g|R] \\
= S^{-1}[Ax_1|Ax_2|Ax_3|\ldots|Ax_g]|AR] \\
= S^{-1}[\lambda x_1|\lambda x_2|\lambda x_3|\ldots|\lambda x_g]|AR] \\
= [S^{-1}\lambda x_1|S^{-1}\lambda x_2|S^{-1}\lambda x_3|\ldots|S^{-1}\lambda x_g]|S^{-1}AR] \\
= [\lambda S^{-1}x_1|\lambda S^{-1}x_2|\lambda S^{-1}x_3|\ldots|\lambda S^{-1}x_g]|S^{-1}AR] \\
= [\lambda e_1|\lambda e_2|\lambda e_3|\ldots|\lambda e_g]|S^{-1}AR]
\]

Now imagine computing the characteristic polynomial of $A$ by computing the characteristic polynomial of $S^{-1}AS$ using the form just obtained. The first $g$ columns of $S^{-1}AS$ are all zero, save for a $\lambda$ on the diagonal. So if we compute the determinant by expanding about the first column, successively, we will get successive factors of $(\lambda - x)$. More precisely, let $T$ be the square matrix of size $n - g$ that is formed from the last $n - g$ rows and last $n - g$ columns of $S^{-1}AR$. Then

\[
p_A(x) = p_{S^{-1}AS}(x) = (\lambda - x)^g p_T(x).
\]

This says that $(x - \lambda)$ is a factor of the characteristic polynomial at least $g$ times, so the algebraic multiplicity of $\lambda$ as an eigenvalue of $A$ is greater than or equal to $g$ \("Theorem AME \[452\]."\) In other words,

\[
\gamma_A(\lambda) = g \leq \alpha_A(\lambda)
\]

as desired.

\"Theorem NEM \[473\]" says that the sum of the algebraic multiplicities for all the eigenvalues of $A$ is equal to $n$. Since the algebraic multiplicity is a positive quantity, no single algebraic multiplicity can exceed $n$ without the sum of all of the algebraic multiplicities doing the same. \hfill \blacksquare

**Theorem MNEM**

**Maximum Number of Eigenvalues of a Matrix**

Suppose that $A$ is a square matrix of size $n$. Then $A$ cannot have more than $n$ distinct eigenvalues. \hfill \square
Proof Suppose that $A$ has $k$ distinct eigenvalues, $\lambda_1$, $\lambda_2$, $\lambda_3$, \ldots, $\lambda_k$. Then

$$k = \sum_{i=1}^{k} 1 \leq \sum_{i=1}^{k} \alpha_A (\lambda_i) \leq n$$

Theorem ME 474

Theorem NEM 473

Subsection EHM

Eigenvalues of Hermitian Matrices

Recall that a matrix is Hermitian (or self-adjoint) if $A = (\overline{A})^t$ (Definition HM 255). In the case where $A$ is a matrix whose entries are all real numbers, being Hermitian is identical to being symmetric (Definition SYM 201). Keep this in mind as you read the next two theorems. Their hypotheses could be changed to “suppose $A$ is a real symmetric matrix.”

Theorem HMRE

Hermitian Matrices have Real Eigenvalues

Suppose that $A$ is a Hermitian matrix and $\lambda$ is an eigenvalue of $A$. Then $\lambda \in \mathbb{R}$. □

Proof Let $x \neq 0$ be one eigenvector of $A$ for $\lambda$. Then

$$\lambda \langle x, x \rangle = \langle \lambda x, x \rangle = \langle Ax, x \rangle = \langle (Ax)^t \bar{x} \rangle = x^t A^t \bar{x} = x^t \left((\overline{A})^t\right)^t \bar{x} = x^t A \bar{x} = x^t A \bar{x} = \langle x, \lambda x \rangle = \overline{\lambda} \langle x, x \rangle$$

Since $x \neq 0$, Theorem PIP 188 says that $\langle x, x \rangle \neq 0$, so we can “cancel” $\langle x, x \rangle$ from both sides of this equality. This leaves $\lambda = \overline{\lambda}$, so $\lambda$ has a complex part equal to zero, and therefore is a real number. □

Look back and compare Example ESMS4 453 and Example CEMS6 455. In Example CEMS6 455 the matrix has only real entries, yet the characteristic polynomial has roots that are complex numbers, and so the matrix has complex eigenvalues. However, in Example ESMS4 453, the matrix has only real entries, but is also symmetric. So by Theorem HMRE 476, we were guaranteed eigenvalues that are real numbers.
In many physical problems, a matrix of interest will be real and symmetric, or Hermitian. Then if the eigenvalues are to represent physical quantities of interest, Theorem HMRE guarantees that these values will not be complex numbers.

The eigenvectors of a Hermitian matrix also enjoy a pleasing property that we will exploit later.

**Theorem HMOE**

**Hermitian Matrices have Orthogonal Eigenvectors**

Suppose that $A$ is a Hermitian matrix and $x$ and $y$ are two eigenvectors of $A$ for different eigenvalues. Then $x$ and $y$ are orthogonal vectors.

**Proof** Let $x \neq 0$ be an eigenvector of $A$ for $\lambda$ and let $y \neq 0$ be an eigenvector of $A$ for $\rho$. By Theorem HMRE, we know that $\rho$ must be a real number. Then

$$\lambda \langle x, y \rangle = \langle \lambda x, y \rangle = \langle Ax, y \rangle = (Ax)^t \bar{y} = x^t A^t \bar{y} = x^t \left( (A^t)^t \right)^t \bar{y} = x^t A y = x^t \bar{A} y = x^t \bar{A} y = \langle x, Ay \rangle = \langle x, \rho y \rangle = \rho \langle x, y \rangle = \rho \langle x, y \rangle$$

Since $\lambda \neq \rho$, we conclude that $\langle x, y \rangle = 0$ and so $x$ and $y$ are orthogonal vectors.

---

**Subsection READ**

**Reading Questions**

1. How can you identify a nonsingular matrix just by looking at its eigenvalues?

2. How many different eigenvalues may a square matrix of size $n$ have?

3. What is amazing about the eigenvalues of a Hermitian matrix and why is it amazing?
Subsection EXC

Exercises

T10 Suppose that $A$ is a square matrix. Prove that the constant term of the characteristic polynomial of $A$ is equal to the determinant of $A$.
Contributed by Robert Beezer Solution

T20 Suppose that $A$ is a square matrix. Prove that a single vector may not be an eigenvector of $A$ for two different eigenvalues.
Contributed by Robert Beezer Solution

T30 Theorem DCP tells us that the characteristic polynomial of a square matrix of size $n$ has degree $n$. By suitably augmenting the proof of Theorem DCP prove that the coefficient of $x^n$ in the characteristic polynomial is $(-1)^n$.
Contributed by Robert Beezer
Subsection SOL
Solutions

\textbf{T10} Contributed by Robert Beezer Statement 479
Suppose that the characteristic polynomial of $A$ is
\[ p_A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \]
Then\[ a_0 = a_0 + a_1(0) + a_2(0)^2 + \cdots + a_n(0)^n \]
\[ = p_A(0) \]
\[ = \det (A - 0I_n) \quad \text{Definition CP 449} \]
\[ = \det (A) \]

\textbf{T20} Contributed by Robert Beezer Statement 479
Suppose that the vector $x \neq 0$ is an eigenvector of $A$ for the two eigenvalues $\lambda$ and $\rho$, where $\lambda \neq \rho$. Then $\lambda - \rho \neq 0$, so
\[ 0 \neq (\lambda - \rho)x \quad \text{Theorem SMEZV 319} \]
\[ = \lambda x - \rho x \quad \text{Property DSAC 91} \]
\[ = Ax - Ax \quad \lambda, \rho \text{ eigenvalues of } A \]
\[ = 0 \quad \text{Property AIC 91} \]
which is a contradiction.
Section SD
Similarity and Diagonalization

This section’s topic will perhaps seem out of place at first, but we will make the connection soon with eigenvalues and eigenvectors. This is also our first look at one of the central ideas of Chapter R [587].

Subsection SM
Similar Matrices

The notion of matrices being “similar” is a lot like saying two matrices are row-equivalent. Two similar matrices are not equal, but they share many important properties. This section, and later sections in Chapter R [587] will be devoted in part to discovering just what these common properties are.

First, the main definition for this section.

Definition SIM
Similar Matrices

Suppose \( A \) and \( B \) are two square matrices of size \( n \). Then \( A \) and \( B \) are similar if there exists a nonsingular matrix of size \( n \), \( S \), such that

\[
A = S^{-1}BS.
\]

We will say “\( A \) is similar to \( B \) via \( S \)” when we want to emphasize the role of \( S \) in the relationship between \( A \) and \( B \). Also, it doesn’t matter if we say \( A \) is similar to \( B \), or \( B \) is similar to \( A \). If one statement is true then so is the other, as can be seen by using \( S^{-1} \) in place of \( S \) (see Theorem SER [485] for the careful proof). Finally, we will refer to \( S^{-1}BS \) as a similarity transformation when we want to emphasize the way \( S \) changes \( B \). OK, enough about language, let’s build a few examples.

Example SMS5
Similar matrices of size 5

If you wondered if there are examples of similar matrices, then it won’t be hard to convince you they exist. Define

\[
B = \begin{bmatrix}
-4 & 1 & -3 & -2 & 2 \\
1 & 2 & -1 & 3 & -2 \\
-4 & 1 & 3 & 2 & 2 \\
-3 & 4 & -2 & -1 & -3 \\
3 & 1 & -1 & 1 & -4
\end{bmatrix}
\]

\[
S = \begin{bmatrix}
1 & 2 & -1 & 1 & 1 \\
0 & 1 & -1 & -2 & -1 \\
1 & 3 & -1 & 1 & 1 \\
-2 & -3 & 3 & 1 & -2 \\
1 & 3 & -1 & 2 & 1
\end{bmatrix}
\]

Check that \( S \) is nonsingular and then compute

\[
A = S^{-1}BS
\]

\[
= \begin{bmatrix}
10 & 1 & 0 & 2 & -5 \\
-1 & 0 & 1 & 0 & 0 \\
3 & 0 & 2 & 1 & -3 \\
0 & 0 & -1 & 0 & 1 \\
-4 & -1 & 1 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
-4 & 1 & -3 & -2 & 2 \\
1 & 2 & -1 & 3 & -2 \\
-4 & 1 & 3 & 2 & 2 \\
-3 & 4 & -2 & -1 & -3 \\
3 & 1 & -1 & 1 & -4
\end{bmatrix}
\begin{bmatrix}
1 & 2 & -1 & 1 & 1 \\
0 & 1 & -1 & -2 & -1 \\
1 & 3 & -1 & 1 & 1 \\
-2 & -3 & 3 & 1 & -2 \\
1 & 3 & -1 & 2 & 1
\end{bmatrix}
\]
Let’s do that again.

Example SMS3
Similar matrices of size 3
Define
\[
B = \begin{bmatrix}
-13 & -8 & -4 \\
12 & 7 & 4 \\
24 & 16 & 7
\end{bmatrix}, \quad S = \begin{bmatrix}
1 & 1 & 2 \\
-2 & -1 & -3 \\
1 & -2 & 0
\end{bmatrix}
\]

Check that \(S\) is nonsingular and then compute
\[
A = S^{-1}BS
\]
\[
= \begin{bmatrix}
-6 & -4 & -1 \\
-3 & -2 & -1 \\
5 & 3 & 1
\end{bmatrix} \begin{bmatrix}
13 & -8 & -4 \\
12 & 7 & 4 \\
24 & 16 & 7
\end{bmatrix} \begin{bmatrix}
1 & 1 & 2 \\
-2 & -1 & -3 \\
1 & -2 & 0
\end{bmatrix}
\]
\[
= \begin{bmatrix}
-1 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

So by this construction, we know that \(A\) and \(B\) are similar. But before we move on, look at how pleasing the form of \(A\) is. Not convinced? Then consider that several computations related to \(A\) are especially easy. For example, in the spirit of Example DUTM \[419\], \[\det (A) = (-1)(3)(-1) = 3.\] Similarly, the characteristic polynomial is straightforward to compute by hand, \[p_A(x) = (-1-x)(3-x)(-1-x) = -(x-3)(x+1)^2\] and since the result is already factored, the eigenvalues are transparently \(\lambda = 3, -1\). Finally, the eigenvectors of \(A\) are just the standard unit vectors (Definition SUV \[234\]).

Subsection PSM
Properties of Similar Matrices

Similar matrices share many properties and it is these theorems that justify the choice of the word “similar.” First we will show that similarity is an equivalence relation. Equivalence relations are important in the study of various algebras and can always be regarded as a kind of weak version of equality. Sort of alike, but not quite equal. The notion of two matrices being row-equivalent is an example of an equivalence relation we have been working with since the beginning of the course (see Exercise RREF.T11 \[45\]). Row-equivalent matrices are not equal, but they are a lot alike. For example, row-equivalent matrices have the same rank. Formally, an equivalence relation requires three conditions hold: reflexive, symmetric and transitive. We will illustrate these as we prove that similarity is an equivalence relation.
Theorem SER
Similarity is an Equivalence Relation
Suppose $A$, $B$ and $C$ are square matrices of size $n$. Then
1. $A$ is similar to $A$. (Reflexive)
2. If $A$ is similar to $B$, then $B$ is similar to $A$. (Symmetric)
3. If $A$ is similar to $B$ and $B$ is similar to $C$, then $A$ is similar to $C$. (Transitive)

□

Proof
To see that $A$ is similar to $A$, we need only demonstrate a nonsingular matrix that effects a similarity transformation of $A$ to $A$. $I_n$ is nonsingular (since it row-reduces to the identity matrix, Theorem NMRRI [77]), and

$$I_n^{-1}AI_n = I_nA = A$$

If we assume that $A$ is similar to $B$, then we know there is a nonsingular matrix $S$ so that $A = S^{-1}BS$ by Definition SIM [483]. By Theorem MIMI [240], $S^{-1}$ is invertible, and by Theorem NI [251] is therefore nonsingular. So

$$(S^{-1})^{-1}A(S^{-1}) = SAS^{-1}$$

$$= SS^{-1}BSS^{-1}$$

$$= (SS^{-1})B(SS^{-1})$$

$$= I_nBI_n$$

$$= B$$

and we see that $B$ is similar to $A$.

Assume that $A$ is similar to $B$, and $B$ is similar to $C$. This gives us the existence of two nonsingular matrices, $S$ and $R$, such that $A = S^{-1}BS$ and $B = R^{-1}CR$, by Definition SIM [483]. (Notice how we have to assume $S \neq R$, as will usually be the case.) Since $S$ and $R$ are invertible, so too $RS$ is invertible by Theorem SS [239] and then nonsingular by Theorem NI [251]. Now

$$(RS)^{-1}C(RS) = S^{-1}R^{-1}CRS$$

$$= S^{-1}(R^{-1}CR)S$$

$$= S^{-1}BS$$

$$= A$$

so $A$ is similar to $C$ via the nonsingular matrix $RS$. □

Here’s another theorem that tells us exactly what sorts of properties similar matrices share.

Theorem SMEE
Similar Matrices have Equal Eigenvalues
Suppose $A$ and $B$ are similar matrices. Then the characteristic polynomials of $A$ and $B$ are equal, that is $p_A(x) = p_B(x)$.

Proof
Suppose $A$ and $B$ have size $n$ and are similar via the nonsingular matrix $S$, so $A = S^{-1}BS$ by Definition SIM [483].

$$p_A(x) = \det (A - xI_n)$$

Definition CP [449]
= \det (S^{-1}BS - xI_n)
= \det (S^{-1}BS - S^{-1}S^{-1}I_nS)
= \det (S^{-1}BS - S^{-1}xI_nS)
= \det (S^{-1} (B - xI_n) S)
= \det (S^{-1}) \det (B - xI_n) \det (S)
= \det (S^{-1}) \det (S) \det (B - xI_n)
= \det (S^{-1}S) \det (B - xI_n)
= \det (S^{-1}S) \det (B - xI_n)
= 1 \det (B - xI_n)
= p_B (x)
\text{Definition SIM} \ 483
\text{Theorem MMIM} \ 218
\text{Theorem MMSMM} \ 219
\text{Theorem MMDAA} \ 219
\text{Theorem DRMM} \ 434
\text{Property MCCN} \ 688
\text{Theorem DRMM} \ 434
\text{Definition MI} \ 232
\text{Definition DM} \ 414
\text{Definition CP} \ 449

So similar matrices not only have the same set of eigenvalues, the algebraic multiplicities of these eigenvalues will also be the same. However, be careful with this theorem. It is tempting to think the converse is true, and argue that if two matrices have the same eigenvalues, then they are similar. Not so, as the following example illustrates.

\textbf{Example EENS}
\textbf{Equal eigenvalues, not similar}

Define

\[
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

and check that

\[
p_A (x) = p_B (x) = 1 - 2x + x^2 = (x - 1)^2
\]

and so \(A\) and \(B\) have equal characteristic polynomials. If the converse of \textbf{Theorem SMEE} \ 485 were true, then \(A\) and \(B\) would be similar. Suppose this is the case. In other words, there is a nonsingular matrix \(S\) so that \(A = S^{-1}BS\). Then

\[
A = S^{-1}BS = S^{-1}I_2S = S^{-1}S = I_2 \neq A
\]

and this contradiction tells us that the converse of \textbf{Theorem SMEE} \ 485 is false.

\textbf{Subsection D}
\textbf{Diagonalization}

Good things happen when a matrix is similar to a diagonal matrix. For example, the eigenvalues of the matrix are the entries on the diagonal of the diagonal matrix. And it can be a much simpler matter to compute high powers of the matrix. Diagonalizable matrices are also of interest in more abstract settings. Here are the relevant definitions, then our main theorem for this section.

\textbf{Definition DIM}
\textbf{Diagonal Matrix}

Suppose that \(A\) is a square matrix. Then \(A\) is a diagonal matrix if \([A]_{ij} = 0\) whenever \(i \neq j\).
Definition DZM

Diagonalizable Matrix

Suppose $A$ is a square matrix. Then $A$ is diagonalizable if $A$ is similar to a diagonal matrix. $\triangle$

Example DAB

Diagonalization of Archetype B

Archetype B [726] has a $3 \times 3$ coefficient matrix

$$B = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$$

and is similar to a diagonal matrix, as can be seen by the following computation with the nonsingular matrix $S$,

$$S^{-1}BS = \begin{bmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Example SMS3 [484] provides yet another example of a matrix that is subjected to a similarity transformation and the result is a diagonal matrix. Alright, just how would we find the magic matrix $S$ that can be used in a similarity transformation to produce a diagonal matrix? Before you read the statement of the next theorem, you might study the eigenvalues and eigenvectors of Archetype B [726] and compute the eigenvalues and eigenvectors of the matrix in Example SMS3 [484].

Theorem DC

Diagonalization Characterization

Suppose $A$ is a square matrix of size $n$. Then $A$ is diagonalizable if and only if there exists a linearly independent set $S$ that contains $n$ eigenvectors of $A$. $\square$

Proof (⇐) Let $S = \{x_1, x_2, x_3, \ldots, x_n\}$ be a linearly independent set of eigenvectors of $A$ for the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$. Recall Definition SUV [234] and define

$$R = [x_1 | x_2 | x_3 | \ldots | x_n]$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\lambda_1e_1 | \lambda_2e_2 | \lambda_3e_3 | \ldots | \lambda_ne_n]$$
The columns of $R$ are the vectors of the linearly independent set $S$ and so by Theorem NMLIC 151 the matrix $R$ is nonsingular. By Theorem NI 251 we know $R^{-1}$ exists.

\[
R^{-1} AR = R^{-1} A [x_1 | x_2 | x_3 | \ldots | x_n] = R^{-1}[Ax_1 | Ax_2 | Ax_3 | \ldots | Ax_n]
\]

\[
= R^{-1}[\lambda_1 x_1 | \lambda_2 x_2 | \lambda_3 x_3 | \ldots | \lambda_n x_n]
\]

\[
= R^{-1}[\lambda_1 x_1 | \lambda_2 x_2 | \lambda_3 x_3 | \ldots | \lambda_n x_n]
\]

\[
= R^{-1}[\lambda_1 Re_1 | \lambda_2 Re_2 | \lambda_3 Re_3 | \ldots | \lambda_n Re_n]
\]

\[
= R^{-1}[R(\lambda_1 e_1)|R(\lambda_2 e_2)|R(\lambda_3 e_3)|\ldots|R(\lambda_n e_n)]
\]

\[
= R^{-1}R[\lambda_1 e_1 | \lambda_2 e_2 | \lambda_3 e_3 | \ldots | \lambda_n e_n]
\]

\[
= I_n D
\]

\[
= D
\]

This says that $A$ is similar to the diagonal matrix $D$ via the nonsingular matrix $R$. Thus $A$ is diagonalizable (Definition DZM 487).

$(\Rightarrow)$ Suppose that $A$ is diagonalizable, so there is a nonsingular matrix of size $n$

\[
T = [y_1 | y_2 | y_3 | \ldots | y_n]
\]

and a diagonal matrix (recall Definition SUV 234)

\[
E = \begin{bmatrix}
d_1 & 0 & 0 & \ldots & 0 \\
0 & d_2 & 0 & \ldots & 0 \\
0 & 0 & d_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & d_n
\end{bmatrix} = [d_1 e_1 | d_2 e_2 | d_3 e_3 | \ldots | d_n e_n]
\]

such that $T^{-1}AT = E$. Then consider,

\[
[Ay_1 | Ay_2 | Ay_3 | \ldots | Ay_n] = A[y_1 | y_2 | y_3 | \ldots | y_n]
\]

\[
= AT
\]

\[
= I_n AT
\]

\[
= TT^{-1}AT
\]

\[
= TE
\]

\[
= T[d_1 e_1 | d_2 e_2 | d_3 e_3 | \ldots | d_n e_n]
\]

\[
= [T(d_1 e_1)|T(d_2 e_2)|T(d_3 e_3)|\ldots|T(d_n e_n)]
\]

\[
= [d_1 Te_1 | d_2 Te_2 | d_3 Te_3 | \ldots | d_n Te_n]
\]

\[
= [d_1 y_1 | d_2 y_2 | d_3 y_3 | \ldots | d_n y_n]
\]

This equality of matrices (Definition ME 197) allows us to conclude that the individual columns are equal vectors (Definition CVE 88). That is, $Ay_i = d_i y_i$ for $1 \leq i \leq n$. In other words, $y_i$ is an eigenvector of $A$ for the eigenvalue $d_i$, $1 \leq i \leq n$. (Why can’t $y_i = 0$?) Because $T$ is nonsingular, the set containing $T$’s columns, $S = \{y_1, y_2, y_3, \ldots, y_n\}$, is a linearly independent set (Theorem NMLIC 151). So the set $S$ has all the required properties. 

Version 0.92
Notice that the proof of [Theorem DC 487] is constructive. To diagonalize a matrix, we need only locate \( n \) linearly independent eigenvectors. Then we can construct a nonsingular matrix using the eigenvectors as columns \( (R) \) so that \( R^{-1}AR \) is a diagonal matrix \( (D) \). The entries on the diagonal of \( D \) will be the eigenvalues of the eigenvectors used to create \( R \), in the same order as the eigenvectors appear in \( R \). We illustrate this by diagonalizing some matrices.

**Example DMS3**

**Diagonalizing a matrix of size 3**

Consider the matrix

\[
F = \begin{bmatrix}
-13 & -8 & -4 \\
12 & 7 & 4 \\
24 & 16 & 7
\end{bmatrix}
\]

of [Example CPMS3 449], [Example EMS3 450] and [Example ESMS3 451]. \( F \)'s eigenvalues and eigenspaces are

\[
\lambda = 3 \quad \mathcal{E}_F(3) = \left\{ \begin{bmatrix}
\frac{1}{3} \\
1
\end{bmatrix}, \begin{bmatrix}
\frac{1}{3} \\
0
\end{bmatrix} \right\}
\]

\[
\lambda = -1 \quad \mathcal{E}_F(-1) = \left\{ \begin{bmatrix}
-\frac{2}{3} \\
1
\end{bmatrix}, \begin{bmatrix}
-\frac{1}{3} \\
0
\end{bmatrix} \right\}
\]

Define the matrix \( S \) to be the \( 3 \times 3 \) matrix whose columns are the three basis vectors in the eigenspaces for \( F \),

\[
S = \begin{bmatrix}
-\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\
\frac{1}{2} & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

Check that \( S \) is nonsingular (row-reduces to the identity matrix, [Theorem NMRRI 77] or has a nonzero determinant, [Theorem SMZD 431]). Then the three columns of \( S \) are a linearly independent set ([Theorem NMLIC 151]). By [Theorem DC 487] we now know that \( F \) is diagonalizable. Furthermore, the construction in the proof of [Theorem DC 487] tells us that if we apply the matrix \( S \) to \( F \) in a similarity transformation, the result will be a diagonal matrix with the eigenvalues of \( F \) on the diagonal. The eigenvalues appear on the diagonal of the matrix in the same order as the eigenvectors appear in \( S \). So,

\[
S^{-1}FS = \begin{bmatrix}
-\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\
\frac{1}{2} & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}^{-1} \begin{bmatrix}
-13 & -8 & -4 \\
12 & 7 & 4 \\
24 & 16 & 7
\end{bmatrix} \begin{bmatrix}
-\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\
\frac{1}{2} & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
6 & 4 & 2 \\
-3 & -1 & -1 \\
-6 & -4 & -1
\end{bmatrix} \begin{bmatrix}
-13 & -8 & -4 \\
12 & 7 & 4 \\
24 & 16 & 7
\end{bmatrix} \begin{bmatrix}
\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\
\frac{1}{2} & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
3 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

Note that the above computations can be viewed two ways. The proof of [Theorem DC 487] tells us that the four matrices \( (F, S, F^{-1} \text{ and the diagonal matrix}) \) will interact the way
we have written the equation. Or as an example, we can actually perform the computations to verify what the theorem predicts.

The dimension of an eigenspace can be no larger than the algebraic multiplicity of the eigenvalue by Theorem ME 474. When every eigenvalue’s eigenspace is this large, then we can diagonalize the matrix, and only then. Three examples we have seen so far in this section, Example SMS5 483, Example DAB 487 and Example DMS3 489, illustrate the diagonalization of a matrix, with varying degrees of detail about just how the diagonalization is achieved. However, in each case, you can verify that the geometric and algebraic multiplicities are equal for every eigenvalue. This is the substance of the next theorem.

**Theorem DMFE**

**Diagonalizable Matrices have Full Eigenspaces**

Suppose $A$ is a square matrix. Then $A$ is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for every eigenvalue $\lambda$ of $A$. □

**Proof** Suppose $A$ has size $n$ and $k$ distinct eigenvalues, $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$.

$(\Leftarrow)$ Let $S_i = \{x_{i1}, x_{i2}, x_{i3}, \ldots, x_{i\gamma_A(\lambda_i)}\}$ be a basis for the eigenspace of $\lambda_i$, $\mathcal{E}_A(\lambda_i)$, $1 \leq i \leq k$. Then

$$S = S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_k$$

is a set of eigenvectors for $A$. A vector cannot be an eigenvector for two different eigenvalues (see Exercise EE.T20 461) so the sets $S_i$ have no vectors in common. Thus the size of $S$ is

$$\sum_{i=1}^{k} \gamma_A(\lambda_i) = \sum_{i=1}^{k} \alpha_A(\lambda_i) \quad \text{Hypothesis}$$

$$= n \quad \text{Theorem NEM 473}$$

We now want to show that $S$ is a linearly independent set. So we will begin with a relation of linear dependence on $S$, using doubly-subscripted scalars and eigenvectors,

$$0 = (a_{11}x_{11} + a_{12}x_{12} + \cdots + a_{1\gamma_A(\lambda_1)}x_{1\gamma_A(\lambda_1)}) + (a_{21}x_{21} + a_{22}x_{22} + \cdots + a_{2\gamma_A(\lambda_2)}x_{2\gamma_A(\lambda_2)}) + \cdots + (a_{k1}x_{k1} + a_{k2}x_{k2} + \cdots + a_{k\gamma_A(\lambda_k)}x_{k\gamma_A(\lambda_k)})$$

Define the vectors $y_i$, $1 \leq i \leq k$ by

$$y_1 = (a_{11}x_{11} + a_{12}x_{12} + a_{13}x_{13} + \cdots + a_{1\gamma_A(\lambda_1)}x_{1\gamma_A(\lambda_1)})$$

$$y_2 = (a_{21}x_{21} + a_{22}x_{22} + a_{23}x_{23} + \cdots + a_{2\gamma_A(\lambda_2)}x_{2\gamma_A(\lambda_2)})$$

$$y_3 = (a_{31}x_{31} + a_{32}x_{32} + a_{33}x_{33} + \cdots + a_{3\gamma_A(\lambda_3)}x_{3\gamma_A(\lambda_3)})$$

$$\vdots$$

$$y_k = (a_{k1}x_{k1} + a_{k2}x_{k2} + a_{k3}x_{k3} + \cdots + a_{k\gamma_A(\lambda_k)}x_{k\gamma_A(\lambda_k)})$$

Then the relation of linear dependence becomes

$$0 = y_1 + y_2 + y_3 + \cdots + y_k$$

Since the eigenspace $\mathcal{E}_A(\lambda_i)$ is closed under vector addition and scalar multiplication, $y_i \in \mathcal{E}_A(\lambda_i)$, $1 \leq i \leq k$. Thus, for each $i$, the vector $y_i$ is an eigenvector of $A$ for $\lambda_i$, or
is the zero vector. Recall that sets of eigenvectors whose eigenvalues are distinct form a linearly independent set by Theorem EDELI. Should any (or some) \( y_i \) be nonzero, the previous equation would provide a nontrivial relation of linear dependence on a set of eigenvectors with distinct eigenvalues, contradicting Theorem EDELI. Thus \( y_i = 0 \), 1 \( \leq i \leq k \).

Each of the \( k \) equations, \( y_i = 0 \) is a relation of linear dependence on the corresponding set \( S_i \), a set of basis vectors for the eigenspace \( \mathcal{E}_A (\lambda_i) \), which is therefore linearly independent. From these relations of linear dependence on linearly independent sets we conclude that the scalars are all zero, more precisely, \( a_{ij} = 0 \), 1 \( \leq j \leq \gamma_A (\lambda_i) \) for 1 \( \leq i \leq k \). This establishes that our original relation of linear dependence on \( S \) has only the trivial relation of linear dependence, and hence \( S \) is a linearly independent set.

We have determined that \( S \) is a set of \( n \) linearly independent eigenvectors for \( A \), and so by Theorem DC is diagonalizable.

\[ (\Rightarrow) \text{ Now we assume that } A \text{ is diagonalizable. Aiming for a contradiction (Technique CD), suppose that there is at least one eigenvalue, say } \lambda_i, \text{ such that } \gamma_A (\lambda_i) \neq \alpha_A (\lambda_i). \text{ By Theorem ME we must have } \gamma_A (\lambda_i) < \alpha_A (\lambda_i), \text{ and } \gamma_A (\lambda_i) \leq \alpha_A (\lambda_i) \text{ for } 1 \leq i \leq k, i \neq t. \]

Since \( A \) is diagonalizable, Theorem DC guarantees a set of \( n \) linearly independent vectors, all of which are eigenvectors of \( A \). Let \( n_i \) denote the number of eigenvectors in \( S \) that are eigenvectors for \( \lambda_i \), and recall that a vector cannot be an eigenvector for two different eigenvalues (Exercise EE.T20). \( S \) is a linearly independent set, so the the subset \( S_i \) containing the \( n_i \) eigenvectors for \( \lambda_i \) must also be linearly independent. Because the eigenspace \( \mathcal{E}_A (\lambda_i) \) has dimension \( \gamma_A (\lambda_i) \) and \( S_i \) is a linearly independent subset in \( \mathcal{E}_A (\lambda_i) \), \( n_i \leq \gamma_A (\lambda_i) \), 1 \( \leq i \leq k \). Now,

\[
\begin{align*}
n &= n_1 + n_2 + n_3 + \cdots + n_t + \cdots + n_k & \text{Size of } S \\
&\leq \gamma_A (\lambda_1) + \gamma_A (\lambda_2) + \gamma_A (\lambda_3) + \cdots + \gamma_A (\lambda_t) + \cdots + \gamma_A (\lambda_k) & S_i \text{ linearly independent} \\
&< \alpha_A (\lambda_1) + \alpha_A (\lambda_2) + \alpha_A (\lambda_3) + \cdots + \alpha_A (\lambda_t) + \cdots + \alpha_A (\lambda_k) & \text{Assumption about } \lambda_i \\
&= n & \text{Theorem NEM}
\end{align*}
\]

This is a contradiction (we can’t have \( n < n! \)) and so our assumption that some eigenspace had less than full dimension was false.

Example SEE, Example CAEHW, Example ESMS3, Example ESMS4, Example DEMS5, Archetype B, Archetype F, Archetype K and Archetype L are all examples of matrices that are diagonalizable and that illustrate Theorem DMFE. While we have provided many examples of matrices that are diagonalizable, especially among the archetypes, there are many matrices that are not diagonalizable. Here’s one now.

Example NDMS4

A non-diagonalizable matrix of size 4

In Example EMMS3 the matrix

\[
B = \begin{bmatrix}
-2 & 1 & -2 & -4 \\
12 & 1 & 4 & 9 \\
6 & 5 & -2 & -4 \\
3 & -4 & 5 & 10
\end{bmatrix}
\]

was determined to have characteristic polynomial

\[ p_B (x) = (x - 1)(x - 2)^3 \]
and an eigenspace for $\lambda = 2$ of

$$\mathcal{E}_B (2) = \left\langle \begin{bmatrix} \frac{-1}{2} \\ \frac{-1}{2} \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$$

So the geometric multiplicity of $\lambda = 2$ is $\gamma_B (2) = 1$, while the algebraic multiplicity is $\alpha_B (2) = 3$. By Theorem DMFE [490], the matrix $B$ is not diagonalizable.

**Archetype A** [721] is the lone archetype with a square matrix that is not diagonalizable, as the algebraic and geometric multiplicities of the eigenvalue $\lambda = 0$ differ. **Example HMEM5** [454] is another example of a matrix that cannot be diagonalized due to the difference between the geometric and algebraic multiplicities of $\lambda = 2$, as is **Example CEMS6** [455] which has two complex eigenvalues, each with differing multiplicities. Likewise, **Example EMMS4** [453] has an eigenvalue with different algebraic and geometric multiplicities and so cannot be diagonalized.

**Theorem DED**

**Distinct Eigenvalues implies Diagonalizable**

Suppose $A$ is a square matrix of size $n$ with $n$ distinct eigenvalues. Then $A$ is diagonalizable.

**Proof** Let $\lambda_1$, $\lambda_2$, $\lambda_3$, $\ldots$, $\lambda_n$ denote the $n$ distinct eigenvalues of $A$. Then by **Theorem NEM** [473] we have $n = \sum_{i=1}^{n} \alpha_A (\lambda_i)$, which implies that $\alpha_A (\lambda_i) = 1$, $1 \leq i \leq n$. From **Theorem ME** [474] it follows that $\gamma_A (\lambda_i) = 1$, $1 \leq i \leq n$. So $\gamma_A (\lambda_i) = \alpha_A (\lambda_i)$, $1 \leq i \leq n$ and **Theorem DMFE** [490] says $A$ is diagonalizable.

**Example DEHD**

**Distinct eigenvalues, hence diagonalizable**

In **Example DEMS5** [457] the matrix

$$H = \begin{bmatrix} 15 & 18 & -8 & 6 & -5 \\ 5 & 3 & 1 & -1 & -3 \\ 0 & -4 & 5 & -4 & -2 \\ -43 & -46 & 17 & -14 & 15 \\ 26 & 30 & -12 & 8 & -10 \end{bmatrix}$$

has characteristic polynomial

$$p_H (x) = x(x - 2)(x - 1)(x + 1)(x + 3)$$

and so is a $5 \times 5$ matrix with 5 distinct eigenvalues. By **Theorem DED** [492] we know $H$ must be diagonalizable. But just for practice, we exhibit the diagonalization itself. The matrix $S$ contains eigenvectors of $H$ as columns, one from each eigenspace, guaranteeing linear independent columns and thus the nonsingularity of $S$. The diagonal matrix has the eigenvalues of $H$ in the same order that their respective eigenvectors appear as the columns of $S$. Notice that we are using the versions of the eigenvectors from **Example DEMS5** [457] that have integer entries.

$$S^{-1}HS$$
Archetype B \[726\] is another example of a matrix that has as many distinct eigenvalues as its size, and is hence diagonalizable by Theorem DED \[492\].

Powers of a diagonal matrix are easy to compute, and when a matrix is diagonalizable, it is almost as easy. We could state a theorem here perhaps, but we will settle instead for an example that makes the point just as well.

**Example HPDM**

**High power of a diagonalizable matrix**

Suppose that

\[
A = \begin{bmatrix}
-19 & 0 & 6 & 13 \\
-33 & -1 & -9 & -21 \\
21 & -4 & 12 & 21 \\
-36 & 2 & -14 & -28
\end{bmatrix}
\]

and we wish to compute \(A^{20}\). Normally this would require 19 matrix multiplications, but since \(A\) is diagonalizable, we can simplify the computations substantially. First, we diagonalize \(A\). With

\[
S = \begin{bmatrix}
-1 & 2 & -1 \\
-3 & 3 & 3 \\
1 & 1 & 3 \\
-2 & 1 & -4
\end{bmatrix}
\]

we find

\[
D = S^{-1}AS = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix}
\]
Now we find an alternate expression for $A^{20}$,

$$
A^{20} = AAA \ldots A = I_n AI_n AI_n \ldots I_n AI_n
$$

$$
= (SS^{-1}) A (SS^{-1}) A (SS^{-1}) A (SS^{-1}) \ldots (SS^{-1}) A (SS^{-1})
$$

$$
= S (S^{-1}AS) (S^{-1}AS) (S^{-1}AS) \ldots (S^{-1}AS) S^{-1}
$$

$$
= SDDD \ldots DS
$$

and since $D$ is a diagonal matrix, powers are much easier to compute,

$$
S = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
= S
$$

and since we effectively replaced the twentieth power of $A$ by the twentieth power of $D$, and how a high power of a diagonal matrix is just a collection of powers of scalars on the diagonal. The price we pay for this simplification is the need to diagonalize the matrix (by computing eigenvalues and eigenvectors) and finding the inverse of the matrix of eigenvectors. And we still need to do two matrix products. But the higher the power, the greater the savings.

Subsection OD
Orthonormal Diagonalization

Every Hermitian matrix (Definition HM [255]) is diagonalizable (Definition DZM [487]), and the similarity transformation that accomplishes the diagonalization is a unitary matrix (Definition OM [??]). This means that for every Hermitian matrix of size $n$ there is a basis of $\mathbb{C}^n$ that is composed entirely of eigenvectors for the matrix and also forms an orthonormal set (Definition ONS [193]). Notice that for matrices with only real entries, we only need the hypothesis that the matrix is symmetric (Definition SYM [201]) to
reach this conclusion (Example ESMS4 [453]). Can you imagine a prettier basis for use with a matrix? I can’t. Eventually we’ll include the precise statement of this result with a proof.

**Subsection READ**

**Reading Questions**

1. What is an equivalence relation?

2. State a condition that is equivalent to a matrix being diagonalizable, but is not the definition.

3. Find a diagonal matrix similar to

\[
A = \begin{bmatrix} -5 & 8 \\ -4 & 7 \end{bmatrix}
\]
C20 Consider the matrix $A$ below. First, show that $A$ is diagonalizable by computing the geometric multiplicities of the eigenvalues and quoting the relevant theorem. Second, find a diagonal matrix $D$ and a nonsingular matrix $S$ so that $S^{-1}AS = D$. (See Exercise EE.C20 for some of the necessary computations.)

$$A = \begin{bmatrix} 18 & -15 & 33 & -15 \\ -4 & 8 & -6 & 6 \\ -9 & 9 & -16 & 9 \\ 5 & -6 & 9 & -4 \end{bmatrix}$$

Contributed by Robert Beezer Solution 499

C21 Determine if the matrix $A$ below is diagonalizable. If the matrix is diagonalizable, then find a diagonal matrix $D$ that is similar to $A$, and provide the invertible matrix $S$ that performs the similarity transformation. You should use your calculator to find the eigenvalues of the matrix, but try only using the row-reducing function of your calculator to assist with finding eigenvectors.

$$A = \begin{bmatrix} 1 & 9 & 9 & 24 \\ -3 & -27 & -29 & -68 \\ 1 & 11 & 13 & 26 \\ 1 & 7 & 7 & 18 \end{bmatrix}$$

Contributed by Robert Beezer Solution 499

C22 Consider the matrix $A$ below. Find the eigenvalues of $A$ using a calculator and use these to construct the characteristic polynomial of $A$, $p_A(x)$. State the algebraic multiplicity of each eigenvalue. Find all of the eigenspaces for $A$ by computing expressions for null spaces, only using your calculator to row-reduce matrices. State the geometric multiplicity of each eigenvalue. Is $A$ diagonalizable? If not, explain why. If so, find a diagonal matrix $D$ that is similar to $A$.

$$A = \begin{bmatrix} 19 & 25 & 30 & 5 \\ -23 & -30 & -35 & -5 \\ 7 & 9 & 10 & 1 \\ -3 & -4 & -5 & -1 \end{bmatrix}$$

Contributed by Robert Beezer Solution 500

T15 Suppose that $A$ and $B$ are similar matrices. Prove that $A^3$ and $B^3$ are similar matrices. Generalize.

Contributed by Robert Beezer Solution 501

T16 Suppose that $A$ and $B$ are similar matrices, with $A$ nonsingular. Prove that $B$ is nonsingular, and that $A^{-1}$ is similar to $B^{-1}$.

Contributed by Robert Beezer
Suppose that $B$ is a nonsingular matrix. Prove that $AB$ is similar to $BA$.

Contributed by Robert Beezer  Solution
Using a calculator, we find that \( A \) has three distinct eigenvalues, \( \lambda = 3, 2, -1 \), with \( \lambda = 2 \) having algebraic multiplicity two, \( \alpha_A(2) = 2 \). The eigenvalues \( \lambda = 3, -1 \) have algebraic multiplicity one, and so by Theorem ME we can conclude that their geometric multiplicities are one as well. Together with the computation of the geometric multiplicity of \( \lambda = 2 \) from Exercise EE.C20, we know

\[
\gamma_A(3) = \alpha_A(3) = 1 \quad \gamma_A(2) = \alpha_A(2) = 2 \quad \gamma_A(-1) = \alpha_A(-1) = 1
\]

This satisfies the hypotheses of Theorem DMFE, and so we can conclude that \( A \) is diagonalizable.

A calculator will give us four eigenvectors of \( A \), the two for \( \lambda = 2 \) being linearly independent presumably. Or, by hand, we could find basis vectors for the three eigenspaces. For \( \lambda = 3, -1 \) the eigenspaces have dimension one, and so any eigenvector for these eigenvalues will be multiples of the ones we use below. For \( \lambda = 2 \) there are many different bases for the eigenspace, so your answer could vary. Our eigenvectors are the basis vectors we would have obtained if we had actually constructed a basis in Exercise EE.C20 rather than just computing the dimension.

By the construction in the proof of Theorem DC, the required matrix \( S \) has columns that are four linearly independent eigenvectors of \( A \) and the diagonal matrix has the eigenvalues on the diagonal (in the same order as the eigenvectors in \( S \)). Here are the pieces, “doing” the diagonalization,

\[
\begin{bmatrix}
-1 & 0 & -3 & 6 \\
-2 & -1 & -1 & 0 \\
0 & 0 & 1 & -3 \\
1 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
18 & -15 & 33 & -15 \\
-4 & 8 & -6 & 6 \\
-9 & 9 & -16 & 9 \\
5 & -6 & 9 & -4
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & -3 & 6 \\
-2 & -1 & -1 & 0 \\
0 & 0 & 1 & -3 \\
1 & 1 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

A calculator will provide the eigenvalues \( \lambda = 2, 1, 0 \), so we can reconstruct the characteristic polynomial as

\[
p_A(x) = (x - 2)^2(x - 1)x
\]

so the algebraic multiplicities of the eigenvalues are

\[
\alpha_A(2) = 2 \quad \alpha_A(1) = 1 \quad \alpha_A(0) = 1
\]

Now compute eigenspaces by hand, obtaining null spaces for each of the three eigenvalues by constructing the correct singular matrix (Theorem EMNS),

\[
A - 2I_4 = \begin{bmatrix}
-1 & 9 & 9 & 24 \\
-3 & -29 & -29 & -68 \\
1 & 11 & 11 & 26 \\
1 & 7 & 7 & 16
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 0 & -\frac{2}{5} \\
0 & 1 & 1 & \frac{5}{2} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
\[ E_A(2) = \mathcal{N}(A - 2I_4) = \left\langle \begin{bmatrix} \frac{3}{2} \\ -5 \\ 0 \\ 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 3 \\ -5 \\ 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\rangle \]

\[ A - 1I_4 = \begin{bmatrix} 0 & 9 & 9 & 24 \\ -3 & -28 & -29 & -68 \\ 1 & 11 & 12 & 26 \\ 1 & 7 & 7 & 17 \end{bmatrix} \rightarrow \text{RREF} \]

\[ E_A(1) = \mathcal{N}(A - I_4) = \left\langle \begin{bmatrix} -\frac{5}{3} \\ \frac{13}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 5 \\ -13 \\ -5 \\ 3 \end{bmatrix} \right\rangle \]

\[ A - 0I_4 = \begin{bmatrix} 1 & 9 & 9 & 24 \\ -3 & -27 & -29 & -68 \\ 1 & 11 & 13 & 26 \\ 1 & 7 & 7 & 18 \end{bmatrix} \rightarrow \text{RREF} \]

\[ E_A(0) = \mathcal{N}(A - I_4) = \left\langle \begin{bmatrix} 3 \\ -5 \\ 2 \\ 1 \end{bmatrix} \right\rangle \]

From this we can compute the dimensions of the eigenspaces to obtain the geometric multiplicities,

\[ \gamma_A(2) = 2 \quad \gamma_A(1) = 1 \quad \gamma_A(0) = 1 \]

For each eigenvalue, the algebraic and geometric multiplicities are equal and so by Theorem DMFE, we now know that \( A \) is diagonalizable. The construction in Theorem DC suggests we form a matrix whose columns are eigenvectors of \( A \)

\[ S = \begin{bmatrix} 3 & 0 & 5 & 3 \\ -5 & -1 & -13 & -5 \\ 0 & 1 & 5 & 2 \\ 2 & 0 & 3 & 1 \end{bmatrix} \]

Since \( \det(S) = -1 \neq 0 \), we know that \( S \) is nonsingular (Theorem SMZD), so the columns of \( S \) are a set of 4 linearly independent eigenvectors of \( A \). By the proof of Theorem SMZD we know

\[ S^{-1}AS = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

a diagonal matrix with the eigenvalues of \( A \) along the diagonal, in the same order as the associated eigenvectors appear as columns of \( S \).
as an eigenvalue of algebraic multiplicity 2 as well. Since eigenvalues are roots of the characteristic polynomial (Theorem EMRCP) we have the factored version

\[ p_A(x) = (x - 0)^2 (x - (-1))^2 = x^2 (x^2 + 2x + 1) = x^4 + 2x^3 + x^2 \]

The eigenspaces are then

\[ \lambda = 0 \]

\[ A - (0)I_4 = \begin{bmatrix} 19 & 25 & 30 & 5 \\ -23 & -30 & -35 & -5 \\ 7 & 9 & 10 & 1 \\ -3 & -4 & -5 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -5 & -5 \\ 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \mathcal{E}_A(0) = \mathcal{N}(C - (0)I_4) = \left\{ \begin{bmatrix} 5 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\} \]

\[ \lambda = -1 \]

\[ A - (-1)I_4 = \begin{bmatrix} 20 & 25 & 30 & 5 \\ -23 & -29 & -35 & -5 \\ 7 & 9 & 11 & 1 \\ -3 & -4 & -5 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \mathcal{E}_A(-1) = \mathcal{N}(C - (-1)I_4) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\} \]

Each eigenspace above is described by a spanning set obtained through an application of Theorem BNS and so is a basis for the eigenspace. In each case the dimension, and therefore the geometric multiplicity, is 2.

For each of the two eigenvalues, the algebraic and geometric multiplicities are equal. Theorem DMFE says that in this situation the matrix is diagonalizable. We know from Theorem DC that when we diagonalize \( A \) the diagonal matrix will have the eigenvalues of \( A \) on the diagonal (in some order). So we can claim that

\[ D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \]

T15 Contributed by Robert Beezer Statement

By Definition SIM we know that there is a nonsingular matrix \( S \) so that \( A = S^{-1}BS \). Then

\[ A^3 = (S^{-1}BS)^3 \\
= (S^{-1}BS)(S^{-1}BS)(S^{-1}BS) \\
= S^{-1}B(SS^{-1})B(SS^{-1})BS \quad \text{Theorem MMA} \]
\[ S^{-1}B(I_3)B(I_3)BS \]
\[ = S^{-1}BBBS \]
\[ = S^{-1}B^3S \]

This equation says that \( A^3 \) is similar to \( B^3 \) (via the matrix \( S \)).

More generally, if \( A \) is similar to \( B \), and \( m \) is a non-negative integer, then \( A^m \) is similar to \( B^m \). This can be proved using induction (Technique I 713).

\[ B^{-1}(BA)B = (B^{-1}B)(AB) \]
\[ = I_nAB \]
\[ = AB \]

Contributed by Robert Beezer Statement 498

The nonsingular (invertible) matrix \( B \) will provide the desired similarity transformation,
Chapter LT
Linear Transformations

In the next linear algebra course you take, the first lecture might be a reminder about what a vector space is (Definition VS [309]), their ten properties, basic theorems and then some examples. The second lecture would likely be all about linear transformations. While it may seem we have waited a long time to present what must be a central topic, in truth we have already been working with linear transformations for some time.

Functions are important objects in the study of calculus, but have been absent from this course until now (well, not really, it just seems that way). In your study of more advanced mathematics it is nearly impossible to escape the use of functions — they are as fundamental as sets are.

Section LT
Linear Transformations

Here comes a key definition.

Subsection LT
Linear Transformations

Definition LT
Linear Transformation

A linear transformation, \( T : U \mapsto V \), is a function that carries elements of the vector space \( U \) (called the domain) to the vector space \( V \) (called the codomain), and which has two additional properties

1. \( T(u_1 + u_2) = T(u_1) + T(u_2) \) for all \( u_1, u_2 \in U \)
2. \( T(\alpha u) = \alpha T(u) \) for all \( u \in U \) and all \( \alpha \in \mathbb{C} \)

(This definition contains Notation LT.)

The two defining conditions in the definition of a linear transformation should “feel linear,” whatever that means. Conversely, these two conditions could be taken as a exactly
what it means to be linear. As every vector space property derives from vector addition and scalar multiplication, so too, every property of a linear transformation derives from these two defining properties. While these conditions may be reminiscent of how we test subspaces, they really are quite different, so do not confuse the two.

Here are two diagrams that convey the essence of the two defining properties of a linear transformation. In each case, begin in the upper left-hand corner, and follow the arrows around the rectangle to the lower-right hand corner, taking two different routes and doing the indicated operations labeled on the arrows. There are two results there. For a linear transformation these two expressions are always equal.

\[
\begin{align*}
\mathbf{u}_1, \mathbf{u}_2 & \xrightarrow{T} T(\mathbf{u}_1), T(\mathbf{u}_2) \\
+ & \\
\mathbf{u}_1 + \mathbf{u}_2 & \xrightarrow{T} T(\mathbf{u}_1) + T(\mathbf{u}_2), \\
& \\
\mathbf{u} & \xrightarrow{T} T(\mathbf{u}) \\
\alpha & \\
\alpha \mathbf{u} & \xrightarrow{T} \alpha T(\mathbf{u}), \\
& \\
& \xrightarrow{T} \alpha T(\mathbf{u}), \\
\end{align*}
\]

A couple of words about notation. \( T \) is the name of the linear transformation, and should be used when we want to discuss the function as a whole. \( T(\mathbf{u}) \) is how we talk about the output of the function, it is a vector in the vector space \( V \). When we write \( T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) \), the plus sign on the left is the operation of vector addition in the vector space \( U \), since \( \mathbf{x} \) and \( \mathbf{y} \) are elements of \( U \). The plus sign on the right is the operation of vector addition in the vector space \( V \), since \( T(\mathbf{x}) \) and \( T(\mathbf{y}) \) are elements of the vector space \( V \). These two instances of vector addition might be wildly different.

Let’s examine several examples and begin to form a catalog of known linear transformations to work with.

**Example ALT**

**A linear transformation**

Define \( T : \mathbb{C}^3 \rightarrow \mathbb{C}^2 \) by describing the output of the function for a generic input with the formula

\[
T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 + x_3 \\ -4x_2 \end{bmatrix}
\]

and check the two defining properties.

\[
T(\mathbf{x} + \mathbf{y}) = T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right) \\
= T \left( \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix} \right) \\
= \begin{bmatrix} 2(x_1 + y_1) + (x_3 + y_3) \\ -4(x_2 + y_2) \end{bmatrix}
\]
\[
\begin{align*}
&= \begin{bmatrix}
2x_1 + x_3 \\
-4x_2 + (-4)y_2
\end{bmatrix} + \begin{bmatrix}
2y_1 + y_3 \\
-4y_2
\end{bmatrix} \\
&= T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + T \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\
&= T(x) + T(y)
\end{align*}
\]

and

\[
T(\alpha x) = T \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix} \\
= T \begin{bmatrix} 2(\alpha x_1) + (\alpha x_3) \\ -4(\alpha x_2) \end{bmatrix} \\
= \alpha \begin{bmatrix} 2x_1 + x_3 \\ -4x_2 \end{bmatrix} \\
= \alpha T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \alpha T(x)
\]

So by Definition LT 503, \( T \) is a linear transformation.

It can be just as instructive to look at functions that are not linear transformations. Since the defining conditions must be true for all vectors and scalars, it is enough to find just one situation where the properties fail.

**Example NLT**

**Not a linear transformation**

Define \( S: \mathbb{C}^3 \rightarrow \mathbb{C}^3 \) by

\[
S \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 + 2x_2 \\ 0 \\ x_1 + 3x_3 - 2 \end{bmatrix}
\]

This function “looks” linear, but consider

\[
3 S \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 8 \\ 0 \\ 24 \end{bmatrix} = \begin{bmatrix} 24 \\ 0 \\ 24 \end{bmatrix}
\]
while

$$S \left( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) = S \left( \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right) = \begin{bmatrix} 24 \\ 0 \\ 28 \end{bmatrix}$$

So the second required property fails for the choice of \( \alpha = 3 \) and \( x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) and by Definition LT \[503\], \( S \) is not a linear transformation. It is just about as easy to find an example where the first defining property fails (try it!). Notice that it is the “-2” in the third component of the definition of \( S \) that prevents the function from being a linear transformation.

**Example LTPM**

**Linear transformation, polynomials to matrices**

Define a linear transformation \( T: P_3 \mapsto M_{22} \) by

\[
T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a + b & a - 2c \\ d & b - d \end{bmatrix}
\]

We verify the two defining conditions of a linear transformations.

\[
T(x + y) = T((a_1 + b_1x + c_1x^2 + d_1x^3) + (a_2 + b_2x + c_2x^2 + d_2x^3))
= T((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2 + (d_1 + d_2)x^3)
= \begin{bmatrix} (a_1 + a_2) + (b_1 + b_2) & (a_1 + a_2) - 2(c_1 + c_2) \\ d_1 + d_2 & (b_1 + b_2) - (d_1 + d_2) \end{bmatrix}
= \begin{bmatrix} a_1 + b_1 & a_1 - 2c_1 \\ d_1 & b_1 - d_1 \end{bmatrix} + \begin{bmatrix} a_2 + b_2 & a_2 - 2c_2 \\ d_2 & b_2 - d_2 \end{bmatrix}
= T(a_1 + b_1x + c_1x^2 + d_1x^3) + T(a_2 + b_2x + c_2x^2 + d_2x^3)
= T(x) + T(y)
\]

and

\[
T(\alpha x) = T(\alpha(a + bx + cx^2 + dx^3))
= T((\alpha a) + (\alpha b)x + (\alpha c)x^2 + (\alpha d)x^3)
= \begin{bmatrix} \alpha(a + b) & \alpha(a - 2c) \\ \alpha d & \alpha(b - d) \end{bmatrix}
= \alpha \begin{bmatrix} a + b & a - 2c \\ d & b - d \end{bmatrix}
= \alpha T(a + bx + cx^2 + dx^3)
= \alpha T(x)
\]
Example LTPP

Linear transformation, polynomials to polynomials

Define a function $S: P_4 \rightarrow P_5$ by

$$S(p(x)) = (x - 2)p(x)$$

Then

$$S(p(x) + q(x)) = (x - 2)(p(x) + q(x)) = (x - 2)p(x) + (x - 2)q(x) = S(p(x)) + S(q(x))$$

$$S(\alpha p(x)) = (x - 2)(\alpha p(x)) = (x - 2)\alpha p(x) = \alpha(x - 2)p(x) = \alpha S(p(x))$$

So by Definition LT, $S$ is a linear transformation.

Linear transformations have many amazing properties, which we will investigate through the next few sections. However, as a taste of things to come, here is a theorem we can prove now and put to use immediately.

**Theorem LTTZZ**

Linear Transformations Take Zero to Zero

Suppose $T: U \rightarrow V$ is a linear transformation. Then $T(0) = 0$.

**Proof** The two zero vectors in the conclusion of the theorem are different. The first is from $U$ while the second is from $V$. We will subscript the zero vectors in this proof to highlight the distinction. Think about your objects. (This proof is contributed by Mark Shoemaker).

$$T(0_U) = T(00_U) = 0T(0_U) = 0_U$$

So by Definition LT, $S$ is a linear transformation.

Subsection MLT

Matrices and Linear Transformations

If you give me a matrix, then I can quickly build you a linear transformation. Always. First a motivating example and then the theorem.

**Example LTM**

Linear transformation from a matrix
Let
\[ A = \begin{bmatrix} 3 & -1 & 8 & 1 \\ 2 & 0 & 5 & -2 \\ 1 & 1 & 3 & -7 \end{bmatrix} \]
and define a function \( P : \mathbb{C}^4 \rightarrow \mathbb{C}^3 \) by
\[ P(x) = Ax \]

So we are using an old friend, the matrix-vector product (Definition MVP [211]) as a way to convert a vector with 4 components into a vector with 3 components. Applying Definition MVP [211] allows us to write the defining formula for \( P \) in a slightly different form,
\[ P(x) = Ax = \begin{bmatrix} 3 & -1 & 8 & 1 \\ 2 & 0 & 5 & -2 \\ 1 & 1 & 3 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix} \]

So we recognize the action of the function \( P \) as using the components of the vector \((x_1, x_2, x_3, x_4)\) as scalars to form the output of \( P \) as a linear combination of the four columns of the matrix \( A \), which are all members of \( \mathbb{C}^3 \), so the result is a vector in \( \mathbb{C}^3 \). We can rearrange this expression further, using our definitions of operations in \( \mathbb{C}^3 \) (Section VO [87]).
\[ P(x) = Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix} \]

You might recognize this final expression as being similar in style to some previous examples (Example ALT [504]) and some linear transformations defined in the archetypes (Archetype M [775] through Archetype R [790]). But the expression that says the output of this linear transformation is a linear combination of the columns of \( A \) is probably the most powerful way of thinking about examples of this type.

Almost forgot — we should verify that \( P \) is indeed a linear transformation. This is easy with two matrix properties from Section MM [211].
\[ P(x + y) = A(x + y) = Ax + Ay = P(x) + P(y) \]

and
\[ P(\alpha x) = A(\alpha x) \]
Theorem MBLT

Matrices Build Linear Transformations

Suppose that $A$ is an $m \times n$ matrix. Define a function $T: \mathbb{C}^n \rightarrow \mathbb{C}^m$ by $T(x) = Ax$. Then $T$ is a linear transformation.

Proof

$T(x + y) = A(x + y)$

$= Ax + Ay$ \hspace{1cm} \text{Theorem MMDAA 219}

$= T(x) + T(y)$ \hspace{1cm} \text{Definition of $T$}

and

$T(\alpha x) = A(\alpha x)$

$= \alpha (Ax)$ \hspace{1cm} \text{Theorem MMSMM 219}

$= \alpha T(x)$ \hspace{1cm} \text{Definition of $T$}

So by Definition LT 503, $T$ is a linear transformation.

So the multiplication of a vector by a matrix “transforms” the input vector into an output vector, possibly of a different size, by performing a linear combination. And this transformation happens in a “linear” fashion. This “functional” view of the matrix-vector product is the most important shift you can make right now in how you think about linear algebra. Here’s the theorem, whose proof is very nearly an exact copy of the verification in the last example.

Example MFLT

Matrix from a linear transformation

Define the function $R: \mathbb{C}^3 \rightarrow \mathbb{C}^4$ by

$$R\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 + 4x_3 \\ x_1 + x_2 + x_3 \\ -x_1 + 5x_2 - 3x_3 \\ x_2 - 4x_3 \end{bmatrix}.$$
\[
\begin{bmatrix}
2x_1 \\
x_1 \\
-x_1 \\
0
\end{bmatrix} + \begin{bmatrix}
-3x_2 \\
x_2 \\
5x_2 \\
x_2
\end{bmatrix} + \begin{bmatrix}
4x_3 \\
x_3 \\
-3x_3 \\
-4x_3
\end{bmatrix}
\]

Definition CVA [89]

\[
x_1 \begin{bmatrix}
2 \\
1 \\
-1 \\
0
\end{bmatrix} + x_2 \begin{bmatrix}
-3 \\
1 \\
5 \\
1
\end{bmatrix} + x_3 \begin{bmatrix}
4 \\
x_3 \\
-3 \\
-4
\end{bmatrix}
\]

Definition CVSM [89]

So if we define the matrix

\[
B = \begin{bmatrix}
2 & -3 & 4 \\
1 & 1 & 1 \\
-1 & 5 & -3 \\
0 & 1 & -4
\end{bmatrix}
\]

then \(R(x) = Bx\). By Theorem MBLT [509], we can easily recognize \(R\) as a linear transformation since it has the form described in the hypothesis of the theorem.

Example MFLT [509] was not accident. Consider any one of the archetypes where both the domain and codomain are sets of column vectors (Archetype M [775] through Archetype R [790]) and you should be able to mimic the previous example. Here’s the theorem, which is notable since it is our first occasion to use the full power of the defining properties of a linear transformation when our hypothesis includes a linear transformation.

**Theorem MLTCV**

**Matrix of a Linear Transformation, Column Vectors**

Suppose that \(T : \mathbb{C}^n \rightarrow \mathbb{C}^m\) is a linear transformation. Then there is an \(m \times n\) matrix \(A\) such that \(T(x) = Ax\).

**Proof** The conclusion says a certain matrix exists. What better way to prove something exists than to actually build it? So our proof will be constructive, and the procedure that we will use abstractly in the proof can be used concretely in specific examples.

Let \(e_1, e_2, e_3, \ldots, e_n\) be the columns of the identity matrix of size \(n\), \(I_n\) (Definition SUV [234]). Evaluate the linear transformation \(T\) with each of these standard unit vectors as an input, and record the result. In other words, define \(n\) vectors in \(\mathbb{C}^m\), \(A_i\), \(1 \leq i \leq n\) by

\[
A_i = T(e_i)
\]

Then package up these vectors as the columns of a matrix

\[
A = [A_1|A_2|A_3|\ldots|A_n]
\]

Does \(A\) have the desired properties? First, \(A\) is clearly an \(m \times n\) matrix. Then

\[
T(x) = T(I_n x)
\]

\[
= T(e_1|e_2|e_3|\ldots|e_n)x
\]

\[
= T([x_1]e_1 + [x_2]e_2 + [x_3]e_3 + \ldots + [x_n]e_n)
\]

Theorem MMIM [218]

Definition SUV [234]

Definition MVP [211]
\[ T([x]_1 e_1) + T([x]_2 e_2) + T([x]_3 e_3) + \cdots + T([x]_n e_n) \]  
\[ [x]_1 T(e_1) + [x]_2 T(e_2) + [x]_3 T(e_3) + \cdots + [x]_n T(e_n) \]  
\[ = [x]_1 A_1 + [x]_2 A_2 + [x]_3 A_3 + \cdots + [x]_n A_n \]  
\[ = Ax \]

as desired. \[\blacksquare\]

So if we were to restrict our study of linear transformations to those where the domain and codomain are both vector spaces of column vectors (Definition VSCV 87), every matrix leads to a linear transformation of this type (Theorem MBLT 509), while every such linear transformation leads to a matrix (Theorem MLTCV 510). So matrices and linear transformations are fundamentally the same. We call the matrix \( A \) of Theorem MLTCV 510 the **matrix representation** of \( T \).

We have defined linear transformations for more general vector spaces than just \( \mathbb{C}^m \), can we extend this correspondence between linear transformations and matrices to more general linear transformations (more general domains and codomains)? Yes, and this is the main theme of Chapter R 587. Stay tuned. For now, let’s illustrate Theorem MLTCV 510 with an example.

**Example MOLT**

**Matrix of a linear transformation**

Suppose \( S: \mathbb{C}^3 \rightarrow \mathbb{C}^4 \) is defined by

\[
S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - 2x_2 + 5x_3 \\ x_1 + x_2 + x_3 \\ 9x_1 - 2x_2 + 5x_3 \\ 4x_2 \end{bmatrix}
\]

Then

\[
C_1 = S(e_1) = S\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \\ 9 \\ 0 \end{bmatrix}
\]

\[
C_2 = S(e_2) = S\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \\ -2 \\ 4 \end{bmatrix}
\]

\[
C_3 = S(e_3) = S\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 1 \\ 5 \\ 0 \end{bmatrix}
\]

so define

\[
C = [C_1|C_2|C_3] = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 1 & 1 \\ 9 & -2 & 5 \\ 0 & 4 & 0 \end{bmatrix}
\]

and Theorem MLTCV 510 guarantees that \( S(x) = Cx \).
As an illuminating exercise, let \( z = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} \) and compute \( S(z) \) two different ways. First, return to the definition of \( S \) and evaluate \( S(z) \) directly. Then do the matrix-vector product \( Cz \). In both cases you should obtain the vector \( S(z) = \begin{bmatrix} 27 \\ 2 \\ 39 \\ -12 \end{bmatrix} \).

**Subsection LTLC**

**Linear Transformations and Linear Combinations**

It is the interaction between linear transformations and linear combinations that lies at the heart of many of the important theorems of linear algebra. The next theorem distills the essence of this. The proof is not deep, the result is hardly startling, but it will be referenced frequently. We have already passed by one occasion to employ it, in the proof of Theorem MLTCV [510]. Paraphrasing, this theorem says that we can “push” linear transformations “down into” linear combinations, or “pull” linear transformations “up out” of linear combinations. We’ll have opportunities to both push and pull.

**Theorem LTLC**

**Linear Transformations and Linear Combinations**

Suppose that \( T: U \rightarrow V \) is a linear transformation, \( u_1, u_2, u_3, \ldots, u_t \) are vectors from \( U \) and \( a_1, a_2, a_3, \ldots, a_t \) are scalars from \( \mathbb{C} \). Then

\[
T(a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_tu_t) = a_1T(u_1) + a_2T(u_2) + a_3T(u_3) + \cdots + a_tT(u_t)
\]

**Proof**

\[
T(a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_tu_t) \\
= T(a_1u_1) + T(a_2u_2) + T(a_3u_3) + \cdots + T(a_tu_t) \\
= a_1T(u_1) + a_2T(u_2) + a_3T(u_3) + \cdots + a_tT(u_t)
\]

Our next theorem says, informally, that it is enough to know how a linear transformation behaves for inputs from a basis of the domain, and all other outputs are described by a linear combination of these values. Again, the theorem and its proof are not remarkable, but the insight that goes along with it is fundamental.

**Theorem LTDB**

**Linear Transformation Defined on a Basis**

Suppose that \( T: U \rightarrow V \) is a linear transformation, \( U = \{u_1, u_2, u_3, \ldots, u_n\} \) is a basis for \( U \) and \( w \) is a vector from \( U \). Let \( a_1, a_2, a_3, \ldots, a_n \) be the scalars from \( \mathbb{C} \) such that

\[
w = a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_nu_n
\]
Then
\[ T(w) = a_1T(u_1) + a_2T(u_2) + a_3T(u_3) + \cdots + a_nT(u_n) \]

\[ \square \]

**Proof** For any \( w \in U \), [Theorem VRRB][355] says there are (unique) scalars such that \( w \) is a linear combination of the basis vectors in \( B \). The result then follows from a straightforward application of [Theorem LTLC][512] to the linear combination. ■

**Example LTDB1**
Linear transformation defined on a basis

Suppose you are told that \( T: \mathbb{C}^3 \mapsto \mathbb{C}^2 \) is a linear transformation and given the three values,

\[
T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad T \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}
\]

Because

\[ B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \]

is a basis for \( \mathbb{C}^3 \) (Theorem SUVB[363]), [Theorem LTDB][512] says we can compute any output of \( T \) with just this information. For example, consider,

\[
w = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = (2) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

so

\[
T(w) = (2) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-3) \begin{bmatrix} -1 \\ 4 \end{bmatrix} + (1) \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 13 \\ -10 \end{bmatrix}
\]

Doing it again,

\[
w = \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix} = (5) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

so

\[
T(w) = (5) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (2) \begin{bmatrix} -1 \\ 4 \end{bmatrix} + (-3) \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 \\ 13 \end{bmatrix}
\]

Any other value of \( T \) could be computed in a similar manner. So rather than being given a formula for the outputs of \( T \), the requirement that \( T \) behave as a linear transformation, along with its values on a handful of vectors (the basis), are just as sufficient as a formula for computing any value of the function. You might notice some parallels between this example and Example MOLT[511] or Theorem MLTCV[510]. ✠

**Example LTDB2**
Linear transformation defined on a basis

Suppose you are told that \( R: \mathbb{C}^3 \mapsto \mathbb{C}^2 \) is a linear transformation and given the three values,

\[
R \left( \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \quad R \left( \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 4 \end{bmatrix} \quad R \left( \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}
\]
You can check that
\[
D = \begin{Bmatrix}
\begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix},
\begin{bmatrix}
-1 \\
5 \\
1
\end{bmatrix},
\begin{bmatrix}
3 \\
1 \\
4
\end{bmatrix}
\end{Bmatrix}
\]
is a basis for \( \mathbb{C}^3 \) (make the vectors the columns of a square matrix and check that the matrix is nonsingular, Theorem CNMB \[369\]). By Theorem LTDB \[512\] we can compute any output of \( R \) with just this information. However, we have to work just a bit harder to take an input vector and express it as a linear combination of the vectors in \( D \). For example, consider,
\[
y = \begin{bmatrix}
8 \\
-3 \\
5
\end{bmatrix}
\]
Then we must first write \( y \) as a linear combination of the vectors in \( D \) and solve for the unknown scalars, to arrive at
\[
y = \begin{bmatrix}
8 \\
-3 \\
5
\end{bmatrix} = (3) \begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix} + (-2) \begin{bmatrix}
-1 \\
5 \\
1
\end{bmatrix} + (1) \begin{bmatrix}
3 \\
1 \\
4
\end{bmatrix}
\]
Then Theorem LTDB \[512\] gives us
\[
R(y) = (3) \begin{bmatrix}
5 \\
-1
\end{bmatrix} + (-2) \begin{bmatrix}
0 \\
4
\end{bmatrix} + (1) \begin{bmatrix}
2 \\
3
\end{bmatrix} = \begin{bmatrix}
17 \\
-8
\end{bmatrix}
\]
Any other value of \( R \) could be computed in a similar manner. 

Here is a third example of a linear transformation defined by its action on a basis, only with more abstract vector spaces involved.

**Example LTDB3**

**Linear transformation defined on a basis**

The set \( W = \{ p(x) \in P_3 \mid p(1) = 0, p(3) = 0 \} \subseteq P_3 \) is a subspace of the vector space of polynomials \( P_3 \). This subspace has \( C = \{ 3 - 4x + x^2, 12 - 13x + x^3 \} \) as a basis (check this!). Suppose we define a linear transformation \( S: P_3 \rightarrow M_{22} \) by the values
\[
S(3 - 4x + x^2) = \begin{bmatrix}
1 & -3 \\
2 & 0
\end{bmatrix} \quad S(12 - 13x + x^3) = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]
To illustrate a sample computation of \( S \), consider \( q(x) = 9 - 6x - 5x^2 + 2x^3 \). Verify that \( q(x) \) is an element of \( W \) (does it have roots at \( x = 1 \) and \( x = 3 \?) , then find the scalars needed to write it as a linear combination of the basis vectors in \( C \). Because
\[
q(x) = 9 - 6x - 5x^2 + 2x^3 = (-5)(3 - 4x + x^2) + (2)(12 - 13x + x^3)
\]
Theorem LTDB \[512\] gives us
\[
S(q) = (-5) \begin{bmatrix}
1 & -3 \\
2 & 0
\end{bmatrix} + (2) \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} = \begin{bmatrix}
-5 & 17 \\
-8 & 0
\end{bmatrix}
\]
And all the other outputs of \( S \) could be computed in the same manner. Every output of \( S \) will have a zero in the second row, second column. Can you see why this is so?
The definition of a function requires that for each input in the domain there is *exactly* one output in the codomain. However, the correspondence does not have to behave the other way around. A member of the codomain might have many inputs from the domain that create it, or it may have none at all. To formalize our discussion of this aspect of linear transformations, we define the pre-image.

**Definition PI Pre-Image**

Suppose that \(T: U \mapsto V\) is a linear transformation. For each \(v\), define the pre-image of \(v\) to be the subset of \(U\) given by

\[
T^{-1}(v) = \{ u \in U \mid T(u) = v \}
\]

In other words, \(T^{-1}(v)\) is the set of all those vectors in the domain \(U\) that get “sent” to the vector \(v\).

TODO: All preimages form a partition of \(U\), an equivalence relation is about. Maybe to exercises.

**Example SPIAS**

**Sample pre-images, Archetype S**

Archetype S 793 is the linear transformation defined by

\[
T: \mathbb{C}^3 \mapsto M_{22}, \quad T \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a - b & 2a + 2b + c \\ 3a + b + c & -2a - 6b - 2c \end{bmatrix}
\]

We could compute a pre-image for every element of the codomain \(M_{22}\). However, even in a free textbook, we do not have the room to do that, so we will compute just two.

Choose

\[
v = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix} \in M_{22}
\]

for no particular reason. What is \(T^{-1}(v)\)? Suppose \(u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in T^{-1}(v)\). That \(T(u) = v\) becomes

\[
\begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} = v = T(u) = T \left( \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \right) = \begin{bmatrix} u_1 - u_2 & 2u_1 + 2u_2 + u_3 \\ 3u_1 + u_2 + u_3 & -2u_1 - 6u_2 - 2u_3 \end{bmatrix}
\]

Using matrix equality (Definition ME 197), we arrive at a system of four equations in the three unknowns \(u_1, u_2, u_3\) with an augmented matrix that we can row-reduce in the hunt for solutions,
We recognize this system as having infinitely many solutions described by the single free variable $u_3$. Eventually obtaining the vector form of the solutions (Theorem VF-SLS \[107\]), we can describe the preimage precisely as,

$$T^{-1}(v) = \{ u \in \mathbb{C}^3 \mid T(u) = v \}$$

$$= \left\{ \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \ \begin{array}{l} u_1 = \frac{5}{4} - \frac{1}{4} u_3, u_2 = -\frac{3}{4} - \frac{1}{4} u_3 \end{array} \right\}$$

$$= \left\{ \begin{bmatrix} \frac{5}{4} - \frac{1}{4} u_3 \\ -\frac{3}{4} - \frac{1}{4} u_3 \\ u_3 \end{bmatrix} \mid u_3 \in \mathbb{C}^3 \right\}$$

$$= \left\{ \begin{bmatrix} \frac{5}{4} \\ -\frac{3}{4} \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix} \mid u_3 \in \mathbb{C}^3 \right\}$$

This last line is merely a suggestive way of describing the set on the previous line. You might create three or four vectors in the preimage, and evaluate $T$ with each. Was the result what you expected? For a hint of things to come, you might try evaluating $T$ with just the lone vector in the spanning set above. What was the result? Now take a look back at Theorem PSPHS \[113\]. Hmmmm.

OK, let’s compute another preimage, but with a different outcome this time. Choose

$$v = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \in M_{22}$$

What is $T^{-1}(v)$? Suppose $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in T^{-1}(v)$. That $T(u) = v$ becomes

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = T(u) = T \left( \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \right) = \begin{bmatrix} u_1 - u_2 & 2u_1 + 2u_2 + u_3 \\ 3u_1 + u_2 + u_3 & -2u_1 - 6u_2 - 2u_3 \end{bmatrix}$$

Using matrix equality (Definition ME \[197\]), we arrive at a system of four equations in the three unknowns $u_1, u_2, u_3$ with an augmented matrix that we can row-reduce in the hunt for solutions,

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 2 & 1 & 1 \\ 3 & 1 & 1 & 2 \\ -2 & -6 & -2 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem RCLS \[54\] we recognize this system as inconsistent. So no vector $u$ is a member of $T^{-1}(v)$ and so

$$T^{-1}(v) = \emptyset$$
Subsection LT.NLTFO New Linear Transformations From Old 517

The preimage is just a set, it is almost never a subspace of $U$ (you might think about just when $T^{-1}(v)$ is a subspace, see Exercise ILT.T10 [541]). We will describe its properties going forward, and it will be central to the main ideas of this chapter.

Subsection NLTFO New Linear Transformations From Old

We can combine linear transformations in natural ways to create new linear transformations. So we will define these combinations and then prove that the results really are still linear transformations. First the sum of two linear transformations.

**Definition LTA Linear Transformation Addition**

Suppose that $T: U \rightarrow V$ and $S: U \rightarrow V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T + S: U \rightarrow V$ whose outputs are defined by

$$(T + S)(u) = T(u) + S(u)$$

Notice that the first plus sign in the definition is the operation being defined, while the second one is the vector addition in $V$. (Vector addition in $U$ will appear just now in the proof that $T + S$ is a linear transformation.) **Definition LTA** [517] only provides a function. It would be nice to know that when the constituents $(T, S)$ are linear transformations, then so too is $T + S$.

**Theorem SLTLT Sum of Linear Transformations is a Linear Transformation**

Suppose that $T: U \rightarrow V$ and $S: U \rightarrow V$ are two linear transformations with the same domain and codomain. Then $T + S: U \rightarrow V$ is a linear transformation.

**Proof** We simply check the defining properties of a linear transformation (**Definition LT** [503]). This is a good place to consistently ask yourself which objects are being combined with which operations.

\[
(T + S)(x + y) = T(x + y) + S(x + y) \quad \text{Definition LTA [517]}
\]

\[
= T(x) + T(y) + S(x) + S(y) \quad \text{Definition LT [503]}
\]

\[
= T(x) + S(x) + T(y) + S(y) \quad \text{Property C [310] in V}
\]

\[
= (T + S)(x) + (T + S)(y) \quad \text{Definition LTA [517]}
\]

and

\[
(T + S)(\alpha x) = T(\alpha x) + S(\alpha x) \quad \text{Definition LTA [517]}
\]

\[
= \alpha T(x) + \alpha S(x) \quad \text{Definition LT [503]}
\]

\[
= \alpha (T(x) + S(x)) \quad \text{Property DVA [310] in V}
\]

\[
= \alpha (T + S)(x) \quad \text{Definition LTA [517]}
\]
Example STLT
Sum of two linear transformations
Suppose that \( T : \mathbb{C}^2 \rightarrow \mathbb{C}^3 \) and \( S : \mathbb{C}^2 \rightarrow \mathbb{C}^3 \) are defined by
\[
T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 - 4x_2 \\ 5x_1 + 2x_2 \end{bmatrix} \quad \quad S \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 4x_1 - x_2 \\ x_1 + 3x_2 \\ -7x_1 + 5x_2 \end{bmatrix}
\]
Then by Definition LTA [517], we have
\[
(T+S) \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) + S \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 - 4x_2 \\ 5x_1 + 2x_2 \end{bmatrix} + \begin{bmatrix} 4x_1 - x_2 \\ x_1 + 3x_2 \\ -7x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 + x_2 \\ 4x_1 - x_2 \\ -2x_1 + 7x_2 \end{bmatrix}
\]
and by Theorem SLTLT [517] we know \( T + S \) is also a linear transformation from \( \mathbb{C}^2 \) to \( \mathbb{C}^3 \).

Definition LTSM
Linear Transformation Scalar Multiplication
Suppose that \( T : U \rightarrow V \) is a linear transformation and \( \alpha \in \mathbb{C} \). Then the scalar multiple is the function \( \alpha T : U \rightarrow V \) whose outputs are defined by
\[
(\alpha T)(u) = \alpha T(u)
\]

Given that \( T \) is a linear transformation, it would be nice to know that \( \alpha T \) is also a linear transformation.

Theorem MLTLT
Multiple of a Linear Transformation is a Linear Transformation
Suppose that \( T : U \rightarrow V \) is a linear transformation and \( \alpha \in \mathbb{C} \). Then \( (\alpha T) : U \rightarrow V \) is a linear transformation.

Proof We simply check the defining properties of a linear transformation (Definition LT [503]). This is another good place to consistently ask yourself which objects are being combined with which operations.

\[
(\alpha T)(x + y) = \alpha (T(x + y)) \quad \quad \text{Definition LTSM} \ [518]
\]
\[
= \alpha (T(x) + T(y)) \quad \quad \text{Definition LT} \ [503]
\]
\[
= \alpha T(x) + \alpha T(y) \quad \quad \text{Property DVA} \ [310] \text{ in } V
\]
\[
= (\alpha T)(x) + (\alpha T)(y) \quad \quad \text{Definition LTSM} \ [518]
\]

and

\[
(\alpha T)(\beta x) = \alpha T(\beta x) \quad \quad \text{Definition LTSM} \ [518]
\]
\[
= \alpha (\beta T(x)) \quad \quad \text{Definition LT} \ [503]
\]
\[
= (\alpha \beta) T(x) \quad \quad \text{Property SMA} \ [310] \text{ in } V
\]
\[
= (\beta \alpha) T(x) \quad \quad \text{Commutativity in } \mathbb{C}
\]
\[ = \beta (\alpha T (x)) \]

Property SMA 310 in \( V \)

\[ = \beta ((\alpha T) (x)) \]

Definition LTSM 518

\[ \]
Given that \(T\) and \(S\) are linear transformations, it would be nice to know that \(S \circ T\) is also a linear transformation.

**Theorem CLTLT**

**Composition of Linear Transformations is a Linear Transformation**

Suppose that \(T: U \mapsto V\) and \(S: V \mapsto W\) are linear transformations. Then \((S \circ T): U \mapsto W\) is a linear transformation.

**Proof**  We simply check the defining properties of a linear transformation (Definition LT 503).

\[
(S \circ T)(x + y) = S(T(x + y)) \quad \text{Definition LTC 519}
\]
\[
= S(T(x) + T(y)) \quad \text{Definition LT 503 for } T
\]
\[
= S(T(x)) + S(T(y)) \quad \text{Definition LT 503 for } S
\]
\[
= (S \circ T)(x) + (S \circ T)(y) \quad \text{Definition LTC 519}
\]

and

\[
(S \circ T)(\alpha x) = S(T(\alpha x)) \quad \text{Definition LTC 519}
\]
\[
= S(\alpha T(x)) \quad \text{Definition LT 503 for } T
\]
\[
= \alpha S(T(x)) \quad \text{Definition LT 503 for } S
\]
\[
= \alpha(S \circ T)(x) \quad \text{Definition LTC 519}
\]

\[\blacksquare\]

**Example CTLTLT**

**Composition of two linear transformations**

Suppose that \(T: \mathbb{C}^2 \mapsto \mathbb{C}^4\) and \(S: \mathbb{C}^4 \mapsto \mathbb{C}^3\) are defined by

\[
T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 - 4x_2 \\ 5x_1 + 2x_2 \\ 6x_1 - 3x_2 \end{bmatrix}
\]
\[
S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 + x_3 - x_4 \\ 5x_1 - 3x_2 + 8x_3 - 2x_4 \\ -4x_1 + 3x_2 - 4x_3 + 5x_4 \end{bmatrix}
\]

Then by Definition LTC 519

\[
(S \circ T)\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = S\left(T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)\right)
\]
\[
= S\left(\begin{bmatrix} x_1 + 2x_2 \\ 3x_1 - 4x_2 \\ 5x_1 + 2x_2 \\ 6x_1 - 3x_2 \end{bmatrix}\right)
\]
\[
= \begin{bmatrix} 2(x_1 + 2x_2) - (3x_1 - 4x_2) + (5x_1 + 2x_2) - (6x_1 - 3x_2) \\ 5(x_1 + 2x_2) - 3(3x_1 - 4x_2) + 8(5x_1 + 2x_2) - 2(6x_1 - 3x_2) \\ -4(x_1 + 2x_2) + 3(3x_1 - 4x_2) - 4(5x_1 + 2x_2) + 5(6x_1 - 3x_2) \end{bmatrix}
\]
\[
= \begin{bmatrix} -2x_1 + 13x_2 \\ 24x_1 + 44x_2 \\ 15x_1 - 43x_2 \end{bmatrix}
\]
Subsection LT.READ  
Reading Questions  

and by Theorem CLTLT 520 \( S \circ T \) is a linear transformation from \( \mathbb{C}^2 \) to \( \mathbb{C}^3 \).

Here is an interesting exercise that will presage an important result later. In Example STLT 518 compute (via Theorem MLTCV 510) the matrix of \( T, S \) and \( T + S \). Do you see a relationship between these three matrices?

In Example SMLT 519 compute (via Theorem MLTCV 510) the matrix of \( T \) and \( 2T \). Do you see a relationship between these two matrices?

Here’s the tough one. In Example CTLT 520 compute (via Theorem MLTCV 510) the matrix of \( T, S \) and \( S \circ T \). Do you see a relationship between these three matrices???

Subsection READ  
Reading Questions  

1. Is the function below a linear transformation? Why or why not?

\[
T: \mathbb{C}^3 \rightarrow \mathbb{C}^2, \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 - x_2 + x_3 \\ 8x_2 - 6 \end{pmatrix}
\]

2. Determine the matrix representation of the linear transformation \( S \) below.

\[
S: \mathbb{C}^2 \rightarrow \mathbb{C}^3, \quad S \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + 5x_2 \\ 8x_1 - 3x_2 \\ -4x_1 \end{pmatrix}
\]

3. Theorem LTLC 512 has a fairly simple proof. Yet the result itself is very powerful. Comment on why we might say this.
Subsection EXC
Exercises

C15  The archetypes below are all linear transformations whose domains and codomains are vector spaces of column vectors (Definition VSCV [87]). For each one, compute the matrix representation described in the proof of Theorem MLTCV [510].

Archetype M  775
Archetype N  778
Archetype O  781
Archetype P  784
Archetype Q  786
Archetype R  790
Contributed by Robert Beezer

C20  Let \( \mathbf{w} = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} \). Referring to Example MOLT 511, compute \( S(\mathbf{w}) \) two different ways. First use the definition of \( S \), then compute the matrix-vector product \( C \mathbf{w} \) (Definition MVP [211]).
Contributed by Robert Beezer  Solution 525

C25  Define the linear transformation

\[
T : \mathbb{C}^3 \rightarrow \mathbb{C}^2, \quad T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{bmatrix}
\]

Verify that \( T \) is a linear transformation.
Contributed by Robert Beezer  Solution 525

C26  Verify that the function below is a linear transformation.

\[
T : P_2 \rightarrow \mathbb{C}^2, \quad T \left( a + bx + cx^2 \right) = \begin{bmatrix} 2a - b \\ b + c \end{bmatrix}
\]

Contributed by Robert Beezer  Solution 525

C30  Define the linear transformation

\[
T : \mathbb{C}^3 \rightarrow \mathbb{C}^2, \quad T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{bmatrix}
\]

Compute the preimages, \( T^{-1} \left( \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) \) and \( T^{-1} \left( \begin{bmatrix} 4 \\ -8 \end{bmatrix} \right) \).
Contributed by Robert Beezer  Solution 526

C31  Consider the linear transformation \( S \), and compute the following pre-images,
\[ S^{-1} \begin{pmatrix} -2 \\ 5 \\ 3 \end{pmatrix} \text{ and } S^{-1} \begin{pmatrix} -5 \\ 5 \\ 7 \end{pmatrix} \].

\[ S: \mathbb{C}^3 \rightarrow \mathbb{C}^3, \quad S \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a - 2b - c \\ 3a - b + 2c \\ a + b + 2c \end{pmatrix} \]

Contributed by Robert Beezer
Solution 526

**M10** Define two linear transformations, \( T: \mathbb{C}^4 \mapsto \mathbb{C}^3 \) and \( S: \mathbb{C}^3 \mapsto \mathbb{C}^2 \) by

\[ S \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_2 + 3x_3 \\ 5x_1 + 4x_2 + 2x_3 \end{pmatrix} \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_1 + 3x_2 + x_3 + 9x_4 \\ 2x_1 + x_3 + 7x_4 \\ 4x_1 + 2x_2 + x_3 + 2x_4 \end{pmatrix} \]

Using the proof of [Theorem MLTCV] 510 compute the matrix representations of the three linear transformations \( T, S \) and \( S \circ T \). Discover and comment on the relationship between these three matrices.

Contributed by Robert Beezer
Solution 527
Subsection SOL

Solutions

C20  Contributed by Robert Beezer  Statement 523

In both cases the result will be $S(w) = \begin{bmatrix} 9 \\ 2 \\ -9 \\ 4 \end{bmatrix}$.

C25  Contributed by Robert Beezer  Statement 523

We can rewrite $T$ as follows:

$$T \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right) = \begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -10 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 5 \\ -4 & 2 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and Theorem MBLT 509 tell us that any function of this form is a linear transformation.

C26  Contributed by Robert Beezer  Statement 523

Check the two conditions of Definition LT 503.

$$T(u + v) = T \left( (a + bx + cx^2) + (d + ex + fx^2) \right)$$

$$= T \left( (a + d) + (b + e)x + (c + f)x^2 \right)$$

$$= \begin{bmatrix} 2(a + d) - (b + e) \\ (b + e) + (c + f) \end{bmatrix}$$

$$= \begin{bmatrix} (2a - b) + (2d - e) \\ (b + c) + (e + f) \end{bmatrix}$$

$$= \begin{bmatrix} 2a - b \\ b + c \end{bmatrix} + \begin{bmatrix} 2d - e \\ e + f \end{bmatrix}$$

$$= T(u) + T(v)$$

and

$$T(\alpha u) = T \left( \alpha (a + bx + cx^2) \right)$$

$$= T \left( (\alpha a) + (\alpha b)x + (\alpha c)x^2 \right)$$

$$= \begin{bmatrix} 2(\alpha a) - (\alpha b) \\ (\alpha b) + (\alpha c) \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 2a - b \\ b + c \end{bmatrix}$$

$$= \alpha T(u)$$

So $T$ is indeed a linear transformation.

C30  Contributed by Robert Beezer  Statement 523
For the first pre-image, we want \( \mathbf{x} \in \mathbb{C}^3 \) such that \( T(\mathbf{x}) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \). This becomes,

\[
\begin{bmatrix}
2x_1 - x_2 + 5x_3 \\
-4x_1 + 2x_2 - 10x_3
\end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}
\]

Vector equality gives a system of two linear equations in three variables, represented by the augmented matrix

\[
\begin{bmatrix}
2 & -1 & 5 & 2 \\
-4 & 2 & -10 & 3
\end{bmatrix}
\]

so the system is inconsistent and the pre-image is the empty set. For the second pre-image the same procedure leads to an augmented matrix with a different vector of constants

\[
\begin{bmatrix}
2 & -1 & 5 & 4 \\
-4 & 2 & -10 & -8
\end{bmatrix}
\]

This system is consistent and has infinitely many solutions, as we can see from the presence of the two free variables \( x_2 \) and \( x_3 \) both to zero. We apply Theorem VFSLS \( 107 \) to obtain

\[
T^{-1} \left( \begin{bmatrix} 4 \\ -8 \end{bmatrix} \right) = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| x_2, x_3 \in \mathbb{C} \right\}
\]

C31 Contributed by Robert Beezer  Statement 523

We work from the definition of the pre-image, Definition PI 515. Setting

\[
S \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}
\]

we arrive at a system of three equations in three variables, with an augmented matrix that we row-reduce in a search for solutions,

\[
\begin{bmatrix}
1 & -2 & -1 & -2 \\
3 & -1 & 2 & 5 \\
1 & 1 & 2 & 3
\end{bmatrix}
\]

so the system is inconsistent (Theorem RCLS 54), and there are no values of \( a, b \) and \( c \) that will create an element of the pre-image. So the preimage is the empty set.

We work from the definition of the pre-image, Definition PI 515. Setting

\[
S \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} -5 \\ 5 \\ 7 \end{bmatrix}
\]

we arrive at a system of three equations in three variables, with an augmented matrix that we row-reduce in a search for solutions,

\[
\begin{bmatrix}
1 & -2 & -1 & -5 \\
3 & -1 & 2 & 5 \\
1 & 1 & 2 & 7
\end{bmatrix}
\]

Version 0.92
The solution set to this system, which is also the desired pre-image, can be expressed using the vector form of the solutions (Theorem VFSLS \[107\])

\[
S^{-1} \begin{pmatrix} -5 \\ 5 \\ 7 \end{pmatrix} = \left\{ \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \mid c \in \mathbb{C} \right\} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -1 \\ \end{pmatrix} \right\rangle
\]

Does the final expression for this set remind you of Theorem KPI \[535\]?

M10 Contributed by Robert Beezer Statement 524

\[
\begin{bmatrix} 1 & -2 & 3 \\ 5 & 4 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 2 \\ 1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 3 & 1 & 9 \\ 2 & 0 & 1 & 7 \\ 4 & 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 2 \\ 1 \\ 11 \\ 19 \\ 11 \\ 77 \end{bmatrix}
\]
Section ILT
Injective Linear Transformations

Some linear transformations possess one, or both, of two key properties, which go by the names injective and surjective. We will see that they are closely related to ideas like linear independence and spanning, and subspaces like the null space and the column space. In this section we will define an injective linear transformation and analyze the resulting consequences. The next section will do the same for the surjective property. In the final section of this chapter we will see what happens when we have the two properties simultaneously.

As usual, we lead with a definition.

Definition ILT
Injective Linear Transformation

Suppose $T: U \rightarrow V$ is a linear transformation. Then $T$ is injective if whenever $T(x) = T(y)$, then $x = y$.

Given an arbitrary function, it is possible for two different inputs to yield the same output (think about the function $f(x) = x^2$ and the inputs $x = 3$ and $x = -3$). For an injective function, this never happens. If we have equal outputs ($T(x) = T(y)$) then we must have achieved those equal outputs by employing equal inputs ($x = y$). Some authors prefer the term one-to-one where we use injective, and we will sometimes refer to an injective linear transformation as an injection.

Subsection EILT
Examples of Injective Linear Transformations

It is perhaps most instructive to examine a linear transformation that is not injective first.

Example NIAQ
Not injective, Archetype Q

Archetype Q [786] is the linear transformation

$$T: \mathbb{C}^5 \rightarrow \mathbb{C}^5, \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_1 + 3x_2 + 3x_3 - 6x_4 + 3x_5 \\ -16x_1 + 9x_2 + 12x_3 - 28x_4 + 28x_5 \\ -19x_1 + 7x_2 + 14x_3 - 32x_4 + 37x_5 \\ -21x_1 + 9x_2 + 15x_3 - 35x_4 + 39x_5 \\ -9x_1 + 5x_2 + 7x_3 - 16x_4 + 16x_5 \end{pmatrix}$$

Notice that for

$$x = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 4 \end{pmatrix}, \quad y = \begin{pmatrix} 4 \\ 7 \\ 0 \\ 5 \\ 7 \end{pmatrix}$$
we have

\[
T \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 55 \\ 72 \\ 77 \\ 31 \end{pmatrix} \quad \text{and} \quad T \begin{pmatrix} 4 \\ 7 \\ 0 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ 55 \\ 72 \\ 77 \\ 31 \end{pmatrix}
\]

So we have two vectors from the domain, \( \mathbf{x} \neq \mathbf{y} \), yet \( T(\mathbf{x}) = T(\mathbf{y}) \), in violation of Definition ILT [529]. This is another example where you should not concern yourself with how \( \mathbf{x} \) and \( \mathbf{y} \) were selected, as this will be explained shortly. However, do understand why these two vectors provide enough evidence to conclude that \( T \) is not injective. ☒

To show that a linear transformation is not injective, it is enough to find a single pair of inputs that get sent to the identical output, as in Example NIAQ [529]. However, to show that a linear transformation is injective we must establish that this coincidence of outputs never occurs. Here is an example that shows how to establish this.

Example IAR

Injective, Archetype R

Archetype R [790] is the linear transformation

\[
T: \mathbb{C}^5 \mapsto \mathbb{C}^5, \quad T \left( \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \right) = \begin{pmatrix} -65x_1 + 128x_2 + 10x_3 - 262x_4 + 40x_5 \\ 36x_1 - 73x_2 - x_3 + 151x_4 - 16x_5 \\ -44x_1 + 88x_2 + 5x_3 - 180x_4 + 24x_5 \\ 34x_1 - 68x_2 - 3x_3 + 140x_4 - 18x_5 \\ 12x_1 - 24x_2 - x_3 + 49x_4 - 5x_5 \end{pmatrix}
\]

To establish that \( R \) is injective we must begin with the assumption that \( T(\mathbf{x}) = T(\mathbf{y}) \) and somehow arrive from this at the conclusion that \( \mathbf{x} = \mathbf{y} \). Here we go,

\[
T(\mathbf{x}) = T(\mathbf{y})
\]

\[
T \left( \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \right) = T \left( \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \right)
\]

\[
\begin{pmatrix}
-65x_1 + 128x_2 + 10x_3 - 262x_4 + 40x_5 \\
36x_1 - 73x_2 - x_3 + 151x_4 - 16x_5 \\
-44x_1 + 88x_2 + 5x_3 - 180x_4 + 24x_5 \\
34x_1 - 68x_2 - 3x_3 + 140x_4 - 18x_5 \\
12x_1 - 24x_2 - x_3 + 49x_4 - 5x_5 \\
\end{pmatrix} = \begin{pmatrix}
-65y_1 + 128y_2 + 10y_3 - 262y_4 + 40y_5 \\
36y_1 - 73y_2 - y_3 + 151y_4 - 16y_5 \\
-44y_1 + 88y_2 + 5y_3 - 180y_4 + 24y_5 \\
34y_1 - 68y_2 - 3y_3 + 140y_4 - 18y_5 \\
12y_1 - 24y_2 - y_3 + 49y_4 - 5y_5 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
-65(x_1 - y_1) + 128(x_2 - y_2) + 10(x_3 - y_3) - 262(x_4 - y_4) + 40(x_5 - y_5) \\
36(x_1 - y_1) - 73(x_2 - y_2) - (x_3 - y_3) + 151(x_4 - y_4) - 16(x_5 - y_5) \\
-44(x_1 - y_1) + 88(x_2 - y_2) + 5(x_3 - y_3) - 180(x_4 - y_4) + 24(x_5 - y_5) \\
34(x_1 - y_1) - 68(x_2 - y_2) - 3(x_3 - y_3) + 140(x_4 - y_4) - 18(x_5 - y_5) \\
12(x_1 - y_1) - 24(x_2 - y_2) - (x_3 - y_3) + 49(x_4 - y_4) - 5(x_5 - y_5) \\
\end{pmatrix} = 0
\]

\[
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]
Subsection ILT.EILT  Examples of Injective Linear Transformations 531

Now we recognize that we have a homogeneous system of 5 equations in 5 variables (the terms \(x_i - y_i\) are the variables), so we row-reduce the coefficient matrix to

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

So the only solution is the trivial solution

\[
\begin{align*}
x_1 - y_1 &= 0 \\
x_2 - y_2 &= 0 \\
x_3 - y_3 &= 0 \\
x_4 - y_4 &= 0 \\
x_5 - y_5 &= 0
\end{align*}
\]

and we conclude that indeed \(x = y\). By Definition I LT [529], \(T\) is injective. ☑

Let’s now examine an injective linear transformation between abstract vector spaces.

**Example IAV**

**Injective, Archetype V**

Archetype V [801] is defined by

\[
T: P_3 \mapsto M_{22}, \quad T (a + bx + cx^2 + dx^3) = \begin{bmatrix}
a + b & a - 2c \\
d & b - d
\end{bmatrix}
\]

To establish that the linear transformation is injective, begin by supposing that two polynomial inputs yield the same output matrix,

\[
T (a_1 + b_1 x + c_1 x^2 + d_1 x^3) = T (a_2 + b_2 x + c_2 x^2 + d_2 x^3)
\]

Then

\[
\mathcal{O} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

= \(T (a_1 + b_1 x + c_1 x^2 + d_1 x^3) - T (a_2 + b_2 x + c_2 x^2 + d_2 x^3)\)  \quad \text{Hypothesis}

= \(T ((a_1 - a_2) + (b_1 - b_2)x + (c_1 - c_2)x^2 + (d_1 - d_2)x^3)\)  \quad \text{Definition I LT [503]}

= \((a_1 - a_2) + (b_1 - b_2) (a_1 - a_2) - 2(c_1 - c_2)
\]

\((d_1 - d_2)\) \quad \text{Operations in } P_3

This single matrix equality translates to the homogeneous system of equations in the variables \(a_i - b_i\),

\[
\begin{align*}
(a_1 - a_2) + (b_1 - b_2) &= 0 \\
(a_1 - a_2) - 2(c_1 - c_2) &= 0 \\
(d_1 - d_2) &= 0
\end{align*}
\]
\[ (b_1 - b_2) - (d_1 - d_2) = 0 \]

This system of equations can be rewritten as the matrix equation
\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & -2 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
(a_1 - a_2) \\
(b_1 - b_2) \\
(c_1 - c_2) \\
(d_1 - d_2)
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

Since the coefficient matrix is nonsingular (check this) the only solution is trivial, i.e.
\[ a_1 - a_2 = 0 \quad b_1 - b_2 = 0 \quad c_1 - c_2 = 0 \quad d_1 - d_2 = 0 \]
so that
\[ a_1 = a_2 \quad b_1 = b_2 \quad c_1 = c_2 \quad d_1 = d_2 \]
so the two inputs must be equal polynomials. By Definition ILT 529, \( T \) is injective. \( \square \)

### Subsection KLT
#### Kernel of a Linear Transformation

For a linear transformation \( T: U \mapsto V \), the kernel is a subset of the domain \( U \). Informally, it is the set of all inputs that the transformation sends to the zero vector of the codomain. It will have some natural connections with the null space of a matrix, so we will keep the same notation, and if you think about your objects, then there should be little confusion. Here’s the careful definition.

**Definition KLT**

**Kernel of a Linear Transformation**

Suppose \( T: U \mapsto V \) is a linear transformation. Then the kernel of \( T \) is the set
\[
\mathcal{K}(T) = \{ u \in U \mid T(u) = 0 \}
\]

(This definition contains Notation KLT.) \( \triangle \)

Notice that the kernel of \( T \) is just the preimage of \( 0 \), \( T^{-1}(0) \) (Definition PI 515). Here’s an example.

**Example NKAO**

**Nontrivial kernel, Archetype O**

Archetype O 781 is the linear transformation

\[
T: \mathbb{C}^3 \mapsto \mathbb{C}^5, \quad T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} -x_1 + x_2 - 3x_3 \\ -x_1 + 2x_2 - 4x_3 \\ x_1 + x_2 + x_3 \\ 2x_1 + 3x_2 + x_3 \\ x_1 + 2x_3 \end{bmatrix}
\]
To determine the elements of $C^3$ in $K(T)$, find those vectors $u$ such that $T(u) = 0$, that is,

$$T(u) = 0$$

\[
\begin{bmatrix}
-u_1 + u_2 - 3u_3 \\
-u_1 + 2u_2 - 4u_3 \\
u_1 + u_2 + u_3 \\
2u_1 + 3u_2 + u_3 \\
u_1 + 2u_3 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

Vector equality (Definition CVE 88) leads us to a homogeneous system of 5 equations in the variables $u_i$:

\[-u_1 + u_2 - 3u_3 = 0\]
\[-u_1 + 2u_2 - 4u_3 = 0\]
\[u_1 + u_2 + u_3 = 0\]
\[2u_1 + 3u_2 + u_3 = 0\]
\[u_1 + 2u_3 = 0\]

Row-reducing the coefficient matrix gives

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

The kernel of $T$ is the set of solutions to this homogeneous system of equations, which by Theorem BNS 154 can be expressed as

$$\mathcal{K}(T) = \left\langle \begin{bmatrix}
-2 \\
1 \\
1 \\
\end{bmatrix} \right\rangle$$

We know that the span of a set of vectors is always a subspace (Theorem SSS 332), so the kernel computed in Example NKAO 532 is also a subspace. This is no accident, the kernel of a linear transformation is \textit{always} a subspace.

**Theorem KLTS**

**Kernel of a Linear Transformation is a Subspace**

Suppose that $T: U \rightarrow V$ is a linear transformation. Then the kernel of $T$, $\mathcal{K}(T)$, is a subspace of $U$.

**Proof** We can apply the three-part test of Theorem TSS 327. First $T(0_U) = 0_V$ by Theorem LTTZZ 507, so $0_U \in \mathcal{K}(T)$ and we know that the kernel is non-empty.

Suppose we assume that $x, y \in \mathcal{K}(T)$. Is $x + y \in \mathcal{K}(T)$?

$$T(x + y) = T(x) + T(y) \quad \text{Definition LT 503}$$

$$= 0 + 0 \quad x, y \in \mathcal{K}(T)$$

Version 0.92
This qualifies $x + y$ for membership in $K(T)$. So we have additive closure.

Suppose we assume that $\alpha \in \mathbb{C}$ and $x \in K(T)$. Is $\alpha x \in K(T)$?

$$T(\alpha x) = \alpha T(x) = \alpha 0$$

This qualifies $\alpha x$ for membership in $K(T)$. So we have scalar closure and Theorem TSS tells us that $K(T)$ is a subspace of $U$.

Let’s compute another kernel, now that we know in advance that it will be a subspace.

**Example TKAP**

**Trivial kernel, Archetype P**

Archetype P is the linear transformation

$$T: \mathbb{C}^3 \mapsto \mathbb{C}^5, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 + x_3 \\ -x_1 + 2x_2 + 2x_3 \\ x_1 + x_2 + 3x_3 \\ 2x_1 + 3x_2 + x_3 \\ -2x_1 + x_2 + 3x_3 \end{bmatrix}$$

To determine the elements of $\mathbb{C}^3$ in $K(T)$, find those vectors $u$ such that $T(u) = 0$, that is,

$$T(u) = 0$$

$$\begin{bmatrix} -u_1 + u_2 + u_3 \\ -u_1 + 2u_2 + 2u_3 \\ u_1 + u_2 + 3u_3 \\ 2u_1 + 3u_2 + u_3 \\ -2u_1 + u_2 + 3u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Vector equality leads us to a homogeneous system of 5 equations in the variables $u_i$,

$$-u_1 + u_2 + u_3 = 0$$
$$-u_1 + 2u_2 + 2u_3 = 0$$
$$u_1 + u_2 + 3u_3 = 0$$
$$2u_1 + 3u_2 + u_3 = 0$$
$$-2u_1 + u_2 + 3u_3 = 0$$

Row-reducing the coefficient matrix gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
The kernel of $T$ is the set of solutions to this homogeneous system of equations, which is simply the trivial solution $u = 0$, so

$$\mathcal{K}(T) = \{0\} = \langle\{\}\rangle$$

Our next theorem says that if a preimage is a non-empty set then we can construct it by picking any one element and adding on elements of the kernel.

**Theorem KPI**

**Kernel and Pre-Image**

Suppose $T: U \mapsto V$ is a linear transformation and $v \in V$. If the preimage $T^{-1}(v)$ is non-empty, and $u \in T^{-1}(v)$ then

$$T^{-1}(v) = \{u + z \mid z \in \mathcal{K}(T)\} = u + \mathcal{K}(T)$$

**Proof**  Let $M = \{u + z \mid z \in \mathcal{K}(T)\}$. First, we show that $M \subseteq T^{-1}(v)$. Suppose that $w \in M$, so $w$ has the form $w = u + z$, where $z \in \mathcal{K}(T)$. Then

$$T(w) = T(u + z) = T(u) + T(z) = v + 0$$

which qualifies $w$ for membership in the preimage of $v$, $w \in T^{-1}(v)$.

For the opposite inclusion, suppose $x \in T^{-1}(v)$. Then,

$$T(x - u) = T(x) - T(u) = v - v = 0$$

This qualifies $x - u$ for membership in the kernel of $T$, $\mathcal{K}(T)$. So there is a vector $z \in \mathcal{K}(T)$ such that $x - u = z$. Rearranging this equation gives $x = u + z$ and so $x \in M$. So $T^{-1}(v) \subseteq M$ and we see that $M = T^{-1}(v)$, as desired.

This theorem, and its proof, should remind you very much of Theorem PSPHS [113]. Additionally, you might go back and review Example SPIAS [515]. Can you tell now which is the only preimage to be a subspace?

The next theorem is one we will cite frequently, as it characterizes injections by the size of the kernel.

**Theorem KILT**

**Kernel of an Injective Linear Transformation**

Suppose that $T: U \mapsto V$ is a linear transformation. Then $T$ is injective if and only if the kernel of $T$ is trivial, $\mathcal{K}(T) = \{0\}$. □

**Proof**  ($\Rightarrow$) Suppose $x \in \mathcal{K}(T)$. Then by Definition KLT [532], $T(x) = 0$. By Theorem LTTZZ [507], $T(0) = 0$. Now, since $T(x) = T(0)$, we can apply Definition ILT [529] to conclude that $x = 0$. Therefore $\mathcal{K}(T) = \{0\}$. □
To establish that \( T \) is injective, appeal to Definition ILT 529 and begin with the assumption that \( T(x) = T(y) \). Then
\[
0 = T(x) - T(y) = T(x - y)
\]
Hypothesis
so by Definition KLT 532 and the hypothesis that the kernel is trivial,
\[
x - y \in \mathcal{K}(T) = \{0\}
\]
which means that
\[
0 = x - y
\]
x = y
thus establishing that \( T \) is injective.

Example NIAQR
Not injective, Archetype Q, revisited
We are now in a position to revisit our first example in this section, Example NIAQ 529. In that example, we showed that Archetype Q 786 is not injective by constructing two vectors, which when used to evaluate the linear transformation provided the same output, thus violating Definition ILT 529. Just where did those two vectors come from?
The key is the vector
\[
z = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 3 \\ 3 \end{bmatrix}
\]
which you can check is an element of \( \mathcal{K}(T) \) for Archetype Q 786. Choose a vector \( x \) at random, and then compute \( y = x + z \) (verify this computation back in Example NIAQ 529). Then
\[
T(y) = T(x + z)
\]
\[
= T(x) + T(z)
\]
\[
= T(x) + 0
\]
\[
= T(x)
\]
Definition LT 503
\( z \in \mathcal{K}(T) \)
Property Z 310
Whenever the kernel of a linear transformation is non-trivial, we can employ this device and conclude that the linear transformation is not injective. This is another way of viewing Theorem KILT 535. For an injective linear transformation, the kernel is trivial and our only choice for \( z \) is the zero vector, which will not help us create two different inputs for \( T \) that yield identical outputs. For every one of the archetypes that is not injective, there is an example presented of exactly this form.

Example NIAO
Not injective, Archetype O
In Example NKAO 532 the kernel of Archetype O 781 was determined to be
\[
\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}
\]
a subspace of $\mathbb{C}^3$ with dimension 1. Since the kernel is not trivial, Theorem KILT tells us that $T$ is not injective.

Example IAP
Injective, Archetype P

In Example TKAP it was shown that the linear transformation in Archetype P has a trivial kernel. So by Theorem KILT, $T$ is injective.

Subsection ILTLI
Injective Linear Transformations and Linear Independence

There is a connection between injective linear transformations and linear independent sets that we will make precise in the next two theorems. However, more informally, we can get a feel for this connection when we think about how each property is defined. A set of vectors is linearly independent if the only relation of linear dependence is the trivial one. A linear transformation is injective if the only way two input vectors can produce the same output is if the trivial way, when both input vectors are equal.

Theorem ILTLI
Injective Linear Transformations and Linear Independence

Suppose that $T: U \mapsto V$ is an injective linear transformation and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t\}$ is a linearly independent subset of $U$. Then $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \ldots, T(\mathbf{u}_t)\}$ is a linearly independent subset of $V$.

Proof
Begin with a relation of linear dependence on $S$ (Definition RLD, Definition LI),

\[ a_1 T(\mathbf{u}_1) + a_2 T(\mathbf{u}_2) + a_3 T(\mathbf{u}_3) + \ldots + a_t T(\mathbf{u}_t) = 0 \]

Theorem LTLC,

\[ T(a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + a_3 \mathbf{u}_3 + \ldots + a_t \mathbf{u}_t) = 0 \]

Theorem KLT,

\[ a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + a_3 \mathbf{u}_3 + \ldots + a_t \mathbf{u}_t \in \mathcal{K}(T) \]

Definition KLT,

\[ a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + a_3 \mathbf{u}_3 + \ldots + a_t \mathbf{u}_t \in \{0\} \]

Theorem KILT,

\[ a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 + a_3 \mathbf{u}_3 + \ldots + a_t \mathbf{u}_t = 0 \]

Since this is a relation of linear dependence on the linearly independent set $S$, we can conclude that

\[ a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad \ldots, \quad a_t = 0 \]

and this establishes that $R$ is a linearly independent set.

Theorem ILTB
Injective Linear Transformations and Bases

Suppose that $T: U \mapsto V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m\}$ is a basis of $U$. Then $T$ is injective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \ldots, T(\mathbf{u}_m)\}$ is a linearly independent subset of $V$.

Proof
Assume $T$ is injective. Since $B$ is a basis, we know $B$ is linearly independent (Definition B). Then Theorem ILTLI says that $C$ is a linearly independent subset of $V$. 
Section ILT Injective Linear Transformations

(⇐) Assume that \( C \) is linearly independent. To establish that \( T \) is injective, we will show that the kernel of \( T \) is trivial (Theorem KILT [535]). Suppose that \( u \in \mathcal{K}(T) \). As an element of \( U \), we can write \( u \) as a linear combination of the basis vectors in \( B \) (uniquely). So there are are scalars, \( a_1, a_2, a_3, \ldots, a_m \), such that

\[
\begin{align*}
    u &= a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_mu_m
\end{align*}
\]

Then,

\[
\begin{align*}
    0 &= T(u) & u \in \mathcal{K}(T) \\
    &= T(a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_mu_m) & B \text{ spans } U \\
    &= a_1T(u_1) + a_2T(u_2) + a_3T(u_3) + \cdots + a_mT(u_m) & \text{Theorem LTLC 512}
\end{align*}
\]

This is a relation of linear dependence (Definition RLD 345) on the linearly independent set \( C \), so the scalars are all zero: \( a_1 = a_2 = a_3 = \cdots = a_m = 0 \). Then

\[
\begin{align*}
    u &= a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_mu_m \\
    &= 0u_1 + 0u_2 + 0u_3 + \cdots + 0u_m & \text{Theorem ZSSM 317} \\
    &= 0 + 0 + 0 + \cdots + 0 \\
    &= 0 & \text{Property Z 310}
\end{align*}
\]

Since \( u \) was chosen as an arbitrary vector from \( \mathcal{K}(T) \), we have \( \mathcal{K}(T) = \{0\} \) and Theorem KILT [535] tells us that \( T \) is injective.

### Subsection ILTD
Injective Linear Transformations and Dimension

#### Theorem ILTD
Injective Linear Transformations and Dimension

Suppose that \( T: U \to V \) is an injective linear transformation. Then \( \dim(U) \leq \dim(V) \).

#### Proof
Suppose to the contrary that \( m = \dim(U) > \dim(V) = t \). Let \( B \) be a basis of \( U \), which will then contain \( m \) vectors. Apply \( T \) to each element of \( B \) to form a set \( C \) that is a subset of \( V \). By Theorem ILTB [537], \( C \) is linearly independent and therefore must contain \( m \) distinct vectors. So we have found a set of \( m \) linearly independent vectors in \( V \), a vector space of dimension \( t \), with \( m > t \). However, this contradicts Theorem G [398], so our assumption is false and \( \dim(U) \leq \dim(V) \).

#### Example NIDAU
Not injective by dimension, Archetype U

The linear transformation in Archetype U [798] is

\[
T: M_{23} \to \mathbb{C}^4, \quad T\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a + 2b + 12c - 3d + e + 6f \\ 2a - b - c + d - 11f \\ a + b + 7c + 2d + e - 3f \\ a + 2b + 12c + 5e - 5f \end{bmatrix}
\]
Since \( \dim(M_{23}) = 6 > 4 = \dim(\mathbb{C}^4) \), \( T \) cannot be injective for then \( T \) would violate Theorem ILTD \[538\].

Notice that the previous example made no use of the actual formula defining the function. Merely a comparison of the dimensions of the domain and codomain are enough to conclude that the linear transformation is not injective. Archetype M \[775\] and Archetype N \[778\] are two more examples of linear transformations that have “big” domains and “small” codomains, resulting in “collisions” of outputs and thus are non-injective linear transformations.

Subsection CILT
Composition of Injective Linear Transformations

In Subsection LT.NLTFO \[517\] we saw how to combine linear transformations to build new linear transformations, specifically, how to build the composition of two linear transformations (Definition LTC \[519\]). It will be useful later to know that the composition of injective linear transformations is again injective, so we prove that here.

**Theorem CILTI**
Composition of Injective Linear Transformations is Injective

Suppose that \( T: U \to V \) and \( S: V \to W \) are injective linear transformations. Then \((S \circ T): U \to W\) is an injective linear transformation. \(\Box\)

**Proof** That the composition is a linear transformation was established in Theorem CLTLT \[520\], so we need only establish that the composition is injective. Applying Definition ILT \[529\], choose \( x, y \) from \( U \). Then

Assume \((S \circ T)(x) = (S \circ T)(y)\)

\[ S(T(x)) = S(T(y)) \quad \text{Definition LTC \[519\]} \]

\[ \Rightarrow T(x) = T(y) \quad \text{Definition ILT \[529\] for } S \]

\[ \Rightarrow x = y \quad \text{Definition ILT \[529\] for } T \]

\[\blacksquare\]

Subsection READ
Reading Questions

1. Suppose \( T: \mathbb{C}^8 \to \mathbb{C}^5 \) is a linear transformation. Why can’t \( T \) be injective?

2. Describe the kernel of an injective linear transformation.

3. Theorem KPI \[535\] should remind you of Theorem PSPHS \[113\]. Why do we say this?
Subsection EXC
Exercises

C10 Each archetype below is a linear transformation. Compute the kernel for each.
Archetype M 775
Archetype N 778
Archetype O 781
Archetype P 784
Archetype Q 786
Archetype R 790
Archetype S 793
Archetype T 796
Archetype U 798
Archetype V 801
Archetype W 803
Archetype X 806

Contributed by Robert Beezer

C20 The linear transformation $T : \mathbb{C}^4 \mapsto \mathbb{C}^3$ is not injective. Find two inputs $x, y \in \mathbb{C}^4$ that yield the same output (that is $T(x) = T(y)$.)

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 + x_2 + x_3 \\ -x_1 + 3x_2 + x_3 - x_4 \\ 3x_1 + x_2 + 2x_3 - 2x_4 \end{bmatrix}$$

Contributed by Robert Beezer  Solution 543

C25 Define the linear transformation

$$T : \mathbb{C}^3 \mapsto \mathbb{C}^2, \quad T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{bmatrix}$$

Find a basis for the kernel of $T$, $\mathcal{K}(T)$. Is $T$ injective?
Contributed by Robert Beezer  Solution 543

C40 Show that the linear transformation $R$ is not injective by finding two different elements of the domain, $x$ and $y$, such that $R(x) = R(y)$. ($S_{22}$ is the vector space of symmetric $2 \times 2$ matrices.)

$$R : S_{22} \mapsto P_1, \quad R \left( \begin{bmatrix} a & b \\ b & c \end{bmatrix} \right) = (2a - b + c) + (a + b + 2c)x$$

Contributed by Robert Beezer  Solution 544

T10 Suppose $T : U \mapsto V$ is a linear transformation. For which vectors $v \in V$ is $T^{-1}(v)$ a subspace of of $U$?
Contributed by Robert Beezer
T15 Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations. Prove the following relationship between null spaces.

$$\mathcal{K}(T) \subseteq \mathcal{K}(S \circ T)$$

Contributed by Robert Beezer Solution 544

T20 Suppose that $A$ is an $m \times n$ matrix. Define the linear transformation $T$ by

$$T: \mathbb{C}^n \mapsto \mathbb{C}^m, \quad T(\mathbf{x}) = A\mathbf{x}$$

Prove that the kernel of $T$ equals the null space of $A$, $\mathcal{N}(A) = \mathcal{K}(T)$.

Contributed by Andy Zimmer Solution 545
Subsection SOL

Solutions

C20 Contributed by Robert Beezer Statement 541

A linear transformation that is not injective will have a non-trivial kernel (Theorem KILT 535), and this is the key to finding the desired inputs. We need one non-trivial element of the kernel, so suppose that \( z \in \mathbb{C}^4 \) is an element of the kernel,

\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = 0 = T(z) = \begin{bmatrix}
2z_1 + z_2 + z_3 \\
-2z_1 + 3z_2 + z_3 - z_4 \\
3z_1 + 2z_2 + 2z_3 - 2z_4
\end{bmatrix}
\]

Vector equality Definition CVE 88 leads to the homogeneous system of three equations in four variables,

\[
\begin{align*}
2z_1 + z_2 + z_3 &= 0 \\
-z_1 + 3z_2 + z_3 - z_4 &= 0 \\
3z_1 + 2z_2 + 2z_3 - 2z_4 &= 0
\end{align*}
\]

The coefficient matrix of this system row-reduces as

\[
\begin{bmatrix}
2 & 1 & 1 & 0 \\
-1 & 3 & 1 & -1 \\
3 & 1 & 2 & -2
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & -3
\end{bmatrix}
\]

From this we can find a solution (we only need one), that is an element of \( \mathcal{K}(T) \),

\[
z = \begin{bmatrix}
-1 \\
-1 \\
3 \\
1
\end{bmatrix}
\]

Now, we choose a vector \( x \) at random and set \( y = x + z \),

\[
x = \begin{bmatrix}
2 \\
3 \\
4 \\
-2
\end{bmatrix} \quad y = x + z = \begin{bmatrix}
2 \\
3 \\
4 \\
-2
\end{bmatrix} + \begin{bmatrix}
-1 \\
-1 \\
3 \\
1
\end{bmatrix} = \begin{bmatrix}
1 \\
2 \\
7 \\
-1
\end{bmatrix}
\]

and you can check that

\[
T(x) = \begin{bmatrix}
11 \\
13 \\
21
\end{bmatrix} = T(y)
\]

A quicker solution is to take two elements of the kernel (in this case, scalar multiples of \( z \)) which both get sent to \( \mathbf{0} \) by \( T \). Quicker yet, take \( \mathbf{0} \) and \( z \) as \( x \) and \( y \), which also both get sent to \( \mathbf{0} \) by \( T \).

C25 Contributed by Robert Beezer Statement 541

To find the kernel, we require all \( x \in \mathbb{C}^3 \) such that \( T(x) = \mathbf{0} \). This condition is

\[
\begin{bmatrix}
2x_1 - x_2 + 5x_3 \\
-4x_1 + 2x_2 - 10x_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
This leads to a homogeneous system of two linear equations in three variables, whose coefficient matrix row-reduces to
\[
\begin{bmatrix}
1 & -\frac{1}{2} & \frac{5}{2} \\
0 & 0 & 0
\end{bmatrix}
\]
With two free variables, Theorem BNS yields the basis for the null space:
\[
\left\{ \begin{bmatrix}
-\frac{5}{2} \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
\frac{1}{2} \\
1 \\
0
\end{bmatrix} \right\}
\]
With \( n(T) \neq 0 \), \( \mathcal{K}(T) \neq \{0\} \), so Theorem KILT says \( T \) is not injective.

C40 Contributed by Robert Beezer Statement

We choose \( x \) to be any vector we like. A particularly cocky choice would be to choose \( x = 0 \), but we will instead choose
\[
x = \begin{bmatrix}
2 \\
-1 \\
4
\end{bmatrix}
\]
Then \( R(x) = 9 + 9x \). Now compute the kernel of \( R \), which by Theorem KILT we expect to be nontrivial. Setting \( R \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) \) equal to the zero vector, \( 0 = 0 + 0x \), and equating coefficients leads to a homogenous system of equations. Row-reducing the coefficient matrix of this system will allow us to determine the values of \( a, b \) and \( c \) that create elements of the null space of \( R \),
\[
\begin{bmatrix}
2 & -1 & 1 \\
1 & 1 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix}
\]
We only need a single element of the null space of this coefficient matrix, so we will not compute a precise description of the whole null space. Instead, choose the free variable \( c = 2 \). Then
\[
z = \begin{bmatrix}
-2 \\
-2 \\
2
\end{bmatrix}
\]
is the corresponding element of the kernel. We compute the desired \( y \) as
\[
y = x + z = \begin{bmatrix}
2 \\
-1 \\
4
\end{bmatrix} + \begin{bmatrix}
-2 \\
-2 \\
2
\end{bmatrix} = \begin{bmatrix}
0 \\
-3 \\
6
\end{bmatrix}
\]
Then check that \( R(y) = 9 + 9x \).

T15 Contributed by Robert Beezer Statement

We are asked to prove that \( \mathcal{K}(T) \) is a subset of \( \mathcal{K}(S \circ T) \). Employing Definition SSET, choose \( x \in \mathcal{K}(T) \). Then we know that \( T(x) = 0 \). So
\[
(S \circ T)(x) = S(T(x)) = S(0) = 0
\]
This qualifies \( x \) for membership in \( \mathcal{K}(S \circ T) \).
This is an equality of sets, so we want to establish two subset conditions (Definition SE 694).

First, show \( \mathcal{N}(A) \subseteq \mathcal{K}(T) \). Choose \( x \in \mathcal{N}(A) \). Check to see if \( x \in \mathcal{K}(T) \),

\[
T(x) = Ax \\
= 0 \quad \text{for } x \in \mathcal{N}(A)
\]

So by Definition KLT 532, \( x \in \mathcal{K}(T) \) and thus \( \mathcal{N}(A) \subseteq \mathcal{N}(T) \).

Now, show \( \mathcal{K}(T) \subseteq \mathcal{N}(A) \). Choose \( x \in \mathcal{K}(T) \). Check to see if \( x \in \mathcal{N}(A) \),

\[
Ax = T(x) \\
= 0 \quad \text{for } x \in \mathcal{K}(T)
\]

So by Definition NSM 68, \( x \in \mathcal{N}(A) \) and thus \( \mathcal{N}(T) \subseteq \mathcal{N}(A) \).
The companion to an injection is a surjection. Surjective linear transformations are closely related to spanning sets and ranges. So as you read this section reflect back on Section ILT and note the parallels and the contrasts. In the next section, Section IVLT, we will combine the two properties.

As usual, we lead with a definition.

**Definition SLT**

**Surjective Linear Transformation**

Suppose $T: U \rightarrow V$ is a linear transformation. Then $T$ is **surjective** if for every $v \in V$ there exists a $u \in U$ so that $T(u) = v$. $\triangle$

Given an arbitrary function, it is possible for there to be an element of the codomain that is not an output of the function (think about the function $y = f(x) = x^2$ and the codomain element $y = -3$). For a surjective function, this never happens. If we choose any element of the codomain ($v \in V$) then there must be an input from the domain ($u \in U$) which will create the output when used to evaluate the linear transformation ($T(u) = v$). Some authors prefer the term **onto** where we use surjective, and we will sometimes refer to a surjective linear transformation as a **surjection**.

**Subsection ESLT**

**Examples of Surjective Linear Transformations**

It is perhaps most instructive to examine a linear transformation that is not surjective first.

**Example NSAQ**

**Not surjective, Archetype Q**

Archetype Q is the linear transformation

$$T: \mathbb{C}^5 \rightarrow \mathbb{C}^5, \quad T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \right) = \begin{bmatrix} -2x_1 + 3x_2 + 3x_3 - 6x_4 + 3x_5 \\ -16x_1 + 9x_2 + 12x_3 - 28x_4 + 28x_5 \\ -19x_1 + 7x_2 + 14x_3 - 32x_4 + 37x_5 \\ -21x_1 + 9x_2 + 15x_3 - 35x_4 + 39x_5 \\ -9x_1 + 5x_2 + 7x_3 - 16x_4 + 16x_5 \end{bmatrix}$$

We will demonstrate that

$$v = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \\ 4 \end{bmatrix}$$
is an unobtainable element of the codomain. Suppose to the contrary that \( u \) is an element of the domain such that \( T(u) = v \). Then
\[
\begin{bmatrix}
-1 \\
2 \\
3 \\
-1 \\
4
\end{bmatrix}
= v = T(u) = T
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5
\end{bmatrix}
\]
\[
= \begin{bmatrix}
-2u_1 + 3u_2 + 3u_3 - 6u_4 + 3u_5 \\
-16u_1 + 9u_2 + 12u_3 - 28u_4 + 28u_5 \\
-19u_1 + 7u_2 + 14u_3 - 32u_4 + 37u_5 \\
-21u_1 + 9u_2 + 15u_3 - 35u_4 + 39u_5 \\
-9u_1 + 5u_2 + 7u_3 - 16u_4 + 16u_5
\end{bmatrix}
\]
\[
= \begin{bmatrix}
-2 & 3 & 3 & -6 & 3 \\
-16 & 9 & 12 & -28 & 28 \\
-19 & 7 & 14 & -32 & 37 \\
-21 & 9 & 15 & -35 & 39 \\
-9 & 5 & 7 & -16 & 16
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5
\end{bmatrix}
\]

Now we recognize the appropriate input vector \( u \) as a solution to a linear system of equations. Form the augmented matrix of the system, and row-reduce to
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & -\frac{3}{2} & 0 \\
0 & 0 & 1 & 0 & -\frac{1}{3} & 0 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

With a leading 1 in the last column, Theorem RCLS [54] tells us the system is inconsistent. From the absence of any solutions we conclude that no such vector \( u \) exists, and by Definition SLT [547], \( T \) is not surjective.

Again, do not concern yourself with how \( v \) was selected, as this will be explained shortly. However, do understand why this vector provides enough evidence to conclude that \( T \) is not surjective.

To show that a linear transformation is not surjective, it is enough to find a single element of the codomain that is never created by any input, as in Example NSAQ [547]. However, to show that a linear transformation is surjective we must establish that every element of the codomain occurs as an output of the linear transformation for some appropriate input.

**Example SAR**

**Surjective, Archetype R**

Archetype R [790] is the linear transformation
\[
T: \mathbb{C}^5 \rightarrow \mathbb{C}^5, \quad T
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix}
= \begin{bmatrix}
-65x_1 + 128x_2 + 10x_3 - 262x_4 + 40x_5 \\
36x_1 - 73x_2 - x_3 + 151x_4 - 16x_5 \\
-44x_1 + 88x_2 + 5x_3 - 180x_4 + 24x_5 \\
34x_1 - 68x_2 - 3x_3 + 140x_4 - 18x_5 \\
12x_1 - 24x_2 - x_3 + 49x_4 - 5x_5
\end{bmatrix}
\]
To establish that $R$ is surjective we must begin with a totally arbitrary element of the codomain, $v$ and somehow find an input vector $u$ such that $T(u) = v$. We desire,

$$T(u) = v$$

$$\begin{bmatrix} -65u_1 + 128u_2 + 10u_3 - 262u_4 + 40u_5 \\ 36u_1 - 73u_2 - u_3 + 151u_4 - 16u_5 \\ -44u_1 + 88u_2 + 5u_3 - 180u_4 + 24u_5 \\ 34u_1 - 68u_2 - 3u_3 + 140u_4 - 18u_5 \\ 12u_1 - 24u_2 - u_3 + 49u_4 - 5u_5 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

We recognize this equation as a system of equations in the variables $u_i$, but our vector of constants contains symbols. In general, we would have to row-reduce the augmented matrix by hand, due to the symbolic final column. However, in this particular example, the $5 \times 5$ coefficient matrix is nonsingular and so has an inverse (Theorem NI [251], Definition MI [232]).

$$\begin{bmatrix} -65 & 128 & 10 & -262 & 40 \\ 36 & -73 & -1 & 151 & -16 \\ -44 & 88 & 5 & -180 & 24 \\ 34 & -68 & -3 & 140 & -18 \\ 12 & -24 & -1 & 49 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} -47 & 92 & 1 & -181 & -14 \\ 27 & -55 & \frac{7}{2} & \frac{221}{2} & 11 \\ -32 & 64 & -1 & -126 & -12 \\ 25 & -50 & \frac{3}{2} & \frac{199}{2} & 9 \\ 9 & -18 & \frac{1}{2} & \frac{71}{2} & 4 \end{bmatrix}$$

so we find that

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} -47v_1 + 92v_2 + v_3 - 181v_4 - 14v_5 \\ 27v_1 - 55v_2 + \frac{7}{2}v_3 + \frac{221}{2}v_4 + 11v_5 \\ -32v_1 + 64v_2 - v_3 - 126v_4 - 12v_5 \\ 25v_1 - 50v_2 + \frac{3}{2}v_3 + \frac{199}{2}v_4 + 9v_5 \\ 9v_1 - 18v_2 + \frac{1}{2}v_3 + \frac{71}{2}v_4 + 4v_5 \end{bmatrix}$$

This establishes that if we are given any output vector $v$, we can use its components in this final expression to formulate a vector $u$ such that $T(u) = v$. So by Definition SLT [547] we now know that $T$ is surjective. You might try to verify this condition in its full generality (i.e. evaluate $T$ with this final expression and see if you get $v$ as the result), or test it more specifically for some numerical vector $v$.

Let’s now examine a surjective linear transformation between abstract vector spaces.

Example SAV

Surjective, Archetype V
Archetype V \[801\] is defined by
\[
T: P_3 \mapsto M_{22}, \quad T (a + bx + cx^2 + dx^3) = \begin{bmatrix} a + b & a - 2c \\ d & b - d \end{bmatrix}
\]

To establish that the linear transformation is surjective, begin by choosing an arbitrary output. In this example, we need to choose an arbitrary \(2 \times 2\) matrix, say
\[
v = \begin{bmatrix} x & y \\ z & w \end{bmatrix}
\]

and we would like to find an input polynomial
\[
u = a + bx + cx^2 + dx^3
\]
so that \(T(u) = v\). So we have,
\[
\begin{bmatrix} x & y \\ z & w \end{bmatrix} = v
\]
\[
= T(u)
\]
\[
= T (a + bx + cx^2 + dx^3)
\]
\[
= \begin{bmatrix} a + b & a - 2c \\ d & b - d \end{bmatrix}
\]
Matrix equality leads us to the system of four equations in the four unknowns, \(x, y, z, w\),
\[
a + b = x
\]
\[
a - 2c = y
\]
\[
d = z
\]
\[
b - d = w
\]
which can be rewritten as a matrix equation,
\[
\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}
\]
The coefficient matrix is nonsingular, hence it has an inverse,
\[
\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}
\]
so we have
\[
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}
\]
Subsection SLT.RLT Range of a Linear Transformation

So the input polynomial \( u = (x - z - w) + (z + w)x + \frac{1}{2}(x - y - z - w)x^2 + zx^3 \) will yield the output matrix \( v \), no matter what form \( v \) takes. This means by [Definition SLT][547] that \( T \) is surjective. All the same, let’s do a concrete demonstration and evaluate \( T \) with \( u \).

\[
T(u) = T\left( (x - z - w) + (z + w)x + \frac{1}{2}(x - y - z - w)x^2 + zx^3 \right)
= \begin{bmatrix}
(x - z - w) + (z + w) & (x - z - w) - 2\left(\frac{1}{2}(x - y - z - w)\right) \\
(z + w) & (z + w) - z
\end{bmatrix}
= \begin{bmatrix}
x & y \\
z & w
\end{bmatrix}
= v
\]

\[\square\]

Subsection RLT
Range of a Linear Transformation

For a linear transformation \( T : U \rightarrow V \), the range is a subset of the codomain \( V \). Informally, it is the set of all outputs that the transformation creates when fed every possible input from the domain. It will have some natural connections with the column space of a matrix, so we will keep the same notation, and if you think about your objects, then there should be little confusion. Here’s the careful definition.

**Definition RLT**

**Range of a Linear Transformation**

Suppose \( T : U \rightarrow V \) is a linear transformation. Then the **range** of \( T \) is the set

\[
\mathcal{R}(T) = \{ T(u) \mid u \in U \}
\]

(This definition contains Notation RLT.)

**Example RAO**

**Range, Archetype O**

[Archetype O][781] is the linear transformation

\[
T : \mathbb{C}^3 \rightarrow \mathbb{C}^5, \quad T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix}
-x_1 + x_2 - 3x_3 \\
-x_1 + 2x_2 - 4x_3 \\
x_1 + x_2 + x_3 \\
2x_1 + 3x_2 + 3x_3 \\
x_1 + 2x_3
\end{bmatrix}
\]
To determine the elements of \( C^5 \) in \( \mathcal{R}(T) \), find those vectors \( v \) such that \( T(u) = v \) for some \( u \in C^3 \),

\[
v = T(u) = \begin{bmatrix} -u_1 + u_2 - 3u_3 \\ -u_1 + 2u_2 - 4u_3 \\ u_1 + u_2 + u_3 \\ 2u_1 + 3u_2 + u_3 \\ u_1 + 2u_3 \end{bmatrix}
\]

\[
= \begin{bmatrix} -u_1 \\ -u_1 \\ u_1 \\ 2u_1 \\ u_1 \end{bmatrix} + \begin{bmatrix} u_2 \\ 2u_2 \\ u_2 \\ 3u_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -3u_3 \\ -4u_3 \\ u_3 \\ u_3 \\ 2u_3 \end{bmatrix}
\]

\[
= u_1 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} -3 \\ -4 \\ 1 \\ 1 \\ 2 \end{bmatrix}
\]

This says that every output of \( T(v) \) can be written as a linear combination of the three vectors

\[
\begin{bmatrix} -1 \\ -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ -4 \\ 1 \\ 1 \\ 2 \end{bmatrix}
\]

using the scalars \( u_1, u_2, u_3 \). Furthermore, since \( u \) can be any element of \( C^3 \), every such linear combination is an output. This means that

\[
\mathcal{R}(T) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}
\]

The three vectors in this spanning set for \( \mathcal{R}(T) \) form a linearly dependent set (check this!). So we can find a more economical presentation by any of the various methods from Section CRS \( \text{[261]} \) and Section FS \( \text{[283]} \). We will place the vectors into a matrix as rows, row-reduce, toss out zero rows and appeal to Theorem BRS \( \text{[271]} \), so we can describe the range of \( T \) with a basis,

\[
\mathcal{R}(T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ -7 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \\ 1 \end{bmatrix} \right\}
\]
We know that the span of a set of vectors is always a subspace (Theorem SSS [332]), so the range computed in Example RAO [551] is also a subspace. This is no accident, the range of a linear transformation is always a subspace.

**Theorem RLTS**

**Range of a Linear Transformation is a Subspace**

Suppose that \( T: U \rightarrow V \) is a linear transformation. Then the range of \( T, \mathcal{R}(T) \), is a subspace of \( V \).

**Proof** We can apply the three-part test of Theorem TSS [327]. First, \( 0_U \in U \) and \( T(0_U) = 0_V \) by Theorem LTTZZ [507], so \( 0_V \in \mathcal{R}(T) \) and we know that the range is non-empty.

Suppose we assume that \( x, y \in \mathcal{R}(T) \). Is \( x + y \in \mathcal{R}(T) \)? If \( x, y \in \mathcal{R}(T) \) then we know there are vectors \( w, z \in U \) such that \( T(w) = x \) and \( T(z) = y \). Because \( U \) is a vector space, additive closure (Property AC [309]) implies that \( w + z \in U \). Then

\[
T(w + z) = T(w) + T(z) \quad \text{Definition LT [503]}
\]

So we have found an input, \( w + z \), which when fed into \( T \) creates \( x + y \) as an output. This qualifies \( x + y \) for membership in \( \mathcal{R}(T) \). So we have additive closure.

Suppose we assume that \( \alpha \in \mathbb{C} \) and \( x \in \mathcal{R}(T) \). Is \( \alpha x \in \mathcal{R}(T) \)? If \( x \in \mathcal{R}(T) \), then there is a vector \( w \in U \) such that \( T(w) = x \). Because \( U \) is a vector space, scalar closure implies that \( \alpha w \in U \). Then

\[
T(\alpha w) = \alpha T(w) \quad \text{Definition LT [503]}
\]

So we have found an input \( \alpha w \) which when fed into \( T \) creates \( \alpha x \) as an output. This qualifies \( \alpha x \) for membership in \( \mathcal{R}(T) \). So we have scalar closure and Theorem TSS [327] tells us that \( \mathcal{R}(T) \) is a subspace of \( V \).

Let’s compute another range, now that we know in advance that it will be a subspace.

**Example FRAN**

**Full range, Archetype N**

Archetype N [778] is the linear transformation

\[
T: \mathbb{C}^5 \rightarrow \mathbb{C}^3, \quad T\left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \right) = \begin{bmatrix} 2x_1 + x_2 + 3x_3 - 4x_4 + 5x_5 \\ x_1 - 2x_2 + 3x_3 - 9x_4 + 3x_5 \\ 3x_1 + 4x_3 - 6x_4 + 5x_5 \end{bmatrix}
\]

To determine the elements of \( \mathbb{C}^3 \) in \( \mathcal{R}(T) \), find those vectors \( v \) such that \( T(u) = v \) for some \( u \in \mathbb{C}^5 \),

\[ v = T(u) \]
This says that every output of \( T(v) \) can be written as a linear combination of the five vectors

\[
\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -4 \\ -9 \\ -6 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}
\]

using the scalars \( u_1, u_2, u_3, u_4, u_5 \). Furthermore, since \( u \) can be any element of \( \mathbb{C}^5 \), every such linear combination is an output. This means that

\[
\mathcal{R}(T) = \langle \left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -4 \\ -9 \\ -6 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix} \right\} \rangle = \mathbb{C}^3
\]
Example NSAQR

Not surjective, Archetype Q, revisited

We are now in a position to revisit our first example in this section, Example NSAQ [547]. In that example, we showed that Archetype Q [786] is not surjective by constructing a vector in the codomain where no element of the domain could be used to evaluate the linear transformation to create the output, thus violating Definition SLT [547]. Just where did this vector come from?

The short answer is that the vector
\[ v = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \\ 4 \end{bmatrix} \]
was constructed to lie outside of the range of \( T \). How was this accomplished? First, the range of \( T \) is given by
\[ \mathcal{R}(T) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\rangle \]

Suppose an element of the range \( v^* \) has its first 4 components equal to \(-1, 2, 3, -1\), in that order. Then to be an element of \( \mathcal{R}(T) \), we would have
\[ v^* = (-1) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + (2) \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + (3) \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -8 \end{bmatrix} \]
So the only vector in the range with these first four components specified, must have \(-8\) in the fifth component. To set the fifth component to any other value (say, 4) will result in a vector (\( v \) in Example NSAQ [547]) outside of the range. Any attempt to find an input for \( T \) that will produce \( v \) as an output will be doomed to failure.

Whenever the range of a linear transformation is not the whole codomain, we can employ this device and conclude that the linear transformation is not surjective. This is another way of viewing Theorem RSLT [554]. For a surjective linear transformation, the range is all of the codomain and there is no choice for a vector \( v \) that lies in \( V \), yet not in the range. For every one of the archetypes that is not surjective, there is an example presented of exactly this form.

Example NSAO

Not surjective, Archetype O

In Example RAO [551] the range of Archetype O [781] was determined to be
\[ \mathcal{R}(T) = \left\langle \begin{bmatrix} 1 \\ 0 \\ -3 \\ -2 \\ 1 \\ 2 \\ -7 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\rangle \]
a subspace of dimension 2 in \( \mathbb{C}^5 \). Since \( \mathcal{R}(T) \neq \mathbb{C}^5 \), Theorem RSLT 554 says \( T \) is not surjective.

Example SAN

Surjective, Archetype N

The range of Archetype N 778 was computed in Example FRAN 553 to be

\[
\mathcal{R}(T) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

Since the basis for this subspace is the set of standard unit vectors for \( \mathbb{C}^3 \) (Theorem SUVB 363), we have \( \mathcal{R}(T) = \mathbb{C}^3 \) and by Theorem RSLT 554, \( T \) is surjective.

Subsection SSSLT

Spanning Sets and Surjective Linear Transformations

Just as injective linear transformations are allied with linear independence (Theorem ILTLI 537, Theorem ILTB 537), surjective linear transformations are allied with spanning sets.

Theorem SSRLT

Spanning Set for Range of a Linear Transformation

Suppose that \( T: U \rightarrow V \) is a linear transformation and \( S = \{u_1, u_2, u_3, \ldots, u_t\} \) spans \( U \). Then \( R = \{T(u_1), T(u_2), T(u_3), \ldots, T(u_t)\} \) spans \( \mathcal{R}(T) \).

Proof

We need to establish that every element of \( \mathcal{R}(T) \) can be written as a linear combination of the vectors in \( R \). To this end, choose \( v \in \mathcal{R}(T) \). Then there exists a vector \( u \in U \), such that \( T(u) = v \) (Definition RLT 551).

Because \( S \) spans \( U \) there are scalars, \( a_1, a_2, a_3, \ldots, a_t \), such that

\[
u = a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_tu_t
\]

Then

\[
\begin{align*}
v &= T(u) \\
&= T(a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_tu_t) \\
&= a_1T(u_1) + a_2T(u_2) + a_3T(u_3) + \cdots + a_tT(u_t)
\end{align*}
\]

which establishes that \( R \) spans the range of \( T, \mathcal{R}(T) \).

Theorem SSRLT 556 provides an easy way to begin the construction of a basis for the range of a linear transformation, since the construction of a spanning set requires simply evaluating the linear transformation on a spanning set of the domain. In practice the best choice for a spanning set of the domain would be as small as possible, in other words, a basis. The resulting spanning set for the codomain may not be linearly independent, so to find a basis for the range might require toss out redundant vectors from the spanning set. Here’s an example.
Example BRLT
A basis for the range of a linear transformation

Define the linear transformation $T: M_{22} \mapsto P_2$ by

$$T\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a + 2b + 8c + d) x + (a + b + 5c) x^2$$

A convenient spanning set for $M_{22}$ is the basis

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

So by Theorem SSRLT 556, a spanning set for $\mathcal{R}(T)$ is

$$\mathcal{R}(T) = \left\{ T\left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right), T\left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right), T\left( \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right), T\left( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \right\}$$

$$= \left\{ 1 - 3x + x^2, 2 + 2x + x^2, 8 + 5x^2, 1 + 5x \right\}$$

The set $R$ is not linearly independent, so if we desire a basis for $\mathcal{R}(T)$, we need to eliminate some redundant vectors. Two particular relations of linear dependence on $R$ are

$$(-2)(1 - 3x + x^2) + (-3)(2 + 2x + x^2) + (8 + 5x^2) = 0 + 0x + 0x^2 = 0$$

$$(1 - 3x + x^2) + (-1)(2 + 2x + x^2) + (1 + 5x) = 0 + 0x + 0x^2 = 0$$

These, individually, allow us to remove $8 + 5x^2$ and $1 + 5x$ from $R$ with out destroying the property that $R$ spans $\mathcal{R}(T)$. The two remaining vectors are linearly independent (check this!), so we can write

$$\mathcal{R}(T) = \langle \{1 - 3x + x^2, 2 + 2x + x^2\} \rangle$$

and see that $\dim(\mathcal{R}(T)) = 2$. ☑

Elements of the range are precisely those elements of the codomain with non-empty preimages.

Theorem RPI
Range and Pre-Image

Suppose that $T: U \mapsto V$ is a linear transformation. Then

$$v \in \mathcal{R}(T) \text{ if and only if } T^{-1}(v) \neq \emptyset$$

Proof $(\Rightarrow)$ If $v \in \mathcal{R}(T)$, then there is a vector $u \in U$ such that $T(u) = v$. This qualifies $u$ for membership in $T^{-1}(v)$, and thus the preimage of $v$ is not empty.

$(\Leftarrow)$ Suppose the preimage of $v$ is not empty, so we can choose a vector $u \in U$ such that $T(u) = v$. Then $v \in \mathcal{R}(T)$. ■

Theorem SLTB
Surjective Linear Transformations and Bases

Suppose that $T: U \mapsto V$ is a linear transformation and $B = \{u_1, u_2, u_3, \ldots, u_m\}$ is a
basis of $U$. Then $T$ is surjective if and only if $C = \{ T(u_1), T(u_2), T(u_3), \ldots, T(u_m) \}$ is a spanning set for $V$.

**Proof**  $(\Rightarrow)$ Assume $T$ is surjective. Since $B$ is a basis, we know $B$ is a spanning set of $U$ (Definition B [363]). Then Theorem SSRLT [556] says that $C$ spans $\mathcal{R}(T)$. But the hypothesis that $T$ is surjective means $V = \mathcal{R}(T)$ (Theorem RSLT [554]), so $C$ spans $V$.

$(\Leftarrow)$ Assume that $C$ spans $V$. To establish that $T$ is surjective, we will show that every element of $V$ is an output of $T$ for some input (Definition SLT [547]). Suppose that $v \in V$. As an element of $V$, we can write $v$ as a linear combination of the spanning set $C$.

So, given any choice of a vector $v \in V$, we can design an input $u \in U$ to produce $v$ as an output of $T$. Thus, by Definition SLT [547], $T$ is surjective.

---

**Example NSDAT**

Not surjective by dimension, Archetype T

The linear transformation in Archetype T [796] is

$$T: P_4 \mapsto P_5, \quad T(p(x)) = (x - 2)p(x)$$

Since $\dim(P_4) = 5 < 6 = \dim(P_5)$, $T$ cannot be surjective for then it would violate Theorem SLTD [558].

---

Version 0.92
Notice that the previous example made no use of the actual formula defining the function. Merely a comparison of the dimensions of the domain and codomain are enough to conclude that the linear transformation is not surjective. Archetype O and Archetype P are two more examples of linear transformations that have “small” domains and “big” codomains, resulting in an inability to create all possible outputs and thus they are non-surjective linear transformations.

Subsection CSLT
Composition of Surjective Linear Transformations

In Subsection LT.NLTFO we saw how to combine linear transformations to build new linear transformations, specifically, how to build the composition of two linear transformations (Definition LTC). It will be useful later to know that the composition of surjective linear transformations is again surjective, so we prove that here.

Theorem CSLTS
Composition of Surjective Linear Transformations is Surjective

Suppose that $T: U \to V$ and $S: V \to W$ are surjective linear transformations. Then $(S \circ T): U \to W$ is a surjective linear transformation.

Proof

That the composition is a linear transformation was established in Theorem CLTLT, so we need only establish that the composition is surjective. Applying Definition SLT, choose $w \in W$. Because $S$ is surjective, there must be a vector $v \in V$, such that $S(v) = w$. With the existence of $v$ established, that $T$ is surjective guarantees a vector $u \in U$ such that $T(u) = v$. Now,

$$(S \circ T)(u) = S(T(u)) = S(v) = w$$

This establishes that any element of the codomain ($w$) can be created by evaluating $S \circ T$ with the right input ($u$). Thus, by Definition SLT, $S \circ T$ is surjective.

Subsection READ
Reading Questions

1. Suppose $T: \mathbb{C}^5 \to \mathbb{C}^8$ is a linear transformation. Why can’t $T$ be surjective?
2. What is the relationship between a surjective linear transformation and its range?
3. Compare and contrast injective and surjective linear transformations.
Subsection EXC Exercises

C10 Each archetype below is a linear transformation. Compute the range for each.

Archetype M 775
Archetype N 778
Archetype O 781
Archetype P 784
Archetype Q 786
Archetype R 790
Archetype S 793
Archetype T 796
Archetype U 798
Archetype V 801
Archetype W 803
Archetype X 806

Contributed by Robert Beezer

C20 Example SAR 548 concludes with an expression for a vector \( u \in \mathbb{C}^5 \) that we believe will create the vector \( v \in \mathbb{C}^5 \) when used to evaluate \( T \). That is, \( T(u) = v \). Verify this assertion by actually evaluating \( T \) with \( u \). If you don’t have the patience to push around all these symbols, try choosing a numerical instance of \( v \), compute \( u \), and then compute \( T(u) \), which should result in \( v \).

Contributed by Robert Beezer

C22 The linear transformation \( S : \mathbb{C}^4 \mapsto \mathbb{C}^3 \) is not surjective. Find an output \( w \in \mathbb{C}^3 \) that has an empty pre-image (that is \( S^{-1}(w) = \emptyset \)).

\[
S \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 + 3x_3 - 4x_4 \\ x_1 + 3x_2 + 4x_3 + 3x_4 \\ -x_1 + 2x_2 + x_3 + 7x_4 \end{pmatrix}
\]

Contributed by Robert Beezer Solution 563

C25 Define the linear transformation

\[
T : \mathbb{C}^3 \mapsto \mathbb{C}^2, \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{pmatrix}
\]

Find a basis for the range of \( T \), \( \mathcal{R}(T) \). Is \( T \) surjective?

Contributed by Robert Beezer Solution 563

C40 Show that the linear transformation \( T \) is not surjective by finding an element of the codomain, \( v \), such that there is no vector \( u \) with \( T(u) = v \). (15 points)

\[
T : \mathbb{C}^3 \mapsto \mathbb{C}^3, \quad T \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a + 3b - c \\ 2b - 2c \\ a - b + 2c \end{pmatrix}
\]
T15 Suppose that that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations. Prove the following relationship between ranges. (15 points)

$$\mathcal{R}(S \circ T) \subseteq \mathcal{R}(S)$$

T20 Suppose that $A$ is an $m \times n$ matrix. Define the linear transformation $T$ by

$$T: \mathbb{C}^n \mapsto \mathbb{C}^m, \quad T(x) = Ax$$

Prove that the range of $T$ equals the column space of $A$, $\mathcal{C}(A) = \mathcal{R}(T)$. 

Contributed by Robert Beezer  Solution 564

Contributed by Andy Zimmer  Solution 564
To find an element of $\mathbb{C}^3$ with an empty pre-image, we will compute the range of the linear transformation $\mathcal{R}(S)$ and then find an element outside of this set.

By Theorem SSRLT we can evaluate $S$ with the elements of a spanning set of the domain and create a spanning set for the range.

$$
S \left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \\ -1 \\ \end{bmatrix},
S \left( \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 3 \\ 2 \\ \end{bmatrix},
S \left( \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 4 \\ 1 \\ \end{bmatrix},
S \left( \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \end{bmatrix} \right) = \begin{bmatrix} -4 \\ 3 \\ 7 \\ \end{bmatrix}
$$

So

$$\mathcal{R}(S) = \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \\ \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \\ \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 7 \\ \end{bmatrix} \right\}$$

This spanning set is obviously linearly dependent, so we can reduce it to a basis for $\mathcal{R}(S)$ using Theorem BRS, where the elements of the spanning set are placed as the rows of a matrix. The result is that

$$\mathcal{R}(S) = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ \end{bmatrix} \right\}$$

Therefore, the unique vector in $\mathcal{R}(S)$ with a first slot equal to 6 and a second slot equal to 15 will be the linear combination

$$6 \begin{bmatrix} 1 \\ 0 \\ -1 \\ \end{bmatrix} + 15 \begin{bmatrix} 0 \\ 1 \\ 1 \\ \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \\ 9 \\ \end{bmatrix}$$

So, any vector with first two components equal to 6 and 15, but with a third component different from 9, such as

$$w = \begin{bmatrix} 6 \\ 15 \\ -63 \\ \end{bmatrix}$$

will not be an element of the range of $S$ and will therefore have an empty pre-image. Another strategy on this problem is to guess. Almost any vector will lie outside the range of $T$, you have to be unlucky to randomly choose an element of the range. This is because the codomain has dimension 3, while the range is “much smaller” at a dimension of 2. You still need to check that your guess lies outside of the range, which generally will involve solving a system of equations that turns out to be inconsistent.
Each of these vectors is a scalar multiple of the others, so we can toss two of them in reducing the spanning set to a linearly independent set (or be more careful and apply Theorem BCS on a matrix with these three vectors as columns). The result is the basis of the range, \[
\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}
\]

With \( r(T) \neq 2 \), \( R(T) \neq \mathbb{C}^2 \), so Theorem RSLT says \( T \) is not surjective.

Contributed by Robert Beezer

We wish to find an output vector \( \mathbf{v} \) that has no associated input. This is the same as requiring that there is no solution to the equality \[
\mathbf{v} = T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a + 3b - c \\ 2b - 2c \\ a - b + 2c \end{bmatrix} = a \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + c \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}
\]

In other words, we would like to find an element of \( \mathbb{C}^3 \) not in the set \[
Y = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}
\]

If we make these vectors the rows of a matrix, and row-reduce, Theorem BRS provides an alternate description of \( Y \),

\[
Y = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -5 \end{bmatrix} \right\}
\]

If we add these vectors together, and then change the third component of the result, we will create a vector that lies outside of \( Y \), say \( \mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 9 \end{bmatrix} \).

Contributed by Andy Zimmer

This question asks us to establish that one set (\( R(S \circ T) \)) is a subset of another (\( R(S) \)). Choose an element in the “smaller” set, say \( \mathbf{w} \in R(S \circ T) \). Then we know that there is a vector \( \mathbf{u} \in U \) such that \( \mathbf{w} = (S \circ T)(\mathbf{u}) = S(T(\mathbf{u})) \)

Now define \( \mathbf{v} = T(\mathbf{u}) \), so that then \( S(\mathbf{v}) = S(T(\mathbf{u})) = \mathbf{w} \)

This statement is sufficient to show that \( \mathbf{w} \in R(S) \), so \( \mathbf{w} \) is an element of the “larger” set, and \( R(S \circ T) \subseteq R(S) \).

Contributed by Andy Zimmer

This is an equality of sets, so we want to establish two subset conditions (Definition SE). First, show \( C(A) \subseteq R(T) \). Choose \( \mathbf{y} \in C(A) \). Then by Definition CSM and Definition MVP there is a vector \( \mathbf{x} \in \mathbb{C}^n \) such that \( A\mathbf{x} = \mathbf{y} \). Then \( T(\mathbf{x}) = A\mathbf{x} \).
This statement qualifies $y$ as a member of $\mathcal{R}(T)$ (Definition RLT 551), so $\mathcal{C}(A) \subseteq \mathcal{R}(T)$.

Now, show $\mathcal{R}(T) \subseteq \mathcal{C}(A)$. Choose $y \in \mathcal{R}(T)$. Then by Definition RLT 551, there is a vector $x$ in $\mathbb{C}^n$ such that $T(x) = y$. Then

\[
Ax = T(x) \quad \text{Definition of } T
\]

\[
= y
\]

So by Definition CSM 261 and Definition MVP 211, $y$ qualifies for membership in $\mathcal{C}(A)$ and so $\mathcal{R}(T) \subseteq \mathcal{C}(A)$. 
Section IVLT
Invertible Linear Transformations

In this section we will conclude our introduction to linear transformations by bringing together the twin properties of injectivity and surjectivity and consider linear transformations with both of these properties.

Subsection IVLT
Invertible Linear Transformations

One preliminary definition, and then we will have our main definition for this section.

Definition IDLT
Identity Linear Transformation

The identity linear transformation on the vector space \( W \) is defined as

\[
I_W : W \mapsto W, \quad I_W(w) = w
\]

Informally, \( I_W \) is the “do-nothing” function. You should check that \( I_W \) is really a linear transformation, as claimed, and then compute its kernel and range to see that it is both injective and surjective. All of these facts should be straightforward to verify. With this in hand we can make our main definition.

Definition IVLT
Invertible Linear Transformations

Suppose that \( T : U \mapsto V \) is a linear transformation. If there is a function \( S : V \mapsto U \) such that

\[
S \circ T = I_U \quad \text{and} \quad T \circ S = I_V
\]

then \( T \) is invertible. In this case, we call \( S \) the inverse of \( T \) and write \( S = T^{-1} \). 

Informally, a linear transformation \( T \) is invertible if there is a companion linear transformation, \( S \), which “undoes” the action of \( T \). When the two linear transformations are applied consecutively (composition), in either order, the result is to have no real effect. It is entirely analogous to squaring a positive number and then taking its (positive) square root.

Here is an example of a linear transformation that is invertible. As usual at the beginning of a section, do not be concerned with where \( S \) came from, just understand how it illustrates Definition IVLT 567.

Example AIVLT
An invertible linear transformation

Archetype V [801] is the linear transformation

\[
T : P_3 \mapsto M_{22}, \quad T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a + b & a - 2c \\ d & b - d \end{bmatrix}
\]
Define the function $S: M_{22} \mapsto P_3$ defined by
\[ S \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a - c - d) + (c + d)x + \frac{1}{2}(a - b - c - d)x^2 + cx^3 \]

Then
\[ (T \circ S) \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = T \left( S \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \]
\[ = T \left( (a - c - d) + (c + d)x + \frac{1}{2}(a - b - c - d)x^2 + cx^3 \right) \]
\[ = \begin{bmatrix} (a - c - d) + (c + d)x & (a - c - d) - 2(\frac{1}{2}(a - b - c - d)) \\ c & (c + d) - c \end{bmatrix} \]
\[ = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]
\[ = I_{M_{22}} \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \]

And
\[ (S \circ T) \left( a + bx + cx^2 + dx^3 \right) = S \left( T \left( a + bx + cx^2 + dx^3 \right) \right) \]
\[ = S \left( \begin{bmatrix} a + b & a - 2c \\ d & b - d \end{bmatrix} \right) \]
\[ = ((a + b) - d - (b - d)) + (d + (b - d))x \]
\[ + \left( \frac{1}{2}((a + b) - (a - 2c) - d - (b - d)) \right)x^2 + (d)x^3 \]
\[ = a + bx + cx^2 + dx^3 \]
\[ = I_{P_3} \left( a + bx + cx^2 + dx^3 \right) \]

For now, understand why these computations show that $T$ is invertible, and that $S = T^{-1}$.

Maybe even be amazed by how $S$ works so perfectly in concert with $T$! We will see later just how to arrive at the correct form of $S$ (when it is possible).

**Example ANILT**

**A non-invertible linear transformation**

Consider the linear transformation $T: \mathbb{C}^3 \mapsto M_{22}$ defined by
\[ T \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a - b & 2a + 2b + c \\ 3a + b + c & -2a - 6b - 2c \end{bmatrix} \]

Suppose we were to search for an inverse function $S: M_{22} \mapsto \mathbb{C}^3$.

First verify that the $2 \times 2$ matrix $A = \begin{bmatrix} 5 & 3 \\ 8 & 2 \end{bmatrix}$ is not in the range of $T$. This will amount to finding an input to $T$, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, such that
\[ a - b = 5 \]
\[2a + 2b + c = 3\]
\[3a + b + c = 8\]
\[-2a - 6b - 2c = 2\]

As this system of equations is inconsistent, there is no input column vector, and \(A \not\in \mathcal{R}(T)\). How should we define \(S(A)\)? Note that

\[T(S(A)) = (T \circ S)(A) = I_{M_{22}}(A) = A\]

So any definition we would provide for \(S(A)\) must then be a column vector that \(T\) sends to \(A\) and we would have \(A \in \mathcal{R}(T)\), contrary to the definition of \(T\). This is enough to see that there is no function \(S\) that will allow us to conclude that \(T\) is invertible, since we cannot provide a consistent definition for \(S(A)\) if we assume \(T\) is invertible.

Even though we now know that \(T\) is not invertible, let’s not leave this example just yet. Check that \(T\left(\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}\right) = B\) and \(T\left(\begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}\right) = B\). How would we define \(S(B)\)?

\[S(B) = S(T\left(\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}\right)) = (S \circ T)\left(\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}\right) = I_{C^3}\left(\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}\]

or

\[S(B) = S(T\left(\begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}\right)) = (S \circ T)\left(\begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}\right) = I_{C^3}\left(\begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}\]

Which definition should we provide for \(S(B)\)? Both are necessary. But then \(S\) is not a function. So we have a second reason to know that there is no function \(S\) that will allow us to conclude that \(T\) is invertible. It happens that there are infinitely many column vectors that \(S\) would have to take to \(B\). Construct the kernel of \(T\),

\[\mathcal{K}(T) = \left\langle \left\{ \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} \right\} \right\rangle\]

Now choose either of the two inputs used above for \(T\) and add to it a scalar multiple of the basis vector for the kernel of \(T\). For example,

\[x = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + (-2) \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}\]

then verify that \(T(x) = B\). Practice creating a few more inputs for \(T\) that would be sent to \(B\), and see why it is hopeless to think that we could ever provide a reasonable definition for \(S(B)\)! There is a “whole subspace’s worth” of values that \(S(B)\) would have to take on.
In [Example ANILT 568] you may have noticed that $T$ is not surjective, since the matrix $A$ was not in the range of $T$. And $T$ is not injective since there are two different input column vectors that $T$ sends to the matrix $B$. Linear transformations $T$ that are not surjective lead to putative inverse functions $S$ that are undefined on inputs outside of the range of $T$. Linear transformations $T$ that are not injective lead to putative inverse functions $S$ that are multiply-defined on each of their inputs. We will formalize these ideas in [Theorem IILTIS 571].

But first notice in [Definition IVLT 567] that we only require the inverse (when it exists) to be a function. When it does exist, it too is a linear transformation.

**Theorem ILTLT**

_Inverse of a Linear Transformation is a Linear Transformation_

Suppose that $T: U \rightarrow V$ is an invertible linear transformation. Then the function $T^{-1}: V \rightarrow U$ is a linear transformation.

**Proof** We work through verifying [Definition LT 503] for $T^{-1}$, employing as we go properties of $T$ given by [Definition LT 503]. To this end, suppose $x, y \in V$ and $\alpha \in \mathbb{C}$.

\[
T^{-1}(x + y) = T^{-1}(T(T^{-1}(x)) + T(T^{-1}(y))) = T^{-1}(T(T^{-1}(x) + T^{-1}(y))) = T^{-1}(x) + T^{-1}(y)
\]

Now check the second defining property of a linear transformation for $T^{-1}$,

\[
T^{-1}(\alpha x) = T^{-1}(\alpha T(T^{-1}(x))) = T^{-1}(T(\alpha T^{-1}(x))) = \alpha T^{-1}(x)
\]

So $T^{-1}$ fulfills the requirements of [Definition LT 503] and is therefore a linear transformation. So when $T$ has an inverse, $T^{-1}$ is also a linear transformation. Additionally, $T^{-1}$ is invertible and its inverse is what you might expect.

**Theorem IILT**

_Inverse of an Invertible Linear Transformation_

Suppose that $T: U \hookrightarrow V$ is an invertible linear transformation. Then $T^{-1}$ is an invertible linear transformation and $(T^{-1})^{-1} = T$.

**Proof** Because $T$ is invertible, [Definition IVLT 567] tells us there is a function $T^{-1}: V \rightarrow U$ such that

\[
T^{-1} \circ T = I_U \quad \text{and} \quad T \circ T^{-1} = I_V
\]

Additionally, [Theorem ILTLT 570] tells us that $T^{-1}$ is more than just a function, it is a linear transformation. Now view these two statements as properties of the linear transformation $T^{-1}$. In light of [Definition IVLT 567], they together say that $T^{-1}$ is invertible (let $T$ play the role of $S$ in the statement of the definition). Furthermore, the inverse of $T^{-1}$ is then $T$, i.e. $(T^{-1})^{-1} = T$. 

Version 0.92
Subsection IV
Invertibility

We now know what an inverse linear transformation is, but just which linear transformations have inverses? Here is a theorem we have been preparing for all chapter.

**Theorem ILTIS**

**Invertible Linear Transformations are Injective and Surjective**

Suppose $T: U \mapsto V$ is a linear transformation. Then $T$ is invertible if and only if $T$ is injective and surjective.

**Proof** ($\Rightarrow$) Since $T$ is presumed invertible, we can employ its inverse, $T^{-1}$ (Definition IVLT 567). To see that $T$ is injective, suppose $x, y \in U$ and assume that $T(x) = T(y)$,

\[
x = I_U(x) = (T^{-1} \circ T)(x) = T^{-1}(T(x)) = T^{-1}(T(y)) = (T^{-1} \circ T)(y) = I_U(y) = y
\]

So by Definition ILT 529 $T$ is injective. To check that $T$ is surjective, suppose $v \in V$. Employ $T^{-1}$ again by defining $u = T^{-1}(v)$. Then

\[
T(u) = T(T^{-1}(v)) = (T \circ T^{-1})(v) = I_V(v) = v
\]

So there is an input to $T$, $u$, that produces the chosen output, $v$, and hence $T$ is surjective by Definition SLT 547.

($\Leftarrow$) Now assume that $T$ is both injective and surjective. We will build a function $S: V \mapsto U$ that will establish that $T$ is invertible. To this end, choose any $v \in V$. Since $T$ is surjective, Theorem RSLT 554 says $R(T) = V$, so we have $v \in R(T)$. Theorem RPI 557 says that the pre-image of $v$, $T^{-1}(v)$, is nonempty. So we can choose a vector from the pre-image of $v$, say $u$. In other words, there exists $u \in T^{-1}(v)$.

Since $T^{-1}(v)$ is non-empty, Theorem KPI 535 then says that

\[
T^{-1}(v) = \{ u + z \mid z \in \mathcal{K}(T) \}
\]

However, because $T$ is injective, by Theorem KILT 535 the kernel is trivial, $\mathcal{K}(T) = \{0\}$. So the pre-image is a set with just one element, $T^{-1}(v) = \{u\}$. Now we can define $S$ by $S(v) = u$. This is the key to this half of this proof. Normally the preimage of a vector from the codomain might be an empty set, or an infinite set. But surjectivity requires that the preimage not be empty, and then injectivity limits the preimage to a singleton.
Since our choice of $v$ was arbitrary, we know that every pre-image for $T$ is a set with a single element. This allows us to construct $S$ as a function. Now that it is defined, verifying that it is the inverse of $T$ will be easy. Here we go.

Choose $u \in U$. Define $v = T(u)$. Then $T^{-1}(v) = \{u\}$, so that $S(v) = u$ and,

$$(S \circ T)(u) = S(T(u)) = S(v) = u = I_U(u)$$

and since our choice of $u$ was arbitrary we have function equality, $S \circ T = I_U$.

Now choose $v \in V$. Define $u$ to be the single vector in the set $T^{-1}(v)$, in other words, $u = S(v)$. Then $T(u) = v$, so

$$(T \circ S)(v) = T(S(v)) = T(u) = v = I_V(v)$$

and since our choice of $v$ was arbitrary we have function equality, $T \circ S = I_V$.

We will make frequent use of this characterization of invertible linear transformations. The next theorem is a good example of this, and we will use it often, too.

**Theorem CIVLT**

**Composition of Invertible Linear Transformations**

Suppose that $T : U \mapsto V$ and $S : V \mapsto W$ are invertible linear transformations. Then the composition, $(S \circ T) : U \mapsto W$ is an invertible linear transformation.

**Proof** Since $S$ and $T$ are both linear transformations, $S \circ T$ is also a linear transformation by Theorem CLTLT \[520\]. Since $S$ and $T$ are both invertible, Theorem ILTIS \[571\] says that $S$ and $T$ are both injective and surjective. Then Theorem CILTI \[539\] says $S \circ T$ is injective, and Theorem CSLTS \[559\] says $S \circ T$ is surjective. Now apply the “other half” of Theorem ILTIS \[571\] and conclude that $S \circ T$ is invertible.

When a composition is invertible, the inverse is easy to construct.

**Theorem ICLT**

**Inverse of a Composition of Linear Transformations**

Suppose that $T : U \mapsto V$ and $S : V \mapsto W$ are invertible linear transformations. Then $S \circ T$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

**Proof** Compute, for all $w \in W$

$$(S \circ T) \circ (T^{-1} \circ S^{-1})(w) = S \left( T \left( T^{-1} \left( S^{-1}(w) \right) \right) \right)$$

$$= S \left( I_V \left( S^{-1}(w) \right) \right)$$

$$= S \left( S^{-1}(w) \right)$$

$$= w$$

$$= I_W(w)$$

so $(S \circ T) \circ (T^{-1} \circ S^{-1}) = I_W$ and also

$$(T^{-1} \circ S^{-1}) \circ (S \circ T)(u) = T^{-1} \left( S^{-1} \left( S \left( T(u) \right) \right) \right)$$

$$= T^{-1} \left( I_V \left( T(u) \right) \right)$$

$$= T^{-1}(T(u))$$

$$= u$$

Download Version 0.92
so \((T^{-1} \circ S^{-1}) \circ (S \circ T) = I_U\). By Definition IVLT [567], \(S \circ T\) is invertible and \((S \circ T)^{-1} = T^{-1} \circ S^{-1}\).

Notice that this theorem not only establishes what the inverse of \(S \circ T\) is, it also duplicates the conclusion of Theorem CIVLT [572] and also establishes the invertibility of \(S \circ T\). But somehow, the proof of Theorem CIVLT [572] is nicer way to get this property.

Does Theorem ICLT [572] remind you of the flavor of any theorem we have seen about matrices? (Hint: Think about getting dressed.) Hmmm.

**Subsection SI**

**Structure and Isomorphism**

A vector space is defined (Definition VS [309]) as a set of objects (“vectors”) endowed with a definition of vector addition (+) and a definition of scalar multiplication (juxtaposition). Many of our definitions about vector spaces involve linear combinations (Definition LC [331]), such as the span of a set (Definition SS [332]) and linear independence (Definition LI [345]). Other definitions are built up from these ideas, such as bases (Definition B [363]) and dimension (Definition D [379]). The defining properties of a linear transformation require that a function “respect” the operations of the two vector spaces that are the domain and the codomain (Definition LT [503]). Finally, an invertible linear transformation is one that can be “undone” — it has a companion that reverses its effect. In this subsection we are going to begin to roll all these ideas into one.

A vector space has “structure” derived from definitions of the two operations and the requirement that these operations interact in ways that satisfy the ten properties of Definition VS [309]. When two different vector spaces have an invertible linear transformation defined between them, then we can translate questions about linear combinations (spans, linear independence, bases, dimension) from the first vector space to the second. The answers obtained in the second vector space can then be translated back, via the inverse linear transformation, and interpreted in the setting of the first vector space. We say that these invertible linear transformations “preserve structure.” And we say that the two vector spaces are “structurally the same.” The precise term is “isomorphic,” from Greek meaning “of the same form.” Let’s begin to try to understand this important concept.

**Definition IVS**

**Isomorphic Vector Spaces**

Two vector spaces \(U\) and \(V\) are isomorphic if there exists an invertible linear transformation \(T\) with domain \(U\) and codomain \(V\), \(T: U \rightarrow V\). In this case, we write \(U \cong V\), and the linear transformation \(T\) is known as an isomorphism between \(U\) and \(V\).

A few comments on this definition. First, be careful with your language (Technique L [700]). Two vector spaces are isomorphic, or not. It is a yes/no situation and the term only applies to a pair of vector spaces. Any invertible linear transformation can be called an isomorphism, it is a term that applies to functions. Second, a given pair of vector spaces there might be several different isomorphisms between the two vector spaces.
But it only takes the existence of one to call the pair isomorphic. Third, \( U \) isomorphic to \( V \), or \( V \) isomorphic to \( U \)? Doesn’t matter, since the inverse linear transformation will provide the needed isomorphism in the “opposite” direction. Being “isomorphic to” is an equivalence relation on the set of all vector spaces (see Theorem SER 485 for a reminder about equivalence relations).

**Example IVSAV**

**Isomorphic vector spaces, Archetype V**

Archetype V [801] is a linear transformation from \( P_3 \) to \( M_{22} \),

\[
T: P_3 \rightarrow M_{22}, \quad T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a + b & a - 2c \\ d & b - d \end{bmatrix}
\]

Since it is injective and surjective, Theorem ILTIS 571 tells us that it is an invertible linear transformation. By Definition IVS 573 we say \( P_3 \) and \( M_{22} \) are isomorphic.

At a basic level, the term “isomorphic” is nothing more than a codeword for the presence of an invertible linear transformation. However, it is also a description of a powerful idea, and this power only becomes apparent in the course of studying examples and related theorems. In this example, we are led to believe that there is nothing “structurally” different about \( P_3 \) and \( M_{22} \). In a certain sense they are the same. Not equal, but the same. One is as good as the other. One is just as interesting as the other.

Here is an extremely basic application of this idea. Suppose we want to compute the following linear combination of polynomials in \( P_3 \),

\[
5(2 + 3x - 4x^2 + 5x^3) + (-3)(3 - 5x + 3x^2 + x^3)
\]

Rather than doing it straight-away (which is very easy), we will apply the transformation \( T \) to convert into a linear combination of matrices, and then compute in \( M_{22} \) according to the definitions of the vector space operations there (Example VSM 311),

\[
T \left( 5(2 + 3x - 4x^2 + 5x^3) + (-3)(3 - 5x + 3x^2 + x^3) \right) = 5T \left( 2 + 3x - 4x^2 + 5x^3 \right) + (-3)T \left( 3 - 5x + 3x^2 + x^3 \right) \quad \text{Theorem LTLC 512}
\]

\[
= 5 \begin{bmatrix} 5 & 10 \\ 5 & -2 \end{bmatrix} + (-3) \begin{bmatrix} -2 & -3 \\ 1 & -6 \end{bmatrix} \quad \text{Definition of } T
\]

\[
= \begin{bmatrix} 31 & 59 \\ 22 & 8 \end{bmatrix} \quad \text{Operations in } M_{22}
\]

Now we will translate our answer back to \( P_3 \) by applying \( T^{-1} \), which we found in Example AIVLT 567,

\[
T^{-1}: M_{22} \rightarrow P_3, \quad T^{-1} \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a - c - d) + (c + d)x + \frac{1}{2}(a - b - c - d)x^2 + cx^3
\]

We compute,

\[
T^{-1} \left( \begin{bmatrix} 31 & 59 \\ 22 & 8 \end{bmatrix} \right) = 1 + 30x - 29x^2 + 22x^3
\]

which is, as expected, exactly what we would have computed for the original linear combination had we just used the definitions of the operations in \( P_3 \) (Example VSP 312).

\[\ddagger\]
Checking the dimensions of two vector spaces can be a quick way to establish that they are not isomorphic. Here’s the theorem.

**Theorem IVSED**  
**Isomorphic Vector Spaces have Equal Dimension**

Suppose $U$ and $V$ are isomorphic vector spaces. Then $\dim(U) = \dim(V)$. □

**Proof**. If $U$ and $V$ are isomorphic, there is an invertible linear transformation $T: U \rightarrow V$ (Definition IVS 573). $T$ is injective by Theorem ILTIS 571 and so by Theorem ILTD 538, $\dim(U) \leq \dim(V)$. Similarly, $T$ is surjective by Theorem ILTIS 571 and so by Theorem SLTD 558, $\dim(U) \geq \dim(V)$. The net effect of these two inequalities is that $\dim(U) = \dim(V)$. ■

The contrapositive of Theorem IVSED 575 says that if $U$ and $V$ have different dimensions, then they are not isomorphic. Dimension is the simplest “structural” characteristic that will allow you to distinguish non-isomorphic vector spaces. For example $P_6$ is not isomorphic to $M_{34}$ since their dimensions (7 and 12, respectively) are not equal. With tools developed in Section VR 587 we will be able to establish that the converse of Theorem IVSED 575 is true. Think about that one for a moment.

**Subsection RNLT**  
**Rank and Nullity of a Linear Transformation**

Just as a matrix has a rank and a nullity, so too do linear transformations. And just like the rank and nullity of a matrix are related (they sum to the number of columns, Theorem RPNC 387) the rank and nullity of a linear transformation are related. Here are the definitions and theorems, see the Archetypes (Appendix A 717) for loads of examples.

**Definition ROLT**
**Rank Of a Linear Transformation**

Suppose that $T: U \rightarrow V$ is a linear transformation. Then the rank of $T$, $\text{r}(T)$, is the dimension of the range of $T$,

$$\text{r}(T) = \dim(\mathcal{R}(T))$$

(This definition contains Notation ROLT.) △

**Definition NOLT**
**Nullity Of a Linear Transformation**

Suppose that $T: U \rightarrow V$ is a linear transformation. Then the nullity of $T$, $\text{n}(T)$, is the dimension of the kernel of $T$,

$$\text{n}(T) = \dim(\mathcal{K}(T))$$

(This definition contains Notation NOLT.) △

Here are two quick theorems.
Theorem ROSLT
Rank Of a Surjective Linear Transformation
Suppose that \( T: U \rightarrow V \) is a linear transformation. Then the rank of \( T \) is the dimension of \( V \), \( r (T) = \dim (V) \), if and only if \( T \) is surjective.

\[ r (T) = \dim (V) \]

\[ \square \]

Proof By Theorem RSLT \[ 554 \], \( T \) is surjective if and only if \( \mathcal{R}(T) = V \). Applying Definition ROLT \[ 575 \], \( \mathcal{R}(T) = V \) if and only if \( r (T) = \dim (\mathcal{R}(T)) = \dim (V) \).

\[ \square \]

Theorem NOILT
Nullity Of an Injective Linear Transformation
Suppose that \( T: U \rightarrow V \) is an injective linear transformation. Then the nullity of \( T \) is zero, \( n(T) = 0 \), if and only if \( T \) is injective.

\[ n(T) = 0 \]

\[ \square \]

Proof By Theorem KILT \[ 535 \], \( T \) is injective if and only if \( \mathcal{K}(T) = \{ 0 \} \). Applying Definition NOLT \[ 575 \], \( \mathcal{K}(T) = \{ 0 \} \) if and only if \( n(T) = 0 \).

\[ \square \]

Just as injectivity and surjectivity come together in invertible linear transformations, there is a clear relationship between rank and nullity of a linear transformation. If one is big, the other is small.

Theorem RPNDD
Rank Plus Nullity is Domain Dimension
Suppose that \( T: U \rightarrow V \) is a linear transformation. Then

\[ r (T) + n (T) = \dim (U) \]

\[ \square \]

Proof Let \( r = r (T) \) and \( s = n (T) \). Suppose that \( \mathcal{R} = \{ v_1, v_2, v_3, \ldots, v_r \} \subseteq V \) is a basis of the range of \( T \), \( \mathcal{R}(T) \), and \( \mathcal{K} = \{ u_1, u_2, u_3, \ldots, u_s \} \subseteq U \) is a basis of the kernel of \( T \), \( \mathcal{K}(T) \). Note that \( \mathcal{R} \) and \( \mathcal{K} \) are possibly empty, which means that some of the sums in this proof are “empty” and are equal to the zero vector.

Because the elements of \( \mathcal{R} \) are all in the range of \( T \), each must have a non-empty pre-image by Theorem RPI \[ 557 \]. Choose vectors \( w_i \in U \), \( 1 \leq i \leq r \) such that \( w_i \in T^{-1}(v_i) \). So \( T(w_i) = v_i \), \( 1 \leq i \leq r \). Consider the set

\[ B = \{ u_1, u_2, u_3, \ldots, u_s, w_1, w_2, w_3, \ldots, w_r \} \]

We claim that \( B \) is a basis for \( U \).

To establish linear independence for \( B \), begin with a relation of linear dependence on \( B \). So suppose there are scalars \( a_1, a_2, a_3, \ldots, a_s \) and \( b_1, b_2, b_3, \ldots, b_r \)

\[ 0 = a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_s u_s + b_1 w_1 + b_2 w_2 + b_3 w_3 + \cdots + b_r w_r \]

Then

\[ 0 = T (0) \]

\[ = T (a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_s u_s + b_1 w_1 + b_2 w_2 + b_3 w_3 + \cdots + b_r w_r ) \]

Substitution

\[ = a_1 T (u_1) + a_2 T (u_2) + a_3 T (u_3) + \cdots + a_s T (u_s) + b_1 T (w_1) + b_2 T (w_2) + b_3 T (w_3) + \cdots + b_r T (w_r ) \]

Theorem LTLC \[ 512 \]

\[ = a_1 0 + a_2 0 + a_3 0 + \cdots + a_s 0 \]
This is a relation of linear dependence on $R$ (Definition RLD [345]), and since $R$ is a linearly independent set (Definition LI [345]), we see that $b_1 = b_2 = b_3 = \ldots = b_r = 0$. Then the original relation of linear dependence on $B$ becomes

\[ 0 = a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_su_s + 0w_1 + 0w_2 + \ldots + 0w_r \]

\[ = a_1u_1 + a_2u_2 + a_3u_3 + \cdots + a_su_s + 0 + 0 + \ldots + 0 \]

But this is again a relation of linear independence (Definition RLD [345]), now on the set $S$. Since $S$ is linearly independent (Definition LI [345]), we have $a_1 = a_2 = a_3 = \ldots = a_r = 0$. Since we now know that all the scalars in the relation of linear dependence on $B$ must be zero, we have established the linear independence of $S$ through Definition LI [345].

To now establish that $B$ spans $U$, choose an arbitrary vector $u \in U$. Then $T(u) \in R(T)$, so there are scalars $c_1, c_2, c_3, \ldots, c_r$ such that

\[ T(u) = c_1v_1 + c_2v_2 + c_3v_3 + \cdots + c_rv_r \]

Use the scalars $c_1, c_2, c_3, \ldots, c_r$ to define a vector $y \in U$,

\[ y = c_1w_1 + c_2w_2 + c_3w_3 + \cdots + c_rw_r \]

Then

\[ T(u - y) = T(u) - T(y) \]

\[ = T(u) - T(c_1w_1 + c_2w_2 + c_3w_3 + \cdots + c_rw_r) \]

\[ = T(u) - (c_1T(w_1) + c_2T(w_2) + \cdots + c_rT(w_r)) \]

\[ = T(u) - (c_1v_1 + c_2v_2 + c_3v_3 + \cdots + c_rv_r) \]

\[ = T(u) - T(u) \]

\[ = 0 \]

So the vector $u - y$ is sent to the zero vector by $T$ and hence is an element of the kernel of $T$. As such it can be written as a linear combination of the basis vectors for $K(T)$, the elements of the set $S$. So there are scalars $d_1, d_2, d_3, \ldots, d_r$ such that

\[ u - y = d_1u_1 + d_2u_2 + d_3u_3 + \cdots + d_ru_s \]

Then

\[ u = (u - y) + y \]

\[ = d_1u_1 + d_2u_2 + d_3u_3 + \cdots + d_su_s + c_1w_1 + c_2w_2 + c_3w_3 + \cdots + c_rw_r \]
This says that for any vector, \( \mathbf{u} \), from \( U \), there exist scalars \( (d_1, d_2, d_3, \ldots, d_s, c_1, c_2, c_3, \ldots, c_r) \) that form \( \mathbf{u} \) as a linear combination of the vectors in the set \( B \). In other words, \( B \) spans \( U \) (Definition SS [332]).

So \( B \) is a basis (Definition B [363]) of \( U \) with \( s + r \) vectors, and thus

\[
\dim (U) = s + r = n (T) + r (T)
\]

as desired. ■

Theorem RPNC [387] said that the rank and nullity of a matrix sum to the number of columns of the matrix. This result is now an easy consequence of Theorem RPNDD [576] when we consider the linear transformation \( T: \mathbb{C}^n \rightarrow \mathbb{C}^m \) defined with the \( m \times n \) matrix \( A \) by \( T(\mathbf{x}) = A\mathbf{x} \). The range and kernel of \( T \) are identical to the column space and null space of the matrix \( A \) (can you prove this?), so the rank and nullity of the matrix \( A \) are identical to the rank and nullity of the linear transformation \( T \). The dimension of the domain of \( T \) is the dimension of \( \mathbb{C}^n \), exactly the number of columns for the matrix \( A \).

This theorem can be especially useful in determining basic properties of linear transformations. For example, suppose that \( T: \mathbb{C}^6 \rightarrow \mathbb{C}^6 \) is a linear transformation and you are able to quickly establish that the kernel is trivial. Then \( n(T) = 0 \). First this means that \( T \) is injective by Theorem NOILT [576]. Also, Theorem RPNDD [576] becomes

\[
6 = \dim (\mathbb{C}^6) = r(T) + n(T) = r(T) + 0 = r(T)
\]

So the rank of \( T \) is equal to the rank of the codomain, and by Theorem ROSLT [575] we know \( T \) is surjective. Finally, we know \( T \) is invertible by Theorem ILTIS [571]. So from the determination that the kernel is trivial, and consideration of various dimensions, the theorems of this section allow us to conclude the existence of an inverse linear transformation for \( T \).

Similarly, Theorem RPNN [576] can be used to provide alternative proofs for Theorem ILTD [538], Theorem SLTD [558] and Theorem IVSED [575]. It would be an interesting exercise to construct these proofs.

It would be instructive to study the archetypes that are linear transformations and see how many of their properties can be deduced just from considering the dimensions of the domain and codomain, and possibly with just the nullity or rank. The table preceding all of the archetypes could be a good place to start this analysis.

### Subsection SLELT

**Systems of Linear Equations and Linear Transformations**

This subsection does not really belong in this section, or any other section, for that matter. It is just the right time to have a discussion about the connections between the central topic of linear algebra, linear transformations, and our motivating topic from Chapter SLE [3], systems of linear equations. We will discuss several theorems we have seen already, but we will also make some forward-looking statements that will be justified in Chapter R [587].

Archetype D [735] and Archetype E [739] are ideal examples to illustrate connections
with linear transformations. Both have the same coefficient matrix,

\[ D = \begin{bmatrix}
2 & 1 & 7 & -7 \\
-3 & 4 & -5 & -6 \\
1 & 1 & 4 & -5
\end{bmatrix} \]

To apply the theory of linear transformations to these two archetypes, employ matrix multiplication (Definition MM [215]) and define the linear transformation,

\[ T: \mathbb{C}^4 \rightarrow \mathbb{C}^3, \quad T(x) = Dx = x_1 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -6 \\ -5 \end{bmatrix} \]

Theorem MBLT 509 tells us that \( T \) is indeed a linear transformation. Archetype D 735 asks for solutions to \( LS(D, b) \), where \( b = \begin{bmatrix} 8 \\ -12 \\ -4 \end{bmatrix} \). In the language of linear transformations this is equivalent to asking for \( T^{-1}(b) \). In the language of vectors and matrices it asks for a linear combination of the four columns of \( D \) that will equal \( b \). One solution listed is \( w = \begin{bmatrix} 7 \\ 8 \\ 1 \\ 3 \end{bmatrix} \). With a non-empty preimage, Theorem KPI 535 tells us that the complete solution set of the linear system is the preimage of \( b \),

\[ w + \mathcal{K}(T) = \{ w + z \mid z \in \mathcal{K}(T) \} \]

The kernel of the linear transformation \( T \) is exactly the null space of the matrix \( D \) (see Exercise ILT.T20 [542]), so this approach to the solution set should be reminiscent of Theorem PSPHS 113. The kernel of the linear transformation is the preimage of the zero vector, exactly equal to the solution set of the homogeneous system \( LS(D, 0) \). Since \( D \) has a null space of dimension two, every preimage (and in particular the preimage of \( b \)) is as “big” as a subspace of dimension two (but is not a subspace). Archetype E 739 is identical to Archetype D 735 but with a different vector of constants, \( d = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \). We can use the same linear transformation \( T \) to discuss this system of equations since the coefficient matrix is identical. Now the set of solutions to \( LS(D, d) \) is the pre-image of \( d \), \( T^{-1}(d) \). However, the vector \( d \) is not in the range of the linear transformation (nor is it in the column space of the matrix, since these two sets are equal by Exercise SLT.T20 [562]). So the empty pre-image is equivalent to the inconsistency of the linear system.

These two archetypes each have three equations in four variables, so either the resulting linear systems are inconsistent, or they are consistent and application of Theorem CMVEI 57 tells us that the system has infinitely many solutions. Considering these same parameters for the linear transformation, the dimension of the domain, \( \mathbb{C}^4 \), is four, while the codomain, \( \mathbb{C}^3 \), has dimension three. Then

\[ n(T) = \dim(\mathbb{C}^4) - r(T) \]
\[ = 4 - \dim(\mathcal{R}(T)) \]

Theorem RPNDD 576

Definition ROLT 575

Version 0.92
\[ \geq 4 - 3 \quad \Rightarrow \quad \mathcal{R}(T) \text{ subspace of } \mathbb{C}^3 \]

So the kernel of \( T \) is nontrivial simply by considering the dimensions of the domain (number of variables) and the codomain (number of equations). Pre-images of elements of the codomain that are not in the range of \( T \) are empty (inconsistent systems). For elements of the codomain that are in the range of \( T \) (consistent systems), Theorem KPI \([535]\) tells us that the pre-images are built from the kernel, and with a non-trivial kernel, these pre-images are infinite (infinitely many solutions).

When do systems of equations have unique solutions? Consider the system of linear equations \( LS(C, f) \) and the linear transformation \( S(x) = Cx \). If \( S \) has a trivial kernel, then pre-images will either be empty or be finite sets with single elements. Correspondingly, the coefficient matrix \( C \) will have a trivial null space and solution sets will either be empty (inconsistent) or contain a single solution (unique solution). Should the matrix be square and have a trivial null space then we recognize the matrix as being nonsingular. A square matrix means that the corresponding linear transformation, \( T \), has equal-sized domain and codomain. With a nullity of zero, \( T \) is injective, and also Theorem RPNDD \([576]\) tells us that rank of \( T \) is equal to the dimension of the domain, which in turn is equal to the dimension of the codomain. In other words, \( T \) is surjective. Injective and surjective, and Theorem ILTIS \([571]\) tells us that \( T \) is invertible. Just as we can use the inverse of the coefficient matrix to find the unique solution of any linear system with a nonsingular coefficient matrix (Theorem SNCM \([252]\)), we can use the inverse of the linear transformation to construct the unique element of any pre-image (proof of Theorem ILTIS \([571]\)).

The executive summary of this discussion is that to every coefficient matrix of a system of linear equations we can associate a natural linear transformation. Solution sets for systems with this coefficient matrix are preimages of elements of the codomain of the linear transformation. For every theorem about systems of linear equations there is an analogue about linear transformations. The theory of linear transformations provides all the tools to recreate the theory of solutions to linear systems of equations.

We will continue this adventure in Chapter R \([587]\).

Subsection READ

Reading Questions

1. What conditions allow us to easily determine if a linear transformation is invertible?
2. What does it mean to say two vector spaces are isomorphic? Both technically, and informally?
3. How do linear transformations relate to systems of linear equations?
Subsection EXC

Exercises

C10  The archetypes below are linear transformations of the form $T: U \rightarrow V$ that are invertible. For each, the inverse linear transformation is given explicitly as part of the archetype’s description. Verify for each linear transformation that

$$T^{-1} \circ T = I_U \quad \quad T \circ T^{-1} = I_V$$

Archetype R [790], Archetype V [801], Archetype W [803]  
Contributed by Robert Beezer

C20  Determine if the linear transformation $T: P_2 \rightarrow M_{22}$ is (a) injective, (b) surjective, (c) invertible.

$$T (a + bx + cx^2) = \begin{bmatrix} a + 2b - 2c & 2a + 2b \\ -a + b - 4c & 3a + 2b + 2c \end{bmatrix}$$

Contributed by Robert Beezer  Solution [583]

C21  Determine if the linear transformation $S: P_3 \rightarrow M_{22}$ is (a) injective, (b) surjective, (c) invertible.

$$S (a + bx + cx^2 + dx^3) = \begin{bmatrix} -a + 4b + c + 2d & 4a - b + 6c - d \\ a + 5b - 2c + 2d & a + 2c + 5d \end{bmatrix}$$

Contributed by Robert Beezer  Solution [583]

C50  Consider the linear transformation $S: M_{12} \rightarrow P_1$ from the set of $1 \times 2$ matrices to the set of polynomials of degree at most 1, defined by

$$S \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = (3a + b) + (5a + 2b)x$$

Prove that $S$ is invertible. Then show that the linear transformation $R: P_1 \rightarrow M_{12}$, $R (r + sx) = \begin{bmatrix} (2r - s) \\ (-5r + 3s) \end{bmatrix}$ is the inverse of $S$, that is $S^{-1} = R$.

Contributed by Robert Beezer  Solution [584]

M30  The linear transformation $S$ below is invertible. Find a formula for the inverse linear transformation, $S^{-1}$.

$$S: P_1 \rightarrow M_{1,2}, \quad S (a + bx) = \begin{bmatrix} 3a + b & 2a + b \end{bmatrix}$$

Contributed by Robert Beezer  Solution [584]

M31  The linear transformation $R: M_{12} \rightarrow M_{21}$ is invertible. Determine a formula for the inverse linear transformation $R^{-1}: M_{21} \rightarrow M_{12}$, (15 points)

$$R \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} a + 3b \\ 4a + 11b \end{bmatrix}$$

Contributed by Robert Beezer  Solution [584]
T15 Suppose that $T: U \rightarrow V$ is a surjective linear transformation and $\dim(U) = \dim(V)$. Prove that $T$ is injective.

T16 Suppose that $T: U \rightarrow V$ is an injective linear transformation and $\dim(U) = \dim(V)$. Prove that $T$ is surjective.
Subsection SOL
Solutions

\textbf{C20} Contributed by Robert Beezer Statement 581
(a) We will compute the kernel of $T$. Suppose that $a + bx + cx^2 \in \mathcal{K}(T)$. Then
\[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} = T (a + bx + cx^2) = \begin{bmatrix}
a + 2b - 2c & 2a + 2b \\
-a + b - 4c & 3a + 2b + 2c
\end{bmatrix}
\]
and matrix equality (Theorem ME 474) yields the homogeneous system of four equations in three variables,
\[
\begin{align*}
a + 2b - 2c &= 0 \\
2a + 2b &= 0 \\
-a + b - 4c &= 0 \\
3a + 2b + 2c &= 0
\end{align*}
\]
The coefficient matrix of this system row-reduces as
\[
\begin{bmatrix}
1 & 2 & -2 \\
2 & 2 & 0 \\
-1 & 1 & -4 \\
3 & 2 & 2
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
From the existence of non-trivial solutions to this system, we can infer non-zero polynomials in $\mathcal{K}(T)$. By Theorem KILT 535, we then know that $T$ is not injective.
(b) Since $3 = \dim (P_2) < \dim (M_{22}) = 4$, by Theorem SLTD 558 $T$ is not surjective.
(c) Since $T$ is not surjective, it is not invertible by Theorem ILTIS 571.

\textbf{C21} Contributed by Robert Beezer Statement 581
(a) To check injectivity, we compute the kernel of $S$. To this end, suppose that $a + bx + cx^2 + dx^3 \in \mathcal{K}(S)$, so
\[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} = S (a + bx + cx^2 + dx^3) = \begin{bmatrix}
-a + 4b + c + 2d & 4a - b + 6c - d \\
a + 5b - 2c + 2d & a + 2c + 5d
\end{bmatrix}
\]
this creates the homogeneous system of four equations in four variables,
\[
\begin{align*}
-a + 4b + c + 2d &= 0 \\
4a - b + 6c - d &= 0 \\
a + 5b - 2c + 2d &= 0 \\
a + 2c + 5d &= 0
\end{align*}
\]
The coefficient matrix of this system row-reduces as,
\[
\begin{bmatrix}
-1 & 4 & 1 & 2 \\
4 & -1 & 6 & -1 \\
1 & 5 & -2 & 2 \\
1 & 0 & 2 & 5
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
We recognize the coefficient matrix as being nonsingular, so the only solution to the system is \(a = b = c = d = 0\), and the kernel of \(S\) is trivial, \(\mathcal{K}(S) = \{0 + 0x + 0x^2 + 0x^3\}\). By [Theorem KILT 535](#), we see that \(S\) is injective.

(b) We can establish that \(S\) is surjective by considering the rank and nullity of \(S\).

\[
\begin{align*}
    r(S) &= \dim(P_3) - n(S) \\
    &= 4 - 0 \\
    &= \dim(M_{22})
\end{align*}
\]

So, \(\mathcal{R}(S)\) is a subspace of \(M_{22}\) ([Theorem RLTS 553](#)) whose dimension equals that of \(M_{22}\). By [Theorem EDYES 401](#), we gain the set equality \(\mathcal{R}(S) = M_{22}\). [Theorem RSLT 554](#) then implies that \(S\) is surjective.

(c) Since \(S\) is both injective and surjective, [Theorem ILTIS 571](#) says \(S\) is invertible.

---

**Contributed by Robert Beezer Statement 581**

Determine the kernel of \(S\) first. The condition that \(S(\begin{bmatrix}a & b\end{bmatrix}) = 0\) becomes \((3a + b) + (5a + 2b)x = 0 + 0x\). Equating coefficients of these polynomials yields the system

\[
\begin{align*}
    3a + b &= 0 \\
    5a + 2b &= 0
\end{align*}
\]

This homogeneous system has a nonsingular coefficient matrix, so the only solution is \(a = 0, b = 0\) and thus

\[
\mathcal{K}(S) = \left\{ \begin{bmatrix} 0 & 0 \end{bmatrix} \right\}
\]

By [Theorem KILT 535](#), we know \(S\) is injective. With \(n(S) = 0\) we employ [Theorem RPNDD 576](#) to find

\[
\begin{align*}
    r(S) &= r(S) + 0 = r(S) + n(S) = \dim(M_{12}) = 2 = \dim(P_1)
\end{align*}
\]

Since \(\mathcal{R}(S) \subseteq P_1\) and \(\dim(\mathcal{R}(S)) = \dim(P_1)\), we can apply [Theorem EDYES 401](#) to obtain the set equality \(\mathcal{R}(S) = P_1\) and therefore \(S\) is surjective.

One of the two defining conditions of an invertible linear transformation is ([Definition IVLT 567](#))

\[
(S \circ R)(a + bx) = S(R(a + bx))
\]

\[
= S \left( \begin{bmatrix} (2a - b) & -(5a + 3b) \end{bmatrix} \right)
\]

\[
= (3(2a - b) + (-5a + 3b)) + (5(2a - b) + 2(-5a + 3b))x
\]

\[
= (6a - 3b) + (-5a + 3b) + (10a - 5b) + (-10a + 6b))x
\]

\[
= a + bx
\]

\[
= I_{P_1} (a + bx)
\]

That \((R \circ S)(\begin{bmatrix}a & b\end{bmatrix}) = I_{M_{12}} (\begin{bmatrix}a & b\end{bmatrix})\) is similar.

**Contributed by Robert Beezer Statement 581**

Suppose that \(S^{-1}: M_{1,2} \mapsto P_1\) has a form given by

\[
S^{-1}(z \ w) = (rz + sw) + (pz + qw)x
\]

where \(r, s, p, q\) are unknown scalars. Then

\[
a + bx = S^{-1}(S(a + bx))
\]
= S^{-1} \left( \begin{bmatrix} 3a + b & 2a + b \end{bmatrix} \right)
= (r(3a + b) + s(2a + b)) + (p(3a + b) + q(2a + b)) x
= ((3r + 2s)a + (r + s)b) + ((3p + 2q)a + (p + q)b) x

Equating coefficients of these two polynomials, and then equating coefficients on \( a \) and 
\( b \), gives rise to 4 equations in 4 variables,

\[
\begin{align*}
3r + 2s &= 1 \\
r + s &= 0 \\
3p + 2q &= 0 \\
p + q &= 1
\end{align*}
\]

This system has a unique solution: \( r = 1, \ s = -1, \ p = -2, \ q = 3 \). So the desired inverse 
linear transformation is

\[
S^{-1}(z \ w) = (z - w) + (-2z + 3w) x
\]

Notice that the system of 4 equations in 4 variables could be split into two systems, 
each with two equations in two variables (and identical coefficient matrices). After mak-
ing this split, the solution might feel like computing the inverse of a matrix (Theo-
rem CINM 237). Hmmmm.

M31 Contributed by Robert Beezer Statement 581

We are given that \( R \) is invertible. The inverse linear transformation can be formulated 
by considering the pre-image of a generic element of the codomain. With injectivity and 
surjectivity, we know that the pre-image of any element will be a set of size one — it is 
this lone element that will be the output of the inverse linear transformation.

Suppose that we set \( \mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} \) as a generic element of the codomain, \( M_{21} \). Then if
\[
\begin{bmatrix} r \\ s \end{bmatrix} = \mathbf{w} \in R^{-1}(\mathbf{v}),
\]

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{v} = R(\mathbf{w}) = \begin{bmatrix} r + 3s \\ 4r + 11s \end{bmatrix}
\]

So we obtain the system of two equations in the two variables \( r \) and \( s \),

\[
\begin{align*}
r + 3s &= x \\
4r + 11s &= y
\end{align*}
\]

With a nonsingular coefficient matrix, we can solve the system using the inverse of the 
coefficient matrix,

\[
\begin{align*}
r &= -11x + 3y \\
s &= 4x - y
\end{align*}
\]

So we define,

\[
R^{-1}(\mathbf{v}) = R^{-1}(\begin{bmatrix} x \\ y \end{bmatrix}) = \mathbf{w} = \begin{bmatrix} r \\ s \end{bmatrix} = [-11x + 3y \ 4x - y]
\]
If $T$ is surjective, then Theorem RSLT \[554\] says $\mathcal{R}(T) = V$, so $r(T) = \dim(V)$. In turn, the hypothesis gives $r(T) = \dim(U)$. Then, using Theorem RPNDD \[576\],

$$n(T) = (r(T) + n(T)) - r(T) = \dim(U) - \dim(U) = 0$$

With a null space of zero dimension, $\mathcal{K}(T) = \{0\}$, and by Theorem KILT \[535\] we see that $T$ is injective. $T$ is both injective and surjective so by Theorem ILTIS \[571\], $T$ is invertible.
Chapter R
Representations

Previous work with linear transformations may have convinced you that we can convert most questions about linear transformations into questions about systems of equations or properties of subspaces of \( \mathbb{C}^m \). In this section we begin to make these vague notions precise. We have used the word “representation” prior, but it will get a heavy workout in this chapter. In many ways, everything we have studied so far was in preparation for this chapter.

Section VR
Vector Representations

We begin by establishing an invertible linear transformation between any vector space \( V \) of dimension \( m \) and \( \mathbb{C}^m \). This will allow us to “go back and forth” between the two vector spaces, no matter how abstract the definition of \( V \) might be.

**Definition VR**
Vector Representation

Suppose that \( V \) is a vector space with a basis \( B = \{ v_1, v_2, v_3, \ldots, v_n \} \). Define a function \( \rho_B: V \mapsto \mathbb{C}^n \) as follows. For \( w \in V \), find scalars \( a_1, a_2, a_3, \ldots, a_n \) so that

\[
  w = a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_n v_n
\]

then

\[
  [\rho_B(w)]_i = a_i \quad 1 \leq i \leq n
\]

We need to show that \( \rho_B \) is really a function (since “find scalars” sounds like it could be accomplished in many ways, or perhaps not at all) and right now we want to establish that \( \rho_B \) is a linear transformation. We will wrap up both objectives in one theorem, even though the first part is working backwards to make sure that \( \rho_B \) is well-defined.

**Theorem VRLT**

Vector Representation is a Linear Transformation

The function \( \rho_B \) (Definition VR 587) is a linear transformation. \qed
Proof The definition of \( \rho_B \) (Definition VR 587) appears to allow considerable latitude in selecting the scalars \( a_1, a_2, a_3, \ldots, a_n \). However, since \( B \) is a basis for \( V \), Theorem VRRB 355 says this can be done, and done uniquely. So despite appearances, \( \rho_B \) is indeed a function.

Suppose that \( \mathbf{x} \) and \( \mathbf{y} \) are two vectors in \( V \) and \( \alpha \in \mathbb{C} \). Then the vector space properties (Definition VS 309) assure us that the vectors \( \mathbf{x} + \mathbf{y} \) and \( \alpha \mathbf{x} \) are also vectors in \( V \). Theorem VRRB 355 then provides the following sets of scalars for the four vectors \( \mathbf{x}, \mathbf{y}, \mathbf{x} + \mathbf{y} \) and \( \alpha \mathbf{x} \), and tells us that each set of scalars is the only way to express the given vector as a linear combination of the basis vectors in \( B \).

\[
\mathbf{x} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \cdots + a_n \mathbf{v}_n \\
\mathbf{y} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + b_3 \mathbf{v}_3 + \cdots + b_n \mathbf{v}_n \\
\mathbf{x} + \mathbf{y} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 + \cdots + c_n \mathbf{v}_n \\
\alpha \mathbf{x} = d_1 \mathbf{v}_1 + d_2 \mathbf{v}_2 + d_3 \mathbf{v}_3 + \cdots + d_n \mathbf{v}_n
\]

Then these coefficients are related, as we now show.

\[
\mathbf{x} + \mathbf{y} = (a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \cdots + a_n \mathbf{v}_n) \\
\quad + (b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + b_3 \mathbf{v}_3 + \cdots + b_n \mathbf{v}_n) \\
= a_1 \mathbf{v}_1 + b_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + b_2 \mathbf{v}_2 + \cdots + a_n \mathbf{v}_n + b_n \mathbf{v}_n \\
= (a_1 + b_1) \mathbf{v}_1 + (a_2 + b_2) \mathbf{v}_2 + \cdots + (a_n + b_n) \mathbf{v}_n
\]

By the uniqueness of the expression of \( \mathbf{x} + \mathbf{y} \) as a linear combination of the vectors in \( B \) (Theorem VRRB 355), we conclude that \( c_i = a_i + b_i \), \( 1 \leq i \leq n \).

Similarly,

\[
\alpha \mathbf{x} = \alpha (a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \cdots + a_n \mathbf{v}_n) \\
= \alpha a_1 \mathbf{v}_1 + \alpha a_2 \mathbf{v}_2 + \alpha a_3 \mathbf{v}_3 + \cdots + \alpha a_n \mathbf{v}_n \\
= (\alpha a_1) \mathbf{v}_1 + (\alpha a_2) \mathbf{v}_2 + (\alpha a_3) \mathbf{v}_3 + \cdots + (\alpha a_n) \mathbf{v}_n
\]

By the uniqueness of the expression of \( \alpha \mathbf{x} \) as a linear combination of the vectors in \( B \) (Theorem VRRB 355), we conclude that \( d_i = \alpha a_i \), \( 1 \leq i \leq n \).

Now, for \( 1 \leq i \leq n \), we have

\[
[\rho_B (\mathbf{x} + \mathbf{y})]_i = c_i \\
= a_i + b_i \\
= [\rho_B (\mathbf{x})]_i + [\rho_B (\mathbf{y})]_i \\
= [\rho_B (\mathbf{x}) + \rho_B (\mathbf{y})]_i
\]

Thus the vectors \( \rho_B (\mathbf{x} + \mathbf{y}) \) and \( \rho_B (\mathbf{x}) + \rho_B (\mathbf{y}) \) are equal in each entry and Definition CVE 88 tells us that \( \rho_B (\mathbf{x} + \mathbf{y}) = \rho_B (\mathbf{x}) + \rho_B (\mathbf{y}) \). This is the first necessary property for \( \rho_B \) to be a linear transformation (Definition LT 503).

Similarly, for \( 1 \leq i \leq n \), we have

\[
[\rho_B (\alpha \mathbf{x})]_i = d_i
\]
\[ \alpha \alpha_i = \alpha \begin{bmatrix} \rho_B(x) \end{bmatrix}_i \]
\[ = \begin{bmatrix} \alpha \rho_B(x) \end{bmatrix}_i \]

and so, the vectors \( \rho_B(\alpha x) \) and \( \alpha \rho_B(x) \) are equal in each entry and therefore by Definition CVE [88] we have the vector equality \( \rho_B(\alpha x) = \alpha \rho_B(x) \). This establishes the second property of a linear transformation (Definition LT [503]) so we can conclude that \( \rho_B \) is a linear transformation.

\[ \blacksquare \]

**Example VRC4**

**Vector representation in \( \mathbb{C}^4 \)**

Consider the vector \( y \in \mathbb{C}^4 \)

\[ y = \begin{bmatrix} 6 \\ 14 \\ 6 \\ 7 \end{bmatrix} \]

We will find several coordinate representations of \( y \) in this example. Notice that \( y \) never changes, but the representations of \( y \) do change.

One basis for \( \mathbb{C}^4 \) is

\[ B = \{ u_1, u_2, u_3, u_4 \} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} \right\} \]

as can be seen by making these vectors the columns of a matrix, checking that the matrix is nonsingular and applying Theorem CNMB [369]. To find \( \rho_B(y) \), we need to find scalars, \( a_1, a_2, a_3, a_4 \) such that

\[ y = a_1 u_1 + a_2 u_2 + a_3 u_3 + a_4 u_4 \]

By Theorem SLSLC [100] the desired scalars are a solution to the linear system of equations with a coefficient matrix whose columns are the vectors in \( B \) and with a vector of constants \( y \). With a nonsingular coefficient matrix, the solution is unique, but this is no surprise as this is the content of Theorem VRRB [355]. This unique solution is

\[ a_1 = 2 \quad a_2 = -1 \quad a_3 = -3 \quad a_4 = 4 \]

Then by Definition VR [587], we have

\[ \rho_B(y) = \begin{bmatrix} 2 \\ -1 \\ -3 \\ -4 \end{bmatrix} \]

Suppose now that we construct a representation of \( y \) relative to another basis of \( \mathbb{C}^4 \),

\[ C = \left\{ \begin{bmatrix} -15 \\ 9 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 16 \\ -14 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} -26 \\ 14 \\ -6 \\ -3 \end{bmatrix}, \begin{bmatrix} 14 \\ -13 \\ 4 \end{bmatrix} \right\} \]
As with $B$, it is easy to check that $C$ is a basis. Writing $y$ as a linear combination of the vectors in $C$ leads to solving a system of four equations in the four unknown scalars with a nonsingular coefficient matrix. The unique solution can be expressed as

$$
y = \begin{bmatrix} 6 \\ 14 \\ 6 \\ 7 \end{bmatrix} = (-28) \begin{bmatrix} -15 \\ 9 \\ -4 \\ -2 \end{bmatrix} + (-8) \begin{bmatrix} 16 \\ -14 \\ 5 \\ 2 \end{bmatrix} + 11 \begin{bmatrix} -26 \\ 14 \\ -6 \\ -3 \end{bmatrix} + 0 \begin{bmatrix} 14 \\ 0 \\ 4 \\ 6 \end{bmatrix}
$$

so that Definition VR 587 gives

$$
\rho_C(y) = \begin{bmatrix} -28 \\ -8 \\ 11 \\ 0 \end{bmatrix}
$$

We often perform representations relative to standard bases, but for vectors in $\mathbb{C}^m$ its a little silly. Let’s find the vector representation of $y$ relative to the standard basis (Theorem SUVB 363),

$$D = \{e_1, e_2, e_3, e_4\}$$

Then, without any computation, we can check that

$$
y = \begin{bmatrix} 6 \\ 14 \\ 6 \\ 7 \end{bmatrix} = 6e_1 + 14e_2 + 6e_3 + 7e_4
$$

so by Definition VR 587,

$$
\rho_D(y) = \begin{bmatrix} 6 \\ 14 \\ 6 \\ 7 \end{bmatrix}
$$

which is not very exciting. Notice however that the order in which we place the vectors in the basis is critical to the representation. Let’s keep the standard unit vectors as our basis, but rearrange the order we place them in the basis. So a fourth basis is

$$E = \{e_3, e_4, e_2, e_1\}$$

Then,

$$
y = \begin{bmatrix} 6 \\ 14 \\ 6 \\ 7 \end{bmatrix} = 6e_3 + 7e_4 + 14e_2 + 6e_1
$$

so by Definition VR 587,

$$
\rho_E(y) = \begin{bmatrix} 6 \\ 7 \\ 14 \\ 6 \end{bmatrix}
$$

So for every possible basis of $\mathbb{C}^4$ we could construct a different representation of $y$. ☑
Vector representations are most interesting for vector spaces that are not $\mathbb{C}^n$.

**Example VRP2**

**Vector representations in $P_2$**

Consider the vector $\mathbf{u} = 15 + 10x - 6x^2 \in P_2$ from the vector space of polynomials with degree at most 2 (Example VSP 312). A nice basis for $P_2$ is

$$B = \{1, x, x^2\}$$

so that

$$\mathbf{u} = 15 + 10x - 6x^2 = 15(1) + 10(x) + (-6)(x^2)$$

so by Definition VR 587

$$\rho_B(\mathbf{u}) = \begin{bmatrix} 15 \\ 10 \\ -6 \end{bmatrix}$$

Another nice basis for $P_2$ is

$$B = \{1, 1 + x, 1 + x + x^2\}$$

so that now it takes a bit of computation to determine the scalars for the representation. We want $a_1, a_2, a_3$ so that

$$15 + 10x - 6x^2 = a_1(1) + a_2(1 + x) + a_3(1 + x + x^2)$$

Performing the operations in $P_2$ on the right-hand side, and equating coefficients, gives the three equations in the three unknown scalars,

$$15 = a_1 + a_2 + a_3$$
$$10 = a_2 + a_3$$
$$-6 = a_3$$

The coefficient matrix of this system is nonsingular, leading to a unique solution (no surprise there, see Theorem VRRB 355),

$$a_1 = 5 \quad a_2 = 16 \quad a_3 = -6$$

so by Definition VR 587

$$\rho_C(\mathbf{u}) = \begin{bmatrix} 5 \\ 16 \\ -6 \end{bmatrix}$$

While we often form vector representations relative to “nice” bases, nothing prevents us from forming representations relative to “nasty” bases. For example, the set

$$D = \{-2 - x + 3x^2, 1 - 2x^2, 5 + 4x + x^2\}$$

can be verified as a basis of $P_2$ by checking linear independence with Definition LI 345 and then arguing that 3 vectors from $P_2$, a vector space of dimension 3 (Theorem DP 383), must also be a spanning set (Theorem G 398). Now we desire scalars $a_1, a_2, a_3$ so that

$$15 + 10x - 6x^2 = a_1(-2 - x + 3x^2) + a_2(1 - 2x^2) + a_3(5 + 4x + x^2)$$
Performing the operations in \( P_2 \) on the right-hand side, and equating coefficients, gives the three equations in the three unknown scalars,

\[
\begin{align*}
15 &= -2a_1 + a_2 + 5a_3 \\
10 &= -a_1 + 4a_3 \\
-6 &= 3a_1 - 2a_2 + a_3
\end{align*}
\]

The coefficient matrix of this system is nonsingular, leading to a unique solution (no surprise there, see Theorem VRRB 355),

\[
\begin{align*}
a_1 &= -2 \\
a_2 &= 1 \\
a_3 &= 2
\end{align*}
\]

so by Definition VR 587

\[
\rho_B(u) = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}
\]

\[\Box\]

**Theorem VRI**

**Vector Representation is Injective**

The function \( \rho_B \) (Definition VR 587) is an injective linear transformation.  \(\square\)

**Proof**  We will appeal to Theorem KILT 535. Suppose \( U \) is a vector space of dimension \( n \), so vector representation is of the form \( \rho_B : U \mapsto \mathbb{C}^n \). Let \( B = \{ u_1, u_2, u_3, \ldots, u_n \} \) be the basis of \( U \) used in the definition of \( \rho_B \). Suppose \( u \in \mathcal{K}(\rho_B) \). Finally, since \( B \) is a basis for \( U \), by Theorem VRRB 355 there are (unique) scalars, \( a_1, a_2, a_3, \ldots, a_n \) such that

\[
u = a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_n u_n\]

Then for \( 1 \leq i \leq n \)

\[
a_i = [\rho_B(u)]_i = [0]_i, \quad u \in \mathcal{K}(\rho_B)
\]

So

\[
u = a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_n u_n = 0u_1 + 0u_2 + 0u_3 + \cdots + 0u_n = 0 + 0 + 0 + \cdots + 0 = 0\]

Thus an arbitrary vector, \( u \), from the kernel \( \mathcal{K}(\rho_B) \), must equal the zero vector of \( U \). So \( \mathcal{K}(\rho_B) = \{ 0 \} \) and by Theorem KILT 535, \( \rho_B \) is injective.  \(\Box\)

**Theorem VRS**

**Vector Representation is Surjective**

The function \( \rho_B \) (Definition VR 587) is a surjective linear transformation.  \(\square\)
Proof We will appeal to Theorem RSLT. Suppose $U$ is a vector space of dimension $n$, so vector representation is of the form $\rho_B : U \rightarrow \mathbb{C}^n$. Let $B = \{u_1, u_2, u_3, \ldots, u_n\}$ be the basis of $U$ used in the definition of $\rho_B$. Suppose $v \in \mathbb{C}^n$. Define the vector $u$ by

$$u = [v]_1 u_1 + [v]_2 u_2 + [v]_3 u_3 + \cdots + [v]_n u_n$$

Then for $1 \leq i \leq n$

$$[\rho_B(u)]_i = [\rho_B([v]_1 u_1 + [v]_2 u_2 + [v]_3 u_3 + \cdots + [v]_n u_n)]_i = [v]_i$$

so the entries of vectors $\rho_B(u)$ and $v$ are equal and Definition CVE yields the vector equality $\rho_B(u) = v$. This demonstrates that $v \in \mathcal{R}(\rho_B)$, so $\mathbb{C}^n \subseteq \mathcal{R}(\rho_B)$. Since $\mathcal{R}(\rho_B) \subseteq \mathbb{C}^n$ by Definition RLT, we have $\mathcal{R}(\rho_B) = \mathbb{C}^n$ and Theorem RSLT says $\rho_B$ is surjective.

We will have many occasions later to employ the inverse of vector representation, so we will record the fact that vector representation is an invertible linear transformation.

**Theorem VRILT**

**Vector Representation is an Invertible Linear Transformation**

The function $\rho_B$ (Definition VR) is an invertible linear transformation. □

Proof The function $\rho_B$ (Definition VR) is a linear transformation (Theorem VRLT) that is injective (Theorem VRI) and surjective (Theorem VRS) with domain $V$ and codomain $\mathbb{C}^n$. By Theorem ILTIS we then know that $\rho_B$ is an invertible linear transformation. ■

Informally, we will refer to the application of $\rho_B$ as coordinatizing a vector, while the application of $\rho_B^{-1}$ will be referred to as un-coordinatizing a vector.

Subsection CVS

**Characterization of Vector Spaces**

Limiting our attention to vector spaces with finite dimension, we now describe every possible vector space. All of them. Really.

**Theorem CFDVS**

**Characterization of Finite Dimensional Vector Spaces**

Suppose that $V$ is a vector space with dimension $n$. Then $V$ is isomorphic to $\mathbb{C}^n$. □

Proof Since $V$ has dimension $n$ we can find a basis of $V$ of size $n$ (Definition D) which we will call $B$. The linear transformation $\rho_B$ is an invertible linear transformation from $V$ to $\mathbb{C}^n$, so by Definition IVS, we have that $V$ and $\mathbb{C}^n$ are isomorphic. ■

Theorem CFDVS is the first of several surprises in this chapter, though it might be a bit demoralizing too. It says that there really are not all that many different (finite dimensional) vector spaces, and none are really any more complicated than $\mathbb{C}^n$. Hmmm. The following examples should make this point.
Example TIVS
Two isomorphic vector spaces

The vector space of polynomials with degree 8 or less, \( P_8 \), has dimension 9 (Theorem DP \[383\]). By Theorem CFDVS \[593\], \( P_8 \) is isomorphic to \( \mathbb{C}^9 \).

Example CVSR
Crazy vector space revealed

The crazy vector space, \( C \) of Example CVS \[314\], has dimension 2 by Example DC \[385\]. By Theorem CFDVS \[593\], \( C \) is isomorphic to \( \mathbb{C}^2 \). Hmmmm. Not really so crazy after all?

Example ASC
A subspace characterized

In Example DSP4 \[384\] we determined that a certain subspace \( W \) of \( P_4 \) has dimension 4. By Theorem CFDVS \[593\], \( W \) is isomorphic to \( \mathbb{C}^4 \).

Theorem IFDVS
Isomorphism of Finite Dimensional Vector Spaces

Suppose \( U \) and \( V \) are both finite-dimensional vector spaces. Then \( U \) and \( V \) are isomorphic if and only if \( \dim (U) = \dim (V) \).

Proof

\((\Rightarrow)\) This is just the statement proved in Theorem IVSED \[575\].

\((\Leftarrow)\) This is the advertised converse of Theorem IVSED \[575\]. We will assume \( U \) and \( V \) have equal dimension and discover that they are isomorphic vector spaces. Let \( n \) be the common dimension of \( U \) and \( V \). Then by Theorem CFDVS \[593\] there are isomorphisms \( T: U \rightarrow \mathbb{C}^n \) and \( S: V \rightarrow \mathbb{C}^n \).

\( T \) is therefore an invertible linear transformation by Definition IVS \[573\]. Similarly, \( S \) is an invertible linear transformation, and so \( S^{-1} \) is an invertible linear transformation (Theorem IILT \[570\]). The composition of invertible linear transformations is again invertible (Theorem CIVLT \[572\]) so the composition of \( S^{-1} \) with \( T \) is invertible. Then \( (S^{-1} \circ T): U \rightarrow V \) is an invertible linear transformation from \( U \) to \( V \) and Definition IVS \[573\] says \( U \) and \( V \) are isomorphic.

Example MIVS
Multiple isomorphic vector spaces

\( \mathbb{C}^{10}, P_9, M_{2,5} \) and \( M_{5,2} \) are all vector spaces and each has dimension 10. By Theorem IFDVS \[594\], each is isomorphic to any other.

The subspace of \( M_{4,4} \) that contains all the symmetric matrices (Definition SYM \[201\]) has dimension 10, so this subspace is also isomorphic to each of the four vector spaces above.

Subsection CP
Coordinatization Principle

With \( \rho_B \) available as an invertible linear transformation, we can translate between vectors in a vector space \( U \) of dimension \( m \) and \( \mathbb{C}^m \). Furthermore, as a linear transformation, \( \rho_B \) respects the addition and scalar multiplication in \( U \), while \( \rho_B^{-1} \) respects the addition...
and scalar multiplication in $\mathbb{C}^n$. Since our definitions of linear independence, spans, bases and dimension are all built up from linear combinations, we will finally be able to translate fundamental properties between abstract vector spaces ($U$) and concrete vector spaces ($\mathbb{C}^n$).

**Theorem CLI**

**Coordinatization and Linear Independence**

Suppose that $U$ is a vector space with a basis $B$ of size $n$. Then $S = \{u_1, u_2, u_3, \ldots, u_k\}$ is a linearly independent subset of $U$ if and only if $R = \{\rho_B (u_1), \rho_B (u_2), \rho_B (u_3), \ldots, \rho_B (u_k)\}$ is a linearly independent subset of $\mathbb{C}^n$.

**Proof** The linear transformation $\rho_B$ is an isomorphism between $U$ and $\mathbb{C}^n$ (Theorem VRILT [593]). As an invertible linear transformation, $\rho_B$ is an injective linear transformation (Theorem ILTIS [571]), and $\rho_B^{-1}$ is also an injective linear transformation (Theorem ILTI [570], Theorem ILTIS [571]).

($\Rightarrow$) Since $\rho_B$ is an injective linear transformation and $S$ is linearly independent, Theorem ILTLI [537] says that $R$ is linearly independent.

($\Leftarrow$) If we apply $\rho_B^{-1}$ to each element of $R$, we will create the set $S$. Since we are assuming $R$ is linearly independent and $\rho_B^{-1}$ is injective, Theorem ILTLI [537] says that $S$ is linearly independent.  

**Theorem CSS**

**Coordinatization and Spanning Sets**

Suppose that $U$ is a vector space with a basis $B$ of size $n$. Then $u \in \{\{u_1, u_2, u_3, \ldots, u_k\}\}$ if and only if $\rho_B (u) \in \{\{\rho_B (u_1), \rho_B (u_2), \rho_B (u_3), \ldots, \rho_B (u_k)\}\}$.

**Proof** ($\Rightarrow$) Suppose $u \in \{\{u_1, u_2, u_3, \ldots, u_k\}\}$. Then there are scalars, $a_1, a_2, a_3, \ldots, a_k$, such that

$$u = a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_k u_k$$

Then,

$$\rho_B (u) = \rho_B (a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_k u_k)$$

$$= a_1 \rho_B (u_1) + a_2 \rho_B (u_2) + a_3 \rho_B (u_3) + \cdots + a_k \rho_B (u_k)$$

which says that $\rho_B (u) \in \{\{\rho_B (u_1), \rho_B (u_2), \rho_B (u_3), \ldots, \rho_B (u_k)\}\}$.

($\Leftarrow$) Suppose that $\rho_B (u) \in \{\{\rho_B (u_1), \rho_B (u_2), \rho_B (u_3), \ldots, \rho_B (u_k)\}\}$. Then there are scalars $b_1, b_2, b_3, \ldots, b_k$ such that

$$\rho_B (u) = b_1 \rho_B (u_1) + b_2 \rho_B (u_2) + b_3 \rho_B (u_3) + \cdots + b_k \rho_B (u_k)$$

Recall that $\rho_B$ is invertible (Theorem VRILT [593]), so

$$u = I_U (u)$$

$$= (\rho_B^{-1} \circ \rho_B) (u)$$

$$= \rho_B^{-1} (\rho_B (u))$$

$$= \rho_B^{-1} (b_1 \rho_B (u_1) + b_2 \rho_B (u_2) + b_3 \rho_B (u_3) + \cdots + b_k \rho_B (u_k))$$

$$= b_1 \rho_B^{-1} (\rho_B (u_1)) + b_2 \rho_B^{-1} (\rho_B (u_2)) + b_3 \rho_B^{-1} (\rho_B (u_3))$$

$$+ \cdots + b_k \rho_B^{-1} (\rho_B (u_k))$$

$$= b_1 I_U (u_1) + b_2 I_U (u_2) + b_3 I_U (u_3) + \cdots + b_k I_U (u_k)$$
\[ b_1 u_1 + b_2 u_2 + b_3 u_3 + \cdots + b_k u_k \]

which says that \( u \in \langle \{u_1, u_2, u_3, \ldots, u_k\} \rangle \).

Here’s a fairly simple example that illustrates a very, very important idea.

**Example CP2**

**Coordinatezizing in \( P_2 \)**

In Example VRP2 we needed to know that

\[ D = \{-2 - x + 3x^2, 1 - 2x^2, 5 + 4x + x^2\} \]

is a basis for \( P_2 \). With Theorem CLI and Theorem CSS this task is much easier. First, choose a known basis for \( P_2 \), a basis that forms vector representations easily.

We will choose

\[ B = \{1, x, x^2\} \]

Now, form the subset of \( \mathbb{C}^3 \) that is the result of applying \( \rho_B \) to each element of \( D \),

\[ F = \{\rho_B (-2 - x + 3x^2), \rho_B (1 - 2x^2), \rho_B (5 + 4x + x^2)\} = \begin{pmatrix} -2 & 1 & 5 \\ -1 & 0 & 4 \\ 3 & -2 & 1 \end{pmatrix} \]

and ask if \( F \) is a linearly independent spanning set for \( \mathbb{C}^3 \). This is easily seen to be the case by forming a matrix \( A \) whose columns are the vectors of \( F \), row-reducing \( A \) to the identity matrix \( I_3 \), and then using the nonsingularity of \( A \) to assert that \( F \) is a basis for \( \mathbb{C}^3 \) (Theorem CNMB). Now, since \( F \) is a basis for \( \mathbb{C}^3 \), Theorem CLI and Theorem CSS tell us that \( D \) is also a basis for \( P_2 \).

Example CP2 illustrates the broad notion that computations in abstract vector spaces can be reduced to computations in \( \mathbb{C}^n \). You may have noticed this phenomenon as you worked through examples in Chapter VS or Chapter LT employing vector spaces of matrices or polynomials. These computations seemed to invariably result in systems of equations or the like from Chapter SLE or Chapter V and Chapter M. It is vector representation, \( \rho_B \), that allows us to make this connection formal and precise.

Knowing that vector representation allows us to translate questions about linear combinations, linear independence and spans from general vector spaces to \( \mathbb{C}^n \) allows us to prove a great many theorems about how to translate other properties. Rather than prove these theorems, each of the same style as the other, we will offer some general guidance about how to best employ Theorem VRLT, Theorem CLI and Theorem CSS. This comes in the form of a “principle”: a basic truth, but most definitely not a theorem (hence, no proof).

**The Coordinatization Principle**

Suppose that \( U \) is a vector space with a basis \( B \) of size \( n \). Then any question about \( U \), or its elements, which ultimately depends on the vector addition or scalar multiplication in \( U \), or depends on linear independence or spanning, may be translated into the same question in \( \mathbb{C}^n \) by application of the linear transformation \( \rho_B \) to the relevant vectors. Once the question is answered in \( \mathbb{C}^n \), the answer may be translated back to \( U \) (if necessary) through application of the inverse linear transformation \( \rho_B^{-1} \).
Example CM32

Coordinatization in $M_{32}$

This is a simple example of the Coordinatization Principle [596], depending only on the fact that coordinatizing is an invertible linear transformation (Theorem VRILT [593]). Suppose we have a linear combination to perform in $M_{32}$, the vector space of $3 \times 2$ matrices, but we are adverse to doing the operations of $M_{32}$ (Definition MA [198], Definition MSM [198]). More specifically, suppose we are faced with the computation

$$
\begin{bmatrix}
3 & 7 \\
-2 & 4 \\
0 & -3
\end{bmatrix}
+ 2
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}
= \begin{bmatrix}
16 & 48 \\
-4 & 40 \\
-8 &
\end{bmatrix}
$$

We choose a nice basis for $M_{32}$ (or a nasty basis if we are so inclined),

$$
B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
$$

and apply $\rho_B$ to each vector in the linear combination. This gives us a new computation, now in the vector space $C^6$, 

$$
\begin{bmatrix}
3 \\
-2 \\
0 \\
7 \\
4 \\
-3
\end{bmatrix}
+ 2
\begin{bmatrix}
-1 \\
4 \\
-2 \\
3 \\
8 \\
-5
\end{bmatrix}
$$

which we can compute with the operations of $C^6$ (Definition CVA [89], Definition CVSM [89]), to arrive at

$$
\begin{bmatrix}
16 \\
-4 \\
-4 \\
48 \\
40 \\
-8
\end{bmatrix}
$$

We are after the result of a computation in $M_{32}$, so we now can apply $\rho_B^{-1}$ to obtain a $3 \times 2$ matrix,

$$
16 \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + (-4) \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + (-4) \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + 48 \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + 40 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} + (-8) \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 48 \\ -4 & 40 \\ -4 & -8 \end{bmatrix}
$$

which is exactly the matrix we would have computed had we just performed the matrix operations in the first place.

Subsection READ

Reading Questions

1. The vector space of $3 \times 5$ matrices, $M_{3,5}$, is isomorphic to what fundamental vector space?
2. A basis for $\mathbb{C}^3$ is

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Compute $\rho_B \left( \begin{bmatrix} 5 \\ 8 \\ -1 \end{bmatrix} \right)$.

3. What is the first “surprise,” and why is it surprising?
Subsection EXC

Exercises

C10  In the vector space $\mathbb{C}^3$, compute the vector representation $\rho_B(v)$ for the basis $B$ and vector $v$ below.

$$B = \left\{ \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \right\} \quad v = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$$

Contributed by Robert Beezer  Solution 601

C20  Rework Example CM32 597 replacing the basis $B$ by the basis

$$C = \left\{ \begin{bmatrix} -14 & -9 \\ 10 & 10 \\ -6 & -2 \end{bmatrix}, \begin{bmatrix} -7 & -4 \\ 5 & 5 \\ -3 & -1 \end{bmatrix}, \begin{bmatrix} -3 & -1 \\ 0 & -2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -7 & -4 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ -3 & -3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & -2 \\ 1 & 1 \end{bmatrix} \right\}$$

Contributed by Robert Beezer  Solution 601

M10  Prove that the set $S$ below is a basis for the vector space of $2 \times 2$ matrices, $M_{22}$. Do this choosing a natural basis for $M_{22}$ and coordinatizing the elements of $S$ with respect to this basis. Examine the resulting set of column vectors from $\mathbb{C}^4$ and apply the Coordinatization Principle 596.

$$S = \left\{ \begin{bmatrix} 33 & 99 \\ 78 & -9 \end{bmatrix}, \begin{bmatrix} -16 & -47 \\ -36 & 2 \end{bmatrix}, \begin{bmatrix} 10 & 27 \\ 17 & 3 \end{bmatrix}, \begin{bmatrix} -2 & -7 \\ -6 & 4 \end{bmatrix} \right\}$$

Contributed by Andy Zimmer
Subsection SOL
Solutions

C10  Contributed by Robert Beezer  Statement 599
We need to express the vector \( \mathbf{v} \) as a linear combination of the vectors in \( B \). Theorem VRRB 355 tells us we will be able to do this, and do it uniquely. The vector equation

\[
\begin{bmatrix}
2 & 1 & 3 \\
-2 & 3 & 1 \\
2 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix}
= 
\begin{bmatrix}
11 \\
5 \\
8
\end{bmatrix}
\]

becomes (via Theorem SLSLC 100) a system of linear equations with augmented matrix,

\[
\begin{bmatrix}
2 & 1 & 3 & 11 \\
-2 & 3 & 5 & 5 \\
2 & 1 & 2 & 8
\end{bmatrix}
\]

This system has the unique solution \( a_1 = 2, a_2 = -2, a_3 = 3 \). So by Definition VR 587,

\[
\rho_B(\mathbf{v}) = \rho_B \left( \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix} \right) = \rho_B \left( 2 \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}
\]

C20  Contributed by Robert Beezer  Statement 599
The following computations replicate the computations given in Example CM32 597, only using the basis \( C \).

\[
\rho_C \left( \begin{bmatrix} 3 & 7 \\ -2 & 4 \\ 0 & -3 \end{bmatrix} \right) = \begin{bmatrix} -9 \\ 12 \\ -6 \\ 7 \\ -2 \\ -1 \end{bmatrix}
\]

\[
\rho_C \left( \begin{bmatrix} -1 & 3 \\ 4 & 8 \\ -2 & 5 \end{bmatrix} \right) = \begin{bmatrix} -11 \\ 34 \\ -4 \\ -1 \\ 16 \\ 5 \end{bmatrix}
\]

\[
6 \begin{bmatrix} -9 \\ 12 \\ -6 \\ 7 \\ -2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -11 \\ 34 \\ -4 \\ -1 \\ 16 \\ 5 \end{bmatrix} = \begin{bmatrix} -76 \\ 140 \\ -44 \\ 40 \\ 20 \\ 4 \end{bmatrix}
\]

\[
\rho_C^{-1} \left( \begin{bmatrix} -76 \\ 140 \\ -44 \\ 40 \\ 20 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 16 & 48 \\ -4 & 30 \\ -4 & -8 \end{bmatrix}
\]
Section MR
Matrix Representations

We have seen that linear transformations whose domain and codomain are vector spaces of columns vectors have a close relationship with matrices (Theorem MBLT 509, Theorem MLTCV 510). In this section, we will extend the relationship between matrices and linear transformations to the setting of linear transformations between abstract vector spaces.

Definition MR
Matrix Representation
Suppose that \( T: U \mapsto V \) is a linear transformation, \( B = \{u_1, u_2, u_3, \ldots, u_n\} \) is a basis for \( U \) of size \( n \), and \( C \) is a basis for \( V \) of size \( m \). Then the matrix representation of \( T \) relative to \( B \) and \( C \) is the \( m \times n \) matrix,

\[
M_{B,C}^T = [\rho_C(T(u_1)) | \rho_C(T(u_2)) | \rho_C(T(u_3)) | \ldots | \rho_C(T(u_n))] 
\]

Example OLTTR
One linear transformation, three representations
Consider the linear transformation

\[
S: P_3 \mapsto M_{22}, \quad S(a + bx + cx^2 + dx^3) = \begin{bmatrix}
3a + 7b - 2c - 5d & 8a + 14b - 2c - 11d \\
-4a - 8b + 2c + 6d & 12a + 22b - 4c - 17d
\end{bmatrix}
\]

First, we build a representation relative to the bases,

\[
B = \{1 + 2x + x^2 - x^3, 1 + 3x + x^2 + x^3, -1 - 2x + 2x^3, 2 + 3x + 2x^2 - 5x^3\}
\]

\[
C = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & -4 \\ -2 & -4 \end{bmatrix} \right\}
\]

We evaluate \( S \) with each element of the basis for the domain, \( B \), and coordinatize the result relative to the vectors in the basis for the codomain, \( C \).

\[
\rho_C(S(1 + 2x + x^2 - x^3)) = \rho_C\begin{bmatrix} 20 \\ -24 \end{bmatrix}
\]

\[
= \rho_C\begin{bmatrix} -90 \\ 37 \end{bmatrix}
\]

\[
\rho_C(S(1 + 3x + x^2 + x^3)) = \rho_C\begin{bmatrix} 17 \\ -20 \end{bmatrix}
\]

\[
= \rho_C\begin{bmatrix} -72 \\ -34 \end{bmatrix}
\]
\[
\rho_C \left( S (-1 - 2x + 2x^3) \right) = \rho_C \left( \begin{bmatrix} -27 & -58 \\ 32 & -90 \end{bmatrix} \right)
\]
\[
= \rho_C \left( 114 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + (-46) \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix} + 54 \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} + (-5) \begin{bmatrix} -1 & -4 \\ -2 & -4 \end{bmatrix} \right) = \begin{bmatrix} 114 \\ -46 \\ 54 \\ -5 \end{bmatrix}
\]
\[
\rho_C \left( S (2 + 3x + 2x^2 - 5x^3) \right) = \rho_C \left( \begin{bmatrix} 48 & 109 \\ -58 & 167 \end{bmatrix} \right)
\]
\[
= \rho_C \left( (-220) \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + 91 \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix} + (-96) \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} + 10 \begin{bmatrix} -1 & -4 \\ -2 & -4 \end{bmatrix} \right) = \begin{bmatrix} -220 \\ 91 \\ -96 \\ 10 \end{bmatrix}
\]

Thus, employing Definition MR 603

\[
M_{B,C}^S = \begin{bmatrix} -90 & -72 & 114 & -220 \\ 37 & 29 & -46 & 91 \\ -40 & -34 & 54 & -96 \\ 4 & 3 & -5 & 10 \end{bmatrix}
\]

Often we use “nice” bases to build matrix representations and the work involved is much easier. Suppose we take bases

\[D = \{1, x, x^2, x^3\} \quad E = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \]

The evaluation of \(S\) at the elements of \(D\) is easy and coordinatization relative to \(E\) can be done on sight,

\[
\rho_E (S (1)) = \rho_E \left( \begin{bmatrix} 3 & 8 \\ -4 & 12 \end{bmatrix} \right)
\]
\[
= \rho_E \left( 3 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 8 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-4) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 12 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 8 \\ -4 \\ 12 \end{bmatrix}
\]
\[
\rho_E (S (x)) = \rho_E \left( \begin{bmatrix} 7 & 14 \\ -8 & 22 \end{bmatrix} \right)
\]
\[
= \rho_E \left( 7 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 14 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-8) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 22 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 7 \\ 14 \\ -8 \\ 22 \end{bmatrix}
\]
\[
\rho_E (S (x^2)) = \rho_E \left( \begin{bmatrix} -2 & -2 \\ 2 & -4 \end{bmatrix} \right)
\]
\[
= \rho_E \left( (-2) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-2) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-4) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ -2 \\ 2 \\ -4 \end{bmatrix}
\]
\[ \rho_E(S(x^3)) = \rho_E\left( \begin{bmatrix} -5 & -11 \\ 6 & -17 \end{bmatrix} \right) \]

\[ = \rho_E\left( (-5) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-11) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 6 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-17) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} -5 \\ -11 \\ 6 \\ -17 \end{bmatrix} \]

So the matrix representation of \( S \) relative to \( D \) and \( E \) is

\[ M_{S,D,E}^S = \begin{bmatrix} 3 & 7 & -2 & -5 \\ 8 & 14 & -2 & -11 \\ -4 & -8 & 2 & 6 \\ 12 & 22 & -4 & -17 \end{bmatrix} \]

One more time, but now let’s use bases

\[ F = \{1 + x - x^2 + 2x^3, -1 + 2x + 2x^3, 2 + x - 2x^2 + 3x^3, 1 + x + 2x^3\} \]

\[ G = \left\{ \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \right\} \]

and evaluate \( S \) with the elements of \( F \), then coordinatize the results relative to \( G \),

\[ \rho_G(S(1 + x - x^2 + 2x^3)) = \rho_G\left( \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \right) = \rho_G\left( 2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \rho_G(S(-1 + 2x + 2x^3)) = \rho_G\left( \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right) = \rho_G\left( -1 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \]

\[ \rho_G(S(2 + x - 2x^2 + 3x^3)) = \rho_G\left( \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right) = \rho_G\left( 2 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \]

\[ \rho_G(S(1 + x + 2x^3)) = \rho_G\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \rho_G\left( 0 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

So we arrive at an especially economical matrix representation,

\[ M_{S,F,G}^S = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]
We may choose to use whatever terms we want when we make a definition. Some are arbitrary, while others make sense, but only in light of subsequent theorems. Matrix representation is in the latter category. We begin with a linear transformation and produce a matrix. So what? Here’s the theorem that justifies the term “matrix representation.”

**Theorem FTMR**

**Fundamental Theorem of Matrix Representation**

Suppose that \( T : U \rightarrow V \) is a linear transformation, \( B \) is a basis for \( U \), \( C \) is a basis for \( V \) and \( M^T_{B,C} \) is the matrix representation of \( T \) relative to \( B \) and \( C \). Then, for any \( u \in U \),

\[
\rho_C (T(u)) = M^T_{B,C} (\rho_B (u))
\]

or equivalently

\[
T(u) = \rho_C^{-1} \left( M^T_{B,C} (\rho_B (u)) \right)
\]

**Proof** Let \( B = \{u_1, u_2, u_3, \ldots, u_n\} \) be the basis of \( U \). Since \( u \in U \), there are scalars \( a_1, a_2, a_3, \ldots, a_n \) such that

\[
u = a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_n u_n
\]

Then,

\[
M^T_{B,C} \rho_B (u) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}
\]

\[
= \begin{bmatrix} \rho_C (T(u_1)) \\ \rho_C (T(u_2)) \\ \rho_C (T(u_3)) \\ \vdots \\ \rho_C (T(u_n)) \end{bmatrix}
\]

\[
= a_1 \rho_C (T(u_1)) + a_2 \rho_C (T(u_2)) + \cdots + a_n \rho_C (T(u_n))
\]

\[
= \rho_C (a_1 T(u_1) + a_2 T(u_2) + a_3 T(u_3) + \cdots + a_n T(u_n))
\]

\[
= \rho_C (T(a_1 u_1 + a_2 u_2 + a_3 u_3 + \cdots + a_n u_n))
\]

\[
= \rho_C (T(u))
\]

The alternative conclusion is obtained as

\[
T(u) = I_V (T(u))
\]

\[
= \left( \rho_C^{-1} \circ \rho_C \right) (T(u))
\]

\[
= \rho_C^{-1} \left( \rho_C (T(u)) \right)
\]

\[
= \rho_C^{-1} \left( M^T_{B,C} (\rho_B (u)) \right)
\]

This theorem says that we can apply \( T \) to \( u \) and coordinatize the result relative to \( C \) in \( V \), or we can first coordinatize \( u \) relative to \( B \) in \( U \), then multiply by the matrix representation. Either way, the result is the same. So the effect of a linear transformation
can always be accomplished by a matrix-vector product (Definition MVP [211]). That’s important enough to say again. The effect of a linear transformation is a matrix-vector product.

\[
\begin{align*}
\mathbf{u} & \quad T \quad T(\mathbf{u}) \\
\rho_B & \downarrow \\
\rho_B(\mathbf{u}) & \quad M_{B,C}^T \quad \rho_C(T(\mathbf{u})), \\
M_{B,C}^T \rho_B(\mathbf{u}) &
\end{align*}
\]

The alternative conclusion of this result might be even more striking. It says that to effect a linear transformation \((T)\) of a vector \((\mathbf{u})\), coordinatize the input (with \(\rho_B\)), do a matrix-vector product (with \(M_{B,C}^T\)), and un-coordinatize the result (with \(\rho_C^{-1}\)). So, absent some bookkeeping about vector representations, a linear transformation is a matrix.

Here’s an example to illustrate how the “action” of a linear transformation can be effected by matrix multiplication.

**Example ALTMM**

**A linear transformation as matrix multiplication**

In Example OLTTR [603] we found three representations of the linear transformation \(S\). In this example, we will compute a single output of \(S\) in four different ways. First “normally,” then three times over using Theorem FTMR [606].

Choose \(p(x) = 3 - x + 2x^2 - 5x^3\), for no particular reason. Then the straightforward application of \(S\) to \(p(x)\) yields

\[
S(p(x)) = S(3 - x + 2x^2 - 5x^3)
= \begin{bmatrix}
3(3) + 7(-1) - 2(2) - 5(-5) & 8(3) + 14(-1) - 2(2) - 11(-5) \\
-4(3) - 8(-1) + 2(2) + 6(-5) & 12(3) + 22(-1) - 4(2) - 17(-5)
\end{bmatrix}
= \begin{bmatrix}
23 & 61 \\
-30 & 91
\end{bmatrix}
\]

Now use the representation of \(S\) relative to the bases \(B\) and \(C\) and Theorem FTMR [606]. Note that we will employ the following linear combination in moving from the second line to the third,

\[
3 - x + 2x^2 - 5x^3 = 48(1 + 2x + x^2 - x^3) - 20(1 + 3x + x^2 + x^3) - (-1 - 2x + 2x^3) - 13(2 + 3x + 2x^2 - 5x^3)
\]

\[
S(p(x)) = \rho_C^{-1}(M_{B,C}^S \rho_B(p(x)))
= \rho_C^{-1}(M_{B,C}^S \rho_B(3 - x + 2x^2 - 5x^3))
= \rho_C^{-1} \begin{pmatrix}
48 \\
-20 \\
-1 \\
-13
\end{pmatrix}
= \rho_C^{-1} \begin{pmatrix}
-90 & -72 & 114 & -220 \\
37 & 29 & -46 & 91 \\
-40 & -34 & 54 & -96 \\
4 & 3 & -5 & 10
\end{pmatrix}
= \rho_C^{-1} \begin{pmatrix}
-134 \\
59 \\
-46 \\
7
\end{pmatrix}
\]
\[ \begin{pmatrix} -134 & 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} + 59 \begin{pmatrix} 2 & 3 \\ 2 & 5 \end{pmatrix} + (-46) \begin{pmatrix} -1 & -1 \\ 0 & -2 \end{pmatrix} + 7 \begin{pmatrix} -1 & -4 \\ -2 & -4 \end{pmatrix} \]

\[ \begin{pmatrix} 23 & 61 \\ -30 & 91 \end{pmatrix} \]

Again, but now with “nice” bases like \( D \) and \( E \), and the computations are more transparent.

\[
S (p(x)) = \rho_{E}^{-1} \left( M_{D,E}^{S} \rho_{D} (p(x)) \right)
\]

\[
= \rho_{E}^{-1} \left( M_{D,E}^{S} \rho_{D} \left( 3 - x + 2x^2 - 5x^3 \right) \right)
\]

\[
= \rho_{E}^{-1} \left( M_{D,E}^{S} \rho_{D} \left( 3(1) + (-1)(x) + 2(x^2) + (-5)(x^3) \right) \right)
\]

\[
= \rho_{E}^{-1} \left( \begin{pmatrix} 3 & -1 \\ 2 & 0 \\ -5 \end{pmatrix} \right)
\]

\[
= \rho_{E}^{-1} \left( \begin{pmatrix} 3 & 7 & -2 & -5 \\ 8 & 14 & -2 & -11 \\ -4 & -8 & 2 & 6 \\ 12 & 22 & -4 & -17 \end{pmatrix} \right)
\]

\[
= \rho_{E}^{-1} \left( \begin{pmatrix} 23 \\ 61 \\ -30 \\ 91 \end{pmatrix} \right)
\]

\[
= 23 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 61 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (-30) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 91 \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} 23 \\ 61 \\ -30 \\ 91 \end{pmatrix}
\]

OK, last time, now with the bases \( F \) and \( G \). The coordinatizations will take some work this time, but the matrix-vector product (Definition MVP [211]) (which is the actual action of the linear transformation) will be especially easy, given the diagonal nature of the matrix representation, \( M_{F,G}^{S} \). Here we go,

\[
S (p(x)) = \rho_{G}^{-1} \left( M_{F,G}^{S} \rho_{F} (p(x)) \right)
\]

\[
= \rho_{G}^{-1} \left( M_{F,G}^{S} \rho_{F} \left( 32(1 + x - x^2 + 2x^3) - 7(-1 + 2x + 2x^3) - 17(2 + x - 2x^2 + 3x^3) - 2(1 + \right.ight.

\[
= \rho_{G}^{-1} \left( \begin{pmatrix} 32 \\ -7 \\ -17 \\ -2 \end{pmatrix} \right)
\]

\[
= \rho_{G}^{-1} \left( \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -17 \end{pmatrix} \right)
\]

\[
= \rho_{G}^{-1} \left( \begin{pmatrix} 32 \\ -7 \\ -2 \end{pmatrix} \right)
\]
Subsection MR.NRFO New Representations from Old 609

\[
\rho_G^{-1} \left( \begin{bmatrix} 64 \\ 7 \\ -17 \\ 0 \end{bmatrix} \right) \\
= 64 \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix} + (-17) \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix} + 0 \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \\
= \begin{bmatrix} 23 & 61 \\ -30 & 91 \end{bmatrix}
\]

This example is not meant to necessarily illustrate that any one of these four computations is simpler than the others. Instead, it is meant to illustrate the many different ways we can arrive at the same result, with the last three all employing a matrix representation to effect the linear transformation.

We will use Theorem FTMR [606] frequently in the next few sections. A typical application will feel like the linear transformation \( T \) “commutes” with a vector representation, \( \rho_C \), and as it does the transformation morphs into a matrix, \( M_{T,B,C} \), while the vector representation changes to a new basis, \( \rho_B \). Or vice-versa.

Subsection NRFO New Representations from Old

In Subsection LT.NLTFO [517] we built new linear transformations from other linear transformations. Sums, scalar multiples and compositions. These new linear transformations will have matrix representations as well. How do the new matrix representations relate to the old matrix representations? Here are the three theorems.

**Theorem MRSLT**

**Matrix Representation of a Sum of Linear Transformations**

Suppose that \( T: U \rightarrow V \) and \( S: U \rightarrow V \) are linear transformations, \( B \) is a basis of \( U \) and \( C \) is a basis of \( V \). Then

\[
M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S
\]

**Proof** Let \( x \) be any vector in \( \mathbb{C}^n \). Define \( u \in U \) by \( u = \rho_B^{-1}(x) \), so \( x = \rho_B(u) \). Then,

\[
M_{B,C}^{T+S} x = M_{B,C}^{T+S} \rho_B(u) \\
= \rho_C((T+S)(u)) \\
= \rho_C(T(u) + S(u)) \\
= \rho_C(T(u)) + \rho_C(S(u)) \\
= M_{T,B,C}^T(\rho_B(u)) + M_{B,C}^S(\rho_B(u)) \\
= (M_{T,B,C}^T + M_{B,C}^S) \rho_B(u) \\
= (M_{T,B,C}^T + M_{B,C}^S) x
\]

Since the matrices \( M_{B,C}^{T+S} \) and \( M_{T,B,C}^T + M_{B,C}^S \) have equal matrix-vector products for every vector in \( \mathbb{C}^n \), by Theorem EMMVP [214] they are equal matrices. (Now would be a good
Theorem MRMLT
Matrix Representation of a Multiple of a Linear Transformation
Suppose that $T: U \mapsto V$ is a linear transformation, $\alpha \in \mathbb{C}$, $B$ is a basis of $U$ and $C$ is a basis of $V$. Then

$$M^{\alpha T}_{B,C} = \alpha M^T_{B,C}$$

Proof Let $x$ be any vector in $\mathbb{C}^n$. Define $u \in U$ by $u = \rho_B^{-1}(x)$, so $x = \rho_B(u)$. Then,

$$M^T_{B,C}x = \alpha M^T_{B,C} \rho_B(u) \quad \text{Substitution}$$

Since the matrices $M^T_{B,C}$ and $\alpha M^T_{B,C}$ have equal matrix-vector products for every vector in $\mathbb{C}^n$, by Theorem EMMVP [214] they are equal matrices.

The vector space of all linear transformations from $U$ to $V$ is now isomorphic to the vector space of all $m \times n$ matrices.

Theorem MRCLT
Matrix Representation of a Composition of Linear Transformations
Suppose that $T: U \mapsto V$ and $S: V \mapsto W$ are linear transformations, $B$ is a basis of $U$, $C$ is a basis of $V$, and $D$ is a basis of $W$. Then

$$M^{S \circ T}_{B,D} = M^S_{C,D} M^T_{B,C}$$

Proof Let $x$ be any vector in $\mathbb{C}^n$. Define $u \in U$ by $u = \rho_B^{-1}(x)$, so $x = \rho_B(u)$. Then,

$$M^{S \circ T}_{B,D}x = M^{S \circ T}_{B,D} \rho_B(u) \quad \text{Substitution}$$

Since the matrices $M^{S \circ T}_{B,D}$ and $M^S_{C,D} M^T_{B,C}$ have equal matrix-vector products for every vector in $\mathbb{C}^n$, by Theorem EMMVP [214] they are equal matrices.
This is the second great surprise of introductory linear algebra. Matrices are linear transformations (functions, really), and matrix multiplication is function composition! We can form the composition of two linear transformations, then form the matrix representation of the result. Or we can form the matrix representation of each linear transformation separately, then multiply the two representations together via Definition MM. In either case, we arrive at the same result.

Example MPMR

Matrix product of matrix representations

Consider the two linear transformations,

\[ T : \mathbb{C}^2 \mapsto P_2 \]
\[ T \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = (-a + 3b) + (2a + 4b)x + (a - 2b)x^2 \]

\[ S : P_2 \mapsto M_{22} \]
\[ S \left( a + bx + cx^2 \right) = \begin{bmatrix} 2a + b + 2c & a + 4b - c \\ -a + 3c & 3a + b + 2c \end{bmatrix} \]

and bases for \( \mathbb{C}^2 \), \( P_2 \) and \( M_{22} \) (respectively),

\[ B = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \]
\[ C = \{ 1 - 2x + x^2, -1 + 3x, 2x + 3x^2 \} \]
\[ D = \left\{ \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 2 & 2 \end{bmatrix} \right\} \]

Begin by computing the new linear transformation that is the composition of \( T \) and \( S \) (Definition LTC, Theorem CLTLT), \( (S \circ T) : \mathbb{C}^2 \mapsto M_{22} \),

\[ (S \circ T) \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = S \left( T \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) \right) = S \left( (-a + 3b) + (2a + 4b)x + (a - 2b)x^2 \right) = \begin{bmatrix} 2(-a + 3b) + (2a + 4b) + 2(a - 2b) & (-a + 3b) + 4(2a + 4b) - (a - 2b) \\ -(-a + 3b) + 3(a - 2b) & 3(-a + 3b) + (2a + 4b) + 2(a - 2b) \end{bmatrix} = \begin{bmatrix} 2a + 6b & 6a + 21b \\ 4a - 9b & a + 9b \end{bmatrix} \]

Now compute the matrix representations (Definition MR) for each of these three linear transformations \( (T, S, S \circ T) \), relative to the appropriate bases. First for \( T \),

\[ \rho_C \left( T \left( \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) \right) = \rho_C \left( 10x + x^2 \right) = \rho_C \left( 28(1 - 2x + x^2) + 28(-1 + 3x) + (-9)(2x + 3x^2) \right) = \begin{bmatrix} 28 \\ 28 \\ -9 \end{bmatrix} \]

\[ \rho_C \left( T \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \right) = \rho_C \left( 1 + 8x \right) = \rho_C \left( 33(1 - 2x + x^2) + 32(-1 + 3x) + (-11)(2x + 3x^2) \right) = \begin{bmatrix} 33 \\ 32 \\ -11 \end{bmatrix} \]
So we have the matrix representation of $T$,

$$M_{B,C}^T = \begin{bmatrix} 28 & 33 \\ 28 & 32 \\ -9 & -11 \end{bmatrix}$$

Now, a representation of $S$,

$$\rho_D (S (1 - 2x + x^2)) = \rho_D \left( \begin{bmatrix} 2 & -8 \\ 2 & 3 \end{bmatrix} \right)$$

$$= \rho_D \left( \begin{bmatrix} -11 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + (17) \begin{bmatrix} 2 & -3 \\ 2 & 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -11 \\ -21 \\ 0 \\ 17 \end{bmatrix}$$

$$\rho_D (S (-1 + 3x)) = \rho_D \left( \begin{bmatrix} 1 & 11 \\ 1 & 0 \end{bmatrix} \right)$$

$$= \rho_D \left( \begin{bmatrix} 26 & 1 \\ 1 & -2 \\ 1 & -1 \end{bmatrix} + 51 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + (-38) \begin{bmatrix} 2 & -3 \\ 2 & 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 26 \\ 51 \\ 0 \\ -38 \end{bmatrix}$$

$$\rho_D (S (2x + 3x^2)) = \rho_D \left( \begin{bmatrix} 8 & 5 \\ 9 & 8 \end{bmatrix} \right)$$

$$= \rho_D \left( \begin{bmatrix} 34 & 1 \\ 1 & -2 \\ 1 & -1 \end{bmatrix} + 67 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + (-46) \begin{bmatrix} 2 & -3 \\ 2 & 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 34 \\ 67 \\ 1 \\ -46 \end{bmatrix}$$

So we have the matrix representation of $S$,

$$M_{C,D}^S = \begin{bmatrix} -11 & 26 & 34 \\ -21 & 51 & 67 \\ 0 & 0 & 1 \\ 17 & -38 & -46 \end{bmatrix}$$

Finally, a representation of $S \circ T$,

$$\rho_D \left( (S \circ T) \left( \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right) \right) = \rho_D \left( \begin{bmatrix} 12 & 39 \\ 3 & 12 \end{bmatrix} \right)$$

$$= \rho_D \left( 114 \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} + 237 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + (-9) \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} + (-174) \begin{bmatrix} 2 & -3 \\ 2 & 2 \end{bmatrix} \right)$$

$$= \rho_D \left( \begin{bmatrix} -9 & 26 \\ -21 & 39 \\ 0 & 12 \\ 17 & -38 \end{bmatrix} \right)$$
\[ \rho_D \left( (S \circ T) \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \right) = \rho_D \left( \begin{bmatrix} 10 \\ -1 \end{bmatrix} \right) \]
\[ = \rho_D \left( \begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \end{bmatrix} + 202 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix} + (-11) \begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \end{bmatrix} + (-149) \begin{bmatrix} 2 \\ 0 \\ 2 \\ -3 \end{bmatrix} \right) \]
\[ = \begin{bmatrix} 114 \\ 237 \\ -9 \\ -174 \end{bmatrix} \]

So we have the matrix representation of \( S \circ T \),

\[ M_{B,D}^{S \circ T} = \begin{bmatrix} 114 & 95 \\ 237 & 202 \\ -9 & -11 \\ -174 & -149 \end{bmatrix} \]

Now, we are all set to verify the conclusion of Theorem MRCLT, \[610\],

\[ M_C^{S} M_T^{C, B} = \begin{bmatrix} -11 & 26 & 34 \\ -21 & 51 & 67 \\ 0 & 0 & 1 \\ 17 & -38 & -46 \end{bmatrix} \begin{bmatrix} 28 & 33 \\ 28 & 32 \\ -9 & -11 \end{bmatrix} \]
\[ = \begin{bmatrix} 114 & 95 \\ 237 & 202 \\ -9 & -11 \\ -174 & -149 \end{bmatrix} \]
\[ = M_{B,D}^{S \circ T} \]

We have intentionally used non-standard bases. If you were to choose “nice” bases for the three vector spaces, then the result of the theorem might be rather transparent. But this would still be a worthwhile exercise — give it a go.

A diagram, similar to ones we have seen earlier, might make the importance of this theorem clearer,

\[ S, T \xrightarrow{\text{Definition MR}} 603, M_{C,D}^{S}, M_{B,C}^{T} \]
\[ \xrightarrow{\text{Definition LTC}} 519, \]
\[ S \circ T \xrightarrow{\text{Definition MR}} 603, M_{C,D}^{S}, M_{B,C}^{T} \xrightarrow{\text{Definition MM}} 215 \]

One of our goals in the first part of this book is to make the definition of matrix multiplication (Definition MVP, \[211\], Definition MM, \[215\]) seem as natural as possible. However, many are brought up with an entry-by-entry description of matrix multiplication (Theorem ME, \[474\]) as the definition of matrix multiplication, and then theorems...
about columns of matrices and linear combinations follow from that definition. With this unmotivated definition, the realization that matrix multiplication is function composition is quite remarkable. It is an interesting exercise to begin with the question, “What is the matrix representation of the composition of two linear transformations?” and then, without using any theorems about matrix multiplication, finally arrive at the entry-by-entry description of matrix multiplication. Try it yourself (Exercise MR.T80 [627]).

Subsection PMR
Properties of Matrix Representations

It will not be a surprise to discover that the kernel and range of a linear transformation are closely related to the null space and column space of the transformation’s matrix representation. Perhaps this idea has been bouncing around in your head already, even before seeing the definition of a matrix representation. However, with a formal definition of a matrix representation (Definition MR [603]), and a fundamental theorem to go with it (Theorem FTMR [606]) we can be formal about the relationship, using the idea of isomorphic vector spaces (Definition IVS [573]). Here are the twin theorems.

**Theorem KNSI**
Kernel and Null Space Isomorphism

Suppose that $T: U \mapsto V$ is a linear transformation, $B$ is a basis for $U$ of size $n$, and $C$ is a basis for $V$. Then the kernel of $T$ is isomorphic to the null space of $M_{B,C}^T$:

$$\mathcal{K}(T) \cong \mathcal{N}(M_{B,C}^T)$$

**Proof** To establish that two vector spaces are isomorphic, we must find an isomorphism between them, an invertible linear transformation (Definition IVS [573]). The kernel of the linear transformation $T$, $\mathcal{K}(T)$, is a subspace of $U$, while the null space of the matrix representation, $\mathcal{N}(M_{B,C}^T)$, is a subspace of $\mathbb{C}^n$. The function $\rho_B$ is defined as a function from $U$ to $\mathbb{C}^n$, but we can just as well employ the definition of $\rho_B$ as a function from $\mathcal{K}(T)$ to $\mathcal{N}(M_{B,C}^T)$.

We must first insure that if we choose an input for $\rho_B$ from $\mathcal{K}(T)$ that then the output will be an element of $\mathcal{N}(M_{B,C}^T)$. So suppose that $u \in \mathcal{K}(T)$. Then

$$M_{B,C}^T \rho_B(u) = \rho_C(T(u)) = \rho_C(0) = 0$$

This says that $\rho_B(u) \in \mathcal{N}(M_{B,C}^T)$, as desired.

The restriction in the size of the domain and codomain $\rho_B$ will not affect the fact that $\rho_B$ is a linear transformation (Theorem VRLT [587]), nor will it affect the fact that $\rho_B$ is injective (Theorem VRI [592]). Something must be done though to verify that $\rho_B$ is surjective. To this end, appeal to the definition of surjective (Definition SLT [547]), and suppose that we have an element of the codomain, $x \in \mathcal{N}(M_{B,C}^T) \subseteq \mathbb{C}^n$ and we wish...
to find an element of the domain with \( x \) as its image. We now show that the desired element of the domain is \( u = \rho_B^{-1}(x) \). First, verify that \( u \in K(T) \),

\[
T(u) = T\left(\rho_B^{-1}(x)\right) \\
= \rho_C^{-1}(M_{B,C}^T(\rho_B^{-1}(x))) \\
= \rho_C^{-1}(M_{B,C}^T(I_{C^n}(x))) \\
= \rho_C^{-1}(M_{B,C}^T \mathbf{0}) \\
= \rho_C^{-1}(\mathbf{0}) \\
= \mathbf{0}_V
\]

Theorem FTMR \[606\]

Second, verify that the proposed isomorphism, \( \rho_B \), takes \( u \) to \( x \),

\[
\rho_B(u) = \rho_B(\rho_B^{-1}(x)) \\
= I_{C^n}(x) \\
= x
\]

Substitution

Definition IVLT \[567\]

Definition IDLT \[567\]

With \( \rho_B \) demonstrated to be an injective and surjective linear transformation from \( K(T) \) to \( \mathcal{N}(M_{B,C}^T) \). Theorem ILTIS \[571\] tells us \( \rho_B \) is invertible, and so by Definition IVS \[573\], we say \( K(T) \) and \( \mathcal{N}(M_{B,C}^T) \) are isomorphic.

Example KVMR

Kernel via matrix representation

Consider the kernel of the linear transformation

\[
T: M_{22} \mapsto P_2, \quad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (2a - b + c - 5d) + (a + 4b + 5b + 2d)x + (3a - 2b + c - 8d)x^2
\]

We will begin with a matrix representation of \( T \) relative to the bases for \( M_{22} \) and \( P_2 \) (respectively),

\[
B = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ -2 & -4 \end{bmatrix} \right\} \\
C = \left\{ 1 + x + x^2, 2 + 3x, -1 - 2x^2 \right\}
\]

Then,

\[
\rho_C\left(T\left(\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\right)\right) = \rho_C\left(4 + 2x + 6x^2\right) \\
= \rho_C\left(2(1 + x + x^2) + 0(2 + 3x) + (-2)(-1 - 2x^2)\right) \\
= \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}
\]

\[
\rho_C\left(T\left(\begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}\right)\right) = \rho_C\left(18 + 28x^2\right) \\
= \rho_C\left((-24)(1 + x + x^2) + 8(2 + 3x) + (-26)(-1 - 2x^2)\right) \\
= \begin{bmatrix} -24 \\ 8 \\ -26 \end{bmatrix}
\]
\[ \rho_C \left( T \left( \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \right) \right) = \rho_C \left( 10 + 5x + 15x^2 \right) = \rho_C \left( 5(1 + x + x^2) + 0(2 + 3x) + (-5)(-1 - 2x^2) \right) = \begin{bmatrix} 5 \\ 0 \\ -5 \end{bmatrix} \]

\[ \rho_C \left( T \left( \begin{bmatrix} 2 & 5 \\ -2 & -4 \end{bmatrix} \right) \right) = \rho_C \left( 17 + 4x + 26x^2 \right) = \rho_C \left( (-8)(1 + x + x^2) + (4)(2 + 3x) + (-17)(-1 - 2x^2) \right) = \begin{bmatrix} -8 \\ 4 \\ -17 \end{bmatrix} \]

So the matrix representation of \( T \) (relative to \( B \) and \( C \)) is

\[ M^T_{B,C} = \begin{bmatrix} 2 & -24 & 5 & -8 \\ 0 & 8 & 0 & 4 \\ -2 & -26 & -5 & -17 \end{bmatrix} \]

We know from Theorem KNSI 614 that the kernel of the linear transformation \( T \) is isomorphic to the null space of the matrix representation \( M^T_{B,C} \) and by studying the proof of Theorem KNSI 614 we learn that \( \rho_B \) is an isomorphism between these null spaces. Rather than trying to compute the kernel of \( T \) using definitions and techniques from Chapter LT 503 we will instead analyze the null space of \( M^T_{B,C} \) using techniques from way back in Chapter V 87. First row-reduce \( M^T_{B,C} \):

\[ \begin{bmatrix} 2 & -24 & 5 & -8 \\ 0 & 8 & 0 & 4 \\ -2 & -26 & -5 & -17 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & \frac{5}{2} & 2 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

So, by Theorem BNS 154, a basis for \( \mathcal{N}(M^T_{B,C}) \) is

\[ \left\{ \begin{bmatrix} \frac{-5}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\} \]

We can now convert this basis of \( \mathcal{N}(M^T_{B,C}) \) into a basis of \( \mathcal{K}(T) \) by applying \( \rho_B^{-1} \) to each element of the basis,

\[ \rho_B^{-1} \left( \begin{bmatrix} \frac{-5}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) = (-\frac{5}{2}) \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ -1 \\ -4 \\ -4 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 5 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ -\frac{3}{2} \\ 1 \\ -1 \end{bmatrix} \]

\[ \rho_B^{-1} \left( \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right) = (-2) \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} + (-\frac{1}{2}) \begin{bmatrix} 1 \\ -1 \\ -4 \\ -4 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 5 \\ -2 \\ -4 \end{bmatrix} \]
\[
\begin{bmatrix}
-\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & 0
\end{bmatrix}
\]

So the set
\[
\left\{ \begin{bmatrix}
-\frac{3}{2} & -3 \\
\frac{5}{2} & 2
\end{bmatrix}, \begin{bmatrix}
-\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & 0
\end{bmatrix} \right\}
\]
is a basis for \( \mathcal{K}(T) \).

An entirely similar result applies to the range of a linear transformation and the column space of a matrix representation of the linear transformation.

**Theorem RCSI**

**Range and Column Space Isomorphism**

Suppose that \( T: U \rightarrow V \) is a linear transformation, \( B \) is a basis for \( U \) of size \( n \), and \( C \) is a basis for \( V \) of size \( m \). Then the range of \( T \) is isomorphic to the column space of \( M_{B,C}^T \),

\[
\mathcal{R}(T) \cong \mathcal{C}(M_{B,C}^T)
\]

**Proof** To establish that two vector spaces are isomorphic, we must find an isomorphism between them, an invertible linear transformation (Definition IVS [573]). The range of the linear transformation \( T, \mathcal{R}(T) \), is a subspace of \( V \), while the column space of the matrix representation, \( \mathcal{C}(M_{B,C}^T) \) is a subspace of \( \mathbb{C}^m \). The function \( \rho_C \) is defined as a function from \( V \) to \( \mathbb{C}^m \), but we can just as well employ the definition of \( \rho_C \) as a function from \( \mathcal{R}(T) \) to \( \mathcal{C}(M_{B,C}^T) \).

We must first ensure that if we choose an input for \( \rho_C \) from \( \mathcal{R}(T) \) that then the output will be an element of \( \mathcal{C}(M_{B,C}^T) \). So suppose that \( v \in \mathcal{R}(T) \). Then there is a vector \( u \in U \), such that \( T(u) = v \). Consider

\[
M_{B,C}^T \rho_B(u) = \rho_C(T(u)) = \rho_C(v)
\]

This says that \( \rho_C(v) \in \mathcal{C}(M_{B,C}^T) \), as desired.

The restriction in the size of the domain and codomain will not affect the fact that \( \rho_C \) is a linear transformation (Theorem VRLT [587]), nor will it affect the fact that \( \rho_C \) is injective (Theorem VRI [592]). Something must be done though to verify that \( \rho_C \) is surjective. This all gets a bit confusing, since the domain of our isomorphism is the range of the linear transformation, so think about your objects as you go. To establish that \( \rho_C \) is surjective, appeal to the definition of a surjective linear transformation (Definition SLT [547]), and suppose that we have an element of the codomain, \( y \in \mathcal{C}(M_{B,C}^T) \subseteq \mathbb{C}^m \) and we wish to find an element of the domain with \( y \) as its image. Since \( y \in \mathcal{C}(M_{B,C}^T) \), there exists a vector, \( x \in \mathbb{C}^n \) with \( M_{B,C}^T x = y \). We now show that the desired element of the domain is \( v = \rho_C^{-1}(y) \). First, verify that \( v \in \mathcal{R}(T) \) by applying \( T \) to \( u = \rho_B^{-1}(x) \),

\[
T(u) = T(\rho_B^{-1}(x)) = \rho_C^{-1}(M_{B,C}^T(\rho_B(\rho_B^{-1}(x)))) = \rho_C^{-1}(M_{B,C}^T(I_{C^*}(x))) = \rho_C^{-1}(M_{B,C}^T x)
\]

\( \square \)
= \rho_C^{-1}(y) \quad y \in \mathcal{C}(M_{B,C}^T)
= v \quad \text{Substitution}

Second, verify that the proposed isomorphism, \( \rho_C \), takes \( v \) to \( y \),

\[
\rho_C(v) = \rho_C \left( \rho_C^{-1}(y) \right)
= I_{cm}(y)
= y
\]

With \( \rho_C \) demonstrated to be an injective and surjective linear transformation from \( \mathcal{R}(T) \) to \( \mathcal{C}(M_{B,C}^T) \), Theorem ILTIS \[571\] tells us \( \rho_C \) is invertible, and so by Definition IVS \[573\], we say \( \mathcal{R}(T) \) and \( \mathcal{C}(M_{B,C}^T) \) are isomorphic. 

**Example RVMR**

**Range via matrix representation**

In this example, we will recycle the linear transformation \( T \) and the bases \( B \) and \( C \) of Example KVMR \[615\] but now we will compute the range of \( T \),

\( T: M_{22} \rightarrow P_2, \quad T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (2a - b + c - 5d) + (a + 4b + 5b + 2d)x + (3a - 2b + c - 8d)x^2 \)

With bases \( B \) and \( C \),

\[
B = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ -2 & -4 \end{bmatrix} \right\}
\]

\( C = \{1 + x + x^2, 2 + 3x, -1 - 2x^2\} \)

we obtain the matrix representation

\[
M_{B,C}^T = \begin{bmatrix} 2 & -24 & 5 & -8 \\ 0 & 8 & 0 & 4 \\ -2 & -26 & -5 & -17 \end{bmatrix}
\]

We know from Theorem RCSII \[617\] that the range of the linear transformation \( T \) is isomorphic to the column space of the matrix representation \( M_{B,C}^T \) and by studying the proof of Theorem RCSII \[617\] we learn that \( \rho_C \) is an isomorphism between these subspaces. Notice that since the range is a subspace of the codomain, we will employ \( \rho_C \) as the isomorphism, rather than \( \rho_B \), which was the correct choice for an isomorphism between the null spaces of Example KVMR \[615\].

Rather than trying to compute the range of \( T \) using definitions and techniques from Chapter LT \[503\], we will instead analyze the column space of \( M_{B,C}^T \) using techniques from way back in Chapter M \[197\]. First row-reduce \( (M_{B,C}^T)^t \),

\[
\begin{bmatrix} 2 & 0 & -2 \\ -24 & 8 & -26 \\ 5 & 0 & -5 \\ -8 & 4 & -17 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -25/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]
Now employ Theorem CSRST \[273\] and Theorem BRS \[271\] (there are other methods we could choose here to compute the column space, such as Theorem BCS \[264\]) to obtain the basis for \(C(M_{B,C}^T)\),

\[
\begin{bmatrix}
1 \\
0 \\
-1
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
-\frac{25}{4}
\end{bmatrix}
\]

We can now convert this basis of \(C(M_{B,C}^T)\) into a basis of \(\mathcal{R}(T)\) by applying \(\rho_C^{-1}\) to each element of the basis,

\[
\rho_C^{-1}\left(\begin{bmatrix}1 \\ 0 \\ -1\end{bmatrix}\right) = (1 + x + x^2) - (-1 - 2x^2) = 2 + x + 3x^2
\]

\[
\rho_C^{-1}\left(\begin{bmatrix}0 \\ 1 \\ -\frac{25}{4}\end{bmatrix}\right) = (2 + 3x) - \frac{25}{4}(-1 - 2x^2) = \frac{33}{4} + 3x + \frac{31}{2}x^2
\]

So the set

\[
\left\{2 + 3x + 3x^2, \frac{33}{4} + 3x + \frac{31}{2}x^2\right\}
\]

is a basis for \(\mathcal{R}(T)\).

Theorem KNSI \[614\] and Theorem RCSI \[617\] can be viewed as further formal evidence for the Coordinatization Principle \[596\], though they are not direct consequences.

Subsection IVLT
Invertible Linear Transformations

We have seen, both in theorems and in examples, that questions about linear transformations are often equivalent to questions about matrices. It is the matrix representation of a linear transformation that makes this idea precise. Here’s our final theorem that solidifies this connection.

Theorem IMR
Invertible Matrix Representations

Suppose that \(T: U \mapsto V\) is an invertible linear transformation, \(B\) is a basis for \(U\) and \(C\) is a basis for \(V\). Then the matrix representation of \(T\) relative to \(B\) and \(C\), \(M_{B,C}^T\), is an invertible matrix, and

\[
M^{-1}_{C,B} = (M_{B,C}^T)^{-1}
\]

Proof This theorem states that the matrix representation of \(T^{-1}\) can be found by finding the matrix inverse of the matrix representation of \(T\) (with suitable bases in the right places). It also says that the matrix representation of \(T\) is an invertible matrix. We can establish the invertibility, and precisely what the inverse is, by appealing to the definition

\[\text{Version 0.92}\]
of a matrix inverse, [Definition MI][232]. To this end, let $B = \{u_1, u_2, u_3, \ldots, u_n\}$ and $C = \{v_1, v_2, v_3, \ldots, v_n\}$. Then

$$M_{C,B}^{-1}M_{B,C}^T = M_{B,B}^{-1}$$

[Theorem MRCLT][610]

$$= I_n$$

[Definition IM][76]

$$= M_{B,C}^{-1}$$

[Definition IVLT][567]

$$= [\rho_B(I_U(u_1))|\rho_B(I_U(u_2))|\ldots|\rho_B(I_U(u_n))]$$

[Definition MR][603]

$$= [\rho_B(u_1)|\rho_B(u_2)|\rho_B(u_3)|\ldots|\rho_B(u_n)]$$

[Definition IDLT][567]

$$= [e_1|e_2|e_3|\ldots|e_n]$$

[Definition VR][587]

and

$$M_{B,C}^TM_{C,B}^{-1} = M_{C,C}^{-1}$$

[Theorem MRCLT][610]

$$= I_n$$

[Definition IM][76]

$$= [\rho_C(I_V(v_1))|\rho_C(I_V(v_2))|\ldots|\rho_C(I_V(v_n))]$$

[Definition MR][603]

$$= [\rho_C(v_1)|\rho_C(v_2)|\rho_C(v_3)|\ldots|\rho_C(v_n)]$$

[Definition IDLT][567]

$$= [e_1|e_2|e_3|\ldots|e_n]$$

[Definition VR][587]

So by [Definition MI][232], the matrix $M_{B,C}$ has an inverse, and that inverse is $M_{C,B}^{-1}$.

**Example ILTVR**

**Inverse of a linear transformation via a representation**

Consider the linear transformation

$$R: P_3 \mapsto M_{22}, \quad R(a + bx + cx^2 + x^3) = \begin{bmatrix} a + b - c + 2d & 2a + 3b - 2c + 3d \\ a + b + 2d & -a + b + 2c - 5d \end{bmatrix}$$

If we wish to quickly find a formula for the inverse of $R$ (presuming it exists), then choosing “nice” bases will work best. So build a matrix representation of $R$ relative to the bases $B$ and $C$,

$$B = \{1, x, x^2, x^3\}$$

$$C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Then,

$$\rho_C(R(1)) = \rho_C \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\rho_C(R(x)) = \rho_C \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$
\[ \rho_C(R(x^2)) = \rho_C \left( \begin{bmatrix} -1 & -2 \\ 0 & 2 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ -2 \\ 2 \\ 0 \end{bmatrix} \]

\[ \rho_C(R(x^3)) = \rho_C \left( \begin{bmatrix} 2 & 3 \\ 2 & -5 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \\ 2 \\ -5 \end{bmatrix} \]

So a representation of \( R \) is

\[ M_{B,C}^R = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 3 & -2 & 3 \\ 1 & 1 & 0 & 2 \\ -1 & 1 & 2 & -5 \end{bmatrix} \]

The matrix \( M_{B,C}^R \) is invertible (as you can check) so we know by Theorem IMR \( 619 \) that \( R \) is invertible. Furthermore,

\[ M_{C,B}^{-1} = (M_{B,C}^R)^{-1} = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 3 & -2 & 3 \\ 1 & 1 & 0 & 2 \\ -1 & 1 & 2 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} 20 & -7 & -2 & 3 \\ -8 & 3 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ -6 & 2 & 1 & -1 \end{bmatrix} \]

We can use this representation of the inverse linear transformation, in concert with Theorem FTMR \( 606 \), to determine an explicit formula for the inverse itself,

\[ R^{-1} \left( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) = \rho_B^{-1} \left( M_{C,B}^{-1} \rho_C \left( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) \right) \]

\[ = \rho_B^{-1} \left( (M_{B,C}^R)^{-1} \rho_C \left( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) \right) \]

\[ = \rho_B^{-1} \left( \left( M_{B,C}^R \right)^{-1} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) \]

\[ = \rho_B^{-1} \left( \begin{bmatrix} 20 & -7 & -2 & 3 \\ -8 & 3 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ -6 & 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) \]

\[ = \rho_B^{-1} \left( \begin{bmatrix} 20a - 7b - 2c + 3d \\ -8a + 3b + c - d \\ -a + c \\ -6a + 2b + c - d \end{bmatrix} \right) \]

\[ = (20a - 7b - 2c + 3d) + (-8a + 3b + c - d)x \
+ (-a + c)x^2 + (-6a + 2b + c - d)x^3 \]

You might look back at Example AIVLT \( 567 \), where we first witnessed the inverse of a linear transformation and recognize that the inverse \( (S) \) was built from using the method of this example on a matrix representation of \( T \).
Theorem IMILT
Invertible Matrices, Invertible Linear Transformation

Suppose that $A$ is a square matrix of size $n$ and $T: \mathbb{C}^n \to \mathbb{C}^n$ is the linear transformation defined by $T(x) = Ax$. Then $A$ is invertible matrix if and only if $T$ is an invertible linear transformation. \qed

Proof Choose bases $B = C = \{e_1, e_2, e_3, \ldots, e_n\}$ consisting of the standard unit vectors as a basis of $\mathbb{C}^n$ (Theorem SUVB [363]) and build a matrix representation of $T$ relative to $B$ and $C$. Then

$$
\rho_C(T(e_i)) = \rho_C(Ae_i) = \rho_C(A_i)
$$

So then the matrix representation of $T$, relative to $B$ and $C$, is simply $M^T_{B,C} = A$. This is the basic observation that makes the rest of this proof go.

(⇐) Suppose $T$ is invertible. Then $T$ is injective by Theorem ILTIS [571] and

$$
n(A) = \dim(N(A)) = \dim(N(M^T_{B,C})) = \dim(\ker T) = \dim(\{0\}) = 0
$$

Then Theorem RNNM [387] tells us that $A$ is nonsingular, and therefore $A$ is invertible (Theorem NI [251]).

(⇒) Suppose $A$ is a nonsingular matrix, then $A$ is invertible (Theorem NI [251]) and has zero nullity (Theorem RNNM [387]). So

$$
n(T) = \dim(K(T)) = \dim(N(M^T_{B,C})) = \dim(N(A)) = \dim(\{0\}) = 0
$$

So $T$ has zero nullity, and therefore has a trivial kernel and by Theorem KILT [535] $T$ is injective. Furthermore, by Theorem RPNDD [576],

$$
r(T) = \dim(\mathbb{C}^n) - n(T) = n - 0 = n
$$

So $T$ has full rank and therefore the range of $T$ is all of $\mathbb{C}^n$ and by Theorem RSLT [554] $T$ is surjective. Finally, with $T$ known to be injective and surjective, Theorem ILTIS [571] says $T$ is invertible.

This theorem looks like more work than you would imagine it to be. But by now, the connections between matrices and linear transformations should be starting to become more transparent, and you may have already recognized the invertibility of a matrix as being tantamount to the invertibility of the associated matrix representation. See Exercise MR.T60 [627] as well.
We can update the NMEx series of theorems, yet again.

**Theorem NME9**

**Nonsingular Matrix Equivalences, Round 9**

Suppose that $A$ is a square matrix of size $n$. The following are equivalent.

1. $A$ is nonsingular.
2. $A$ row-reduces to the identity matrix.
3. The null space of $A$ contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
4. The linear system $L(A, b)$ has a unique solution for every possible choice of $b$.
5. The columns of $A$ are a linearly independent set.
6. $A$ is invertible.
7. The column space of $A$ is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.
8. The columns of $A$ are a basis for $\mathbb{C}^n$.
9. The rank of $A$ is $n$, $r(A) = n$.
10. The nullity of $A$ is zero, $n(A) = 0$.
11. The determinant of $A$ is nonzero, $\det(A) \neq 0$.
12. $\lambda = 0$ is not an eigenvalue of $A$.
13. The linear transformation $T : \mathbb{C}^n \mapsto \mathbb{C}^n$ defined by $T(x) = Ax$ is invertible.

**Proof** By Theorem IMILT, the new addition to this list is equivalent to the statement that $A$ is invertible so we can expand Theorem NME8.

---

**Subsection READ**

**Reading Questions**

1. Why does Theorem FTMR deserve the moniker “fundamental”?
2. Find the matrix representation, $M_{B,C}^T$ of the linear transformation $T : \mathbb{C}^2 \mapsto \mathbb{C}^2$, $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ 3x_1 + 2x_2 \end{bmatrix}$ relative to the bases $B = \left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$ and $C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.
3. What is the second “surprise,” and why is it surprising?
Subsection EXC
Exercises

C20 Compute the matrix representation of $T$ relative to the bases $B$ and $C$.

$$T: P_3 \mapsto \mathbb{C}^3, \quad T(a + bx + cx^2 + dx^3) = \begin{bmatrix} 2a - 3b + 4c - 2d \\ a + b - c + d \\ 3a + 2c - 3d \end{bmatrix}$$

$$B = \{1, x, x^2, x^3\} \quad C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Contributed by Robert Beezer  Solution [629]

C21 Find a matrix representation of the linear transformation $T$ relative to the bases $B$ and $C$.

$$T: P_2 \mapsto \mathbb{C}^2, \quad T(p(x)) = \begin{bmatrix} p(1) \\ p(3) \end{bmatrix}$$

$$B = \{2 - 5x + x^2, 1 + x - x^2, x^2\} \quad C = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

Contributed by Robert Beezer  Solution [629]

C22 Let $S_{22}$ be the vector space of $2 \times 2$ symmetric matrices. Build the matrix representation of the linear transformation $T: P_2 \mapsto S_{22}$ relative to the bases $B$ and $C$ and then use this matrix representation to compute $T(3 + 5x - 2x^2)$.

$$B = \{1, 1 + x, 1 + x + x^2\} \quad C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$T(a + bx + cx^2) = \begin{bmatrix} 2a - b + c & a + 3b - c \\ a + 3b - c & a - c \end{bmatrix}$$

Contributed by Robert Beezer  Solution [629]

C25 Use a matrix representation to determine if the linear transformation $T: P_3 \mapsto M_{22}$ surjective.

$$T(a + bx + cx^2 + dx^3) = \begin{bmatrix} -a + 4b + c + 2d & 4a - b + 6c - d \\ a + 5b - 2c + 2d & a + 2c + 5d \end{bmatrix}$$

Contributed by Robert Beezer  Solution [630]

C30 Find bases for the kernel and range of the linear transformation $S$ below.

$$S: M_{22} \mapsto P_2, \quad S\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a + 2b + 5c - 4d) + (3a - b + 8c + 2d)x + (a + b + 4c - 2d)x^2$$
Contributed by Robert Beezer  

**C40**  Let $S_{22}$ be the set of $2 \times 2$ symmetric matrices. Verify that the linear transformation $R$ is invertible and find $R^{-1}$.

$$R: S_{22} \mapsto P_2, \quad R \begin{pmatrix} a & b \\ b & c \end{pmatrix} = (a - b) + (2a - 3b - 2c)x + (a - b + c)x^2$$

Contributed by Robert Beezer  

**C41**  Prove that the linear transformation $S$ is invertible. Then find a formula for the inverse linear transformation, $S^{-1}$, by employing a matrix inverse. (15 points)

$$S: P_1 \mapsto M_{1,2}, \quad S(a + bx) = \begin{bmatrix} 3a + b \\ 2a + b \end{bmatrix}$$

Contributed by Robert Beezer  

**C42**  The linear transformation $R: M_{12} \mapsto M_{21}$ is invertible. Use a matrix representation to determine a formula for the inverse linear transformation $R^{-1}: M_{21} \mapsto M_{12}$.

$$R \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a + 3b \\ 4a + 11b \end{bmatrix}$$

Contributed by Robert Beezer  

**C50**  Use a matrix representation to find a basis for the range of the linear transformation $L$. (15 points)

$$L: M_{22} \mapsto P_2, \quad T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a + 2b + 4c + d) + (3a + c - 2d)x + (-a + b + 3c + 3d)x^2$$

Contributed by Robert Beezer  

**C51**  Use a matrix representation to find a basis for the kernel of the linear transformation $L$. (15 points)

$$L: M_{22} \mapsto P_2, \quad T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a + 2b + 4c + d) + (3a + c - 2d)x + (-a + b + 3c + 3d)x^2$$

Contributed by Robert Beezer  

**C52**  Find a basis for the kernel of the linear transformation $T: P_2 \mapsto M_{22}$.

$$T(a + bx + cx^2) = \begin{bmatrix} a + 2b - 2c \\ 2a + 2b \\ -a + b - 4c \\ 3a + 2b + 2c \end{bmatrix}$$

Contributed by Robert Beezer  

**M20**  The linear transformation $D$ performs differentiation on polynomials. Use a matrix representation of $D$ to find the rank and nullity of $D$.

$$D: P_n \mapsto P_n, \quad D(p(x)) = p'(x)$$
T60  Create an entirely different proof of Theorem IMILT 621 that relies on Definition IVLT 567 to establish the invertibility of $T$, and that relies on Definition MI 232 to establish the invertibility of $A$.
Contributed by Robert Beezer

T80  Suppose that $T: U \rightarrow V$ and $S: V \rightarrow W$ are linear transformations, and that $B$, $C$ and $D$ are bases for $U$, $V$, and $W$. Using only Definition MR 603 define matrix representations for $T$ and $S$. Using these two definitions, and Definition MR 603, derive a matrix representation for the composition $S \circ T$ in terms of the entries of the matrices $M_{B,C}^T$ and $M_{C,D}^S$. Explain how you would use this result to motivate a definition for matrix multiplication that is strikingly similar to Theorem ME 474.
Contributed by Robert Beezer
Apply Definition MR 603,

\[ \rho_C(T(1)) = \rho_C \left( \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right) = \rho_C \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \]

\[ \rho_C(T(x)) = \rho_C \left( \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right) = \rho_C \left( (-4) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} \]

\[ \rho_C(T(x^2)) = \rho_C \left( \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} \right) = \rho_C \left( 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} \]

\[ \rho_C(T(x^3)) = \rho_C \left( \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \right) = \rho_C \left( (-3) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 4 \\ -3 \end{bmatrix} \]

These four vectors are the columns of the matrix representation,

\[ M_{B,C}^T = \begin{bmatrix} 1 & -4 & 5 & -3 \\ -2 & 1 & -3 & 4 \\ 3 & 0 & 2 & -3 \end{bmatrix} \]

Applying Definition MR 603,

\[ \rho_C(T(2 - 5x + x^2)) = \rho_C \left( \begin{bmatrix} -2 \\ -4 \end{bmatrix} \right) = \rho_C \left( 2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + (-4) \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \]

\[ \rho_C(T(1 + x - x^2)) = \rho_C \left( \begin{bmatrix} 1 \\ -5 \end{bmatrix} \right) = \rho_C \left( 13 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + (-19) \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 13 \\ -19 \end{bmatrix} \]

\[ \rho_C(T(x^2)) = \rho_C \left( \begin{bmatrix} 1 \\ 9 \end{bmatrix} \right) = \rho_C \left( (-15) \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 23 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} -15 \\ 23 \end{bmatrix} \]

So the resulting matrix representation is

\[ M_{B,C}^T = \begin{bmatrix} 2 & 13 & -15 \\ -4 & -19 & 23 \end{bmatrix} \]

Input to \( T \) the vectors of the basis \( B \) and coordinatize the outputs relative to \( C \),

\[ \rho_C(T(1)) = \rho_C \left( \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right) = \rho_C \left( 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \]
\[ \rho_C(T(1+x)) = \rho_C \left( \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \right) = \rho_C \left( 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 4 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \]

\[ \rho_C(T(1+x+x^2)) = \rho_C \left( \begin{bmatrix} 2 & 3 \\ 3 & 0 \end{bmatrix} \right) = \rho_C \left( 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

Applying Definition MR 603 we have the matrix representation

\[ M_{B,C}^T = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 4 & 3 \\ 1 & 1 & 0 \end{bmatrix} \]

To compute \( T(3+5x-2x^2) \) employ Theorem FTMR 606,

\[ T(3+5x-2x^2) = \rho_C^{-1} \left( M_{B,C}^T \rho_B \left( 3+5x-2x^2 \right) \right) \]
\[ = \rho_C^{-1} \left( M_{B,C}^T \rho_B \left( (-2)(1) + 7(1+x) + (-2)(1+x+x^2) \right) \right) \]
\[ = \rho_C^{-1} \left( \begin{bmatrix} 2 & 1 & 2 \\ 1 & 4 & 3 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 7 \\ -2 \end{bmatrix} \right) \]
\[ = \rho_C^{-1} \left( \begin{bmatrix} -1 \\ 20 \\ 5 \end{bmatrix} \right) \]
\[ = (-1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 20 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]
\[ = \begin{bmatrix} -1 \\ 20 \\ 5 \end{bmatrix} \]

You can, of course, check your answer by evaluating \( T(3+5x-2x^2) \) directly.
\[
\rho_C(T(x^3)) = \rho_C \left( \begin{bmatrix} 2 & -1 \\ 2 & 5 \end{bmatrix} \right) = \rho_C \left( 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 5 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 5 \end{bmatrix}
\]

Applying Definition MR we have the matrix representation

\[
M_{B,C}^T = \begin{bmatrix} -1 & 4 & 1 & 2 \\ 4 & -1 & 6 & -1 \\ 1 & 5 & -2 & 2 \\ 1 & 0 & 2 & 5 \end{bmatrix}
\]

Properties of this matrix representation will translate to properties of the linear transformation. The matrix representation is nonsingular since it row-reduces to the identity matrix (Theorem NMRR [77]) and therefore has a column space equal to \( \mathbb{C}^4 \) (Theorem CNMB [369]). The column space of the matrix representation is isomorphic to the range of the linear transformation (Theorem RCSI [617]). So the range of \( T \) has dimension 4, equal to the dimension of the codomain \( M_{22} \). By Theorem ROSLT [575], \( T \) is surjective.

C30 Contributed by Robert Beezer Statement

These subspaces will be easiest to construct by analyzing a matrix representation of \( S \). Since we can use any matrix representation, we might as well use natural bases that allow us to construct the matrix representation quickly and easily,

\[
B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \quad C = \{1, x, x^2\}
\]

then we can practically build the matrix representation on sight,

\[
M_{B,C}^S = \begin{bmatrix} 1 & 2 & 5 & -4 \\ 3 & -1 & 8 & 2 \\ 1 & 1 & 4 & -2 \end{bmatrix}
\]

The first step is to find bases for the null space and column space of the matrix representation. Row-reducing the matrix representation we find,

\[
\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\]

So by Theorem BNS [154] and Theorem BCS [264], we have

\[
\mathcal{N}(M_{B,C}^S) = \left\langle \left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle \quad \mathcal{C}(M_{B,C}^S) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\} \right\rangle
\]

Now, the proofs of Theorem KNSI [614] and Theorem RCSI [617] tell us that we can apply \( \rho_B^{-1} \) and \( \rho_C^{-1} \) (respectively) to “un-coordinatize” and get bases for the kernel and range of the linear transformation \( S \) itself,

\[
\mathcal{K}(S) = \left\langle \left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle \quad \mathcal{R}(S) = \left\langle \{1 + 3x + x^2, 2 - x + x^2\} \right\rangle
\]
The analysis of $R$ will be easiest if we analyze a matrix representation of $R$. Since we can use any matrix representation, we might as well use natural bases that allow us to construct the matrix representation quickly and easily,

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \quad C = \{1, x, x^2\}$$

then we can practically build the matrix representation on sight,

$$M_{B,C}^R = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

This matrix representation is invertible (it has a nonzero determinant of $-1$), so Theorem IMR tells us that the linear transformation $S$ is also invertible. To find a formula for $R^{-1}$ we compute,

$$R^{-1} (a + bx + cx^2) = \rho_B^{-1} \left( M_{C,B}^R \rho_C (a + bx + cx^2) \right)$$

$$= \rho_B^{-1} \left( (M_{B,C}^R)^{-1} \rho_C (a + bx + cx^2) \right)$$

$$= \rho_B^{-1} \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right)$$

$$= \rho_B^{-1} \left( \begin{bmatrix} 5 & -1 & -2 \\ 4 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right)$$

$$= \rho_B^{-1} \left( \begin{bmatrix} 5a - b - 2c \\ 4a - b - 2c \\ -a + c \end{bmatrix} \right)$$

$$= \begin{bmatrix} 5a - b - 2c & 4a - b - 2c & -a + c \end{bmatrix}$$

First, build a matrix representation of $S$ (Definition MR). We are free to choose whatever bases we wish, so we should choose ones that are easy to work with, such as

$$B = \{1, x\} \quad C = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix}\}$$

The resulting matrix representation is then

$$M_{B,C}^T = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

this matrix is invertible, since it has a nonzero determinant, so by Theorem IMR the linear transformation $S$ is invertible. We can use the matrix inverse and Theorem IMR to find a formula for the inverse linear transformation,

$$S^{-1} \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = \rho_B^{-1} \left( M_{C,B}^{-1} \rho_C \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) \right)$$
Subsection MR.SOL  |  Solutions  | 633

\[
\rho^{−1}_B \left( (M^s_{B,C})^{-1} \rho_C \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) \right)
\]

\[
= \rho^{−1}_B \left( (M^s_{B,C})^{-1} \begin{bmatrix} a \\ b \end{bmatrix} \right)
\]

\[
= \rho_B^{-1} \left( \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} a \\ b \end{bmatrix}
\]

\[
= \rho_B^{-1} \left( \begin{bmatrix} 1 \\ -2 \\ -1 \\ 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right)
\]

\[
= \rho_B^{-1} \left( \begin{bmatrix} a - b \\ -2a + 3b \end{bmatrix} \right)
\]

\[
= (a - b) + (-2a + 3b)x
\]

\textbf{C42}  Contributed by Robert Beezer  Statement 626

Choose bases \(B\) and \(C\) for \(M_{12}\) and \(M_{21}\) (respectively),

\[
B = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \} \quad C = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \}
\]

The resulting matrix representation is

\[
M^R_{B,C} = \begin{bmatrix} 1 & 3 \\ 4 & 11 \end{bmatrix}
\]

This matrix is invertible (its determinant is nonzero), so by Theorem IMR 619, we can compute the matrix representation of \(R^{-1}\) with a matrix inverse (Theorem TTMI 234),

\[
M^{-1}_{C,B} = \begin{bmatrix} 1 & 3 \\ 4 & 11 \end{bmatrix}^{-1} = \begin{bmatrix} -11 & 3 \\ 4 & -1 \end{bmatrix}
\]

To obtain a general formula for \(R^{-1}\), use Theorem FTMR 606,

\[
R^{-1} \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = \rho_B \left( M^{-1}_{C,B} \rho_C \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) \right) = \rho_B \left( \begin{bmatrix} -11x + 3y \\ 4x - y \end{bmatrix} \right) = \begin{bmatrix} -11x + 3y \\ 4x - y \end{bmatrix}
\]

\textbf{C50}  Contributed by Robert Beezer  Statement 626

As usual, build any matrix representation of \(L\), most likely using a “nice” bases, such as

\[
B = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}
\]

\[
C = \{ 1, x, x^2 \}
\]

Then the matrix representation (Definition MR 603) is,

\[
M^L_{B,C} = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 3 & 0 & 1 & -2 \\ -1 & 1 & 3 & 3 \end{bmatrix}
\]

Theorem RCSI 617 tells us that we can compute the column space of the matrix representation, then use the isomorphism \(\rho_C^{-1}\) to convert the column space of the matrix
representation into the range of the linear transformation. So we first analyze the matrix representation,

\[
\begin{bmatrix}
1 & 2 & 4 & 1 \\
3 & 0 & 1 & -2 \\
-1 & 1 & 3 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

With three nonzero rows in the reduced row-echelon form of the matrix, we know the column space has dimension 3. Since \( P_2 \) has dimension 3 (Theorem DP [383]), the range must be all of \( P_2 \). So any basis of \( P_2 \) would suffice as a basis for the range. For instance, \( C \) itself would be a correct answer.

A more laborious approach would be to use Theorem BCS [264] and choose the first three columns of the matrix representation as a basis for the range of the matrix representation. These could then be “un-coordinatized” with \( \rho_C^{-1} \) to yield a (“not nice”) basis for \( P_2 \).

C52 Contributed by Robert Beezer Statement 626
Choose bases \( B \) and \( C \) for the matrix representation,

\[
B = \{1, x, x^2\}, \quad C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]

Input to \( T \) the vectors of the basis \( B \) and coordinatize the outputs relative to \( C \),

\[
\rho_C(T(1)) = \rho_C \left( \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} \right) = \rho_C \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}
\]

\[
\rho_C(T(x)) = \rho_C \left( \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right) = \rho_C \left( 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}
\]

\[
\rho_C(T(x^2)) = \rho_C \left( \begin{bmatrix} -2 \\ -4 \\ 0 \\ 2 \end{bmatrix} \right) = \rho_C \left( (-2) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-4) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 0 \\ -4 \\ 2 \end{bmatrix}
\]

Applying Definition MR 603 we have the matrix representation

\[
M_{B,C}^T = \begin{bmatrix}
1 & 2 & -2 \\
2 & 2 & 0 \\
-1 & 1 & -4 \\
3 & 2 & 2
\end{bmatrix}
\]

The null space of the matrix representation is isomorphic (via \( \rho_B \)) to the kernel of the linear transformation (Theorem KNSI [614]). So we compute the null space of the matrix representation by first row-reducing the matrix to,

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Version 0.92
Employing Theorem BNS \[154\] we have
\[ \mathcal{N}(M_{B,C}^T) = \left\langle \left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\} \right\rangle \]

We only need to uncoordinatize this one basis vector to get a basis for \( \mathcal{K}(T) \),
\[ \mathcal{K}(T) = \left\langle \left\{ \rho_B^{-1} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\} \right\rangle = \left\langle \{ -2 + 2x + x^2 \} \right\rangle \]

\textbf{M20} Contributed by Robert Beezer Statement \[626\]

Build a matrix representation with the set
\[ B = \{1, x, x^2, \ldots, x^n\} \]
employed as a basis of both the domain and codomain. Then
\[
\begin{align*}
\rho_B (D(1)) &= \rho_B (0) = \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \\
\rho_B (D(x)) &= \rho_B (1) = \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
\rho_B (D(x^2)) &= \rho_B (2x) = \\
&= \begin{bmatrix} 0 \\ 2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
\rho_B (D(x^3)) &= \rho_B (3x^2) = \\
&= \begin{bmatrix} 0 \\ 0 \\ 3 \\ \vdots \\ 0 \end{bmatrix} \\
\vdots \end{align*}
\]

\[ \rho_B (D(x^n)) = \rho_B (nx^{n-1}) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ n \\ 0 \end{bmatrix} \]

and the resulting matrix representation is
\[
M_{B,B}^D = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 2 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 3 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & n & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
\]
This \((n+1) \times (n+1)\) matrix is very close to being in reduced row-echelon form. Multiply row \(i\) by \(\frac{1}{i}\), for \(1 \leq i \leq n\), to convert it to reduced row-echelon form. From this we can see that matrix representation \(M_{B,B}^D\) has rank \(n\) and nullity 1. Applying Theorem RCSI \(\text{617}\) and Theorem KNSI \(\text{614}\) tells us that the linear transformation \(D\) will have the same values for the rank and nullity, as well.
Section CB
Change of Basis

We have seen in Section MR that a linear transformation can be represented by a matrix, once we pick bases for the domain and codomain. How does the matrix representation change if we choose different bases? Which bases lead to especially nice representations? From the infinite possibilities, what is the best possible representation? This section will begin to answer these questions. But first we need to define eigenvalues for linear transformations and the change-of-basis matrix.

Subsection EELT
Eigenvalues and Eigenvectors of Linear Transformations

We now define the notion of an eigenvalue and eigenvector of a linear transformation. It should not be too surprising, especially if you remind yourself of the close relationship between matrices and linear transformations.

Definition EELT
Eigenvalue and Eigenvector of a Linear Transformation
Suppose that \( T: V \rightarrow V \) is a linear transformation. Then a nonzero vector \( \mathbf{v} \in V \) is an eigenvector of \( T \) for the eigenvalue \( \lambda \) if \( T(\mathbf{v}) = \lambda \mathbf{v} \).

We will see shortly the best method for computing the eigenvalues and eigenvectors of a linear transformation, but for now, here are some examples to verify that such things really do exist.

Example ELTBM
Eigenvectors of linear transformation between matrices
Consider the linear transformation \( T: M_{22} \rightarrow M_{22} \) defined by

\[
T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} -17a + 11b + 8c - 11d \\ -14a + 10b + 6c - 10d \end{bmatrix} = -57a + 35b + 24c - 33d \\
\begin{bmatrix} -14a + 10b + 6c - 10d \\ -14a + 10b + 6c - 10d \end{bmatrix} = -41a + 25b + 16c - 23d
\]

and the vectors

\[
\mathbf{x}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \quad \mathbf{x}_4 = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}
\]

Then compute

\[
T(\mathbf{x}_1) = T\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} = 2\mathbf{x}_1
\]

\[
T(\mathbf{x}_2) = T\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} = 2\mathbf{x}_2
\]

\[
T(\mathbf{x}_3) = T\left(\begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}\right) = \begin{bmatrix} -1 & -3 \\ -2 & -3 \end{bmatrix} = -1\mathbf{x}_3
\]
\[
T(x_4) = T \begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix} = \begin{bmatrix} -4 & -12 \\ -2 & -8 \end{bmatrix} = (-2)x_4
\]

So \(x_1, x_2, x_3, x_4\) are eigenvectors of \(T\) with eigenvalues (respectively) \(\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = -1, \lambda_4 = -2\).

Here’s another.

**Example ELTBP**

Eigenvectors of linear transformation between polynomials

Consider the linear transformation \(R: P_2 \mapsto P_2\) defined by

\[
R(a + bx + cx^2) = (15a + 8b - 4c) + (-12a - 6b + 3c)x + (24a + 14b - 7c)x^2
\]

and the vectors

\[
w_1 = 1 - x + x^2 \quad w_2 = x + 2x^2 \quad w_3 = 1 + 4x^2
\]

Then compute

\[
\begin{align*}
R(w_1) &= R(1 - x + x^2) = 3 - 3x + 3x^2 = 3w_1 \\
R(w_2) &= R(x + 2x^2) = 0 + 0x + 0x^2 = 0w_2 \\
R(w_3) &= R(1 + 4x^2) = -1 - 4x^2 = (-1)w_3
\end{align*}
\]

So \(w_1, w_2, w_3\) are eigenvectors of \(R\) with eigenvalues (respectively) \(\lambda_1 = 3, \lambda_2 = 0, \lambda_3 = -1\). Notice how the eigenvalue \(\lambda_2 = 0\) indicates that the eigenvector \(w_2\) is a non-trivial element of the kernel of \(R\), and therefore \(R\) is not injective (Exercise CB.T15 [661]).

Of course, these examples are meant only to illustrate the definition of eigenvectors and eigenvalues for linear transformations, and therefore beg the question, “How would I find eigenvectors?” We’ll have an answer before we finish this section. We need one more construction first.

**Subsection CBM**

**Change-of-Basis Matrix**

Given a vector space, we know we can usually find many different bases for the vector space, some nice, some nasty. If we choose a single vector from this vector space, we can build many different representations of the vector by constructing the representations relative to different bases. How are these different representations related to each other? A change-of-basis matrix answers this question.

**Definition CBM**

**Change-of-Basis Matrix**

Suppose that \(V\) is a vector space, and \(I_V: V \mapsto V\) is the identity linear transformation
on $V$. Let $B = \{v_1, v_2, v_3, \ldots, v_n\}$ and $C$ be two bases of $V$. Then the change-of-basis matrix from $B$ to $C$ is the matrix representation of $I_V$ relative to $B$ and $C$,

$$C_{B,C} = M_{I_V}^{I_V}$$

$$= [\rho_C (I_V (v_1)) | \rho_C (I_V (v_2)) | \rho_C (I_V (v_3)) | \ldots | \rho_C (I_V (v_n))]$$

$$= [\rho_C (v_1) | \rho_C (v_2) | \rho_C (v_3) | \ldots | \rho_C (v_n)]$$

\[\triangle\]

Notice that this definition is primarily about a single vector space ($V$) and two bases of $V$ ($B, C$). The linear transformation ($I_V$) is necessary but not critical. As you might expect, this matrix has something to do with changing bases. Here is the theorem that gives the matrix its name (not the other way around).

**Theorem CB**

**Change-of-Basis**

Suppose that $\mathbf{v}$ is a vector in the vector space $V$ and $B$ and $C$ are bases of $V$. Then

$$\rho_C (\mathbf{v}) = C_{B,C} \rho_B (\mathbf{v})$$

\[\square\]

**Proof**

$$\rho_C (\mathbf{v}) = \rho_C (I_V (\mathbf{v}))$$

Definition IDLT 567

$$= M_{I_V}^{I_V} \rho_B (\mathbf{v})$$

Theorem FTMR 606

$$= C_{B,C} \rho_B (\mathbf{v})$$

Definition CBM 638

So the change-of-basis matrix can be used with matrix multiplication to convert a vector representation of a vector ($\mathbf{v}$) relative to one basis ($\rho_B (\mathbf{v})$) to a representation of the same vector relative to a second basis ($\rho_C (\mathbf{v})$).

**Theorem ICBM**

**Inverse of Change-of-Basis Matrix**

Suppose that $V$ is a vector space, and $B$ and $C$ are bases of $V$. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

\[\square\]

**Proof** The linear transformation $I_V: V \mapsto V$ is invertible, and its inverse is itself, $I_V$ (check this!). So by Theorem IMR 619, the matrix $M_{B,C}^{I_V} = C_{B,C}$ is invertible. Theorem NI 251 says an invertible matrix is nonsingular.

Then

$$C_{B,C}^{-1} = (M_{B,C}^{I_V})^{-1}$$

Definition CBM 638

$$= M_{C,B}^{-1}$$

Theorem IMR 619

$$= M_{C,B}$$

Definition IDLT 567

$$= C_{C,B}$$

Definition CBM 638

Version 0.92
Example CBP
Change of basis with polynomials
The vector space $P_4$ (Example VSP 312) has two nice bases (Example BP 364),
$B = \{1, x, x^2, x^3, x^4\} \quad C = \{1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3, 1 + x + x^2 + x^3 + x^4\}$

To build the change-of-basis matrix between $B$ and $C$, we must first build a vector representation of each vector in $B$ relative to $C$,

$\rho_C(1) = \rho_C((1)(1)) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\rho_C(x) = \rho_C((-1)(1) + (1)(1 + x)) = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\rho_C(x^2) = \rho_C((-1)(1 + x) + (1)(1 + x + x^2)) = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$\rho_C(x^3) = \rho_C((-1)(1 + x + x^2) + (1)(1 + x + x^2 + x^3)) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

$\rho_C(x^4) = \rho_C((-1)(1 + x + x^2 + x^3) + (1)(1 + x + x^2 + x^3 + x^4)) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

Then we package up these vectors as the columns of a matrix,

$C_{B,C} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Now, to illustrate Theorem CB 639, consider the vector $u = 5 - 3x + 2x^2 + 8x^3 - 3x^4$. 
We can build the representation of \( u \) relative to \( B \) easily,

\[
\rho_B(u) = \rho_B\left(5 - 3x + 2x^2 + 8x^3 - 3x^4\right) = \begin{bmatrix}
5 \\
-3 \\
2 \\
8 \\
-3
\end{bmatrix}
\]

Applying Theorem CB \[639\], we obtain a second representation of \( u \), but now relative to \( C \),

\[
\rho_C (u) = C_{B,C} \rho_B (u)
\]

\[
= \begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
5 \\
-3 \\
2 \\
8 \\
-3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
8 \\
-5 \\
-6 \\
11 \\
-3
\end{bmatrix}
\]

We can check our work by unraveling this second representation,

\[
u = \rho_C^{-1} (\rho_C (u))
\]

\[
= \rho_C^{-1} \begin{bmatrix}
8 \\
-5 \\
-6 \\
11 \\
-3
\end{bmatrix}
\]

\[
= 8(1) + (-5)(1 + x) + (-6)(1 + x + x^2)
\]

\[
+ (11)(1 + x + x^2 + x^3) + (-3)(1 + x + x^2 + x^3 + x^4)
\]

\[
= 5 - 3x + 2x^2 + 8x^3 - 3x^4
\]

The change-of-basis matrix from \( C \) to \( B \) is actually easier to build. Grab each vector in the basis \( C \) and form its representation relative to \( B \)

\[
\rho_B (1) = \rho_B ((1)1) = \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\rho_B (1 + x) = \rho_B ((1)1 + (1)x) = \begin{bmatrix}
1 \\
1 \\
0 \\
0 \\
0
\end{bmatrix}
\]
\[
\rho_B (1 + x + x^2) = \rho_B ((1)1 + (1)x + (1)x^2) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}
\]

\[
\rho_B (1 + x + x^2 + x^3) = \rho_B ((1)1 + (1)x + (1)x^2 + (1)x^3) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}
\]

\[
\rho_B (1 + x + x^2 + x^3 + x^4) = \rho_B ((1)1 + (1)x + (1)x^2 + (1)x^3 + (1)x^4) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\]

Then we package up these vectors as the columns of a matrix,

\[
C_{C,B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

We formed two representations of the vector \(\mathbf{u}\) above, so we can again provide a check on our computations by converting from the representation of \(\mathbf{u}\) relative to \(C\) to the representation of \(\mathbf{u}\) relative to \(B\),

\[
\rho_B (\mathbf{u}) = C_{C,B}\rho_C (\mathbf{u})
\]

\[
\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ -5 \\ -6 \\ 11 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 2 \\ 8 \\ -3 \end{bmatrix}
\]

One more computation that is either a check on our work, or an illustration of a theorem. The two change-of-basis matrices, \(C_{B,C}\) and \(C_{C,B}\), should be inverses of each other, according to Theorem ICBM. Here we go,

\[
C_{B,C}C_{C,B} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]
The computations of the previous example are not meant to present any labor-saving devices, but instead are meant to illustrate the utility of the change-of-basis matrix. However, you might have noticed that \( C_{B,C} \) was easier to compute than \( C_{C,B} \). If you needed \( C_{B,C} \), then you could first compute \( C_{C,B} \) and then compute its inverse, which by Theorem ICBM 639, would equal \( C_{B,C} \).

Here’s another illustrative example. We have been concentrating on working with abstract vector spaces, but all of our theorems and techniques apply just as well to \( \mathbb{C}^m \), the vector space of column vectors. We only need to use more complicated bases than the standard unit vectors (Theorem SUVB 363) to make things interesting.

**Example CBCV**

**Change of basis with column vectors**

For the vector space \( \mathbb{C}^4 \) we have the two bases,

\[
B = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 3 \\ 0 \end{bmatrix} \right\} \quad C = \left\{ \begin{bmatrix} 1 \\ -6 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 8 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 9 \end{bmatrix} \right\}
\]

The change-of-basis matrix from \( B \) to \( C \) requires writing each vector of \( B \) as a linear combination the vectors in \( C \),

\[
\rho_C\left( \begin{bmatrix} 1 \\ -2 \\ 1 \\ -2 \end{bmatrix} \right) = \rho_C\left( \begin{bmatrix} 1 \\ -6 \\ -1 \end{bmatrix} \right) + (-2) \begin{bmatrix} -4 \\ 8 \\ -4 \end{bmatrix} + (1) \begin{bmatrix} -5 \\ -5 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}
\]

\[
\rho_C\left( \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} \right) = \rho_C\left( \begin{bmatrix} 1 \\ -6 \\ -1 \end{bmatrix} \right) + (-3) \begin{bmatrix} -4 \\ 8 \\ -5 \end{bmatrix} + (3) \begin{bmatrix} -5 \\ -5 \\ -2 \end{bmatrix} + (0) \begin{bmatrix} 3 \\ -7 \\ -6 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}
\]

\[
\rho_C\left( \begin{bmatrix} 2 \\ 3 \\ 1 \\ -4 \end{bmatrix} \right) = \rho_C\left( \begin{bmatrix} 1 \\ -6 \\ -1 \end{bmatrix} \right) + (-3) \begin{bmatrix} -4 \\ 8 \\ -5 \end{bmatrix} + (1) \begin{bmatrix} -5 \\ -5 \\ -2 \end{bmatrix} + (2) \begin{bmatrix} 3 \\ -7 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]

\[
\rho_C\left( \begin{bmatrix} -1 \\ 3 \\ 3 \\ 0 \end{bmatrix} \right) = \rho_C\left( \begin{bmatrix} 1 \\ -6 \\ -1 \end{bmatrix} \right) + (-2) \begin{bmatrix} -4 \\ 8 \\ -5 \end{bmatrix} + (4) \begin{bmatrix} -5 \\ -5 \\ -2 \end{bmatrix} + (3) \begin{bmatrix} 3 \\ -7 \\ -6 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}
\]

Then we package these vectors up as the change-of-basis matrix,

\[
C_{B,C} = \begin{bmatrix}
1 & 2 & 1 & 2 \\
-2 & -3 & -3 & -2 \\
1 & 3 & 1 & 4 \\
-1 & 0 & -2 & 3
\end{bmatrix}
\]
Now consider a single (arbitrary) vector \( y = \begin{bmatrix} 2 \\ 6 \\ -3 \\ 4 \end{bmatrix} \). First, build the vector representation of \( y \) relative to \( B \). This will require writing \( y \) as a linear combination of the vectors in \( B \),

\[
\rho_B(y) = \begin{bmatrix} 2 \\ 6 \\ -3 \\ 4 \end{bmatrix}
\]

\[
= \rho_B(-21 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + 6 \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + 11 \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix} + (-7) \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}) = \begin{bmatrix} -21 \\ 6 \\ 11 \\ -7 \end{bmatrix}
\]

Now, applying Theorem CB [639] we can convert the representation of \( y \) relative to \( B \) into a representation relative to \( C \),

\[
\rho_C(y) = C_{B,C} \rho_B(y)
\]

\[
= \begin{bmatrix} 1 & 2 & 1 & 2 \\ -2 & -3 & -3 & -2 \\ 1 & 3 & 1 & 4 \\ -1 & 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} -21 \\ 6 \\ 11 \\ -7 \end{bmatrix}
\]

\[
= \begin{bmatrix} -12 \\ 5 \\ -20 \\ -22 \end{bmatrix}
\]

Definition MVP [211]

We could continue further with this example, perhaps by computing the representation of \( y \) relative to the basis \( C \) directly as a check on our work [Exercise CB.C20 [661]]. Or we could choose another vector to play the role of \( y \) and compute two different representations of this vector relative to the two bases \( B \) and \( C \).

\[\Box\]

### Subsection MRS

**Matrix Representations and Similarity**

Here is the main theorem of this section. It looks a bit involved at first glance, but the proof should make you realize it is not all that complicated. In any event, we are more interested in a special case.

**Theorem MRCB**

**Matrix Representation and Change of Basis**

Suppose that \( T: U \rightarrow V \) is a linear transformation, \( B \) and \( C \) are bases for \( U \), and \( D \) and \( E \) are bases for \( V \). Then

\[
M_{B,D}^T = C_{E,D}M_{C,E}^T C_{B,C}
\]
Proof

\[ C_{E,D}M_{C,E}^{T}C_{B,C} = M_{E,D}^{I}M_{C,E}^{T}M_{B,C}^{I} = M_{E,D}^{I}M_{B,E}^{T} = M_{E,D}^{I}M_{B,E}^{T} = M_{B,D}^{I} \]

\[ \text{Definition CBM [638]} \]

\[ \text{Theorem MRCLT [610]} \]

\[ \text{Definition IDLT [567]} \]

\[ \text{Theorem MRCLT [610]} \]

\[ \text{Definition IDLT [567]} \]

We will be most interested in a special case of this theorem (Theorem SCB [647]), but here’s an example that illustrates the full generality of Theorem MRCB [644].

Example MRCM

Matrix representations and change-of-basis matrices

Begin with two vector spaces, \( S_{2} \), the subspace of \( M_{22} \) containing all \( 2 \times 2 \) symmetric matrices, and \( P_{3} \) (Example VSP [312]), the vector space of all polynomials of degree 3 or less. Then define the linear transformation \( Q: S_{2} \rightarrow P_{3} \) by

\[ Q \left( \begin{bmatrix} a & b \\ b & c \end{bmatrix} \right) = (5a - 2b + 6c) + (3a - b + 2c)x + (a + 3b - c)x^{2} + (-4a + 2b + c)x^{3} \]

Here are two bases for each vector space, one nice, one nasty. First for \( S_{2} \),

\[ B = \left\{ \begin{bmatrix} 5 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\} \quad C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \]

and then for \( P_{3} \),

\[ D = \{ 2 + x - 2x^{2} + 3x^{3}, -1 - 2x^{2} + 3x^{3}, -3 - x + x^{3}, -x^{2} + x^{3} \} \quad E = \{ 1, x, x^{2}, x^{3} \} \]

We’ll begin with a matrix representation of \( Q \) relative to \( C \) and \( E \). We first find vector representations of the elements of \( C \) relative to \( E \),

\[ \rho_{E} \left( Q \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \right) = \rho_{E} \left( 5 + 3x + x^{2} - 4x^{3} \right) = \begin{bmatrix} 5 \\ 3 \\ 1 \\ -4 \end{bmatrix} \]

\[ \rho_{E} \left( Q \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \right) = \rho_{E} \left( -2 - x + 3x^{2} + 2x^{3} \right) = \begin{bmatrix} -2 \\ -1 \\ 3 \\ 2 \end{bmatrix} \]

\[ \rho_{E} \left( Q \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \right) = \rho_{E} \left( 6 + 2x - x^{2} + x^{3} \right) = \begin{bmatrix} 6 \\ 2 \\ -1 \\ 1 \end{bmatrix} \]
Now we construct two change-of-basis matrices. First, $C_{B,C}$ requires vector representations of the elements of $B$, relative to $C$. Since $C$ is a nice basis, this is straightforward,

$$
\rho_C \left( \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \right) = \rho_C \left( 5 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (-2) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ -3 \end{bmatrix}
$$

$$
\rho_C \left( \begin{bmatrix} 2 & -3 \\ -3 & 0 \end{bmatrix} \right) = \rho_C \left( 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (0) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}
$$

$$
\rho_C \left( \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \right) = \rho_C \left( 1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (2) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (4) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}
$$

So

$$
C_{B,C} = \begin{bmatrix} 5 & 2 & 1 \\ -3 & -3 & 2 \\ -2 & 0 & 4 \end{bmatrix}
$$

The other change-of-basis matrix we’ll compute is $C_{E,D}$. However, since $E$ is a nice basis (and $D$ is not) we’ll turn it around and instead compute $C_{D,E}$ and apply Theorem ICBM [639] to use an inverse to compute $C_{E,D}$.

$$
\rho_E \left( 2 + x - 2x^2 + 3x^3 \right) = \rho_E \left( (2)1 + (1)x + (-2)x^2 + (3)x^3 \right) = \begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \end{bmatrix}
$$

$$
\rho_E \left( -1 - 2x^2 + 3x^3 \right) = \rho_E \left( (-1)1 + (0)x + (-2)x^2 + (3)x^3 \right) = \begin{bmatrix} -1 \\ 0 \\ -2 \\ 3 \end{bmatrix}
$$

$$
\rho_E \left( -3 - x + x^3 \right) = \rho_E \left( (-3)1 + (0)x + (-1)x^2 + (1)x^3 \right) = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}
$$

$$
\rho_E \left( -x^2 + x^3 \right) = \rho_E \left( (0)1 + (0)x + (-1)x^2 + (1)x^3 \right) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}
$$

So, we can package these column vectors up as a matrix to obtain $C_{D,E}$ and then, $C_{E,D} = (C_{D,E})^{-1}$ Theorem ICBM [639]
We are now in a position to apply Theorem MRCB \[644\]. The matrix representation of \( Q \) relative to \( B \) and \( D \) can be obtained as follows,

\[
M_{Q,B,D}^{C} = C_{E,D}^{-1}M_{Q,C,E}C_{B,C} \quad \text{Theorem MRCB} \[644\]
\]

Now check our work by computing \( M_{Q,B,D}^{C} \) directly (Exercise CB.C21 \[661\]).

Here is a special case of the previous theorem, where we choose \( U \) and \( V \) to be the same vector space, so the matrix representations and the change-of-basis matrices are all square of the same size.

**Theorem SCB**

**Similarity and Change of Basis**

Suppose that \( T: V \rightarrow V \) is a linear transformation and \( B \) and \( C \) are bases of \( V \). Then

\[
M_{T,B,B}^{C} = C_{B,C}^{-1}M_{T,C,C}^{C}C_{B,C} \quad \text{Theorem MRCB} \[644\]
\]

**Proof** In the conclusion of Theorem MRCB \[644\], replace \( D \) by \( B \), and replace \( E \) by \( C \),

\[
M_{T,B,B}^{C} = C_{B,C}^{-1}M_{T,C,C}^{C}C_{B,C} \quad \text{Theorem MRCE} \[644\]
\]

This is the third surprise of this chapter. Theorem SCB \[647\] considers the special case where a linear transformation has the same vector space for the domain and codomain (\( V \)). We build a matrix representation of \( T \) using the basis \( B \) simultaneously for both
the domain and codomain \((M_{B,B}^T)\), and then we build a second matrix representation of \(T\), now using the basis \(C\) for both the domain and codomain \((M_{C,C}^T)\). Then these two representations are related via a similarity transformation (Definition SIM [483]) using a change-of-basis matrix \((C_{B,C})\)!

**Example MRBE**

Matrix representation with basis with eigenvectors

We return to the linear transformation \(T: M_{22} \mapsto M_{22}\) of Example ELTBM [637] defined by

\[
T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -17a + 11b + 8c - 11d & -57a + 35b + 24c - 33d \\ -14a + 10b + 6c - 10d & -41a + 25b + 16c - 23d \end{bmatrix}
\]

In Example ELTBM [637] we showcased four eigenvectors of \(T\). We will now put these four vectors in a set,

\[
\mathcal{B} = \{ \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \} = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix} \right\}
\]

Check that \(\mathcal{B}\) is a basis of \(M_{22}\) by first establishing the linear independence of \(\mathcal{B}\) and then employing Theorem G [398] to get the spanning property easily. Here is a second set of \(2 \times 2\) matrices, which also forms a basis of \(M_{22}\) (Example BM [364]),

\[
\mathcal{C} = \{ \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4 \} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]

We can build two matrix representations of \(T\), one relative to \(\mathcal{B}\) and one relative to \(\mathcal{C}\). Each is easy, but for wildly different reasons. In our computation of the matrix representation relative to \(\mathcal{B}\) we borrow some of our work in Example ELTBM [637]. Here are the representations, then the explanation.

\[
\rho_{\mathcal{B}}(T(\mathbf{x}_1)) = \rho_{\mathcal{B}}(2\mathbf{x}_1) = \rho_{\mathcal{B}}(2\mathbf{x}_1 + 0\mathbf{x}_2 + 0\mathbf{x}_3 + 0\mathbf{x}_4) = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[
\rho_{\mathcal{B}}(T(\mathbf{x}_2)) = \rho_{\mathcal{B}}(2\mathbf{x}_2) = \rho_{\mathcal{B}}(0\mathbf{x}_1 + 2\mathbf{x}_2 + 0\mathbf{x}_3 + 0\mathbf{x}_4) = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}
\]

\[
\rho_{\mathcal{B}}(T(\mathbf{x}_3)) = \rho_{\mathcal{B}}((-1)\mathbf{x}_3) = \rho_{\mathcal{B}}(0\mathbf{x}_1 + 0\mathbf{x}_2 + (-1)\mathbf{x}_3 + 0\mathbf{x}_4) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}
\]

\[
\rho_{\mathcal{B}}(T(\mathbf{x}_4)) = \rho_{\mathcal{B}}((-2)\mathbf{x}_4) = \rho_{\mathcal{B}}(0\mathbf{x}_1 + 0\mathbf{x}_2 + 0\mathbf{x}_3 + (-2)\mathbf{x}_4) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}
\]

So the resulting representation is

\[
M_{B,B}^T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}
\]
Very pretty. Now for the matrix representation relative to \( C \) first compute,

\[
\rho_C(T(y_1)) = \rho_C\left(\begin{bmatrix} -17 & -57 \\ -14 & -41 \end{bmatrix}\right)
= \rho_C\left((-17)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-57)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-14)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-41)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} -17 \\ -57 \\ -14 \\ -41 \end{bmatrix}
\]

\[
\rho_C(T(y_2)) = \rho_C\left(\begin{bmatrix} 11 & 35 \\ 10 & 25 \end{bmatrix}\right)
= \rho_C\left(11\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 35\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 10\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 25\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 11 \\ 35 \\ 10 \\ 25 \end{bmatrix}
\]

\[
\rho_C(T(y_3)) = \rho_C\left(\begin{bmatrix} 8 & 24 \\ 6 & 16 \end{bmatrix}\right)
= \rho_C\left(8\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 24\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 6\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 16\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ 24 \\ 6 \\ 16 \end{bmatrix}
\]

\[
\rho_C(T(y_4)) = \rho_C\left(\begin{bmatrix} -11 & -33 \\ -10 & -23 \end{bmatrix}\right)
= \rho_C\left((-11)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-33)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-10)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-23)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} -11 \\ -33 \\ -10 \\ -23 \end{bmatrix}
\]

So the resulting representation is

\[
M_{C,C}^T = \begin{bmatrix}
-17 & 11 & 8 & -11 \\
-57 & 35 & 24 & -33 \\
-14 & 10 & 6 & -10 \\
-41 & 25 & 16 & -23
\end{bmatrix}
\]

Not quite as pretty. The purpose of this example is to illustrate Theorem SCB [647]. This theorem says that the two matrix representations, \( M_{B,B}^T \) and \( M_{C,C}^T \), of the one linear transformation, \( T \), are related by a similarity transformation using the change-of-basis matrix \( C_{B,C} \). Let’s compute this change-of-basis matrix. Notice that since \( C \) is such a nice basis, this is fairly straightforward,

\[
\rho_C(x_1) = \rho_C\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) = \rho_C\left(0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}
\]

\[
\rho_C(x_2) = \rho_C\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = \rho_C\left(1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}
\]
\[ \rho_C (x_3) = \rho_C \left( \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \right) = \rho_C \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 3 \end{bmatrix} \]

\[ \rho_C (x_4) = \rho_C \left( \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix} \right) = \rho_C \left( \begin{bmatrix} 2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 6 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 6 \\ 1 \\ 4 \end{bmatrix} \]

So we have,

\[ C_{B,C} = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{bmatrix} \]

Now, according to Theorem SCB\textsuperscript{647} we can write,

\[ M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C} \]

This should look and feel exactly like the process for diagonalizing a matrix that was described in Section SD\textsuperscript{483}. And it is.

We can now return to the question of computing an eigenvalue or eigenvector of a linear transformation. For a linear transformation of the form \( T: \mathcal{V} \mapsto \mathcal{V} \), we know that representations relative to different bases are similar matrices. We also know that similar matrices have equal characteristic polynomials by Theorem SMEE\textsuperscript{485}. We will now show that eigenvalues of a linear transformation \( T \) are precisely the eigenvalues of any matrix representation of \( T \). Since the choice of a different matrix representation leads to a similar matrix, there will be no “new” eigenvalues obtained from this second representation. Similarly, the change-of-basis matrix can be used to show that eigenvectors obtained from one matrix representation will be precisely those obtained from any other representation. So we can determine the eigenvalues and eigenvectors of a linear transformation by forming one matrix representation, using \( \text{any} \) basis we please, and analyzing the matrix in the manner of Chapter E\textsuperscript{441}.

**Theorem EER**

**Eigenvalues, Eigenvectors, Representations**

Suppose that \( T: \mathcal{V} \mapsto \mathcal{V} \) is a linear transformation and \( B \) is a basis of \( \mathcal{V} \). Then \( \mathbf{v} \in \mathcal{V} \) is an eigenvector of \( T \) for the eigenvalue \( \lambda \) if and only if \( \rho_B (\mathbf{v}) \) is an eigenvector of \( M_{B,B}^T \) for the eigenvalue \( \lambda \).

**Proof** \((\Rightarrow)\) Assume that \( \mathbf{v} \in \mathcal{V} \) is an eigenvector of \( T \) for the eigenvalue \( \lambda \). Then

\[ M_{B,B}^T \rho_B (\mathbf{v}) = \rho_B (T(\mathbf{v})) = \rho_B (\lambda \mathbf{v}) = \lambda \rho_B (\mathbf{v}) \]

\[ \text{Theorem FTMR}^{606} \]

\[ \text{Hypothesis} \]

\[ \text{Theorem VRLT}^{587} \]
which by Definition EEM \[441\] says that \( \rho_B(v) \) is an eigenvector of the matrix \( M_{B,B}^T \) for the eigenvalue \( \lambda \).

\( \Leftarrow \) Assume that \( \rho_B(v) \) is an eigenvector of \( M_{B,B}^T \) for the eigenvalue \( \lambda \). Then

\[
T(v) = \rho_B^{-1}(T(v)) = \rho_B^{-1}(M_{B,B}^T \rho_B(v)) = \rho_B^{-1}(\lambda \rho_B(v)) = \lambda \rho_B^{-1}(\rho_B(v)) = \lambda v
\]

which by Definition EELT \[637\] says \( v \) is an eigenvector of \( T \) for the eigenvalue \( \lambda \). ■

Subsection CELT
Computing Eigenvectors of Linear Transformations

Knowing that the eigenvalues of a linear transformation are the eigenvalues of any representation, no matter what the choice of the basis \( B \) might be, we could now unambiguously define items such as the characteristic polynomial of a linear transformation, rather than a matrix. We’ll say that again — eigenvalues, eigenvectors, and characteristic polynomials are intrinsic properties of a linear transformation, independent of the choice of a basis used to construct a matrix representation.

As a practical matter, how does one compute the eigenvalues and eigenvectors of a linear transformation of the form \( T: V \mapsto V \)? Choose a nice basis \( B \) for \( V \), one where the vector representations of the values of the linear transformations necessary for the matrix representation are easy to compute. Construct the matrix representation relative to this basis, and find the eigenvalues and eigenvectors of this matrix using the techniques of Chapter E \[441\]. The resulting eigenvalues of the matrix are precisely the eigenvalues of the linear transformation. The eigenvectors of the matrix are column vectors that need to be converted to vectors in \( V \) through application of \( \rho_B^{-1} \).

Now consider the case where the matrix representation of a linear transformation is diagonalizable. The \( n \) linearly independent eigenvectors that must exist for the matrix (Theorem DC \[487\]) can be converted (via \( \rho_B^{-1} \)) into eigenvectors of the linear transformation. A matrix representation of the linear transformation relative to a basis of eigenvectors will be a diagonal matrix — an especially nice representation! Though we did not know it at the time, the diagonalizations of Section SD \[483\] were really finding especially pleasing matrix representations of linear transformations.

Here are some examples.

Example ELTT
Eigenvectors of a linear transformation, twice

Consider the linear transformation \( S: M_{22} \mapsto M_{22} \) defined by

\[
S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} -b - c - 3d & -14a - 15b - 13c + d \\ 18a + 21b + 19c + 3d & -6a - 7b - 7c - 3d \end{bmatrix}
\]

To find the eigenvalues and eigenvectors of \( S \) we will build a matrix representation and analyze the matrix. Since Theorem EER \[650\] places no restriction on the choice of the
basis $B$, we may as well use a basis that is easy to work with. So set 

$$B = \{ x_1, x_2, x_3, x_4 \} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Then to build the matrix representation of $S$ relative to $B$ compute,

$$\rho_B (S(x_1)) = \rho_B \left( \begin{bmatrix} 0 & -14 \\ 18 & -6 \end{bmatrix} \right) = \rho_B (0x_1 + (-14)x_2 + 18x_3 + (-6)x_4) = \begin{bmatrix} 0 \\ -14 \\ 18 \\ -6 \end{bmatrix}$$

$$\rho_B (S(x_2)) = \rho_B \left( \begin{bmatrix} -1 & -15 \\ 21 & -7 \end{bmatrix} \right) = \rho_B ((-1)x_1 + (-15)x_2 + 21x_3 + (-7)x_4) = \begin{bmatrix} -1 \\ -15 \\ 21 \\ -7 \end{bmatrix}$$

$$\rho_B (S(x_3)) = \rho_B \left( \begin{bmatrix} -1 & -13 \\ 19 & -7 \end{bmatrix} \right) = \rho_B ((-1)x_1 + (-13)x_2 + 19x_3 + (-7)x_4) = \begin{bmatrix} -1 \\ -13 \\ 19 \\ -7 \end{bmatrix}$$

$$\rho_B (S(x_4)) = \rho_B \left( \begin{bmatrix} -3 & 1 \\ 3 & -3 \end{bmatrix} \right) = \rho_B ((-3)x_1 + 1x_2 + 3x_3 + (-3)x_4) = \begin{bmatrix} -3 \\ 1 \\ 3 \\ -3 \end{bmatrix}$$

So by Definition MR 603 we have

$$M = M_{B,B}^S = \begin{bmatrix} 0 & -1 & -1 & -3 \\ -14 & -15 & -13 & 1 \\ 18 & 21 & 19 & 3 \\ -6 & -7 & -7 & -3 \end{bmatrix}$$

Now compute eigenvalues and eigenvectors of the matrix representation of $M$ with the techniques of Section EE 441. First the characteristic polynomial,

$$p_M(x) = \det (M - xI_4) = x^4 - x^3 - 10x^2 + 4x + 24 = (x - 3)(x - 2)(x + 2)^2$$

We could now make statements about the eigenvalues of $M$, but in light of Theorem EER 650 we can refer to the eigenvalues of $S$ and mildly abuse (or extend) our notation for multiplicities to write

$$\alpha_S (3) = 1 \quad \alpha_S (2) = 1 \quad \alpha_S (-2) = 2$$

Now compute the eigenvectors of $M$,

$$\lambda = 3 \quad M - 3I_4 = \begin{bmatrix} -3 & -1 & -1 & -3 \\ -14 & -18 & -13 & 1 \\ 18 & 21 & 16 & 3 \\ -6 & -7 & -7 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{E}_M (3) = \mathcal{N}(M - 3I_4) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$
\[ \lambda = 2 \quad M - 2I_4 = \begin{bmatrix} -2 & -1 & -1 & -3 \\ -14 & -13 & -13 & 1 \\ 18 & 21 & 17 & 3 \\ -6 & -7 & -7 & -5 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \mathcal{E}_M(2) = N(M - 2I_4) = \left\{ \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} \right\} \]

\[ \lambda = -2 \quad M - (-2)I_4 = \begin{bmatrix} 2 & -1 & -1 & -3 \\ -14 & -13 & -13 & 1 \\ 18 & 21 & 21 & 3 \\ -6 & -7 & -7 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \mathcal{E}_M(-2) = N(M - (-2)I_4) = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \]

According to Theorem EER \[650\] the eigenvectors just listed as basis vectors for the eigenspaces of \( M \) are vector representations (relative to \( B \)) of eigenvectors for \( S \). So the application of the inverse function \( \rho_B^{-1} \) will convert these column vectors into elements of the vector space \( M_{22} \) (2 \( \times \) 2 matrices) that are eigenvectors of \( S \). Since \( \rho_B \) is an isomorphism (Theorem VRILT \[593\]), so is \( \rho_B^{-1} \). Applying the inverse function will then preserve linear independence and spanning properties, so with a sweeping application of the Coordinatization Principle \[596\] and some extensions of our previous notation for eigenspaces and geometric multiplicities, we can write,

\[ \rho_B^{-1} \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix} = (-1)x_1 + 3x_2 + (-3)x_3 + 1x_4 = \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix} \]

\[ \rho_B^{-1} \begin{bmatrix} -2 \\ 4 \\ -3 \\ 1 \end{bmatrix} = (-2)x_1 + 4x_2 + (-3)x_3 + 1x_4 = \begin{bmatrix} -2 \\ 4 \\ -3 \\ 1 \end{bmatrix} \]

\[ \rho_B^{-1} \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = 0x_1 + (-1)x_2 + 1x_3 + 0x_4 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \]

\[ \rho_B^{-1} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = 1x_1 + (-1)x_2 + 0x_3 + 1x_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \]

So

\[ \mathcal{E}_S(3) = \left\{ \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix} \right\} \]
\[ E_S(2) = \left\langle \begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix} \right\rangle \]
\[ E_S(-2) = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right\rangle \]

with geometric multiplicities given by
\[ \gamma_S(3) = 1 \quad \gamma_S(2) = 1 \quad \gamma_S(-2) = 2 \]

Suppose we now decided to build another matrix representation of \( S \), only now relative to a linearly independent set of eigenvectors of \( S \), such as \( C = \{ \begin{bmatrix} -1 & 3 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \} \)

At this point you should have computed enough matrix representations to predict that the result of representing \( S \) relative to \( C \) will be a diagonal matrix. Computing this representation is an example of how Theorem SCB \[647\] generalizes the diagonalizations from Section SD \[483\]. For the record, here is the diagonal representation,
\[ M_{C,C}^S = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \]

Our interest in this example is not necessarily building nice representations, but instead we want to demonstrate how eigenvalues and eigenvectors are an intrinsic property of a linear transformation, independent of any particular representation. To this end, we will repeat the foregoing, but replace \( B \) by another basis. We will make this basis different, but not extremely so, \( D = \{ y_1, y_2, y_3, y_4 \} = \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \} \)

Then to build the matrix representation of \( S \) relative to \( D \) compute,
\[ \rho_D(S(y_1)) = \rho_D \left( \begin{bmatrix} 0 & -14 \\ 18 & -6 \end{bmatrix} \right) = \rho_D(14y_1 + (-32)y_2 + 24y_3 + (-6)y_4) = \begin{bmatrix} 14 \\ -32 \\ 24 \\ -6 \end{bmatrix} \]
\[ \rho_D(S(y_2)) = \rho_D \left( \begin{bmatrix} -1 & -29 \\ 39 & -13 \end{bmatrix} \right) = \rho_D(28y_1 + (-68)y_2 + 52y_3 + (-13)y_4) = \begin{bmatrix} 28 \\ -68 \\ 52 \\ -13 \end{bmatrix} \]
\[ \rho_D(S(y_3)) = \rho_D \left( \begin{bmatrix} -2 & -42 \\ 58 & -20 \end{bmatrix} \right) = \rho_D(40y_1 + (-100)y_2 + 78y_3 + (-20)y_4) = \begin{bmatrix} 40 \\ -100 \\ 78 \\ -20 \end{bmatrix} \]
\[ \rho_D(S(y_4)) = \rho_D \left( \begin{bmatrix} -5 & -41 \\ 61 & -23 \end{bmatrix} \right) = \rho_D(36y_1 + (-102)y_2 + 84y_3 + (-23)y_4) = \begin{bmatrix} 36 \\ -102 \\ 84 \\ -23 \end{bmatrix} \]
So by Definition MR 603 we have

\[ N = M_{S,D,D}^{S} = \begin{bmatrix}
14 & 28 & 40 & 36 \\
-32 & -68 & -100 & -102 \\
24 & 52 & 78 & 84 \\
-6 & -13 & -20 & -23
\end{bmatrix} \]

Now compute eigenvalues and eigenvectors of the matrix representation of \( N \) with the techniques of Section EE 441. First the characteristic polynomial,

\[ p_N(x) = \det (N - xI_4) = x^4 - x^3 - 10x^2 + 4x + 24 = (x - 3)(x - 2)(x + 2)^2 \]

Of course this is not news. We now know that \( M = M_{B,B}^{S} \) and \( N = M_{D,D}^{S} \) are similar matrices (Theorem SCB 647). But Theorem SMEE 485 told us long ago that similar matrices have identical characteristic polynomials. Now compute eigenvectors for the matrix representation, which will be different than what we found for \( M \),

\[ \begin{align*}
\lambda &= 3 & N - 3I_4 &= \begin{bmatrix}
11 & 28 & 40 & 36 \\
-32 & -71 & -100 & -102 \\
24 & 52 & 75 & 84 \\
-6 & -13 & -20 & -26
\end{bmatrix} & \xrightarrow{\text{RREF}} & \begin{bmatrix}
1 & 0 & 0 & 4 \\
0 & 1 & 0 & -6 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix} \\
E_N(3) &= N(N - 3I_4) = \langle \begin{bmatrix}
-4 \\
6 \\
-4 \\
1
\end{bmatrix} \rangle
\end{align*} \]

\[ \begin{align*}
\lambda &= 2 & N - 2I_4 &= \begin{bmatrix}
12 & 28 & 40 & 36 \\
-32 & -70 & -100 & -102 \\
24 & 52 & 76 & 84 \\
-6 & -13 & -20 & -25
\end{bmatrix} & \xrightarrow{\text{RREF}} & \begin{bmatrix}
1 & 0 & 0 & 6 \\
0 & 1 & 0 & -7 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix} \\
E_N(2) &= N(N - 2I_4) = \langle \begin{bmatrix}
-6 \\
7 \\
-4 \\
1
\end{bmatrix} \rangle
\end{align*} \]

\[ \begin{align*}
\lambda &= -2 & N - (-2)I_4 &= \begin{bmatrix}
16 & 28 & 40 & 36 \\
-32 & -66 & -100 & -102 \\
24 & 52 & 80 & 84 \\
-6 & -13 & -20 & -21
\end{bmatrix} & \xrightarrow{\text{RREF}} & \begin{bmatrix}
1 & 0 & -1 & -3 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \\
E_N(-2) &= N(N - (-2)I_4) = \langle \begin{bmatrix}
1 \\
-2 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
-3 \\
0 \\
1
\end{bmatrix} \rangle
\end{align*} \]

Employing Theorem EER 650 we can apply \( \rho_D^{-1} \) to each of the basis vectors of the eigenspaces of \( N \) to obtain eigenvectors for \( S \) that also form bases for eigenspaces of \( S \),

\[ \rho_D^{-1} \begin{bmatrix}
-4 \\
6 \\
-4 \\
1
\end{bmatrix} = (-4)y_1 + 6y_2 + (-4)y_3 + 1y_4 = \begin{bmatrix}
-1 \\
3 \\
-3 \\
1
\end{bmatrix} \]
\[
\rho_D^{-1} \left( \begin{bmatrix} -6 \\ 7 \\ -4 \\ 1 \end{bmatrix} \right) = (-6)y_1 + 7y_2 + (-4)y_3 + 1y_4 = \begin{bmatrix} -2 \\ 4 \\ -3 \\ 1 \end{bmatrix}
\]

\[
\rho_D^{-1} \left( \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right) = 1y_1 + (-2)y_2 + 1y_3 + 0y_4 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}
\]

\[
\rho_D^{-1} \left( \begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right) = 3y_1 + (-3)y_2 + 0y_3 + 1y_4 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}
\]

The eigenspaces for the eigenvalues of algebraic multiplicity 1 are exactly as before,

\[
\mathcal{E}_S(3) = \left\langle \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\} \rightangle
\]

\[
\mathcal{E}_S(2) = \left\langle \left\{ \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \right\} \rightangle
\]

However, the eigenspace for \( \lambda = -2 \) would at first glance appear to be different. Here are the two eigenspaces for \( \lambda = -2 \), first the eigenspace obtained from \( M = M_{B,B}^{S} \), then followed by the eigenspace obtained from \( M = M_{D,D}^{S} \).

\[
\mathcal{E}_S(-2) = \left\langle \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \rightangle
\]

\[
\mathcal{E}_S(-2) = \left\langle \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\} \rightangle
\]

Subspaces generally have many bases, and that is the situation here. With a careful proof of set equality, you can show that these two eigenspaces are equal sets. The key observation to make such a proof go is that

\[
\begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}
\]

which will establish that the second set is a subset of the first. With equal dimensions, Theorem EDYES \([401]\) will finish the task. So the eigenvalues of a linear transformation are independent of the matrix representation employed to compute them! ☭

Another example, this time a bit larger and with complex eigenvalues.

**Example CELT**

**Complex eigenvectors of a linear transformation**

Consider the linear transformation \( Q: P_4 \mapsto P_4 \) defined by

\[
Q(a + bx + cx^2 + dx^3 + ex^4) = (-6a - 22b + 13c + 5d + e) + (117a + 57b - 32c - 15d - 4e)x + (-69a - 29b + 21c - 7e)x^2 + (159a + 73b - 44c - 13d + 2e)x^3 + (-195a - 87b + 55c + 10d - 13e)x^4
\]
Choose a simple basis to compute with, say
\[ B = \{1, x, x^2, x^3, x^4\} \]

Then it should be apparent that the matrix representation of \( Q \) relative to \( B \) is
\[
M = M_B^Q = \begin{bmatrix}
-46 & -22 & 13 & 5 & 1 \\
117 & 57 & -32 & -15 & -4 \\
-69 & -29 & 21 & 0 & -7 \\
159 & 73 & -44 & -13 & 2 \\
-195 & -87 & 55 & 10 & -13
\end{bmatrix}
\]

Compute the characteristic polynomial, eigenvalues and eigenvectors according to the techniques of Section EE [441],
\[
p_Q(x) = -x^5 + 6x^4 - x^3 - 88x^2 + 252x - 208
\]
\[
= -(x - 2)^2(x + 4)(x^2 - 6x + 13)
\]
\[
= -(x - 2)^2(x + 4)(x - (3 + 2i))(x - (3 - 2i))
\]

\[
\begin{align*}
\alpha_Q(2) &= 2 & \alpha_Q(-4) &= 1 & \alpha_Q(3 + 2i) &= 1 & \alpha_Q(3 - 2i) &= 1
\end{align*}
\]

\[
\lambda = 2
\]
\[
M - (2)I_5 = \begin{bmatrix}
-48 & -22 & 13 & 5 & 1 \\
117 & 55 & -32 & -15 & -4 \\
-69 & -29 & 19 & 0 & -7 \\
159 & 73 & -44 & -15 & 2 \\
-195 & -87 & 55 & 10 & -15
\end{bmatrix}
\]
\[
\text{RREF} \rightarrow \begin{bmatrix}
1 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} \\
0 & 1 & 0 & -\frac{5}{2} & -\frac{5}{2} \\
0 & 0 & 1 & -2 & -6 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
\mathcal{E}_M(2) = \mathcal{N}(M - (2)I_5) = \left\langle \begin{bmatrix} \frac{1}{2} \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{-1}{2} \\ 6 \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 1 \\ 5 \\ 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} \right\rangle
\]

\[
\lambda = -4
\]
\[
M - (-4)I_5 = \begin{bmatrix}
-42 & -22 & 13 & 5 & 1 \\
117 & 61 & -32 & -15 & -4 \\
-69 & -29 & 25 & 0 & -7 \\
159 & 73 & -44 & -9 & 2 \\
-195 & -87 & 55 & 10 & -9
\end{bmatrix}
\]
\[
\text{RREF} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -3 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
\mathcal{E}_M(-4) = \mathcal{N}(M - (-4)I_5) = \left\langle \begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\rangle
\]
\[ \lambda = 3 + 2i \]

\[
M - (3 + 2i)I_5 = \begin{bmatrix} -49 - 2i & -22 & 13 & 5 & 1 \\ 117 & 54 - 2i & -32 & -15 & -4 \\ -69 & -29 & 18 - 2i & 0 & -7 \\ 159 & 73 & -44 & -16 - 2i & 2 \\ -195 & -87 & 55 & 10 & -16 - 2i \end{bmatrix}
\]

\[ \varepsilon_M (3 + 2i) = \mathcal{N}(M - (3 + 2i)I_5) = \left\langle \begin{bmatrix} \frac{3}{4} - \frac{i}{4} \\ -\frac{1}{2} + \frac{i}{2} \\ -\frac{7}{4} + \frac{i}{4} \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 3 - i \\ -7 + i \\ 2 - 2i \\ -7 + i \end{bmatrix} \right\rangle \]

\[ \varepsilon_M (3 - 2i) = \mathcal{N}(M - (3 - 2i)I_5) = \left\langle \begin{bmatrix} \frac{3}{4} + \frac{i}{4} \\ -\frac{1}{2} - \frac{i}{2} \\ -\frac{7}{4} - \frac{i}{4} \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 3 + i \\ -7 - i \\ 2 + 2i \\ -7 - i \end{bmatrix} \right\rangle \]

It is straightforward to convert each of these basis vectors for eigenspaces of \( M \) back to elements of \( P_4 \) by applying the isomorphism \( \rho_B^{-1} \):

\[
\rho_B^{-1} \begin{bmatrix} -1 \\ 5 \\ 4 \\ 2 \\ 0 \end{bmatrix} = -1 + 5x + 4x^2 + 2x^3
\]

\[
\rho_B^{-1} \begin{bmatrix} 1 \\ 5 \\ 12 \\ 0 \\ 2 \end{bmatrix} = 1 + 5x + 12x^2 + 2x^4
\]

\[
\rho_B^{-1} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \\ 1 \end{bmatrix} = -1 + 3x + x^2 + 2x^3 + x^4
\]

\[
\rho_B^{-1} \begin{bmatrix} 3 - i \\ -7 + i \\ 2 - 2i \\ -7 + i \\ 4 \end{bmatrix} = (3 - i) + (-7 + i)x + (2 - 2i)x^2 + (-7 + i)x^3 + 4x^4
\]
\[ \rho_B^{-1} \begin{bmatrix} 3 + i \\ -7 - i \\ 2 + 2i \\ -7 - i \\ 4 \end{bmatrix} = (3 + i) + (-7 - i)x + (2 + 2i)x^2 + (-7 - i)x^3 + 4x^4 \]

So we apply Theorem EER 650 and the Coordinatization Principle 596 to get the eigenspaces for \( Q \),

\[ \mathcal{E}_Q(2) = \langle \{-1 + 5x + 4x^2 + 2x^3, 1 + 5x + 12x^2 + 2x^4\} \rangle \]
\[ \mathcal{E}_Q(-4) = \langle \{-1 + 3x + x^2 + 2x^3 + x^4\} \rangle \]
\[ \mathcal{E}_Q(3 + 2i) = \langle \{(3 - i) + (-7 + i)x + (2 - 2i)x^2 + (-7 + i)x^3 + 4x^4\} \rangle \]
\[ \mathcal{E}_Q(3 - 2i) = \langle \{(3 + i) + (-7 - i)x + (2 + 2i)x^2 + (-7 - i)x^3 + 4x^4\} \rangle \]

with geometric multiplicities

\[ \gamma_Q(2) = 2 \quad \gamma_Q(-4) = 1 \quad \gamma_Q(3 + 2i) = 1 \quad \gamma_Q(3 - 2i) = 1 \]

Subsection READ
Reading Questions

1. The change-of-basis matrix is a matrix representation of which linear transformation?

2. Find the change-of-basis matrix, \( C_{B,C} \), for the two bases of \( \mathbb{C}^2 \)

\[ B = \left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\} \quad C = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \]

3. What is the third “surprise,” and why is it surprising?
Subsection EXC
Exercises

C20  In Example CBCV 643 we computed the vector representation of \( \mathbf{y} \) relative to \( C, \rho_C(\mathbf{y}) \), as an example of Theorem CB 639. Compute this same representation directly. In other words, apply Definition VR 587 rather than Theorem CB 639.
Contributed by Robert Beezer

C21  Perform a check on Example MRCM 645 by computing \( M_{B,D}^Q \) directly. In other words, apply Definition MR 603 rather than Theorem MRCB 644.
Contributed by Robert Beezer  Solution 663

C30  Find a basis for the vector space \( P_3 \) composed of eigenvectors of the linear transformation \( T \). Then find a matrix representation of \( T \) relative to this basis.

\[ T: P_3 \hookrightarrow P_3, \quad T(a + bx + cx^2 + dx^3) = (a+c+d)+(b+c+d)x+(a+b+c)x^2+(a+b+d)x^3 \]

Contributed by Robert Beezer  Solution 663

C40  Let \( S_{22} \) be the vector space of \( 2 \times 2 \) symmetric matrices. Find a basis \( B \) for \( S_{22} \) that yields a diagonal matrix representation of the linear transformation \( R \). (15 points)

\[ R: S_{22} \hookrightarrow S_{22}, \quad R \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} -5a + 2b - 3c & -12a + 5b - 6c \\ -12a + 5b - 6c & 6a - 2b + 4c \end{bmatrix} \]

Contributed by Robert Beezer  Solution 664

C41  Let \( S_{22} \) be the vector space of \( 2 \times 2 \) symmetric matrices. Find a basis for \( S_{22} \) composed of eigenvectors of the linear transformation \( Q: S_{22} \hookrightarrow S_{22} \). (15 points)

\[ Q \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 25a + 18b + 30c & -16a - 11b - 20c \\ -16a - 11b - 20c & -11a - 9b - 12c \end{bmatrix} \]

Contributed by Robert Beezer  Solution 665

T10  Suppose that \( T: V \hookrightarrow V \) is an invertible linear transformation with a nonzero eigenvalue \( \lambda \). Prove that \( \frac{1}{\lambda} \) is an eigenvalue of \( T^{-1} \).
Contributed by Robert Beezer  Solution 666

T15  Suppose that \( V \) is a vector space and \( T: V \hookrightarrow V \) is a linear transformation. Prove that \( T \) is injective if and only if \( \lambda = 0 \) is not an eigenvalue of \( T \).
Contributed by Robert Beezer
Subsection CB.SOL  Solutions  663

C21 Contributed by Robert Beezer Statement 661

Apply Definition MR 603,

\[ \rho_D \left( Q \begin{bmatrix} 5 & -3 \\ -3 & -2 \end{bmatrix} \right) = \rho_D \left( 19 + 14x - 2x^2 - 28x^3 \right) \]

= \rho_D \left( (-39)(2 + x - 2x^2 + 3x^3) + 62(-1 - 2x^2 + 3x^3) + (-53)(-3 - x + x^3) + (-44)(-x^2 + x^3) \right)

= \begin{bmatrix} -39 \\ 62 \\ -53 \\ -44 \end{bmatrix}

\rho_D \left( Q \begin{bmatrix} 2 & -3 \\ -3 & 0 \end{bmatrix} \right) = \rho_D \left( 16 + 9x - 7x^2 - 14x^3 \right)

= \rho_D \left( (-23)(2 + x - 2x^2 + 3x^3) + (34)(-1 - 2x^2 + 3x^3) + (-32)(-3 - x + x^3) + (-15)(-x^2 + x^3) \right)

= \begin{bmatrix} -23 \\ 34 \\ -32 \\ -15 \end{bmatrix}

\rho_D \left( Q \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \right) = \rho_D \left( 25 + 9x + 3x^2 + 4x^3 \right)

= \rho_D \left( (14)(2 + x - 2x^2 + 3x^3) + (12)(-1 - 2x^2 + 3x^3) + 5(-3 - x + x^3) + (-7)(-x^2 + x^3) \right)

= \begin{bmatrix} 14 \\ -12 \\ 5 \\ -7 \end{bmatrix}

These three vectors are the columns of the matrix representation,

\[ M_{Q,B,D}^{Q} = \begin{bmatrix} -39 & -23 & 14 \\ 62 & 34 & -12 \\ -53 & -32 & 5 \\ -44 & -15 & -7 \end{bmatrix} \]

which coincides with the result obtained in Example MRCM 645.

C30 Contributed by Robert Beezer Statement 661

With the domain and codomain being identical, we will build a matrix representation using the same basis for both the domain and codomain. The eigenvalues of the matrix representation will be the eigenvalues of the linear transformation, and we can obtain the eigenvectors of the linear transformation by un-coordinatizing (Theorem EER 650). Since the method does not depend on which basis we choose, we can choose a natural basis for ease of computation, say,

\[ B = \{1, x, x^2, x^3\} \]
The matrix representation is then,

\[
M_{T,B,B}^T = \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{bmatrix}
\]

The eigenvalues and eigenvectors of this matrix were computed in Example ESMS4. A basis for \( \mathbb{C}^4 \), composed of eigenvectors of the matrix representation is,

\[
C = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}
\]

Applying \( \rho_B^{-1} \) to each vector of this set, yields a basis of \( P_3 \) composed of eigenvectors of \( T \),

\[
D = \{1 + x + x^2 + x^3, -1 + x, -x^2 + x^3, -1 - x + x^2 + x^3\}
\]

The matrix representation of \( T \) relative to the basis \( D \) will be a diagonal matrix with the corresponding eigenvalues along the diagonal, so in this case we get

\[
M_{D,D}^T = \begin{bmatrix}
3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

**C40** Contributed by Robert Beezer Statement 661

Begin with a matrix representation of \( R \), any matrix representation, but use the same basis for both instances of \( S_{22} \). We’ll choose a basis that makes it easy to compute vector representations in \( S_{22} \).

\[
B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}
\]

Then the resulting matrix representation of \( R \) (Definition MR) is

\[
M_{B,B}^R = \begin{bmatrix}
-5 & 2 & -3 \\
-12 & 5 & -6 \\
6 & -2 & 4
\end{bmatrix}
\]

Now, compute the eigenvalues and eigenvectors of this matrix, with the goal of diagonalizing the matrix (Theorem DC).

\[
\lambda = 2 \quad \mathcal{E}_{M_{B,B}^R}(2) = \left\langle \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\rangle
\]

\[
\lambda = 1 \quad \mathcal{E}_{M_{B,B}^R}(1) = \left\langle \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\rangle
\]
The three vectors that occur as basis elements for these eigenspaces will together form a linearly independent set (check this!). So these column vectors may be employed in a matrix that will diagonalize the matrix representation. If we “un-coordinatize” these three column vectors relative to the basis $B$, we will find three linearly independent elements of $S_{22}$ that are eigenvectors of the linear transformation $R$ (Theorem EER [650]).

A matrix representation relative to this basis of eigenvectors will be diagonal, with the eigenvalues ($\lambda = 2, 1$) as the diagonal elements. Here we go,

$$
\rho_B^{-1} \left( \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right) = (-1) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-2) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}
$$

$$
\rho_B^{-1} \left( \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right) = (-1) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}
$$

$$
\rho_B^{-1} \left( \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right) = 1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix}
$$

So the requested basis of $S_{22}$, yielding a diagonal matrix representation of $R$, is

$$
\left\{ \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 0 \end{bmatrix} \right\}
$$

C41 Contributed by Robert Beezer Statement [661]

Use a single basis for both the domain and codomain, since they are equal.

$$
B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
$$

The matrix representation of $Q$ relative to $B$ is

$$
M = M_{Q,B} = \begin{bmatrix} 25 & 18 & 30 \\ -16 & -11 & -20 \\ -11 & -9 & -12 \end{bmatrix}
$$

We can analyze this matrix with the techniques of Section EE [441] and then apply Theorem EER [650]. The eigenvalues of this matrix are $\lambda = -2, 1, 3$ with eigenspaces

$$
\mathcal{E}_M(-2) = \left\{ \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix} \right\} \quad \mathcal{E}_M(1) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \mathcal{E}_M(3) = \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\}
$$

Because the three eigenvalues are distinct, the three basis vectors from the three eigenspaces for a linearly independent set (Theorem EDELI [467]). Theorem EER [650] says we can uncoordinatize these eigenvectors to obtain eigenvectors of $Q$. By Theorem ILTLI [537] the resulting set will remain linearly independent. Set

$$
C = \left\{ \rho_B^{-1} \left( \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix} \right), \rho_B^{-1} \left( \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right), \rho_B^{-1} \left( \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right) \right\} = \left\{ \begin{bmatrix} -6 & 4 & 3 \\ -2 & 1 & 1 \\ -3 & 2 & 1 \end{bmatrix} \right\}
$$
Then $C$ is a linearly independent set of size 3 in the vector space $M_{22}$, which has dimension 3 as well. By Theorem G, $C$ is a basis of $M_{22}$.

**T10** Contributed by Robert Beezer  

Let $v$ be an eigenvector of $T$ for the eigenvalue $\lambda$. Then,

$$T^{-1}(v) = \frac{1}{\lambda}T^{-1}(v) \quad \lambda \neq 0$$

$$= \frac{1}{\lambda}T^{-1}(\lambda v) \quad \text{Theorem ILTLT \[570\]}$$

$$= \frac{1}{\lambda}T^{-1}(T(v)) \quad v \text{ eigenvector of } T$$

$$= \frac{1}{\lambda}I_V(v) \quad \text{Definition IVLT \[567\]}$$

$$= \frac{1}{\lambda}v \quad \text{Definition IDLT \[567\]}$$

which says that $\frac{1}{\lambda}$ is an eigenvalue of $T^{-1}$ with eigenvector $v$. Note that it is possible to prove that any eigenvalue of an invertible linear transformation is never zero. So the hypothesis that $\lambda$ be nonzero is just a convenience for this problem.
Section NLT Nilpotent Linear Transformations

This Section Under Construction

We have seen that some matrices are diagonalizable and some are not. Some authors refer to a non-diagonalizable matrix as defective, but we will study them carefully anyway. Examples of such matrices include Example EMMS4 453, Example HMEM5 454, and Example CEMS6 455. Each of these matrices has at least one eigenvalue with geometric multiplicity strictly less than its geometric multiplicity, and therefore Theorem DMFE 490 tells us these matrices are not diagonalizable.

Given a square matrix $A$, it is likely similar to many, many other matrices. Of all these possibilities, which is the best? “Best” is a subjective term, but we might agree that a diagonal matrix is certainly a very nice choice. Unfortunately, as we have seen, this will not always be possible. What form of a matrix is “next-best”? Our goal, which will take us several sections to reach, is to show that every matrix is similar to a matrix that is “nearly-diagonal.” More precisely, every matrix is similar to a matrix with elements on the diagonal, and zeros and ones on the diagonal just above the main diagonal (the “super diagonal”), with zeros everywhere else. In the language of equivalence relations (see Theorem SER 485), we are determining a systematic representative for each equivalence class. Such a representative for a set of similar matrices is called a canonical form.

We have just discussed the determination of a canonical form as a question about matrices. However, we know that every square matrix creates a natural linear transformation (Theorem MBLT 509) and every linear transformation with identical domain and codomain has a square matrix representation for each choice of a basis, with a change of basis creating a similarity transformation (Theorem SCB 647). So we will state, and prove, theorems using the language of linear transformations on abstract vector spaces, while most of our examples will work with square matrices. You can, and should, mentally translate between the two settings frequently and easily.

Subsection NLT Nilpotent Linear Transformations

We will discover that nilpotent linear transformations are the essential obstacle in a non-diagonalizable linear transformation. So we will study them carefully first, both as an object of inherent mathematical interest, but also as the object at the heart of the argument that leads to a pleasing canonical form for any linear transformation. Once we understand these linear transformations thoroughly, we will be able to easily analyze the structure of any linear transformation.

Definition NLT Nilpotent Linear Transformation

Suppose that $T : V \mapsto V$ is a linear transformation such that there is an integer $p > 0$ such that $T^p (v) = 0$ for every $v \in V$. The smallest $p$ for which this condition is met is called the index of $T$. △

Version 0.92
Of course, the linear transformation $T$ defined by $T(v) = 0$ will qualify as nilpotent of index 1. But are there others?

**Example NM64**

**Nilpotent matrix, size 6, index 4**

Recall that our definitions and theorems are being stated for linear transformations on abstract vector spaces, while our examples will work with square matrices (and use the same terms interchangeably). In this case, to demonstrate the existence of nontrivial nilpotent linear transformations, we desire a matrix such that some power of the matrix is the zero matrix. Consider

$$A = \begin{bmatrix}
-3 & 3 & -2 & 5 & 0 & -5 \\
-3 & 5 & -3 & 4 & 3 & -9 \\
-3 & 4 & -2 & 6 & -4 & -3 \\
-3 & 3 & -2 & 5 & 0 & -5 \\
-3 & 3 & -2 & 4 & 2 & -6 \\
-2 & 3 & -2 & 2 & 4 & -7
\end{bmatrix}$$

and compute powers of $A$,

$$A^2 = \begin{bmatrix}
1 & -2 & 1 & 0 & -3 & 4 \\
0 & -2 & 1 & 1 & -3 & 4 \\
3 & 0 & 0 & -3 & 0 & 0 \\
1 & -2 & 1 & 0 & -3 & 4 \\
0 & -2 & 1 & 1 & -3 & 4 \\
-1 & -2 & 1 & 2 & -3 & 4
\end{bmatrix}$$

$$A^3 = \begin{bmatrix}
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0
\end{bmatrix}$$

$$A^4 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Thus we can say that $A$ is nilpotent of index 4.

Because it will presage some upcoming theorems, we will record some extra information about the eigenvalues and eigenvectors of $A$ here. $A$ has just one eigenvalue, $\lambda = 0$, with algebraic multiplicity 6 and geometric multiplicity 2. The eigenspace for this
eigenvalue is
\[ \mathcal{E}_A(0) = \left\langle \begin{bmatrix} 2 \\ 2 \\ 5 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -5 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle \]

If there were degrees of singularity, we might say this matrix was very singular, since zero is an eigenvalue with maximum algebraic multiplicity (Theorem SMZE, Theorem ME). Notice too that \( A \) is “far” from being diagonalizable (Theorem DMFE).

Another example.

**Example NM62**

**Nilpotent matrix, size 6, index 2**

Consider the matrix
\[
B = \begin{bmatrix}
-1 & 1 & -1 & 4 & -3 & -1 \\
1 & 1 & -1 & 2 & -3 & -1 \\
-9 & 10 & -5 & 9 & 5 & -15 \\
-1 & 1 & -1 & 4 & -3 & -1 \\
1 & -1 & 0 & 2 & -4 & 2 \\
4 & -3 & 1 & -1 & -5 & 5
\end{bmatrix}
\]

and compute the second power of \( B \),
\[
B^2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

So \( B \) is nilpotent of index 2. Again, the only eigenvalue of \( B \) is zero, with algebraic multiplicity 6. The geometric multiplicity of the eigenvalue is 3, as seen in the eigenspace,
\[
\mathcal{E}_B(0) = \left\langle \begin{bmatrix} 1 \\ 3 \\ 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ -7 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle
\]

Again, Theorem DMFE tells us that \( B \) is far from being diagonalizable.
the two previous examples were somewhat surprising. But we have seen that matrix
algebra does not always behave the way we expect (Example MMNC [216]), and we
also now recognize matrix products not just as arithmetic, but as function composition
(Theorem MRCLT [610]). We will now turn to some examples of nilpotent matrices
which might be more transparent.

**Definition JB**

**Jordan Block**

Given the scalar \( \lambda \in \mathbb{C} \), the Jordan block \( J_n(\lambda) \) is the \( n \times n \) matrix defined by

\[
J_n(\lambda)_{ij} = \begin{cases} 
\lambda & i = j \\
1 & j = i + 1 \\
0 & \text{otherwise}
\end{cases}
\]

(This definition contains Notation JB.)

**Example JB4**

**Jordan block, size 4**

A simple example of a Jordan block, \( J_4(5) \):

\[
J_4(5) = \begin{bmatrix}
5 & 1 & 0 & 0 \\
0 & 5 & 1 & 0 \\
0 & 0 & 5 & 1 \\
0 & 0 & 0 & 5
\end{bmatrix}
\]

We will return to general Jordan blocks later, but in this section we are just interested
in Jordan blocks where \( \lambda = 0 \). Here’s an example of why we are specializing in these
matrices now.

**Example acronym**

**Nilpotent Jordan block, size 5**

Consider

\[
J_5(0) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

and compute powers,

\[
(J_5(0))^2 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
(J_5(0))^3 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[(J_5(0))^4 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[(J_5(0))^5 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

So \(J_5(0)\) is nilpotent of index 5. As before, we record some information about the eigenvalues and eigenvectors of this matrix. The only eigenvalue is zero, with algebraic multiplicity 5, the maximum possible (Theorem ME [474]). The geometric multiplicity of this eigenvalue is just 1, the minimum possible (Theorem ME [474]), as seen in the eigenspace,

\[E_{J_5(0)}(0) = \langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rangle\]

There should not be any real surprises in this example. We can watch the ones in the powers of \(J_5(0)\) slowly march off to the upper-right hand corner of the powers. In some vague way, the eigenvalues and eigenvectors of this matrix are equally extreme.

We can form combinations of Jordan blocks to build a variety of nilpotent matrices. Simply place Jordan blocks on the diagonal of a matrix with zeros everywhere else, to create a block diagonal matrix.

**Example NM83**
**Nilpotent matrix, size 8, index 3**

Consider the matrix

\[C = \begin{bmatrix}
J_3(0) & \mathcal{O} & \mathcal{O} \\
\mathcal{O} & J_3(0) & \mathcal{O} \\
\mathcal{O} & \mathcal{O} & J_2(0) \\
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
and compute powers,

\[
C^2 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
C^3 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

So \(C\) is nilpotent of index 3. You should notice how block diagonal matrices behave in products (much like diagonal matrices) and that it was the largest Jordan block that determined the index of this combination. All eight eigenvalues are zero, and each of the three Jordan blocks contributes one eigenvector to a basis for the eigenspace, resulting in zero having a geometric multiplicity of 3.

It would appear that nilpotent matrices only have zero as an eigenvalue, so the algebraic multiplicity will be the maximum possible. However, by creating block diagonal matrices with Jordan blocks on the diagonal you should be able to attain any desired geometric multiplicity for this lone eigenvalue. Likewise, the size of the largest Jordan block employed will determine the index of the matrix. So nilpotent matrices with various combinations of index and geometric multiplicities are easy to manufacture. The predictable properties of block diagonal matrices in matrix products and eigenvector computations, along with the next theorem, make this possible.

Subsection PNLT

Properties of Nilpotent Linear Transformations

In this subsection we collect some basic properties of nilpotent linear transformations. After studying the examples in the previous section, some of these will be no surprise.

**Theorem ENLT**

**Eigenvalues of Nilpotent Linear Transformations**

Suppose that \(T: V \rightarrow V\) is a linear transformation and \(\lambda\) is an eigenvalue of \(T\). Then \(\lambda = 0\).

**Proof** Let \(x\) be an eigenvector of \(T\) for the eigenvalue \(\lambda\), and suppose that \(T\) is nilpotent with index \(p\). Then

\[
0 = T^p (x) \quad \text{Definition NLT 667}
\]
Because $x$ is an eigenvector, it is nonzero, and therefore \text{Theorem SMEZV} \ [319] tells us that $\lambda^p = 0$ and so $\lambda = 0$.

Paraphrasing, all of the eigenvalues of a nilpotent linear transformation are zero. So in particular, the characteristic polynomial of a nilpotent linear transformation, $T$, on a vector space of dimension $n$, is simply $p_T(x) = x^n$.

The next theorem is not critical for what follows, but it will explain our interest in nilpotent linear transformations. More specifically, it is the first step in backing up the assertion that nilpotent linear transformations are the essential obstacle in a non-diagonalizable linear transformation.

\textbf{Theorem DNLT}
\textit{Diagonalizable Nilpotent Linear Transformations}

Suppose the linear transformation $T: V \mapsto V$ is nilpotent. Then $T$ is diagonalizable if and only if $T$ is the zero linear transformation. \hfill $\square$

\textbf{Proof} \hspace{1em} We start with the easy direction. Let $n = \dim(V)$.

($\Leftarrow$) The linear transformation $Z: V \mapsto V$ defined by $Z(v) = 0$ for all $v \in V$ is nilpotent of index $p = 1$ and a matrix representation relative to any basis of $V$ is the $n \times n$ zero matrix, $O$. Quite obviously, the zero matrix is a diagonal matrix (Definition \text{DIM} \ [486]) and hence $Z$ is diagonalizable (Definition \text{DZM} \ [487]).

($\Rightarrow$) Assume now that $T$ is diagonalizable, so $\gamma_T(\lambda) = \alpha_T(\lambda)$ for every eigenvalue $\lambda$ (Theorem \text{DMFE} \ [490]). By Theorem \text{ENLT} \ [672], $T$ has only one eigenvalue (zero), which therefore must have algebraic multiplicity $n$ (Theorem \text{NEM} \ [473]). So the geometric multiplicity of zero will be $n$ as well, $\gamma_T(0) = n$.

Let $B$ be a basis for the eigenspace $\mathcal{E}_T(0)$. Then $B$ is a linearly independent subset of $V$ of size $n$, and by Theorem \text{G} \ [398] will be a basis for $V$. For any $x \in B$ we have

\[
T(x) = 0x \quad \text{Definition EM} \ [450] \\
= 0 \quad \text{Theorem ZSSM} \ [317]
\]

So $T$ is identically zero on a basis for $B$, and since the action of a linear transformation on a basis determines all of the values of the linear transformation (Theorem \text{LTDB} \ [512]), it is easy to see that $T(v) = 0$ for every $v \in V$.

So, other than one trivial case (the zero matrix), every nilpotent linear transformation is not diagonalizable. It remains to see what is so “essential” about this broad class of non-diagonalizable linear transformations. For this we now turn to a discussion of kernels of powers of nilpotent linear transformations, beginning with a result about general linear transformations that may not necessarily be nilpotent.

\textbf{Theorem KPLT}
\textit{Kernels of Powers of Linear Transformations}

Suppose $T: V \mapsto V$ is a linear transformation, where $\dim(V) = n$. Then there is an integer $m$, $0 \leq m \leq n$, such that

\[
\{0\} = \mathcal{K}(T^0) \subsetneq \mathcal{K}(T^1) \subsetneq \mathcal{K}(T^2) \subsetneq \cdots \subsetneq \mathcal{K}(T^m) = \mathcal{K}(T^{m+1}) = \mathcal{K}(T^{m+2}) = \cdots
\]

\hspace{1em} $\square$
Proof There are several items to verify in the conclusion as stated. First, we show that $K(T^k) \subseteq K(T^{k+1})$ for any $k$. Choose $z \in K(T^k)$. Then

\[
T^{k+1}(z) = T(T^k(z)) = T(0) = 0
\]

by Definition LTC [519] and Definition KLT [532]. So by Definition KLT [532], $z \in K(T^{k+1})$ and by Definition SSET [693] we have $K(T^k) \subseteq K(T^{k+1})$.

Second, we demonstrate the existence of a power $m$ where consecutive powers result in equal kernels. A by-product will be the condition that $m$ can be chosen so that $m \leq n$.

To the contrary, suppose that $\{0\} = K(T^0) \subsetneq K(T^1) \subsetneq K(T^2) \subsetneq \cdots \subsetneq K(T^{n-1}) \subsetneq K(T^n) \subsetneq K(T^{n+1}) \subsetneq \cdots$

Since $K(T^k) \subsetneq K(T^{k+1})$, Theorem PSSD [400] implies that $\dim(K(T^{k+1})) \geq \dim(K(T^k)) + 1$. Repeated application of this observation yields

\[
\dim(K(T^{n+1})) \geq \dim(K(T^n)) + 1 \\
\geq \dim(K(T^{n-1})) + 2 \\
\vdots \\
\geq \dim(K(T^0)) + (n + 1) \\
= \dim(\{0\}) + n + 1 \\
= n + 1
\]

Thus, $K(T^{n+1})$ has a basis of size at least $n + 1$, which is a linearly independent set of size greater than $n$ in the vector space $V$ of dimension $n$. This contradicts Theorem G [398].

This contradiction yields the existence of an integer $k$ such that $K(T^k) = K(T^{k+1})$, so we can define $m$ to be smallest such integer with this property. From the argument above about dimensions resulting from a strictly increasing chain of subspaces, it should be clear that $m \leq n$.

It remains to show that once two consecutive kernels are equal, then all of the remaining kernels are equal. More formally, if $K(T^m) = K(T^{m+1})$, then $K(T^m) = K(T^{m+j})$ for all $j \geq 1$. We will give a proof by induction on $j$ (Technique 1 [713]). The base case ($j = 1$) is precisely our defining property for $m$.

In the induction step, we assume that $K(T^m) = K(T^{m+j})$ and endeavor to show that $K(T^m) = K(T^{m+j+1})$. At the outset of this proof we established that $K(T^m) \subseteq K(T^{m+j+1})$. So Definition SE [694] requires only that we establish the subset inclusion in the opposite direction. To wit, choose $z \in K(T^{m+j+1})$. Then

\[
0 = T^{m+j+1}(z) = T^{m+j}(T(z)) = T^{m+j}(T(z)) \quad \text{Induction Hypothesis} \\
= T^m(T(z)) = T^m(T(z)) \quad \text{Base Case} \\
= T^{m+1}(z) = T^{m+1}(z) \quad \text{Definition LTC 519} \\
= T^m(z) = T^m(z) \quad \text{Definition LTC 519}
\]

So by Definition KLT [532], $z \in K(T^m)$ as desired. ■
We now specialize Theorem KPLT 673 to the case of nilpotent linear transformations, which buys us just a bit more precision in the conclusion.

**Theorem KPNLT**

Kernels of Powers of Nilpotent Linear Transformations

Suppose $T: V \mapsto V$ is a nilpotent linear transformation with index $p$ and $\dim(V) = n$. Then $0 \leq p \leq n$ and

$$\{0\} = \mathcal{K}(T^0) \subset \mathcal{K}(T^1) \subset \mathcal{K}(T^2) \subset \cdots \subset \mathcal{K}(T^p) = \mathcal{K}(T^{p+1}) = \cdots = V$$

\[\square\]

**Proof** Since $T^p = 0$ it follows that $T^{p+j} = 0$ for all $j \geq 0$ and thus $\ker T^{p+j} = V$ for $j \geq 0$. So the value of $m$ guaranteed by Theorem KPLT 673 is at most $p$. The only remaining aspect of our conclusion that does not follow from Theorem KPLT 673 is that $m = p$. To see this we must show that $\mathcal{K}(T^k) \subset \mathcal{K}(T^{k+1})$ for $0 \leq k \leq p - 1$. If $\mathcal{K}(T^k) = \mathcal{K}(T^{k+1})$ for some $k < p$, then $\mathcal{K}(T^k) = \mathcal{K}(T^p) = V$. This implies that $T^k = 0$, violating the fact that $T$ has index $p$. So the smallest value of $m$ is indeed $p$, and we learn that $p < n$.

The structure of the kernels of powers of nilpotent linear transformations will be crucial to what follows. But immediately we can see a practical benefit. Suppose we are confronted with the question of whether or not an $n \times n$ matrix, $A$, is nilpotent or not. If we don’t quickly find a low power that equals the zero matrix, when do we stop trying higher and higher powers? Theorem KPNLT 675 gives us the answer: if we don’t see a zero matrix by the time we finish computing $A^n$, then it is not going to ever happen. We’ll now take a look at one example of Theorem KPNLT 675 in action.

**Example KPNLT**

Kernels of Powers of a Nilpotent Linear Transformation

We will recycle the nilpotent matrix $A$ of index 4 from Example NM64 668. We now know that would have only needed to look at the first 6 powers of $A$ if the matrix had not been nilpotent. We list bases for the null spaces of the powers of $A$. (Notice how we are using null spaces for matrices interchangeably with kernels of linear transformations, see Theorem KNSI 614 for justification.)

$$\mathcal{N}(A) = \mathcal{N}(\begin{pmatrix} -3 & 3 & -2 & 5 & 0 & -5 \\ -3 & 5 & -3 & 4 & 3 & -9 \\ -3 & 4 & -2 & 6 & -4 & -3 \\ -3 & 3 & -2 & 5 & 0 & -5 \\ -3 & 3 & -2 & 4 & 2 & -6 \\ -2 & 3 & -2 & 2 & 4 & -7 \end{pmatrix}) = \left\{ \begin{pmatrix} 2 \\ 2 \\ 5 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -5 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{N}(A^2) = \mathcal{N}(\begin{pmatrix} 1 & -2 & 1 & 0 & -3 & 4 \\ 0 & -2 & 1 & 1 & -3 & 4 \\ 3 & 0 & 0 & -3 & 0 & 0 \\ 1 & -2 & 1 & 0 & -3 & 4 \\ 0 & -2 & 1 & 1 & -3 & 4 \\ -1 & -2 & 1 & 2 & -3 & 4 \end{pmatrix}) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$
Verify the following, with the exception of some convenience scaling of the basis vectors in \( \mathcal{N}(A^2) \) these are exactly the basis vectors described in \( \text{Theorem BNS} \) \( [154] \). We can see that the dimension of \( \mathcal{N}(A) \) equals the geometric multiplicity of the zero eigenvalue. Why is this not an accident? We can see the dimensions of the kernels consistently increasing, and we can see that \( \mathcal{N}(A^4) = \mathbb{C}^6 \). But \( \text{Theorem KPNLT} \) \( [675] \) says a little more. Each successive kernel should be a superset of the previous one. We ought to be able to begin with a basis of \( \mathcal{N}(A) \) and extend it to a basis of \( \mathcal{N}(A^2) \). Then we should be able to extend a basis of \( \mathcal{N}(A^2) \) into a basis of \( \mathcal{N}(A^3) \), all with repeated applications of \( \text{Theorem ELIS} \) \( [397] \). Verify the following,

\[
\mathcal{N}(A^3) = \mathcal{N}
\begin{bmatrix}
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0
\end{bmatrix}
\leftarrow
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\mathcal{N}(A^4) = \mathcal{N}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\leftarrow
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Do not be concerned at the moment about how these bases were constructed since we are not describing the applications of Theorem ELIS here. Do verify carefully for each alleged basis that, (1) it is a superset of the basis for the previous kernel, (2) the basis vectors really are members of the kernel of the right power of $A$, (3) the basis is a linearly independent set, (4) the size of the basis is equal to the size of the basis found previously for each kernel. With these verifications, Theorem G will tell us that we have successfully demonstrated what Theorem KPNLT guarantees.
Appendix CN
Computation Notes

Section MMA
Mathematica

Computation Note ME.MMA
Matrix Entry

Matrices are input as lists of lists, since a list is a basic data structure in Mathematica. A matrix is a list of rows, with each row entered as a list. Mathematica uses braces ({{ , }}) to delimit lists. So the input

\[
a = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}\}
\]

would create a 3 \times 4 matrix named 'a' that is equal to

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12
\end{bmatrix}
\]

To display a matrix named 'a' "nicely" in Mathematica, type \texttt{MatrixForm[a]}, and the output will be displayed with rows and columns. If you just type 'a', then you will get a list of lists, like how you input the matrix in the first place.

Computation Note RR.MMA
Row Reduce

If 'a' is the name of a matrix in Mathematica, then the command \texttt{RowReduce[a]} will output the reduced row-echelon form of the matrix.
Computation Note LS.MMA

Linear Solve

*Mathematica* will solve a linear system of equations using the `LinearSolve[]` command. The inputs are a matrix with the coefficients of the variables (but not the column of constants), and a list containing the constant terms of each equation. This will look a bit odd, since the lists in the matrix are rows, but the column of constants is also input as a list and so looks like a row rather than a column. The result will be a single solution (even if there are infinitely many), reported as a list, or the statement that there is no solution. When there are infinitely many, the single solution reported is exactly that solution used in the proof of Theorem RCLS [54], where the free variables are all set to zero, and the dependent variables come along with values from the final column of the row-reduced matrix.

As an example, Archetype A [721] is

\[
\begin{align*}
\mathbf{x}_1 - \mathbf{x}_2 + 2\mathbf{x}_3 &= 1 \\
2\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 &= 8 \\
\mathbf{x}_1 + \mathbf{x}_2 &= 5
\end{align*}
\]

To ask *Mathematica* for a solution, enter

```
LinearSolve[ {{1, -1, 2}, {2, 1, 1}, {1, 1, 0}}, {1, 8, 5} ]
```

and you will get back the single solution

\[
\{3, 2, 0\}
\]

We will see later how to coax *Mathematica* into giving us infinitely many solutions for this system (Computation VFSS.MMA [681]).

Computation Note VLC.MMA

Vector Linear Combinations

Contributed by Robert Beezer

Vectors in *Mathematica* are represented as lists, written and displayed horizontally. For example, the vector

\[
\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}
\]

would be entered and named via the command

\[
\mathbf{v} = \{1, 2, 3, 4\}
\]

Vector addition and scalar multiplication are then very natural. If \(\mathbf{u}\) and \(\mathbf{v}\) are two lists of equal length, then

\[
2\mathbf{u} + (-3)\mathbf{v}
\]

will compute the correct vector and return it as a list. If \(\mathbf{u}\) and \(\mathbf{v}\) have different sizes, then *Mathematica* will complain about “objects of unequal length.”
Computation Note NS.MMA
Null Space

Given a matrix $A$, Mathematica will compute a set of column vectors whose span is the null space of the matrix with the `NullSpace[]` command. Perhaps not coincidentally, this set is exactly $\{z_j \mid 1 \leq j \leq n - r\}$. However, Mathematica prefers to output the vectors in the opposite order than one we have chosen. Here’s a small example.

Begin with the $3 \times 4$ matrix $A$, and its row-reduced version $B$,

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & 4 & 1 & -2 \\ -1 & 1 & -5 & 3 \end{bmatrix} \quad \text{RREF} \quad B = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We could extract entries from $B$ to build the vectors $z_1$ and $z_2$ according to Theorem SSNS [129] and describe $\mathcal{N}(A)$ as a span of the set $\{z_1, z_2\}$. Instead, if $a$ has been set to $A$, then executing the command `NullSpace[a]` yields the list of lists (column vectors),

$$\{\{2, -1, 0, 1\}, \{-3, 2, 1, 0\}\}$$

Notice how our $z_1$ is second in the list. To “correct” this we can use a list-processing command from Mathematica, `Reverse[]`, as follows,

`Reverse[NullSpace[a]]`

and receive the output in our preferred order. Give it a try yourself.

Computation Note VFSS.MMA
Vector Form of Solution Set

Suppose that $A$ is an $m \times n$ matrix and $b \in \mathbb{C}^m$ is a column vector. We might wish to find all of the solutions to the linear system $LS(A, b)$. Mathematica’s `LinearSolve[A, b]` will return at most one solution (Computation LS.MMA [680]). However, when the system is consistent, then this one solution reported is exactly the vector $c$, described in the statement of Theorem VFSLS [107].

The vectors $u_j$, $1 \leq j \leq n - r$ of Theorem VFSLS [107] are exactly the output of Mathematica’s `NullSpace[]` command, though Mathematica lists them in the opposite order from the order we have chosen. These are the same vectors listed as $z_j$, $1 \leq j \leq n - r$ in Theorem SSNS [129]. With $c$ produced from the `LinearSolve[]` command, and the $u_j$ coming from the `NullSpace[]` command we can use Mathematica’s symbolic manipulation commands to create an expression that describes all of the solutions.

Begin with the system $LS(A, b)$. Row-reduce $A$ (Computation RR.MMA [679]) and identify the free variables by determining the non-pivot columns. Suppose, for the sake of argument, that we have the three free variables $x_3$, $x_7$ and $x_8$. Then the following command will build an expression for an arbitrary solution:

```
LinearSolve[A, b]+{x8, x7, x3}.NullSpace[A]
```
Be sure to include the “dot” right before the \texttt{NullSpace[]} command — it has the effect of creating a linear combination of the vectors in the null space, using scalars that are symbols reminiscent of the variables.

A concrete example should help here. Suppose we want a solution set for the linear system with coefficient matrix $A$ and vector of constants $b$,

\[
A = \begin{bmatrix}
1 & 2 & 3 & -5 & 1 & -1 & 2 \\
2 & 4 & 0 & 8 & -4 & 1 & -8 \\
3 & 6 & 4 & 0 & -2 & 5 & 7
\end{bmatrix}
\quad b = \begin{bmatrix}
8 \\
1 \\
-5
\end{bmatrix}
\]

If we were to apply Theorem VFSLS \textsuperscript{107}, we would extract the components of $c$ and $u_j$ from the row-reduced version of the augmented matrix of the system (obtained with Mathematica, Computation RR.MMA \textsuperscript{679}),

\[
\begin{bmatrix}
1 & 2 & 0 & 4 & -2 & 0 & -5 & 2 \\
0 & 0 & 1 & -3 & 1 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & -3
\end{bmatrix}
\]

Instead, we will use this augmented matrix in reduced row-echelon form only to identify the free variables. In this example, we locate the non-pivot columns and see that $x_2, x_4, x_5$ and $x_7$ are free. If we have set $a$ to the coefficient matrix and $b$ to the vector of constants, then we execute the Mathematica command,

\[
\text{LinearSolve}[a, b]+\{x7, x5, x4, x2\}.\text{NullSpace}[a]
\]

As output we obtain the column vector (list),

\[
\begin{bmatrix}
2 - 2 x2 & - 4 x4 & + 2 x5 & + 5 x7 \\
x2 \\
1 + 3 x4 & - x5 & - 3 x7 \\
x4 \\
x5 \\
-3 - 2 x7 \\
x7
\end{bmatrix}
\]

\textbf{Computation Note GSP.MMA}

\textbf{Gram-Schmidt Procedure}

Mathematica has a built-in routine that will do the Gram-Schmidt procedure (Theorem GSPCV \textsuperscript{191}). The input is a set of vectors, which must be linearly independent. This is written as a list, containing lists that are the vectors. Let $a$ be such a list of lists, containing the vectors $v_i$, $1 \leq i \leq p$ from the statement of the theorem. You will need to first load the right Mathematica package — execute \texttt{\textless\textless LinearAlgebra'Orthogonalization\textgreater\textgreater} to make this happen. Then execute \texttt{GramSchmidt[a]} . The output will be another list of lists containing the vectors $u_i$, $1 \leq i \leq p$ from the statement of the theorem. Mathematica will complain if you do not provide a linearly independent set as input (try it!).
An example. Suppose our linearly independent set (check this!) is

\[ S = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix} \]

The output of the `GramSchmidt[]` command will be the set,

\[ T = \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} \frac{37}{4\sqrt{685}} \\ \frac{4\sqrt{685}}{29} \end{bmatrix}, \begin{bmatrix} \frac{-337}{6\sqrt{120423}} \\ \frac{-6\sqrt{120423}}{37} \end{bmatrix}, \begin{bmatrix} \frac{23}{\sqrt{879}} \\ \frac{3\sqrt{879}}{26} \end{bmatrix} \]

Ugly, but true. At this stage, you might just as well be encouraged to think of the Gram-Schmidt procedure as a computational black box, linearly independent set in, orthogonal span-preserving set out.

To check that the output set is orthogonal, we can easily check the orthogonality of individual pairs of vectors. Suppose the output was set equal to `b` (say via `b=GramSchmidt[a]`). We can extract the individual vectors of `c` as “parts” with syntax like `c[[3]]`, which would return the third vector in the set. When our vectors have only real number entries, we can accomplish an inner product with a “dot.” So, for example, you should discover that `c[[3]].c[[5]]` will return zero. Try it yourself with another pair of vectors.

**Computation Note TM.MMA**

**Transpose of a Matrix**

Contributed by Robert Beezer

Suppose `a` is the name of a matrix stored in *Mathematica*. Then `Transpose[a]` will create the transpose of `a`.

**Computation Note MM.MMA**

**Matrix Multiplication**

If `A` and `B` are matrices defined in *Mathematica*, then `A.B` will return the product of the two matrices (notice the dot between the matrices). If `A` is a matrix and `v` is a vector, then `A.v` will return the vector that is the matrix-vector product of `A` and `v`. In every case the sizes of the matrices and vectors need to be correct.

Some examples:

\[ \{1, 2\}, \{3, 4\}\cdot\{5, 6, 7\}, \{8, 9, 10\} = \{21, 24, 27\}, \{47, 54, 61\} \]
Understanding the difference between the last two examples will go a long way to explaining how some Mathematica constructs work.

**Computation Note MI.MMA**

**Matrix Inverse**

If $A$ is a matrix defined in Mathematica, then `Inverse[A]` will return the inverse of $A$, should it exist. In the case where $A$ does not have an inverse Mathematica will tell you the matrix is singular (see Theorem NI[251]).

**Section TI86**

**Texas Instruments 86**

**Computation Note ME.TI86**

**Matrix Entry**

On the TI-86, press the **MATRX** key (Yellow-7). Press the second menu key over, F2, to bring up the **EDIT** screen. Give your matrix a name, one letter or many, then press ENTER. You can then change the size of the matrix (rows, then columns) and begin editing individual entries (which are initially zero). ENTER will move you from entry to entry, or the down arrow key will move you to the next row. A menu gives you extra options for editing.

Matrices may also be entered on the home screen as follows. Use brackets ([ , ]) to enclose rows with elements separated by commas. Group rows, in order, into a final set of brackets (with no commas between rows). This can then be stored in a name with the **STO** key. So, for example,

$$[[1, 2, 3, 4], [5, 6, 7, 8], [9, 10, 11, 12]] \rightarrow A$$

will create a matrix named A that is equal to

$$
\begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12
\end{bmatrix}
$$
Computation Note RR.TI86
Row Reduce

If $A$ is the name of a matrix stored in the TI-86, then the command `rref A` will return the reduced row-echelon form of the matrix. This command can also be found by pressing the MATRX key, then F4 for OPS, and finally, F5 for rref.

Note that this command will not work for a matrix with more rows than columns. (Ed. Not sure just why this is!) A work-around is to pad the matrix with extra columns of zeros until the matrix is square.

Computation Note VLC.TI86
Vector Linear Combinations

Contributed by Robert Beezer

Vector operations on the TI-86 can be accessed via the VECTR key, which is Yellow-8. The EDIT tool appears when the F2 key is pressed. After providing a name and giving a “dimension” (the size) then you can enter the individual entries, one at a time. Vectors can also be entered on the home screen using brackets ([, ]). To create the vector

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

use brackets and the store key (STO),

$$[1, 2, 3, 4] \rightarrow v$$

Vector addition and scalar multiplication are then very natural. If $u$ and $v$ are two vectors of equal size, then

$$2 \cdot u + (-3) \cdot v$$

will compute the correct vector and display the result as a vector.

Computation Note TM.TI86
Transpose of a Matrix

Contributed by Eric Fickenscher

Suppose $A$ is the name of a matrix stored in the TI-86. Use the command $A^T$ to transpose $A$. This command can be found by pressing the MATRX key, then F3 for MATH, then F2 for $^T$. 
Section TI83
Texas Instruments 83

Computation Note ME.TI83
Matrix Entry

Contributed by Douglas Phelps
On the TI-83, press the MATRX key. Press the right arrow key twice so that EDIT is highlighted. Move the cursor down so that it is over the desired letter of the matrix and press ENTER. For example, let’s call our matrix B, so press the down arrow once and press ENTER. To enter a 2 × 3 matrix, press 2 ENTER 3 ENTER. To create the matrix
\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
\]
press 1 ENTER 2 ENTER 3 ENTER 4 ENTER 5 ENTER 6 ENTER.

Computation Note RR.TI83
Row Reduce

Contributed by Douglas Phelps
Suppose B is the name of a matrix stored in the TI-83. Press the MATRX key. Press the right arrow key once so that MATH is highlighted. Press the down arrow eleven times so that rref ( is highlighted, then press ENTER. to choose the matrix B, press MATRX, then the down arrow once followed by ENTER. Supply a right parenthesis ( ) and press ENTER.

Note that this command will not work for a matrix with more rows than columns. (Ed. Not sure just why this is!) A work-around is to pad the matrix with extra columns of zeros until the matrix is square.

Computation Note VLC.TI83
Vector Linear Combinations

Contributed by Douglas Phelps
Entering a vector on the TI-83 is the same process as entering a matrix. You press 4 ENTER 3 ENTER for a 4 × 3 matrix. Likewise, you press 4 ENTER 1 ENTER for a vector of size 4. To multiply a vector by 8, press the number 8, then press the MATRX key, then scroll down to the letter you named your vector (A, B, C, etc) and press ENTER.

To add vectors A and B for example, press the MATRX key, then ENTER. Then press the + key. Then press the MATRX key, then the down arrow once, then ENTER. [A] + [B] will appear on the screen. Press ENTER.
Appendix P
Preliminaries

This appendix contains important ideas about complex numbers, sets, and the logic and techniques of forming proofs. It is not meant to be read straight through, but you should head here when you need to review these ideas.

We choose to expand the set of scalars from the real numbers, \( \mathbb{R} \), to the set of complex numbers, \( \mathbb{C} \). So basic operations with complex numbers (like addition and division) will be necessary. This can be safely postponed until your arrival in Section 0 [183], and a refresher before Chapter E [441] would be a good idea as well.

Sets are extremely important in all of mathematics, but maybe you have not had much exposure to the basic operations. Check out Section SET [693]. The text will send you here frequently as well. Visit often.

This book is as much about doing mathematics as it is about linear algebra. The “Proof Techniques” are vignettes about logic, types of theorems, structure of proofs, or just plain old-fashioned advice about how to do advanced mathematics. The text will frequently point to one of these techniques in advance of their first use, and for specific instructions there will be additional references. If you find constructing proofs difficult (we all did once), then head back here and browse through the advice for second or third readings.

Section CNO
Complex Number Operations

In this section we review of the basics of working with complex numbers.

Subsection CNA
Arithmetic with complex numbers

A complex number is a linear combination of 1 and \( i = \sqrt{-1} \), typically written in the form \( a + bi \). Complex numbers can be added, subtracted, multiplied and divided, just like we are used to doing with real numbers, including the restriction on division by zero. We will not define these operations carefully, but instead illustrate with examples.
Example ACN
Arithmetic of complex numbers

\[(2 + 5i) + (6 - 4i) = (2 + 6) + (5 + (-4))i = 8 + i\]
\[(2 + 5i) - (6 - 4i) = (2 - 6) + (5 - (-4))i = -4 + 9i\]
\[(2 + 5i)(6 - 4i) = (2)(6) + (5i)(6) + (2)(-4i) + (5i)(-4i) = 12 + 30i - 8i - 20i^2\]
\[= 12 + 22i - 20(-1) = 32 + 22i\]

Division takes just a bit more care. We multiply the denominator by a complex number chosen to produce a real number and then we can produce a complex number as a result.

\[\frac{2 + 5i}{6 - 4i} = \frac{2 + 5i}{6 - 4i} \cdot \frac{6 + 4i}{6 + 4i} = \frac{-8 + 38i}{52} = -\frac{8}{52} + \frac{38}{52}i = -\frac{2}{13} + \frac{19}{26}i\]

In this example, we used \(6 + 4i\) to convert the denominator in the fraction to a real number. This number is known as the conjugate, which we define in the next section. We will often exploit the basic properties of complex number addition, subtraction, multiplication and division, so we will carefully define the two basic operations, together with a definition of equality, and then collect nine basic properties in a theorem.

Definition CNE
Complex Number Equality

The complex numbers \(\alpha = a + bi\) and \(\beta = c + di\) are equal, denoted \(\alpha = \beta\), if \(a = c\) and \(b = d\).

(This definition contains Notation CNE.)

Definition CNA
Complex Number Addition

The sum of the complex numbers \(\alpha = a + bi\) and \(\beta = c + di\), denoted \(\alpha + \beta\), is \((a + c) + (b + d)i\).

(This definition contains Notation CNA.)

Definition CNM
Complex Number Multiplication

The product of the complex numbers \(\alpha = a + bi\) and \(\beta = c + di\), denoted \(\alpha \beta\), is \((ac - bd) + (ad + bc)i\).

(This definition contains Notation CNM.)

Theorem PCNA
Properties of Complex Number Arithmetic

The operations of addition and multiplication of complex numbers have the following properties.

- ACCN Additive Commutativity, Complex Numbers
  For any \(\alpha, \beta \in \mathbb{C}\), \(\alpha + \beta = \beta + \alpha\).

- MCCN Multiplicative Commutativity, Complex Numbers
  For any \(\alpha, \beta \in \mathbb{C}\), \(\alpha \beta = \beta \alpha\).
• **AACN** Additive Associativity, Complex Numbers
  For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

• **MACN** Multiplicative Associativity, Complex Numbers
  For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha (\beta \gamma) = (\alpha \beta) \gamma$.

• **DCN** Distributivity, Complex Numbers
  For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha (\beta + \gamma) = \alpha \beta + \alpha \gamma$.

• **ZCN** Zero, Complex Numbers
  There is a complex number $0 = 0 + 0i$ so that for any $\alpha \in \mathbb{C}$, $0 + \alpha = \alpha$.

• **OCN** One, Complex Numbers
  There is a complex number $1 = 1 + 0i$ so that for any $\alpha \in \mathbb{C}$, $1 \alpha = \alpha$.

• **AICN** Additive Inverse, Complex Numbers
  For every $\alpha \in \mathbb{C}$ there exists $-\alpha \in \mathbb{C}$ so that $\alpha + (-\alpha) = 0$.

• **MICN** Multiplicative Inverse, Complex Numbers
  For every $\alpha \in \mathbb{C}$, $\alpha \neq 0$ there exists $\frac{1}{\alpha} \in \mathbb{C}$ so that $\frac{1}{\alpha} \alpha = 1$.

**Proof**  We could derive each of these properties of complex numbers with a proof that builds on the identical properties of the real numbers. The only proof that might be at all interesting would be to show [Property MICN](#) since we would need to trot out a conjugate. For this property, and especially for the others, we might be tempted to construct proofs of the identical properties for the reals. This would take us way too far afield, so we will draw a line in the sand right here and just agree that these nine fundamental behaviors are true. OK?

Mostly we have stated these nine properties carefully so that we can make reference to them later in other proofs. So we will be linking back here often.

---

**Subsection CCN**

Conjugates of Complex Numbers

---

**Definition CCN**

Conjugate of a Complex Number

The **conjugate** of the complex number $c = a + bi \in \mathbb{C}$ is the complex number $\overline{c} = a - bi$.

(This definition contains Notation CCN.)

**Example CSCN**

Conjugate of some complex numbers

\[
2 + 3i = 2 - 3i \quad 5 - 4i = 5 + 4i \quad -3 + 0i = -3 + 0i \quad 0 + 0i = 0 + 0i
\]
Notice how the conjugate of a real number leaves the number unchanged. The conjugate enjoys some basic properties that are useful when we work with linear expressions involving addition and multiplication.

**Theorem CCRA**

**Complex Conjugation Respects Addition**

Suppose that $c$ and $d$ are complex numbers. Then $\overline{c + d} = \overline{c} + \overline{d}$.

**Proof** Let $c = a + bi$ and $d = r + si$. Then

$$
\overline{c + d} = (a + r) + (b + s)i = (a + r) - (b + s)i = (a - bi) + (r - si) = \overline{c} + \overline{d}
$$

**Theorem CCRM**

**Complex Conjugation Respects Multiplication**

Suppose that $c$ and $d$ are complex numbers. Then $\overline{cd} = \overline{c}\overline{d}$.

**Proof** Let $c = a + bi$ and $d = r + si$. Then

$$
\overline{cd} = (ar - bs) + (as + br)i = (ar - bs) - (as + br)i = (ar - (-b)(-s)) + (a(-s) + (-b)r)i = (a - bi)(r - si) = \overline{c}\overline{d}
$$

**Theorem CCT**

**Complex Conjugation Twice**

Suppose that $c$ is a complex number. Then $\overline{\overline{c}} = c$.

**Proof** Let $c = a + bi$. Then

$$
\overline{\overline{c}} = a - bi = a - (-bi) = a + bi = c
$$

**Subsection MCN**

**Modulus of a Complex Number**

We define one more operation with complex numbers that may be new to you.

**Definition MCN**

**Modulus of a Complex Number**

The modulus of the complex number $c = a + bi \in \mathbb{C}$, is the nonnegative real number

$$
|c| = \sqrt{\overline{c}c} = \sqrt{a^2 + b^2}.
$$

**Example MSCN**

Modulus of some complex numbers

$$
|2 + 3i| = \sqrt{13} \quad |5 - 4i| = \sqrt{41} \quad |-3 + 0i| = 3 \quad |0 + 0i| = 0
$$
The modulus can be interpreted as a version of the absolute value for complex numbers, as is suggested by the notation employed. You can see this in how $|−3| = |−3 + 0i| = 3$. Notice too how the modulus of the complex zero, $0 + 0i$, has value 0.
Definition SET

Set

A set is an unordered collection of objects. If $S$ is a set and $x$ is an object that is in the set $S$, we write $x \in S$. If $x$ is not in $S$, then we write $x \notin S$. We refer to the objects in a set as its elements.

Example SETM

Set membership

From the set of all possible symbols, construct the following set of three symbols,

$$ S = \{\blacksquare, \Diamond, \star\} $$

Then the statement $\blacksquare \in S$ is true, while the statement $\blacksquare \in S$ is false. However, then the statement $\Diamond \notin S$ is true.

A portion of a set is known as a subset. Notice how the following definition uses an implication (if whenever...then...). Note too how the definition of a subset relies on the definition of a set through the idea of set membership.

Definition SSET

Subset

If $S$ and $T$ are two sets, then $S$ is a subset of $T$, written $S \subseteq T$ if whenever $x \in S$ then $x \in T$.

Example SSET

Subset

If $S = \{\blacksquare, \Diamond, \star\}$, $T = \{\star, \Diamond\}$, $R = \{\Diamond, \star\}$, then

- $T \subseteq S$
- $R \nsubseteq T$
- $\emptyset \subseteq S$
- $T \subset S$
- $S \subseteq S$
- $S \nsubset S$
What does it mean for two sets to be equal? They must be the same. Well, that explanation is not really too helpful, is it? How about: If \( A \subseteq B \) and \( B \subseteq A \), then \( A \) equals \( B \). This gives us something to work with, if \( A \) is a subset of \( B \), and \emph{vice versa}, then they must really be the same set. We will now make the symbol “\( = \)” do double-duty and extend its use to statements like \( A = B \), where \( A \) and \( B \) are sets. Here’s the definition, which we will reference often.

**Definition SE**

**Set Equality**

Two sets, \( S \) and \( T \), are equal, if \( S \subseteq T \) and \( T \subseteq S \). In this case, we write \( S = T \).

(This definition contains Notation SE.)

Sets are typically written inside of braces, as \{ \}, as we have seen above. However, when sets have more than a few elements, a description will typically have two components. The first is a description of the general type of objects contained in a set, while the second is some sort of restriction on the properties the objects have. Every object in the set must be of the type described in the first part and it must satisfy the restrictions in the second part. Conversely, any object of the proper type for the first part, that also meets the conditions of the second part, will be in the set. These two parts are set off from each other somehow, often with a vertical bar (\( | \)) or a colon (\( : \)).

I like to think of sets as clubs. The first part is some description of the type of people who \emph{might} belong to the club, the basic objects. For example, a bicycle club would describe its members as being people who like to ride bicycles. The second part is like a membership committee, it restricts the people who are allowed in the club. Continuing with our bicycle club analogy, we might decide to limit ourselves to “serious” riders and only have members who can document having ridden 100 kilometers or more in a single day at least one time.

The restrictions on membership can migrate around some between the first and second part, and there may be several ways to describe the same set of objects. Here’s a more mathematical example, employing the set of all integers, \( \mathbb{Z} \), to describe the set of even integers.

\[
E = \{ x \in \mathbb{Z} \mid x \text{ is an even number} \} \\
= \{ x \in \mathbb{Z} \mid 2 \text{ divides } x \text{ evenly} \} \\
= \{ 2k \mid k \in \mathbb{Z} \}
\]

Notice how this set tells us that its objects are integer numbers (not, say, matrices or functions, for example) and just those that are even. So we can write that \( 10 \in E \), while \( 17 \notin E \) once we check the membership criteria. We also recognize the question

\[
\begin{bmatrix}
1 & -3 & 5 \\
2 & 0 & 3
\end{bmatrix} \in E?
\]

as being simply ridiculous.
Subsection SC
Set Cardinality

On occasion, we will be interested in the number of elements in a finite set. Here’s the definition and the associated notation.

Definition C
Cardinality
Suppose $S$ is a finite set. Then the number of elements in $S$ is called the **cardinality** or **size** of $S$, and is denoted $|S|$.

(This definition contains Notation C.)

Example CS
Cardinality and Size
If $S = \{\Diamond, \star, \blacksquare\}$, then $|S| = 3$.

Subsection SO
Set Operations

In this subsection we define and illustrate the three most common basic ways to manipulate sets to create other sets. Since much of linear algebra is about sets, we will use these often.

Definition SU
Set Union
Suppose $S$ and $T$ are sets. Then the **union** of $S$ and $T$, denoted $S \cup T$, is the set whose elements are those that are elements of $S$ or of $T$, or both. More formally,

$$x \in S \cup T \text{ if and only if } x \in S \text{ or } x \in T$$

(This definition contains Notation SU.)

Notice that the use of the word “or” in this definition is meant to be non-exclusive. That is, it allows for $x$ to be an element of both $S$ and $T$ and still qualify for membership in $S \cup T$.

Example SU
Set union
If $S = \{\Diamond, \star, \blacksquare\}$ and $T = \{\Diamond, \star, \blacklozenge\}$ then $S \cup T = \{\Diamond, \star, \blacksquare, \blacklozenge\}$.

Definition SI
Set Intersection
Suppose $S$ and $T$ are sets. Then the **intersection** of $S$ and $T$, denoted $S \cap T$, is the set whose elements are only those that are elements of $S$ and of $T$. More formally,

$$x \in S \cap T \text{ if and only if } x \in S \text{ and } x \in T$$
Example SI
Set intersection

If $S = \{\spadesuit, \star, \blacksquare\}$ and $T = \{\spadesuit, \star, \triangle\}$ then $S \cap T = \{\spadesuit, \star\}$.

The union and intersection of sets are operations that begin with two sets and produce a third, new, set. Our final operation is the set complement, which we usually think of as an operation that takes a single set and creates a second, new, set. However, if you study the definition carefully, you will see that it needs to be computed relative to some “universal” set.

Definition SC
Set Complement

Suppose $S$ is a set that is a subset of a universal set $U$. Then the complement of $S$, denoted $\overline{S}$, is the set whose elements are those that are elements of $U$ and not elements of $S$. More formally,

$$x \in \overline{S} \text{ if and only if } x \in U \text{ and } x \notin S$$

Example SC
Set complement

If $U = \{\spadesuit, \star, \blacksquare, \triangle\}$ and $S = \{\spadesuit, \star, \blacksquare\}$ then $\overline{S} = \{\triangle\}$.

There are many more natural operations that can be performed on sets, such as an exclusive-or and the symmetric difference. Many of these can be defined in terms of the union, intersection and complement. We will not have much need of them in this course, and so we will not give precise descriptions here in this preliminary section.

There is also an interesting variety of basic results that describe the interplay of these operations with each other. We mention just two as an example, these are known as DeMorgan’s Laws.

$$(S \cup T) = \overline{S} \cap \overline{T}$$

$$(S \cap T) = \overline{S} \cup \overline{T}$$

Besides having an appealing symmetry, we mention these two facts, since constructing the proofs of each is a useful exercise that will require a solid understanding of all but one of the definitions presented in this section. Give it a try.
Section PT

Proof Techniques

In this section we collect many short essays designed to help you understand how to read, understand and construct proofs. Some are very factual, while others consist of advice. They appear in the order that they are first needed (or advisable) in the text, and are meant to be self-contained. So you should not think of reading through this section in one sitting as you begin this course. But be sure to head back here for a first reading whenever the text suggests it. Also think about returning to browse at various points during the course, and especially as you struggle with becoming an accomplished mathematician who is comfortable with the difficult process of designing new proofs.
Proof Technique D
Definitions

A definition is a made-up term, used as a kind of shortcut for some typically more complicated idea. For example, we say a whole number is even as a shortcut for saying that when we divide the number by two we get a remainder of zero. With a precise definition, we can answer certain questions unambiguously. For example, did you ever wonder if zero was an even number? Now the answer should be clear since we have a precise definition of what we mean by the term even.

A single term might have several possible definitions. For example, we could say that the whole number \( n \) is even if there is another whole number \( k \) such that \( n = 2k \). We say this is an equivalent definition since it categorizes even numbers the same way our first definition does.

Definitions are like two-way streets — we can use a definition to replace something rather complicated by its definition (if it fits) and we can replace a definition by its more complicated description. A definition is usually written as some form of an implication, such as “If something-nice-happens, then blatzo.” However, this also means that “If blatzo, then something-nice-happens,” even though this may not be formally stated. This is what we mean when we say a definition is a two-way street — it is really two implications, going in opposite “directions.”

Anybody (including you) can make up a definition, so long as it is unambiguous, but the real test of a definition’s utility is whether or not it is useful for describing interesting or frequent situations.

We will talk about theorems later (and especially equivalences). For now, be sure not to confuse the notion of a definition with that of a theorem.

In this book, we will display every new definition carefully set-off from the text, and the term being defined will be written thus: definition. Additionally, there is a full list of all the definitions, in order of their appearance located at the front of the book (Definitions). Finally, the acronym for each definition can be found in the index (Index).

Definitions are critical to doing mathematics and proving theorems, so we’ve given you lots of ways to locate a definition should you forget its... uh, well, ... definition.

Can you formulate a precise definition for what it means for a number to be odd? (Don’t just say it is the opposite of even. Act as if you don’t have a definition for even yet.) Can you formulate your definition a second, equivalent, way? Can you employ your definition to test an odd and an even number for “odd-ness”?
Proof Technique T
Theorems

Higher mathematics is about understanding theorems. Reading them, understanding them, applying them, proving them. We are ready to prove our first momentarily. Every theorem is a shortcut — we prove something in general, and then whenever we find a specific instance covered by the theorem we can immediately say that we know something else about the situation by applying the theorem. In many cases, this new information can be gained with much less effort than if we did not know the theorem.

The first step in understanding a theorem is to realize that the statement of every theorem can be rewritten using statements of the form “If something-happens, then something-else-happens.” The “something-happens” part is the hypothesis and the “something-else-happens” is the conclusion. To understand a theorem, it helps to rewrite its statement using this construction. To apply a theorem, we verify that “something-happens” in a particular instance and immediately conclude that “something-else-happens.” To prove a theorem, we must argue based on the assumption that the hypothesis is true, and arrive through the process of logic that the conclusion must then also be true.
Like any science, the language of math must be understood before further study can continue.

Erin Wilson, Student
September, 2004

Mathematics is a language. It is a way to express complicated ideas clearly, precisely, and unambiguously. Because of this, it can be difficult to read. Read slowly, and have pencil and paper at hand. It will usually be necessary to read something several times. While reading can be difficult, it is even harder to speak mathematics, and so that is the topic of this technique.

“Natural” language, in the present case English, is fraught with ambiguity. Consider the possible meanings of the sentence: The fish is ready to eat. One fish, or two fish? Are the fish hungry, or will the fish be eaten? (See Exercise SSLE.M10, Exercise SSLE.M11, Exercise SSLE.M12, Exercise SSLE.M13.) In your daily interactions with others, give some thought to how many misunderstandings arise from the ambiguity of pronouns, modifiers and objects.

I am going to suggest a simple modification to the way you use language that will make it much, much easier to become proficient at speaking mathematics and eventually it will become second nature. Think of it as a training aid or practice drill you might use when learning to become skilled at a sport.

First, eliminate pronouns from your vocabulary when discussing linear algebra, in class or with your colleagues. Do not use: it, that, those, their or similar sources of confusion. This is the single easiest step you can take to make your oral expression of mathematics clearer to others, and in turn, it will greatly help your own understanding.

Now rid yourself of the word “thing” (or variants like “something”). When you are tempted to use this word realize that there is some object you want to discuss, and we likely have a definition for that object (see the discussion at Technique D). Always “think about your objects” and many aspects of the study of mathematics will get easier. Ask yourself: “Am I working with a set, a number, a function, an operation, a differential equation, or what?” Knowing what an object is will allow you to narrow down the procedures you may apply to it. If you have studied an object-oriented computer programming language, then perhaps this advice will be even clearer, since you know that a compiler will often complain with an error message if you confuse your objects.

Third, eliminate the verb “works” (as in “the equation works”) from your vocabulary. This term is used as a substitute when we are not sure just what we are trying to accomplish. Usually we are trying to say that some object fulfills some condition. The condition might even have a definition associated with it, making it even easier to describe.

Last, speak sloooooowly and thoughtfully as you try to get by without all these lazy words. It is hard at first, but you will get better with practice. Especially in class, when the pressure is on and all eyes are on you, don’t succumb to the temptation to use these weak words. Slow down, we’d all rather wait for a slow, well-formed question or answer than a fast, sloppy, incomprehensible one.
You will find the improvement in your ability to *speak* clearly about complicated ideas will greatly improve your ability to *think* clearly about complicated ideas. And I believe that you cannot think clearly about complicated ideas if you cannot formulate questions or answers clearly in the correct language. This is as applicable to the study of law, economics or philosophy as it is to the study of science or mathematics.

So when you come to class, check your pronouns at the door, along with other weak words. And when studying with friends, you might make a game of catching one another using pronouns, “thing,” or “works.” I know I’ll be calling you on it!
Proof Technique GS
Getting Started

“I don’t know how to get started!” is often the lament of the novice proof-builder. Here are a few pieces of advice.

1. As mentioned in Technique T [699], rewrite the statement of the theorem in an “if-then” form. This will simplify identifying the hypothesis and conclusion, which are referenced in the next few items.

2. Ask yourself what kind of statement you are trying to prove. This is always part of your conclusion. Are you being asked to conclude that two numbers are equal, that a function is differentiable or a set is a subset of another? You cannot bring other techniques to bear if you do not know what type of conclusion you have.

3. Write down reformulations of your hypotheses. Interpret and translate each definition properly.

4. Write your hypothesis at the top of a sheet of paper and your conclusion at the bottom. See if you can formulate a statement that precedes the conclusion and also implies it. Work down from your hypothesis, and up from your conclusion, and see if you can meet in the middle. When you are finished, rewrite the proof nicely, from hypothesis to conclusion, with verifiable implications giving each subsequent statement.

5. As you work through your proof, think about what kinds of objects your symbols represent. For example, suppose $A$ is a set and $f(x)$ is a real-valued function. Then the expression $A + f$ might make no sense if we have not defined what it means to “add” a set to a function, so we can stop at that point and adjust accordingly. On the other hand we might understand $2f$ to be the function whose rule is described by $(2f)(x) = 2f(x)$. “Think about your objects” means to always verify that your objects and operations are compatible.
Proof Technique C
Constructive Proofs

Conclusions of proofs come in a variety of types. Often a theorem will simply assert that something exists. The best way, but not the only way, to show something exists is to actually build it. Such a proof is called constructive. The thing to realize about constructive proofs is that the proof itself will contain a procedure that might be used computationally to construct the desired object. If the procedure is not too cumbersome, then the proof itself is as useful as the statement of the theorem. Such is the case with our next theorem.
Proof Technique E
Equivalences

When a theorem uses the phrase “if and only if” (or the abbreviation “iff”) it is a shorthand way of saying that two if-then statements are true. So if a theorem says “P if and only if Q,” then it is true that “if P, then Q” while it is also true that “if Q, then P.” For example, it may be a theorem that “I wear bright yellow knee-high plastic boots if and only if it is raining.” This means that I never forget to wear my super-duper yellow boots when it is raining and I wouldn’t be seen in such silly boots unless it was raining. You never have one without the other. I’ve got my boots on and it is raining or I don’t have my boots on and it is dry.

The upshot for proving such theorems is that it is like a 2-for-1 sale, we get to do two proofs. Assume P and conclude Q, then start over and assume Q and conclude P. For this reason, “if and only if” is sometimes abbreviated by $\iff$, while proofs indicate which of the two implications is being proved by prefacing each with $\rightarrow$ or $\leftarrow$. A carefully written proof will remind the reader which statement is being used as the hypothesis, a quicker version will let the reader deduce it from the direction of the arrow. Tradition dictates we do the “easy” half first, but that’s hard for a student to know until you’ve finished doing both halves! Oh well, if you rewrite your proofs (a good habit), you can then choose to put the easy half first.

Theorems of this type are called equivalences or characterizations, and they are some of the most pleasing results in mathematics. They say that two objects, or two situations, are really the same. You don’t have one without the other, like rain and my yellow boots. The more different P and Q seem to be, the more pleasing it is to discover they are really equivalent. And if P describes a very mysterious solution or involves a tough computation, while Q is transparent or involves easy computations, then we’ve found a great shortcut for better understanding or faster computation. Remember that every theorem really is a shortcut in some form. You will also discover that if proving $P \Rightarrow Q$ is very easy, then proving $Q \Rightarrow P$ is likely to be proportionately harder. Sometimes the two halves are about equally hard. And in rare cases, you can string together a whole sequence of other equivalences to form the one you’re after and you don’t even need to do two halves. In this case, the argument of one half is just the argument of the other half, but in reverse.

One last thing about equivalences. If you see a statement of a theorem that says two things are “equivalent,” translate it first into an “if and only if” statement.
Proof Technique N
Negation

When we construct the contrapositive of a theorem (Technique CP [706]), we need to negate the two statements in the implication. And when we construct a proof by contradiction (Technique CD [708]), we need to negate the conclusion of the theorem. One way to construct a converse (Technique CV [707]) is to simultaneously negate the hypothesis and conclusion of an implication (but remember that this is not guaranteed to be a true statement). So we often have the need to negate statements, and in some situations it can be tricky.

If a statement says that a set is empty, then its negation is the statement that the set is nonempty. That’s straightforward. Suppose a statement says “something-happens” for all $i$, or every $i$, or any $i$. Then the negation is that “something-doesn’t-happen” for at least one value of $i$. If a statement says that there exists at least one “thing,” then the negation is the statement that there is no “thing.” If a statement says that a “thing” is unique, then the negation is that there is zero, or more than one, of the “thing.”

We are not covering all of the possibilities, but we wish to make the point that logical qualifiers like “there exists” or “for every” must be handled with care when negating statements. Studying the proofs which employ contradiction (as listed in Technique CD [708]) is a good first step towards understanding the range of possibilities.
Proof Technique CP
Contra\textit{positions}

The \textit{contrapositive} of an implication $P \Rightarrow Q$ is the implication $\text{not}(Q) \Rightarrow \text{not}(P)$, where “not” means the logical negation, or opposite. An implication is true if and only if its contrapositive is true. In symbols, $(P \Rightarrow Q) \iff (\text{not}(Q) \Rightarrow \text{not}(P))$ is a theorem. Such statements about logic, that are always true, are known as \textit{tautologies}.

For example, it is a theorem that “if a vehicle is a fire truck, then it has big tires and has a siren.” (Yes, I’m sure you can conjure up a counterexample, but play along with me anyway.) The contrapositive is “if a vehicle does not have big tires or does not have a siren, then it is not a fire truck.” Notice how the “and” became an “or” when we negated the conclusion of the original theorem.

It will frequently happen that it is easier to construct a proof of the contrapositive than of the original implication. If you are having difficulty formulating a proof of some implication, see if the contrapositive is easier for you. The trick is to construct the negation of complicated statements accurately. More on that later.
Proof Technique CV
Converses

The converse of the implication $P \Rightarrow Q$ is the implication $Q \Rightarrow P$. There is no guarantee that the truth of these two statements are related. In particular, if an implication has been proven to be a theorem, then do not try to use its converse too, as if it were a theorem. Sometimes the converse is true (and we have an equivalence, see Technique E [704]). But more likely the converse is false, especially if it wasn’t included in the statement of the original theorem.

For example, we have the theorem, “if a vehicle is a fire truck, then it is has big tires and has a siren.” The converse is false. The statement that “if a vehicle has big tires and a siren, then it is a fire truck” is false. A police vehicle for use on a sandy public beach would have big tires and a siren, yet is not equipped to fight fires.

We bring this up now, because Theorem CSRN [56] has a tempting converse. Does this theorem say that if $r < n$, then the system is consistent? Definitely not, as Archetype E [739] has $r = 3 < 4 = n$, yet is inconsistent. This example is then said to be a counterexample to the converse. Whenever you think a theorem that is an implication might actually be an equivalence, it is good to hunt around for a counterexample that shows the converse to be false (the archetypes, Appendix A [717], can be a good hunting ground).
Another proof technique is known as “proof by contradiction” and it can be a powerful (and satisfying) approach. Simply put, suppose you wish to prove the implication, “If $A$, then $B$.” As usual, we assume that $A$ is true, but we also make the additional assumption that $B$ is false. If our original implication is true, then these twin assumptions should lead us to a logical inconsistency. In practice we assume the negation of $B$ to be true (see Technique N [705]). So we argue from the assumptions $A$ and $not(B)$ looking for some obviously false conclusion such as $1 = 6$, or a set is simultaneously empty and nonempty, or a matrix is both nonsingular and singular.

You should be careful about formulating proofs that look like proofs by contradiction, but really aren’t. This happens when you assume $A$ and $not(B)$ and proceed to give a “normal” and direct proof that $B$ is true by only using the assumption that $A$ is true. Your last step is to then claim that $B$ is true and you then appeal to the assumption that $not(B)$ is true, thus getting the desired contradiction. Instead, you could have avoided the overhead of a proof by contradiction and just run with the direct proof. This stylistic flaw is known, quite graphically, as “setting up the strawman to knock him down.”

Here is a simple example of a proof by contradiction. There are direct proofs that are just about as easy, but this will demonstrate the point, while narrowly avoiding knocking down the straw man.

**Theorem:** If $a$ and $b$ are odd integers, then their product, $ab$, is odd.

**Proof:** To begin a proof by contradiction, assume the hypothesis, that $a$ and $b$ are odd. Also assume the negation of the conclusion, in this case, that $ab$ is even. Then there are integers, $j$, $k$, $\ell$ so that $a = 2j + 1$, $b = 2k + 1$, $ab = 2\ell$. Then

\[
0 = ab - ab = (2j + 1)(2k + 1) - (2\ell) = 4jk + 2j + 2k - 2\ell + 1 = 2(2jk + j + k - \ell) + 1
\]

Notice how we used both our hypothesis and the negation of the conclusion in the second line. Now divide the integer on each end of this string of equalities by 2. On the left we get a remainder of 0, while on the right we see that the remainder will be 1. Both remainders cannot be correct, so this is our desired contradiction. Thus, the conclusion (that $ab$ is odd) is true.

Again, we do not offer this example as the best proof of this fact about even and odd numbers, but rather it is a simple illustration of a proof by contradiction. You can find examples of proofs by contradiction in Theorem NMUS [79], Theorem NPNT [249], Theorem RREFU [116], Theorem TTMI [234], Theorem GSPCV [191], Theorem ELIS [397], Theorem EDYES [401], Theorem EMHE [445], Theorem EDELI [467], and Theorem DMFE [490], in addition to several examples and solutions to exercises.
A theorem will sometimes claim that some object, having some desirable property, is unique. In other words, there should be only one such object. To prove this, a standard technique is to assume there are two such objects and proceed to analyze the consequences. The end result may be a contradiction (Technique CD [708]), or the conclusion that the two allegedly different objects really are equal.
A very specialized form of a theorem begins with the statement “The following are equivalent...,” which is then followed by a list of statements. Informally, this lead-in sometimes gets abbreviated by “TFAE.” This formulation means that any two of the statements on the list can be connected with an “if and only if” to form a theorem. So if the list has \( n \) statements then, there are \( \frac{n(n-1)}{2} \) possible equivalences that can be constructed (and are claimed to be true).

Suppose a theorem of this form has statements denoted as \( A, B, C,...,Z \). To prove the entire theorem, we can prove \( A \Rightarrow B, B \Rightarrow C, C \Rightarrow D,\ldots, Y \Rightarrow Z \) and finally, \( Z \Rightarrow A \). This circular chain of \( n \) equivalences would allow us, logically, if not practically, to form any one of the \( \frac{n(n-1)}{2} \) possible equivalences by chasing the equivalences around the circle as far as required.
Many theorems have conclusions that say two objects are equal. Perhaps one object is hard to compute or understand, while the other is easy to compute or understand. This would make for a pleasing theorem. Whether the result is pleasing or not, we take the same approach to formulate a proof. Sometimes we need to employ specialized notions of equality, such as Definition SE [694] or Definition CVE [88], but in other cases we can string together a list of equalities.

The wrong way to prove an identity is to begin by writing it down and then beating on it until it reduces to an obvious identity. The first flaw is that you would be writing down the statement you wish to prove, as if you already believed it to be true. But more dangerous is the possibility that some of your maneuvers are not reversible. Here’s an example. Let’s prove that $3 = -3$.

\[
3 = -3 \quad \text{(This is a bad start)} \\
3^2 = (-3)^2 \quad \text{Square both sides} \\
9 = 9 \\
0 = 0 \quad \text{Subtract 9 from both sides}
\]

So because $0 = 0$ is a true statement, does it follow that $3 = -3$ is a true statement? Nope. Of course, we didn’t really expect a legitimate proof of $3 = -3$, but this attempt should illustrate the dangers of this (incorrect) approach.

What you have just seen in the proof of Theorem VSPCV [91], and what you will see consistently throughout this text, is proofs of the following form. To prove that $A = D$ we write

\[
A = B \quad \text{Theorem, Definition or Hypothesis justifying } A = B \\
= C \quad \text{Theorem, Definition or Hypothesis justifying } B = C \\
= D \quad \text{Theorem, Definition or Hypothesis justifying } C = D
\]

In your scratch work exploring possible approaches to proving a theorem you may massage a variety of expressions, sometimes making connections to various bits and pieces, while some parts get abandoned. Once you see a line of attack, rewrite your proof carefully mimicking this style.
Much of your mathematical upbringing, especially once you began a study of algebra, revolved around simplifying expressions — combining like terms, obtaining common denominators so as to add fractions, factoring in order to solve polynomial equations. However, as often as not, we will do the opposite. Many theorems and techniques will revolve around taking some object and “decomposing” it into some combination of other objects, ostensibly in a more complicated fashion. When we say something can “be written as” something else, we mean that the one object can be decomposed into some combination of other objects. This may seem unnatural at first, but results of this type will give us insight into the structure of the original object by exposing its inner workings. An appropriate analogy might be stripping the wallboards away from the interior of a building to expose the structural members supporting the whole building.

This is a major shift in thinking, so come back here often, especially when we say “can be written as”, or “can be expressed as,” or “can be decomposed as.”
Proof Technique I
Induction

"Induction" or "mathematical induction" is a framework for proving statements that are indexed by integers. In other words, suppose you have a statement to prove that is really multiple statements, one for \( n = 1 \), another for \( n = 2 \), a third for \( n = 3 \), and so on. If there is enough similarity between the statements, then you can use a script (the framework) to prove them all at once.

For example, consider the theorem

**Theorem** \[ 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2} \] for \( n \geq 1 \).

This is shorthand for the many statements \( 1 = \frac{1(1+1)}{2} \), \( 1+2 = \frac{2(2+1)}{2} \), \( 1+2+3 = \frac{3(3+1)}{2} \), \( 1+2+3+4 = \frac{4(4+1)}{2} \), and so on. Forever. You can do the calculations in each of these statements and verify that all four are true. We might not be surprised to learn that the fifth statement is true as well (go ahead and check). However, do we think the theorem is true for \( n = 872 \)? Or \( n = 1,234,529 \)?

To see that these questions are not so ridiculous, consider the following example from Rotman's *Journey into Mathematics*. The statement "\( n^2 - n + 41 \) is prime" is true for integers \( 1 \leq n \leq 40 \) (check a few). However, when we check \( n = 41 \) we find \( 41^2 - 41 + 41 = 41^2 \), which is not prime.

So how do we prove infinitely many statements all at once? More formally, let's denote our statements as \( P(n) \). Then, if we can prove the two assertions

1. \( P(1) \) is true.
2. If \( P(k) \) is true, then \( P(k + 1) \) is true.

then it follows that \( P(n) \) is true for all \( n \geq 1 \). To understand this, I liken the process to climbing an infinitely long ladder with equally spaced rungs. Confronted with such a ladder, suppose I tell you that you are able to step up onto the first rung, and if you are on any particular rung, then you are capable of stepping up to the next rung. It follows that you can climb the ladder as far up as you wish. The first formal assertion above is akin to stepping onto the first rung, and the second formal assertion is akin to assuming that if you are on any one rung then you can always reach the next rung.

In practice, establishing that \( P(1) \) is true is called the "base case" and in most cases is straightforward. Establishing that \( P(k) \Rightarrow P(k + 1) \) is referred to as the "induction step," or in this book (and elsewhere) we will typically refer to the assumption of \( P(k) \) as the "induction hypothesis." This is perhaps the most mysterious part of a proof by induction, since it looks like you are assuming \( P(k) \) what you are trying to prove \( P(n) \). Sometimes it is even worse, since as you get more comfortable with induction, we often don't bother to use a different letter \( (k) \) for the index \( (n) \) in the induction step. Notice that the second formal assertion never says that \( P(k) \) is true, it simply says that if \( P(k) \) were true, what might logically follow. We can establish statements like "If I lived on the moon, then I could pole-vault over a bar 12 meters high." This may be a true statement, but it does not say we live on the moon, and indeed we may never live there.
Enough generalities. Let’s work an example and prove the theorem above about sums of integers. Formally, our statement is \( P(n) : 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2} \).

**Proof:** Base Case. \( P(1) \) is the statement \( 1 = \frac{1(1+1)}{2} \), which we see simplifies to the true statement \( 1 = 1 \).

Induction Step: We will assume \( P(k) \) is true, and will try to prove \( P(k + 1) \). Given what we want to accomplish, it is natural to begin by examining the sum of the first \( k + 1 \) integers.

\[
1 + 2 + 3 + \cdots + (k + 1) \\
= (1 + 2 + 3 + \cdots + k) + (k + 1) \\
= \frac{k(k + 1)}{2} + (k + 1) \quad \text{Induction Hypothesis} \\
= \frac{k^2 + k}{2} = \frac{k^2 + 3k + 2}{2} \\
= \frac{(k + 1)(k + 2)}{2} = \frac{(k + 1)((k + 1) + 1)}{2}
\]

We then recognize the two ends of this chain of equalities as \( P(k+1) \). So, by mathematical induction, the theorem is true for all \( n \).

How do you recognize when to use induction? The first clue is a statement that is really many statements, one for each integer. The second clue would be that you begin a more standard proof and you find yourself using words like “and so on” (as above!) or lots of ellipses (dots) to establish patterns that you are convinced continue on and on forever. However, there are many minor instances where induction might be warranted but we don’t bother.

Induction is important enough, and used often enough, that it appears in various variations. The base case sometimes begins with \( n = 0 \), or perhaps an integer greater than \( n \). Some formulate the induction step as \( P(k - 1) \Rightarrow P(k) \). There is also a “strong form” of induction where we assume all of \( P(1), P(2), P(3), \ldots P(k) \) as a hypothesis for showing the conclusion \( P(k + 1) \).

You can find examples of induction in the proofs of Theorem GSPCV [191], Theorem DER [416], Theorem DT [417], Theorem DIM [430], Theorem EOMP [469], and Theorem DCP [472].
Here is a technique used by many practicing mathematicians when they are teaching themselves new mathematics. As they read a textbook, monograph or research article, they attempt to prove each new theorem themselves, before reading the proof. Often the proofs can be very difficult, so it is wise not to spend too much time on each. Maybe limit your losses and try each proof for 10 or 15 minutes. Even if the proof is not found, it is time well-spent. You become more familiar with the definitions involved, and the hypothesis and conclusion of the theorem. When you do work through the proof, it might make more sense, and you will gain added insight about just how to construct a proof.
Proof Technique LC
Lemmas and Corollaries

Theorems often go by different titles. Two of the most popular being “lemma” and “corollary.” Before we describe the fine distinctions, be aware that lemmas, corollaries, propositions, claims and facts are all just theorems. And every theorem can be rephrased as an “if-then” statement, or perhaps a pair of “if-then” statements expressed as an equivalence (Technique E [704]).

A lemma is a theorem that is not too interesting in its own right, but is important for proving other theorems. It might be a generalization or abstraction of a key step of several different proofs. For this reason you often hear the phrase “technical lemma” though some might argue that the adjective “technical” is redundant.

A corollary is a theorem that follows very easily from another theorem. For this reason, corollaries frequently do not have proofs. You are expected to easily and quickly see how a previous theorem implies the corollary.

A proposition or fact is really just a codeword for a theorem. A claim might be similar, but some authors like to use claims within a proof to organize key steps. In a similar manner, some long proofs are organized as a series of lemmas.

In order to not confuse the novice, we have just called all our theorems theorems. It is also an organizational convenience. With only theorems and definitions, the theoretical backbone of the course is laid bare in the two lists of Definitions [xi] and Theorems [xiii].
Appendix A
Archetypes

The American Heritage Dictionary of the English Language (Third Edition) gives two definitions of the word “archetype”: 1. An original model or type after which other similar things are patterned; a prototype; and 2. An ideal example of a type; quintessence.

Either use might apply here. Our archetypes are typical examples of systems of equations, matrices and linear transformations. They have been designed to demonstrate the range of possibilities, allowing you to compare and contrast them. Several are of a size and complexity that is usually not presented in a textbook, but should do a better job of being “typical.”

We have made frequent reference to many of these throughout the text, such as the frequent comparisons between Archetype A [721] and Archetype B [726]. Some we have left for you to investigate, such as Archetype J [762], which parallels Archetype I [757].

How should you use the archetypes? First, consult the description of each one as it is mentioned in the text. See how other facts about the example might illuminate whatever property or construction is being described in the example. Second, each property has a short description that usually includes references to the relevant theorems. Perform the computations and understand the connections to the listed theorems. Third, each property has a small checkbox in front of it. Use the archetypes like a workbook and chart your progress by “checking-off” those properties that you understand.

The next page has a chart that summarizes some (but not all) of the properties described for each archetype. Notice that while there are several types of objects, there are fundamental connections between them. That some lines of the table do double-duty is meant to convey some of these connections. Consult this table when you wish to quickly find an example of a certain phenomenon.
| Type       | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S | T | U | V | W | X |
| Vars, Cols, Domain | 3 | 3 | 4 | 4 | 4 | 2 | 2 | 7 | 9 | 5 | 5 | 5 | 5 | 5 | 3 | 3 | 5 | 3 | 5 | 6 | 4 | 3 | 4 |
| Eqns, Rows, CoDom | 3 | 3 | 3 | 3 | 4 | 5 | 5 | 4 | 6 | 5 | 5 | 3 | 3 | 5 | 5 | 5 | 4 | 6 | 4 | 4 | 3 | 4 |
| Solution Set | I | U | I | I | N | U | U | N | I | I | I | U | I | U | I | I | U | I | U | I | U | I | U | I | U |
| Rank      | 2 | 3 | 3 | 2 | 2 | 4 | 2 | 2 | 3 | 4 | 5 | 3 | 2 | 3 | 2 | 3 | 4 | 5 | 2 | 5 | 4 | 4 | 3 | 3 |
| Nullity   | 1 | 0 | 1 | 2 | 2 | 0 | 0 | 0 | 4 | 5 | 0 | 2 | 3 | 2 | 1 | 0 | 1 | 0 | 1 | 0 | 2 | 0 | 0 | 1 |
| Injective |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   | X | X | N | Y | N | Y | X | Y | Y | N |
| Surjective|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   | N | Y | X | X | N | Y | X | Y | Y | N |
| Full Rank | N | Y | Y | N | N | Y | Y | N | N | Y | N | Y | N | N | Y | N | N | Y | Y | N | X | X | Y | Y | Y | N |
| Nonsingular | N | Y |   | Y | Y | Y | N | N | N | Y | N | Y | N | N | Y | N | N | Y | Y | N | X | X | Y | Y | Y | N |
| Invertible | N | Y |   | Y | Y | Y | N | N | N | Y | Y | Y | N | N | Y | N | N | Y | Y | N | X | X | Y | Y | Y | N |
| Determinant| 0 | -2| -18| 16| 0 |   |   |   |   |   |   |   |   |   |   |   |   | 2 | -3| 0 |   |   |   |   |   |   |
| Diagonalizable | N | Y |   | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y | Y |

Archetype Facts

S=System of Equations, M=Matrix, L=Linear Transformation
U=Unique solution, I=Infinitely many solutions, N=No solutions
Y=Yes, N=No, X=Impossible, blank=Not Applicable
Archetype A

Summary  Linear system of three equations, three unknowns. Singular coefficient matrix with dimension 1 null space. Integer eigenvalues and a degenerate eigenspace for coefficient matrix.

A system of linear equations (Definition SLE 13):
\[
\begin{align*}
  x_1 - x_2 + 2x_3 &= 1 \\
  2x_1 + x_2 + x_3 &= 8 \\
  x_1 + x_2 &= 5
\end{align*}
\]

Some solutions to the system of linear equations (not necessarily exhaustive):
\[
\begin{align*}
  &x_1 = 2, \quad x_2 = 3, \quad x_3 = 1 \\
  &x_1 = 3, \quad x_2 = 2, \quad x_3 = 0
\end{align*}
\]

Augmented matrix of the linear system of equations (Definition AM 32):
\[
\begin{bmatrix}
  1 & -1 & 2 & 1 \\
  2 & 1 & 1 & 8 \\
  1 & 1 & 0 & 5
\end{bmatrix}
\]

Matrix in reduced row-echelon form, row-equivalent to augmented matrix:
\[
\begin{bmatrix}
  1 & 0 & 1 & 3 \\
  0 & 1 & -1 & 2 \\
  0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix (Notation RREFA 36):
\[
r = 2 \quad D = \{1, 2\} \quad F = \{3, 4\}
\]

Vector form of the solution set to the system of equations (Theorem VFSLS 107). Notice the relationship between the free variables and the set \( F \) above. Also, notice the pattern of 0's and 1's in the entries of the vectors corresponding to elements of the set \( F \) for the larger examples.
Given a system of equations we can always build a new, related, homogeneous system \( \text{[Definition HS [65]]} \) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system:

\[
\begin{align*}
    x_1 - x_2 + 2x_3 &= 0 \\
    2x_1 + x_2 + x_3 &= 0 \\
    x_1 + x_2 &= 0
\end{align*}
\]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):

\[
\begin{align*}
    x_1 &= 0, & x_2 &= 0, & x_3 &= 0 \\
    x_1 &= -1, & x_2 &= 1, & x_3 &= 1 \\
    x_1 &= -5, & x_2 &= 5, & x_3 &= 5
\end{align*}
\]

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system \( \text{[Notation RREFA [36]]} \). Notice the slight variation for the same analysis of the original system only when the original system was consistent:

\[
r = 2 \\
D = \{1, 2\} \\
F = \{3, 4\}
\]

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations:

\[
\begin{bmatrix}
1 & -1 & 2 \\
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]
Matrix brought to reduced row-echelon form:

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\]

Analysis of the row-reduced matrix (Notation RREFA 36):

\[ r = 2 \quad D = \{1, 2\} \quad F = \{3\} \]

Matrix (coefficient matrix) is nonsingular or singular? (Theorem NMRRI 77) at the same time, examine the size of the set \( F \) above. Notice that this property does not apply to matrices that are not square.

Singular.

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS 129, Theorem BNS 154). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VF-SLS 107) to see these vectors arise.

\[
\langle \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \rangle
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \( D \) above. (Theorem BCS 264)

\[
\langle \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \rangle
\]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \( L \) is computed as described in Definition EEF 287. This is followed by the column space described by a set of linearly independent vectors that span the null space of \( L \), computed as according to Theorem FS 290 and Theorem BNS 154. When \( r = m \), the matrix \( L \) has no rows and the column space is all of \( \mathbb{C}^m \).

\[
L = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}
\]
Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRS\textsuperscript{T} \cite{273} and Theorem BRS \cite{271}, and in the style of Example CSROI \cite{273}, this yields a linearly independent set of vectors that span the column space.

\[
\left\langle \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\rangle
\]

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS \cite{271})

\[
\left\langle \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\rangle
\]

Inverse matrix, if it exists. The inverse is not defined for matrices that are not square, and if the matrix is square, then the matrix must be nonsingular. (Definition MI \cite{232}, Theorem NI \cite{251})

Subspace dimensions associated with the matrix. (Definition NOM \cite{385}, Definition ROM \cite{385}) Verify Theorem RPNC \cite{387}

Matrix columns: 3  \quad Rank: 2  \quad Nullity: 1

Determinant of the matrix, which is only defined for square matrices. The matrix is nonsingular if and only if the determinant is nonzero (Theorem SMZD \cite{432}). (Product of all eigenvalues?)

Determinant = 0

Eigenvalues, and bases for eigenspaces. (Definition EEM \cite{441}, Definition EM \cite{450})
\[ \lambda = 0 \quad \mathcal{E}_A(0) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} \]

\[ \lambda = 2 \quad \mathcal{E}_A(2) = \left\{ \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\} \]

- Geometric and algebraic multiplicities. (Definition GME 452, Definition AME 452)

\[ \gamma_A(0) = 1 \quad \alpha_A(0) = 2 \]
\[ \gamma_A(2) = 1 \quad \alpha_A(2) = 1 \]

- Diagonalizable? (Definition DZM 487)

No, \( \gamma_A(0) \neq \alpha_B(0) \), Theorem DMFE 490.
Archetype B


A system of linear equations (Definition SLE 13):

\[ -7x_1 - 6x_2 - 12x_3 = -33 \]
\[ 5x_1 + 5x_2 + 7x_3 = 24 \]
\[ x_1 + 4x_3 = 5 \]

Some solutions to the system of linear equations (not necessarily exhaustive):

\( x_1 = -3, \quad x_2 = 5, \quad x_3 = 2 \)

Augmented matrix of the linear system of equations (Definition AM 32):

\[
\begin{bmatrix}
-7 & -6 & -12 & -33 \\
5 & 5 & 7 & 24 \\
1 & 0 & 4 & 5 \\
\end{bmatrix}
\]

Matrix in reduced row-echelon form, row-equivalent to augmented matrix:

\[
\begin{bmatrix}
1 & 0 & 0 & -3 \\
0 & 1 & 0 & 5 \\
0 & 0 & 1 & 2 \\
\end{bmatrix}
\]

Analysis of the augmented matrix (Notation RREFA 36):

\[ r = 3 \quad D = \{1, 2, 3\} \quad F = \{4\} \]

Vector form of the solution set to the system of equations (Theorem VFSLS 107).
Notice the relationship between the free variables and the set \( F \) above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set \( F \) for the larger examples.
Given a system of equations we can always build a new, related, homogeneous system (Definition HS [65]) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system.

\[
-11x_1 + 2x_2 - 14x_3 = 0 \\
23x_1 - 6x_2 + 33x_3 = 0 \\
14x_1 - 2x_2 + 17x_3 = 0
\]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):
\[x_1 = 0, \quad x_2 = 0, \quad x_3 = 0\]

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system (Notation RREFA [36]). Notice the slight variation for the same analysis of the original system only when the original system was consistent:
\[r = 3, \quad D = \{1, 2, 3\}, \quad F = \{4\}\]

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations.

\[
\begin{bmatrix}
-7 & -6 & -12 \\
5 & 5 & 7 \\
1 & 0 & 4
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Analysis of the row-reduced matrix (Notation RREFA [36]):

\[ r = 3 \quad \quad D = \{1, 2, 3\} \quad \quad F = \{\} \]

Matrix (coefficient matrix) is nonsingular or singular? (Theorem NMRRI [77]) at the same time, examine the size of the set \( F \) above. Notice that this property does not apply to matrices that are not square.

Nonsingular.

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS [129], Theorem BNS [154]). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VF-SLS [107]) to see these vectors arise.

\[ \langle \{\} \rangle \]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \( D \) above. (Theorem BCS [264])

\[ \langle\{ \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix} \rangle \}

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \( L \) is computed as described in Definition EEF [287]. This is followed by the column space described by a set of linearly independent vectors that span the null space of \( L \), computed as according to Theorem FS [290] and Theorem BNS [154]. When \( r = m \), the matrix \( L \) has no rows and the column space is all of \( \mathbb{C}^m \).

\[ L = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST [273] and Theorem BRS [271], and in the
style of Example CSROI \[273\], this yields a linearly independent set of vectors that span
the column space.

\[
\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle
\]

Row space of the matrix, expressed as a span of a set of linearly independent vectors,
obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form.
(Theorem BRS \[271\])

\[
\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle
\]

Inverse matrix, if it exists. The inverse is not defined for matrices that are not square,
and if the matrix is square, then the matrix must be nonsingular. (Definition MI \[232\],
Theorem NI \[251\])

\[
\begin{bmatrix}
-10 & -12 & -9 \\
13 & 8 & 11 \\
5 & 3 & 5 \\
2 & 2 & 2
\end{bmatrix}
\]

Subspace dimensions associated with the matrix. (Definition NOM \[385\], Definition
ROM \[385\]) Verify Theorem RPNC \[387\]

Matrix columns: 3 Rank: 3 Nullity: 0

Determinant of the matrix, which is only defined for square matrices. The matrix is
nonsingular if and only if the determinant is nonzero (Theorem SMZD \[432\]). (Product
of all eigenvalues?)

Determinant = \(-2\)

Eigenvalues, and bases for eigenspaces. (Definition EEM \[441\], Definition EM \[450\])

\[
\lambda = -1 \quad \mathcal{E}_B (-1) = \left\langle \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix} \right\rangle
\]

\[
\lambda = 1 \quad \mathcal{E}_B (1) = \left\langle \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \right\rangle
\]
\( \lambda = 2 \quad \mathcal{E}_B (2) = \left\langle \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\rangle \)

- Geometric and algebraic multiplicities. (Definition GME 452, Definition AME 452)

  \( \gamma_B (-1) = 1 \quad \alpha_B (-1) = 1 \)
  \( \gamma_B (1) = 1 \quad \alpha_B (1) = 1 \)
  \( \gamma_B (2) = 1 \quad \alpha_B (2) = 1 \)

- Diagonalizable? (Definition DZM 487)

  Yes, distinct eigenvalues, Theorem DED 492.

- The diagonalization. (Theorem DC 487)

\[
\begin{bmatrix}
-1 & -1 & -1 \\
2 & 3 & 1 \\
-1 & -2 & 1
\end{bmatrix}
\begin{bmatrix}
-7 & -6 & -12 \\
5 & 5 & 7 \\
1 & 0 & 4
\end{bmatrix}
\begin{bmatrix}
-5 & -3 & -2 \\
3 & 2 & 1 \\
1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]
Archetype C

Summary  System with three equations, four variables. Consistent. Null space of coefficient matrix has dimension 1.

A system of linear equations (Definition SLE [13]):

\[
\begin{align*}
2x_1 - 3x_2 + x_3 - 6x_4 &= -7 \\
4x_1 + x_2 + 2x_3 + 9x_4 &= -7 \\
3x_1 + x_2 + x_3 + 8x_4 &= -8
\end{align*}
\]

Some solutions to the system of linear equations (not necessarily exhaustive):

\[
\begin{align*}
x_1 &= -7, & x_2 &= -2, & x_3 &= 7, & x_4 &= 1 \\
x_1 &= -1, & x_2 &= -7, & x_3 &= 4, & x_4 &= -2
\end{align*}
\]

Augmented matrix of the linear system of equations (Definition AM [32]):

\[
\begin{bmatrix}
2 & -3 & 1 & -6 & -7 \\
4 & 1 & 2 & 9 & -7 \\
3 & 1 & 1 & 8 & -8
\end{bmatrix}
\]

Matrix in reduced row-echelon form, row-equivalent to augmented matrix:

\[
\begin{bmatrix}
1 & 0 & 0 & 2 & -5 \\
0 & 1 & 0 & 3 & 1 \\
0 & 0 & 1 & -1 & 6
\end{bmatrix}
\]

Analysis of the augmented matrix (Notation RREFA [36]):

\[
r = 3 \quad D = \{1, 2, 3\} \quad F = \{4, 5\}
\]

Vector form of the solution set to the system of equations (Theorem VFSLS [107]). Notice the relationship between the free variables and the set F above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set F for the larger examples.
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
-5 \\
1 \\
6 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
-2 \\
-3 \\
1 \\
1
\end{bmatrix}
\]

Given a system of equations we can always build a new, related, homogeneous system (Definition HS [65]) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system.

\[
\begin{align*}
2x_1 - 3x_2 + x_3 - 6x_4 &= 0 \\
4x_1 + x_2 + 2x_3 + 9x_4 &= 0 \\
3x_1 + x_2 + x_3 + 8x_4 &= 0
\end{align*}
\]

\[
\begin{align*}
x_1 &= 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0 \\
x_1 &= -2, \quad x_2 = -3, \quad x_3 = 1, \quad x_4 = 1 \\
x_1 &= -4, \quad x_2 = -6, \quad x_3 = 2, \quad x_4 = 2
\end{align*}
\]

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

\[
\begin{bmatrix}
1 & 0 & 0 & 2 & 0 \\
0 & 1 & 0 & 3 & 0 \\
0 & 0 & 1 & -1 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system (Notation RREFA [36]). Notice the slight variation for the same analysis of the original system only when the original system was consistent:

\[
r = 3 \quad D = \{1, 2, 3\} \quad F = \{4, 5\}
\]

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations.

\[
\begin{bmatrix}
2 & -3 & 1 & -6 \\
4 & 1 & 2 & 9 \\
3 & 1 & 1 & 8
\end{bmatrix}
\]
Matrix brought to reduced row-echelon form:
\[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

Analysis of the row-reduced matrix (Notation RREFA 30):
- \( r = 3 \)
- \( D = \{1, 2, 3\} \)
- \( F = \{4\} \)

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS 129, Theorem BNS 154). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VF-SLS 107) to see these vectors arise.

\[
\left\langle \begin{bmatrix}
-2 \\
-3 \\
1 \\
1
\end{bmatrix} \right\rangle
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \( D \) above. (Theorem BCS 264)

\[
\left\langle \begin{bmatrix}
2 \\
4 \\
3
\end{bmatrix}, \begin{bmatrix}
-3 \\
1 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix} \right\rangle
\]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \( L \) is computed as described in Definition EEF 287. This is followed by the column space described by a set of linearly independent vectors that span the null space of \( L \), computed as according to Theorem FS 290 and Theorem BNS 154. When \( r = m \), the matrix \( L \) has no rows and the column space is all of \( \mathbb{C}^m \).

\[
L = [ ]
\]

\[
\left\langle \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} \right\rangle
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into
reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST \[273\] and Theorem BRS \[271\], and in the style of Example CSROI \[273\], this yields a linearly independent set of vectors that span the column space.

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}
\]

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS \[271\])

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}
\]

Subspace dimensions associated with the matrix. (Definition NOM \[385\], Definition ROM \[385\]) Verify Theorem RPNC \[387\]

Matrix columns: 4
Rank: 3
Nullity: 1
Archetype D

Summary  System with three equations, four variables. Consistent. Null space of coefficient matrix has dimension 2. Coefficient matrix identical to that of Archetype E, vector of constants is different.

- A system of linear equations (Definition SLE [13]):
  \[ 2x_1 + x_2 + 7x_3 - 7x_4 = 8 \]
  \[ -3x_1 + 4x_2 - 5x_3 - 6x_4 = -12 \]
  \[ x_1 + x_2 + 4x_3 - 5x_4 = 4 \]

- Some solutions to the system of linear equations (not necessarily exhaustive):
  \[ x_1 = 0, \quad x_2 = 1, \quad x_3 = 2, \quad x_4 = 1 \]
  \[ x_1 = 4, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0 \]
  \[ x_1 = 7, \quad x_2 = 8, \quad x_3 = 1, \quad x_4 = 3 \]

- Augmented matrix of the linear system of equations (Definition AM [32]):
  \[
  \begin{bmatrix}
  2 & 1 & 7 & -7 & 8 \\
  -3 & 4 & -5 & -6 & -12 \\
  1 & 1 & 4 & -5 & 4
  \end{bmatrix}
  \]

- Matrix in reduced row-echelon form, row-equivalent to augmented matrix:
  \[
  \begin{bmatrix}
  1 & 0 & 3 & -2 & 4 \\
  0 & 1 & 1 & -3 & 0 \\
  0 & 0 & 0 & 0 & 0
  \end{bmatrix}
  \]

- Analysis of the augmented matrix (Notation RREFA [36]):
  \[ r = 2 \]
  \[ D = \{ 1, 2 \} \]
  \[ F = \{ 3, 4, 5 \} \]

- Vector form of the solution set to the system of equations (Theorem VFSLS [107]). Notice the relationship between the free variables and the set \( F \) above. Also, notice the
pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set $F$ for the larger examples.

$$
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 
\end{bmatrix}
= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}
$$

Given a system of equations we can always build a new, related, homogeneous system (Definition HS [65]) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system.

$$
\begin{align*}
2x_1 + x_2 + 7x_3 - 7x_4 &= 0 \\
-3x_1 + 4x_2 - 5x_3 - 6x_4 &= 0 \\
x_1 + x_2 + 4x_3 - 5x_4 &= 0
\end{align*}
$$

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):

- $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$
- $x_1 = -3, x_2 = -1, x_3 = 1, x_4 = 0$
- $x_1 = 2, x_2 = 3, x_3 = 0, x_4 = 1$
- $x_1 = -1, x_2 = 2, x_3 = 1, x_4 = 1$

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

$$
\begin{bmatrix}
1 & 0 & 3 & -2 & 0 \\
0 & 0 & 1 & -3 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Analysis of the augmented matrix for the homogenous system (Notation RREFA [36]). Notice the slight variation for the same analysis of the original system only when the original system was consistent:

$$
r = 2 \quad D = \{1, 2\} \quad F = \{3, 4, 5\}
$$

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of
equations.
\[
\begin{bmatrix}
2 & 1 & 7 & -7 \\
-3 & 4 & -5 & -6 \\
1 & 1 & 4 & -5
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:
\[
\begin{bmatrix}
1 & 0 & 3 & -2 \\
0 & 1 & 1 & -3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis of the row-reduced matrix (Notation RREFA 36):
\[ r = 2 \quad D = \{1, 2\} \quad F = \{3, 4\} \]

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS 129, Theorem BNS 154). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VF-SLS 107) to see these vectors arise.
\[
\langle \begin{bmatrix}
-3 \\
-1 \\
1 \\
0
\end{bmatrix} , \begin{bmatrix}
2 \\
3 \\
0 \\
1
\end{bmatrix} \rangle
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \( D \) above. (Theorem BCS 264)
\[
\langle \begin{bmatrix}
2 \\
-3 \\
1
\end{bmatrix} , \begin{bmatrix}
1 \\
4 \\
1
\end{bmatrix} \rangle
\]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \( L \) is computed as described in Definition EEF 287. This is followed by the column space described by a set of linearly independent vectors that span the null space of \( L \), computed as according to Theorem FS 290 and Theorem BNS 154. When \( r = m \), the matrix \( L \) has no rows and the column space is all of \( \mathbb{C}^m \).
\[
L = \begin{bmatrix}
1 & \frac{1}{7} & -\frac{11}{7}
\end{bmatrix}
\]
Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST \[273\] and Theorem BRS \[271\], and in the style of Example CSROI \[273\], this yields a linearly independent set of vectors that span the column space.

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS \[271\])

Subspace dimensions associated with the matrix. (Definition NOM \[385\], Definition ROM \[385\]) Verify Theorem RPNC \[387\]

Matrix columns: 4  
Rank: 2  
Nullity: 2
\textbf{Archetype E}

\textbf{Summary}  System with three equations, four variables. Inconsistent. Null space of coefficient matrix has dimension 2. Coefficient matrix identical to that of Archetype D, constant vector is different.

\begin{itemize}
  \item A system of linear equations (Definition SLE\textsuperscript{[13]}):
    \begin{align*}
      2x_1 + x_2 + 7x_3 - 7x_4 &= 2 \\
      -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 3 \\
      x_1 + x_2 + 4x_3 - 5x_4 &= 2
    \end{align*}

  \item Some solutions to the system of linear equations (not necessarily exhaustive):
    None. (Why?)

  \item Augmented matrix of the linear system of equations (Definition AM\textsuperscript{[32]}):
    \[
    \begin{bmatrix}
      2 & 1 & 7 & -7 & 2 \\
      -3 & 4 & -5 & -6 & 3 \\
      1 & 1 & 4 & -5 & 2
    \end{bmatrix}
    \]

  \item Matrix in reduced row-echelon form, row-equivalent to augmented matrix:
    \[
    \begin{bmatrix}
      1 & 0 & 3 & -2 & 0 \\
      0 & 1 & 1 & -3 & 0 \\
      0 & 0 & 0 & 0 & 1
    \end{bmatrix}
    \]

  \item Analysis of the augmented matrix (Notation RREFA\textsuperscript{[36]}):
    \begin{align*}
      r &= 3 & D &= \{1, 2, 5\} & F &= \{3, 4\}
    \end{align*}

  \item Vector form of the solution set to the system of equations (Theorem VFSLS\textsuperscript{[107]}).
    Notice the relationship between the free variables and the set $F$ above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set $F$ for the larger examples.
\end{itemize}
Inconsistent system, no solutions exist.

Given a system of equations we can always build a new, related, homogeneous system (Definition HS [65]) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system.

\[
\begin{align*}
2x_1 + x_2 + 7x_3 - 7x_4 &= 0 \\
-3x_1 + 4x_2 - 5x_3 - 6x_4 &= 0 \\
x_1 + x_2 + 4x_3 - 5x_4 &= 0
\end{align*}
\]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):
\[
\begin{align*}
x_1 &= 0, & x_2 &= 0, & x_3 &= 0, & x_4 &= 0 \\
x_1 &= 4, & x_2 &= 13, & x_3 &= 2, & x_4 &= 5
\end{align*}
\]

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

\[
\begin{bmatrix}
1 & 0 & 3 & -2 & 0 \\
0 & 1 & 1 & -3 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system (Notation RREFA [36]). Notice the slight variation for the same analysis of the original system only when the original system was consistent:

\[
\begin{align*}
r &= 2 & D &= \{1, 2\} & F &= \{3, 4, 5\}
\end{align*}
\]

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations.

\[
\begin{bmatrix}
2 & 1 & 7 & -7 \\
-3 & 4 & -5 & -6 \\
1 & 1 & 4 & -5
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:

\[
\begin{bmatrix}
1 & 0 & 3 & -2 \\
0 & 1 & 1 & -3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
Analysis of the row-reduced matrix (Notation RREFA 36):

\[ r = 2 \quad D = \{1, 2\} \quad F = \{3, 4\} \]

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS 129, Theorem BNS 154). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VF-SLS 107) to see these vectors arise.

\[
\left\langle \left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \( D \) above. (Theorem BCS 264)

\[
\left\langle \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \right\} \right\rangle
\]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \( L \) is computed as described in Definition EEF 287. This is followed by the column space described by a set of linearly independent vectors that span the null space of \( L \), computed as according to Theorem FS 290 and Theorem BNS 154. When \( r = m \), the matrix \( L \) has no rows and the column space is all of \( \mathbb{C}^m \).

\[
L = \begin{bmatrix} 1 & \frac{1}{7} & -\frac{11}{7} \end{bmatrix}
\]

\[
\left\langle \left\{ \begin{bmatrix} \frac{11}{7} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{7} \\ 1 \\ 0 \end{bmatrix} \right\} \right\rangle
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST 273 and Theorem BRS 271, and in the style of Example CSROI 273, this yields a linearly independent set of vectors that span the column space.
Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS [271])

\[
\left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 7/11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1/11 \end{bmatrix} \right\} \right\rangle
\]

Subspace dimensions associated with the matrix. (Definition NOM [385], Definition ROM [385]) Verify Theorem RPNC [387]

Matrix columns: 4 Rank: 2 Nullity: 2
Archetype F

Summary  System with four equations, four variables. Nonsingular coefficient matrix. Integer eigenvalues, one has “high” multiplicity.

A system of linear equations (Definition SLE 13):

\[
\begin{align*}
33x_1 - 16x_2 + 10x_3 - 2x_4 &= -27 \\
90x_1 - 47x_2 + 27x_3 - 7x_4 &= -77 \\
78x_1 - 36x_2 + 17x_3 - 6x_4 &= -52 \\
-9x_1 + 2x_2 + 3x_3 + 4x_4 &= 5
\end{align*}
\]

Some solutions to the system of linear equations (not necessarily exhaustive):

\[x_1 = 1, \quad x_2 = 2, \quad x_3 = -2, \quad x_4 = 4\]

Augmented matrix of the linear system of equations (Definition AM 32):

\[
\begin{bmatrix}
33 & -16 & 10 & -2 & -27 \\
99 & -47 & 27 & -7 & -77 \\
78 & -36 & 17 & -6 & -52 \\
-9 & 2 & 3 & 4 & 5 \\
\end{bmatrix}
\]

Matrix in reduced row-echelon form, row-equivalent to augmented matrix:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & -2 \\
0 & 0 & 0 & 1 & 4 \\
\end{bmatrix}
\]

Analysis of the augmented matrix (Notation RREFA 36):

\[r = 4 \quad D = \{1, 2, 3, 4\} \quad F = \{5\}\]

Vector form of the solution set to the system of equations (Theorem VFSLS 107). Notice the relationship between the free variables and the set \(F\) above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set.
Given a system of equations we can always build a new, related, homogeneous system (Definition HS) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system.

\[
\begin{pmatrix}
33x_1 - 16x_2 + 10x_3 - 2x_4 = 0 \\
99x_1 - 47x_2 + 27x_3 - 7x_4 = 0 \\
78x_1 - 36x_2 + 17x_3 - 6x_4 = 0 \\
-9x_1 + 2x_2 + 3x_3 + 4x_4 = 0
\end{pmatrix}
\]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):
\[x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0\]

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

Analysis of the augmented matrix for the homogenous system (Notation RREFA). Notice the slight variation for the same analysis of the original system only when the original system was consistent:
\[r = 4 \quad D = \{1, 2, 3, 4\} \quad F = \{5\}\]

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations.
\[
\begin{pmatrix}
33 & -16 & 10 & -2 \\
99 & -47 & 27 & -7 \\
78 & -36 & 17 & -6 \\
-9 & 2 & 3 & 4
\end{pmatrix}
\]
Matrix brought to reduced row-echelon form:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Analysis of the row-reduced matrix (Notation RREFa [36]):

\[ r = 4 \quad D = \{1, 2, 3, 4\} \quad F = \{\}\]

Matrix (coefficient matrix) is nonsingular or singular? (Theorem NMRRI [77]) at the same time, examine the size of the set \( F \) above. Notice that this property does not apply to matrices that are not square.

Nonsingular.

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS [129], Theorem BNS [154]). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VF-SLS [107]) to see these vectors arise.

\[ \langle \{ \} \rangle \]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \( D \) above. (Theorem BCS [264])

\[ \left\langle \begin{bmatrix} 33 \\ 99 \\ 78 \\ -9 \end{bmatrix}, \begin{bmatrix} -16 \\ -47 \\ -36 \\ 2 \end{bmatrix}, \begin{bmatrix} 10 \\ 27 \\ 17 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -7 \\ -6 \\ 4 \end{bmatrix} \right\rangle \]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \( L \) is computed as described in Definition EEF [287]. This is followed by the column space described by a set of linearly independent vectors that span the null space of \( L \), computed as according to Theorem FS [290] and Theorem BNS [154]. When \( r = m \), the matrix \( L \) has no rows and the column space is all of \( \mathbb{C}^m \).

\[ L = \]
Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST [273] and Theorem BRS [271], and in the style of Example CSROI [273], this yields a linearly independent set of vectors that span the column space.

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS [271])

Inverse matrix, if it exists. The inverse is not defined for matrices that are not square, and if the matrix is square, then the matrix must be nonsingular. (Definition MI [232], Theorem NI [251])

Subspace dimensions associated with the matrix. (Definition NOM [385], Definition ROM [385]) Verify Theorem RPNC [387]

Matrix columns: 4  Rank: 4  Nullity: 0

Determinant of the matrix, which is only defined for square matrices. The matrix is nonsingular if and only if the determinant is nonzero (Theorem SMZD [432]). (Product of all eigenvalues?)
Determinant = −18

- Eigenvalues, and bases for eigenspaces. (Definition EEM 441, Definition EM 450)

\[ \lambda = -1 \quad \mathcal{E}_F(-1) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \]

\[ \lambda = 2 \quad \mathcal{E}_F(2) = \left\{ \begin{bmatrix} 2 \\ 5 \\ 2 \\ 1 \end{bmatrix} \right\} \]

\[ \lambda = 3 \quad \mathcal{E}_F(3) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 17 \\ 45 \\ 21 \\ 0 \end{bmatrix} \right\} \]

- Geometric and algebraic multiplicities. (Definition GME 452, Definition AME 452)

\[ \gamma_F(-1) = 1 \quad \alpha_F(-1) = 1 \]
\[ \gamma_F(2) = 1 \quad \alpha_F(2) = 1 \]
\[ \gamma_F(3) = 2 \quad \alpha_F(3) = 2 \]

- Diagonalizable? (Definition DZM 487)

Yes, full eigenspaces, Theorem DMFE 490.

- The diagonalization. (Theorem DC 487)

\[
\begin{bmatrix}
12 & -5 & 1 & -1 \\
-39 & 18 & -7 & 3 \\
27 & -13 & 6 & -\frac{1}{7} \\
\frac{3}{7} & -\frac{12}{7} & \frac{6}{7} & -\frac{3}{7}
\end{bmatrix}
=\begin{bmatrix}
12 & -5 & 1 & -1 \\
33 & -16 & 10 & -2 \\
99 & -47 & 27 & -7 \\
78 & -36 & 17 & -6
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 1 & 17 \\
2 & 5 & 1 & 45 \\
0 & 2 & 0 & 21 \\
1 & 1 & 7 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix}
\]
Archetype G

Summary System with five equations, two variables. Consistent. Null space of coefficient matrix has dimension 0. Coefficient matrix identical to that of Archetype H, constant vector is different.

A system of linear equations (Definition SLE [13]):

\[
\begin{align*}
2x_1 + 3x_2 &= 6 \\
-x_1 + 4x_2 &= -14 \\
3x_1 + 10x_2 &= -2 \\
3x_1 - x_2 &= 20 \\
6x_1 + 9x_2 &= 18
\end{align*}
\]

Some solutions to the system of linear equations (not necessarily exhaustive):

\[x_1 = 6, \quad x_2 = -2\]

Augmented matrix of the linear system of equations (Definition AM [32]):

\[
\begin{bmatrix}
2 & 3 & 6 \\
-1 & 4 & -14 \\
3 & 10 & -2 \\
3 & -1 & 20 \\
6 & 9 & 18
\end{bmatrix}
\]

Matrix in reduced row-echelon form, row-equivalent to augmented matrix:

\[
\begin{bmatrix}
1 & 0 & 6 \\
0 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix (Notation RREFA [36]):

\[r = 2 \quad D = \{1, 2\} \quad F = \{3\}\]
Vector form of the solution set to the system of equations (Theorem VFSLS [107]). Notice the relationship between the free variables and the set $F$ above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set $F$ for the larger examples.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

Given a system of equations we can always build a new, related, homogeneous system (Definition HS [65]) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system.

$$\begin{align*}
2x_1 + 3x_2 &= 0 \\
-x_1 + 4x_2 &= 0 \\
3x_1 + 10x_2 &= 0 \\
3x_1 - x_2 &= 0 \\
6x_1 + 9x_2 &= 0
\end{align*}$$

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):

$$x_1 = 0, \quad x_2 = 0$$

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

$$\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}$$

Analysis of the augmented matrix for the homogenous system (Notation RREFA [36]). Notice the slight variation for the same analysis of the original system only when the original system was consistent:

$$r = 2 \quad D = \{1, 2\} \quad F = \{3\}$$

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of
equations.
\[
\begin{bmatrix}
2 & 3 \\
-1 & 4 \\
3 & 10 \\
3 & -1 \\
6 & 9
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:
\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

Analysis of the row-reduced matrix (Notation RREFA [36]):
\[ r = 2 \quad D = \{1, 2\} \quad F = \{\}\]

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS [129], Theorem BNS [154]). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VF-SLS [107]) to see these vectors arise.
\[ \langle\{\}\rangle \]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \(D\) above. (Theorem BCS [264])
\[ \left\langle \begin{bmatrix} 2 \\ -1 \\ 3 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 10 \\ -1 \\ 9 \end{bmatrix} \right\rangle \]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \(L\) is computed as described in Definition EEF [287]. This is followed by the column space described by a set of linearly independent vectors that span the null space of \(L\), computed as according to Theorem FS [290] and Theorem BNS [154]. When \(r = m\), the matrix \(L\) has no rows and the column space is all of \(\mathbb{C}^m\).
\[
L = \begin{bmatrix}
1 & 0 & 0 & 0 & -\frac{1}{3}
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 1 & -\frac{1}{3}
0 & 0 & 1 & 1 & -1
\end{bmatrix}
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST \[273\] and Theorem BRS \[271\], and in the style of Example CSROI \[273\], this yields a linearly independent set of vectors that span the column space.

\[
\left\langle \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\rangle
\]

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS \[271\])

\[
\left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\rangle
\]

Subspace dimensions associated with the matrix. (Definition NOM \[385\], Definition ROM \[385\]) Verify Theorem RPNC \[387\]

- Matrix columns: 2
- Rank: 2
- Nullity: 0
Archetype H

Summary  System with five equations, two variables. Inconsistent, overdetermined. Null space of coefficient matrix has dimension 0. Coefficient matrix identical to that of Archetype G, constant vector is different.

A system of linear equations (Definition SLE [13]):

\[
\begin{align*}
2x_1 + 3x_2 &= 5 \\
-x_1 + 4x_2 &= 6 \\
3x_1 + 10x_2 &= 2 \\
3x_1 - x_2 &= -1 \\
6x_1 + 9x_2 &= 3
\end{align*}
\]

Some solutions to the system of linear equations (not necessarily exhaustive):

None. (Why?)

Augmented matrix of the linear system of equations (Definition AM [32]):

\[
\begin{bmatrix}
2 & 3 & 5 \\
-1 & 4 & 6 \\
3 & 10 & 2 \\
3 & -1 & -1 \\
6 & 9 & 3
\end{bmatrix}
\]

Matrix in reduced row-echelon form, row-equivalent to augmented matrix:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix (Notation RREFA [36]):

\[
r = 3 \quad D = \{1, 2, 3\} \quad F = \{\}
\]
Vector form of the solution set to the system of equations (Theorem VFSLS [107]). Notice the relationship between the free variables and the set $F$ above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set $F$ for the larger examples.

Inconsistent system, no solutions exist.

Given a system of equations we can always build a new, related, homogeneous system (Definition HS [65]) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system.

\[
\begin{align*}
2x_1 + 3x_2 &= 0 \\
-x_1 + 4x_2 &= 0 \\
3x_1 + 10x_2 &= 0 \\
3x_1 - x_2 &= 0 \\
6x_1 + 9x_2 &= 0
\end{align*}
\]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):

\[
x_1 = 0, \quad x_2 = 0
\]

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system (Notation RREFA [36]). Notice the slight variation for the same analysis of the original system only when the original system was consistent:

\[
r = 2 \quad D = \{1, 2\} \quad F = \{3\}
\]

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of
Archetype H

equations.

\[
\begin{bmatrix}
2 & 3 \\
-1 & 4 \\
3 & 10 \\
3 & -1 \\
6 & 9 \\
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
\]

Analysis of the row-reduced matrix (Notation RREFA [36]):

\[r = 2\quad D = \{1, 2\} \quad F = \{\}\]

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS [129], Theorem BNS [154]). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VF-SLS [107]) to see these vectors arise.

\([\{\}\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \(D\) above. (Theorem BCS [264])

\[
\left\langle \begin{bmatrix}
2 \\
-1 \\
3 \\
3 \\
6 \\
\end{bmatrix} ,
\begin{bmatrix}
3 \\
4 \\
10 \\
-1 \\
9 \\
\end{bmatrix}
\right\rangle
\]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \(L\) is computed as described in Definition EEF [287]. This is followed by the column space described by a set of linearly independent vectors that span the null space of \(L\), computed as according to Theorem FS [290] and Theorem BNS [154]. When \(r = m\), the matrix \(L\) has no rows and the column space is all of \(\mathbb{C}^m\).
\[ L = \begin{bmatrix}
\frac{1}{3} & 1 \\
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & -1 \\
-1 & 1 \\
1 & 0
\end{bmatrix}\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST \[273\] and Theorem BRS \[271\], and in the style of Example CSROI \[273\], this yields a linearly independent set of vectors that span the column space.

\[ \begin{bmatrix}
1 \\
0 \\
2 \\
1
\end{bmatrix}, \begin{bmatrix}
0 \\
1 \\
1 \\
-1
\end{bmatrix}\]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix \( L \) is computed as described in Definition EEF \[287\]. This is followed by the column space described by a set of linearly independent vectors that span the null space of \( L \), computed as according to Theorem FS \[290\] and Theorem BNS \[154\]. When \( r = m \), the matrix \( L \) has no rows and the column space is all of \( \mathbb{C}^m \).

\[ L = \begin{bmatrix}
1 & 0 & 0 & 0 & -\frac{1}{3} \\
0 & 1 & 0 & 1 & -\frac{1}{3} \\
0 & 0 & 1 & 1 & -1
\end{bmatrix}\]

\[ \begin{bmatrix}
\frac{1}{3} \\
1 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
0 \\
-1 \\
-1 \\
1
\end{bmatrix}\]

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS \[271\])

\[ \begin{bmatrix}
1 \\
0
\end{bmatrix}, \begin{bmatrix}
0 & 1
\end{bmatrix}\]

Subspace dimensions associated with the matrix. (Definition NOM \[385\], Defini-
Archetype H

Matrix columns: 2  Rank: 2  Nullity: 0
Summary  System with four equations, seven variables. Consistent. Null space of coefficient matrix has dimension 4.

A system of linear equations (Definition SLE [13]):

\[
\begin{align*}
    x_1 + 4x_2 - x_4 + 7x_6 - 9x_7 &= 3 \\
    2x_1 + 8x_2 - x_3 + 3x_4 + 9x_5 - 13x_6 + 7x_7 &= 9 \\
    2x_3 - 3x_4 - 4x_5 + 12x_6 - 8x_7 &= 1 \\
    -x_1 - 4x_2 + 2x_3 + 4x_4 + 8x_5 - 31x_6 + 37x_7 &= 4
\end{align*}
\]

Some solutions to the system of linear equations (not necessarily exhaustive):

\[
\begin{align*}
    x_1 &= -25, \ x_2 = 4, \ x_3 = 22, \ x_4 = 29, \ x_5 = 1, \ x_6 = 2, \ x_7 = -3 \\
    x_1 &= -7, \ x_2 = 5, \ x_3 = 7, \ x_4 = 15, \ x_5 = -4, \ x_6 = 2, \ x_7 = 1 \\
    x_1 &= 4, \ x_2 = 0, \ x_3 = 2, \ x_4 = 1, \ x_5 = 0, \ x_6 = 0, \ x_7 = 0
\end{align*}
\]

Augmented matrix of the linear system of equations (Definition AM [32]):

\[
\begin{bmatrix}
1 & 4 & 0 & 0 & 7 & -9 & 3 \\
2 & 8 & -1 & 3 & 9 & -13 & 7 & 9 \\
0 & 0 & 2 & -3 & -4 & 12 & -8 & 1 \\
-1 & -4 & 2 & 4 & 8 & -31 & 37 & 4
\end{bmatrix}
\]

Matrix in reduced row-echelon form, row-equivalent to augmented matrix:

\[
\begin{bmatrix}
1 & 4 & 0 & 0 & 2 & 1 & -3 & 4 \\
0 & 0 & 1 & 0 & 1 & -3 & 5 & 2 \\
0 & 0 & 0 & 1 & 2 & -6 & 6 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix (Notation RREFA [36]):

\[
r = 3 \quad D = \{1, 3, 4\} \quad F = \{2, 5, 6, 7, 8\}
\]

Vector form of the solution set to the system of equations (Theorem VFSLS [107]). Notice the relationship between the free variables and the set \(F\) above. Also, notice the
Archetype I

pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set $F$ for the larger examples.

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
\end{bmatrix} = \begin{bmatrix}
4 \\
0 \\
2 \\
1 \\
0 \\
0 \\
0 \\
\end{bmatrix}
+ \begin{bmatrix}
-4 \\
1 \\
0 \\
0 \\
-1 \\
1 \\
0 \\
\end{bmatrix} + \begin{bmatrix}
-2 \\
0 \\
-1 \\
-2 \\
6 \\
0 \\
\end{bmatrix} + \begin{bmatrix}
-1 \\
0 \\
3 \\
1 \\
-5 \\
1 \\
\end{bmatrix} + \begin{bmatrix}
3 \\
0 \\
0 \\
0 \\
0 \\
1 \\
\end{bmatrix}
\]

Given a system of equations we can always build a new, related, homogeneous system (Definition HS [65]) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system.

\[
\begin{align*}
x_1 + 4x_2 - x_4 + 7x_6 - 9x_7 &= 0 \\
2x_1 + 8x_2 - x_3 + 3x_4 + 9x_5 - 13x_6 + 7x_7 &= 0 \\
2x_3 - 3x_4 - 4x_5 + 12x_6 - 8x_7 &= 0 \\
-x_1 - 4x_2 + 2x_3 + 4x_4 + 8x_5 - 31x_6 + 37x_7 &= 0
\end{align*}
\]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):

\begin{align*}
x_1 &= 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0 \\
x_1 &= 3, x_2 = 0, x_3 = -5, x_4 = -6, x_5 = 0, x_6 = 0, x_7 = 1 \\
x_1 &= -1, x_2 = 0, x_3 = 3, x_4 = 6, x_5 = 0, x_6 = 1, x_7 = 0 \\
x_1 &= -2, x_2 = 0, x_3 = -1, x_4 = -2, x_5 = 1, x_6 = 0, x_7 = 0 \\
x_1 &= -4, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 0, x_6 = 0, x_7 = 0 \\
x_1 &= -4, x_2 = 1, x_3 = -3, x_4 = -2, x_5 = 1, x_6 = 1, x_7 = 1
\end{align*}

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain zeros:

\[
\begin{bmatrix}
1 & 4 & 0 & 0 & 2 & 1 & -3 & 0 \\
0 & 0 & 1 & 0 & 1 & -3 & 5 & 0 \\
0 & 0 & 0 & 1 & 2 & -6 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system (Notation RREFA [36]).
Notice the slight variation for the same analysis of the original system only when the original system was consistent:

\[ r = 3 \quad D = \{1, 3, 4\} \quad F = \{2, 5, 6, 7, 8\} \]

- Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations.

\[
\begin{bmatrix}
1 & 4 & 0 & -1 & 0 & 7 & -9 \\
2 & 8 & -1 & 3 & 9 & -13 & 7 \\
0 & 0 & 2 & -3 & -4 & 12 & -8 \\
-1 & -4 & 2 & 4 & 8 & -31 & 37 \\
\end{bmatrix}
\]

- Matrix brought to reduced row-echelon form:

\[
\begin{bmatrix}
1 & 4 & 0 & 0 & 2 & 1 & -3 \\
0 & 0 & 1 & 0 & -1 & 3 & 5 \\
0 & 0 & 0 & 1 & 2 & -6 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

- Analysis of the row-reduced matrix (Notation RREFA 36):

\[ r = 3 \quad D = \{1, 3, 4\} \quad F = \{2, 5, 6, 7\} \]

- This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS 129, Theorem BNS 154). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VF-SLS 107) to see these vectors arise.

\[
\langle \begin{bmatrix}
-4 \\
1 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}, \begin{bmatrix}
-2 \\
0 \\
-1 \\
0 \\
0 \\
0 \\
\end{bmatrix}, \begin{bmatrix}
-1 \\
0 \\
3 \\
1 \\
0 \\
0 \\
\end{bmatrix}, \begin{bmatrix}
3 \\
0 \\
-5 \\
0 \\
0 \\
1 \\
\end{bmatrix} \rangle
\]

- Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set \( D \) above. (Theorem BCS 264)
The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix $L$ is computed as described in Definition EEF [287]. This is followed by the column space described by a set of linearly independent vectors that span the null space of $L$, computed as according to Theorem FS [290] and Theorem BNS [154]. When $r = m$, the matrix $L$ has no rows and the column space is all of $\mathbb{C}^m$.

$$L = \begin{bmatrix} 1 & -\frac{12}{31} & -\frac{13}{31} & \frac{7}{31} \end{bmatrix}$$

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST [273] and Theorem BRS [271], and in the style of Example CSROI [273], this yields a linearly independent set of vectors that span the column space.

$$\begin{pmatrix} -\frac{7}{31} & \frac{13}{31} & \frac{12}{31} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{1}{7} & 0 & 0 \end{pmatrix}$$

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS [271])

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Subspace dimensions associated with the matrix. (Definition NOM [385], Definition ROM [385]) Verify Theorem RPNC [387]

Matrix columns: 7  Rank: 3  Nullity: 4
Archetype J

Summary  System with six equations, nine variables. Consistent. Null space of coefficient matrix has dimension 5.

A system of linear equations (Definition SLE [13]):

\[
\begin{align*}
    x_1 + 2x_2 - 2x_3 + 9x_4 + 3x_5 - 5x_6 - 2x_7 + x_8 + 27x_9 &= -5 \\
    2x_1 + 4x_2 + 3x_3 + 4x_4 - x_5 + 4x_6 + 10x_7 + 2x_8 - 23x_9 &= 18 \\
    x_1 + 2x_2 + x_3 + 3x_4 + x_5 + 6x_6 + 5x_7 + 2x_8 - 7x_9 &= 6 \\
    2x_1 + 4x_2 + 3x_3 + 4x_4 - 7x_5 + 2x_6 + 4x_7 - 11x_9 &= 20 \\
    x_1 + 2x_2 + 5x_4 + 2x_5 - 4x_6 + 3x_7 + 8x_8 + 13x_9 &= -4 \\
    -3x_1 - 6x_2 - x_3 - 13x_4 + 2x_5 - 5x_6 - 4x_7 + 13x_8 + 10x_9 &= -29
\end{align*}
\]

Some solutions to the system of linear equations (not necessarily exhaustive):

\[
\begin{align*}
    x_1 &= 6, \quad x_2 = 0, \quad x_3 = -1, \quad x_4 = 0, \quad x_5 = -1, \quad x_6 = 2, \quad x_7 = 0, \quad x_8 = 0, \quad x_9 = 0 \\
    x_1 &= 4, \quad x_2 = 1, \quad x_3 = -1, \quad x_4 = 0, \quad x_5 = -1, \quad x_6 = 2, \quad x_7 = 0, \quad x_8 = 0, \quad x_9 = 0 \\
    x_1 &= -17, \quad x_2 = 7, \quad x_3 = 3, \quad x_4 = 2, \quad x_5 = -1, \quad x_6 = 14, \quad x_7 = -1, \quad x_8 = 3, \quad x_9 = 2 \\
    x_1 &= -11, \quad x_2 = -6, \quad x_3 = 1, \quad x_4 = 5, \quad x_5 = -4, \quad x_6 = 7, \quad x_7 = 3, \quad x_8 = 1, \quad x_9 = 1
\end{align*}
\]

Augmented matrix of the linear system of equations (Definition AM [32]):

\[
\begin{bmatrix}
    1 & 2 & -2 & 9 & 3 & -5 & -2 & 1 & 27 & -5 \\
    2 & 4 & 3 & 4 & -1 & 4 & 10 & 2 & -23 & 18 \\
    1 & 2 & 1 & 3 & 1 & 5 & 2 & 7 & 6 \\
    2 & 4 & 3 & 4 & -7 & 2 & 4 & 0 & -11 & 20 \\
    1 & 2 & 0 & 5 & 2 & -4 & 3 & 8 & 13 & -4 \\
    -3 & -6 & -1 & -13 & 2 & -5 & -4 & 13 & 10 & -29
\end{bmatrix}
\]

Matrix in reduced row-echelon form, row-equivalent to augmented matrix:
Analysis of the augmented matrix (Notation RREFA [36]):

\[ r = 4 \quad D = \{1, 3, 5, 6\} \quad F = \{2, 4, 7, 8, 9, 10\} \]

Vector form of the solution set to the system of equations (Theorem VFSLS [107]). Notice the relationship between the free variables and the set \( F \) above. Also, notice the pattern of 0’s and 1’s in the entries of the vectors corresponding to elements of the set \( F \) for the larger examples.

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
  x_6 \\
  x_7 \\
  x_8 \\
  x_9
\end{bmatrix} =
\begin{bmatrix}
  6 \\
  0 \\
 -1 \\
  0 \\
 -1 \\
  2 \\
  0 \\
  0 \\
  0
\end{bmatrix} +
\begin{bmatrix}
  -2 \\
  1 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix} x_2 +
\begin{bmatrix}
  -5 \\
  0 \\
  2 \\
  1 \\
  -1 \\
  0 \\
  0 \\
  0 \\
  0
\end{bmatrix} x_4 +
\begin{bmatrix}
  -1 \\
  0 \\
 -3 \\
  0 \\
 -1 \\
  0 \\
  0 \\
  1 \\
  0
\end{bmatrix} x_7 +
\begin{bmatrix}
  2 \\
  0 \\
 -5 \\
  0 \\
 -1 \\
  2 \\
  0 \\
  1 \\
  0
\end{bmatrix} x_8 +
\begin{bmatrix}
  -3 \\
  0 \\
  6 \\
  0 \\
  1 \\
  3 \\
  0 \\
  0 \\
  1
\end{bmatrix} x_9
\]

Given a system of equations we can always build a new, related, homogeneous system (Definition HS [65]) by converting the constant terms to zeros and retaining the coefficients of the variables. Properties of this new system will have precise relationships with various properties of the original system.

\[
x_1 + 2x_2 - 2x_3 + 9x_4 + 3x_5 - 5x_6 - 2x_7 + x_8 + 27x_9 = 0
\]
\[
2x_1 + 4x_2 + 3x_3 + 4x_4 - x_5 + 4x_6 + 10x_7 + 2x_8 - 23x_9 = 0
\]
\[
x_1 + 2x_2 + 3x_3 + 4x_4 + x_5 + x_6 + 5x_7 + 2x_8 - 7x_9 = 0
\]
\[
2x_1 + 4x_2 + 3x_3 + 4x_4 - 7x_5 + 2x_6 + 4x_7 - 11x_9 = 0
\]
\[
x_1 + 2x_2 + 5x_4 + 2x_5 - 4x_6 + 3x_7 + 8x_8 + 13x_9 = 0
\]
\[
-3x_1 - 6x_2 - x_3 - 13x_4 + 2x_5 - 5x_6 - 4x_7 + 13x_8 + 10x_9 = 0
\]

Some solutions to the associated homogenous system of linear equations (not necessarily exhaustive):

\[
x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0, \quad x_5 = 0, \quad x_6 = 0, \quad x_7 = 0, \quad x_8 = 0, \quad x_9 = 0
\]
\[
x_1 = -2, \quad x_2 = 1, \quad x_3 = 0, \quad x_4 = 0, \quad x_5 = 0, \quad x_6 = 0, \quad x_7 = 0, \quad x_8 = 0, \quad x_9 = 0
\]
\[
x_1 = -23, \quad x_2 = 7, \quad x_3 = 4, \quad x_4 = 2, \quad x_5 = 0, \quad x_6 = 12, \quad x_7 = -1, \quad x_8 = 3, \quad x_9 = 2
\]
\[
x_1 = -17, \quad x_2 = -6, \quad x_3 = 2, \quad x_4 = 5, \quad x_5 = -3, \quad x_6 = 5, \quad x_7 = 3, \quad x_8 = 1, \quad x_9 = 1
\]

Form the augmented matrix of the homogenous linear system, and use row operations to convert to reduced row-echelon form. Notice how the entries of the final column remain
zeros:

\[
\begin{bmatrix}
1 & 2 & 0 & 5 & 0 & 0 & 1 & -2 & 3 & 0 \\
0 & 0 & 1 & -2 & 0 & 0 & 3 & 5 & -6 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis of the augmented matrix for the homogenous system (Notation RREFA [36]). Notice the slight variation for the same analysis of the original system only when the original system was consistent:

\[
r = 4 \quad D = \{1, 3, 5, 6\} \quad F = \{2, 4, 7, 8, 9, 10\}
\]

Coefficient matrix of original system of equations, and of associated homogenous system. This matrix will be the subject of further analysis, rather than the systems of equations.

\[
\begin{bmatrix}
1 & 2 & -2 & 9 & 3 & -5 & -2 & 1 & 27 \\
2 & 4 & 3 & 4 & -1 & 4 & 10 & 2 & -23 \\
1 & 2 & 1 & 3 & 1 & 1 & 5 & 2 & -7 \\
2 & 4 & 3 & 4 & -7 & 2 & 4 & 0 & -11 \\
1 & 2 & 0 & 5 & 2 & -4 & 3 & 8 & 13 \\
-3 & -6 & -1 & -13 & 2 & -5 & -4 & 13 & 10
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:

\[
\begin{bmatrix}
1 & 2 & 0 & 5 & 0 & 0 & 1 & -2 & 3 \\
0 & 0 & 1 & -2 & 0 & 0 & 3 & 5 & -6 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Analysis of the row-reduced matrix (Notation RREFA [36]):

\[
r = 4 \quad D = \{1, 3, 5, 6\} \quad F = \{2, 4, 7, 8, 9\}
\]

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix.
(Theorem SSNS [129], Theorem BNS [154]). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VF-SLS [107]) to see these vectors arise.

\[
\langle \begin{bmatrix}
-2 & -5 & -1 & 2 & -3 \\
1 & 0 & 0 & 0 & 0 \\
0 & 2 & -3 & -5 & 6 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 1 \\
0 & 0 & 0 & 2 & 3 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix},
\begin{bmatrix}
-2 & -5 & -1 & 2 & -3 \\
1 & 0 & 0 & 0 & 0 \\
0 & 2 & -3 & -5 & 6 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 1 \\
0 & 0 & 0 & 2 & 3 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}\rangle
\]

\[\text{Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set } D \text{ above. (Theorem BCS [264])}\]

\[
\langle \begin{bmatrix}
1 & 2 & -2 & 3 & 5 \\
2 & 3 & 1 & 1 & -7 \\
1 & 0 & 2 & -4 & -5 \\
-3 & -1 & 2 & -5 & -1 \\
\end{bmatrix}\rangle
\]

\[\text{The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix } L \text{ is computed as described in Definition EEF [287]. This is followed by the column space described by a set of linearly independent vectors that span the null space of } L, \text{ computed as according to Theorem FS [290] and Theorem BNS [154]. When } r = m, \text{ the matrix } L \text{ has no rows and the column space is all of } \mathbb{C}^m.\]

\[L = \begin{bmatrix}
1 & 0 & \frac{186}{131} & \frac{51}{131} & \frac{-188}{131} & \frac{77}{131} \\
0 & 1 & \frac{-272}{131} & \frac{-45}{131} & \frac{58}{131} & \frac{-14}{131} \\
\end{bmatrix}\]

\[
\langle \begin{bmatrix}
-77 \\
14 \\
131 \\
131 \\
131 \\
0 \\
0 \\
0 \\
1 \\
\end{bmatrix},
\begin{bmatrix}
188 \\
-272 \\
131 \\
131 \\
131 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix},
\begin{bmatrix}
51 \\
45 \\
131 \\
131 \\
131 \\
0 \\
1 \\
0 \\
0 \\
\end{bmatrix},
\begin{bmatrix}
-186 \\
-14 \\
131 \\
131 \\
131 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}\rangle
\]

\[\text{Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST [273] and Theorem BRS [271], and in the style of Example CSROI [273], this yields a linearly independent set of vectors that span}\]
the column space.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-1 & -\frac{11}{2} & 10 & \frac{3}{2} \\
-\frac{29}{7} & -\frac{91}{7} & 22 & 3
\end{pmatrix}
\]

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. 
(Theorem BRS [271])

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
5 & -2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 3 & 1 & 0 \\
-2 & 5 & 1 & -2 \\
3 & -6 & -1 & -3
\end{pmatrix}
\]

Subspace dimensions associated with the matrix. 
(Definition NOM [385], Definition ROM [385]) Verify Theorem RPNC [387]

Matrix columns: 9  Rank: 4  Nullity: 5
**Archetype K**

**Summary**  Square matrix of size 5. Nonsingular. 3 distinct eigenvalues, 2 of multiplicity 2.

A matrix:
\[
\begin{bmatrix}
10 & 18 & 24 & 24 & -12 \\
12 & -2 & -6 & 0 & -18 \\
-30 & -21 & -23 & -30 & 39 \\
27 & 30 & 36 & 37 & -30 \\
18 & 24 & 30 & 30 & -20 \\
\end{bmatrix}
\]

Matrix brought to reduced row-echelon form:
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Analysis of the row-reduced matrix (Notation RREFA 36):
\[
r = 5 \quad D = \{1, 2, 3, 4, 5\} \quad F = \{\}
\]

Matrix (coefficient matrix) is nonsingular or singular? (Theorem NMRRI 77) at the same time, examine the size of the set \( F \) above. Notice that this property does not apply to matrices that are not square.

Nonsingular.

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS 129, Theorem BNS 154). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VF-SLS 107) to see these vectors arise.

\( \langle \{ \} \rangle \)

Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the
set $D$ above. (Theorem BCS 264)

\[
\left\{ \begin{bmatrix} 10 \\ 12 \\ -30 \\ 27 \\ 18 \end{bmatrix}, \begin{bmatrix} 18 \\ -2 \\ -21 \\ 30 \\ 24 \end{bmatrix}, \begin{bmatrix} 24 \\ -3 \\ -23 \\ 36 \\ 30 \end{bmatrix}, \begin{bmatrix} 24 \\ 0 \\ -30 \\ 37 \\ 30 \end{bmatrix}, \begin{bmatrix} -12 \\ -18 \\ -30 \\ -20 \end{bmatrix} \right\}
\]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix $L$ is computed as described in Definition EEF 287. This is followed by the column space described by a set of linearly independent vectors that span the null space of $L$, computed as according to Theorem FS 290 and Theorem BNS 154. When $r = m$, the matrix $L$ has no rows and the column space is all of $\mathbb{C}^m$.

\[
L = \left[ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right]
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST 273 and Theorem BRS 271, and in the style of Example CSROI 273, this yields a linearly independent set of vectors that span the column space.

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form. (Theorem BRS 271)

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

Inverse matrix, if it exists. The inverse is not defined for matrices that are not square,
and if the matrix is square, then the matrix must be nonsingular. (Definition MI [232], Theorem NI [251])

$$\begin{bmatrix}
\frac{1}{2} & -\left(\frac{3}{4}\right) & -\left(\frac{3}{2}\right) & 3 & -6 \\
\frac{21}{2} & \frac{43}{4} & \frac{21}{2} & 9 & -9 \\
-15 & -\left(\frac{31}{2}\right) & -11 & -15 & \frac{39}{2} \\
9 & \frac{15}{4} & \frac{9}{2} & 10 & -15 \\
\frac{9}{2} & \frac{15}{4} & \frac{9}{2} & 6 & -\left(\frac{19}{2}\right)
\end{bmatrix}$$

Subspace dimensions associated with the matrix. (Definition NOM [385], Definition ROM [385]) Verify Theorem RPNC [387]

Matrix columns: 5  
Rank: 5  
Nullity: 0

Determinant of the matrix, which is only defined for square matrices. The matrix is nonsingular if and only if the determinant is nonzero (Theorem SMZD [432]). (Product of all eigenvalues?)

Determinant = 16

Eigenvalues, and bases for eigenspaces. (Definition EEM [441], Definition EM [450])

$$\lambda = -2$$

$$\mathcal{E}_K(-2) = \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\lambda = 1$$

$$\mathcal{E}_K(1) = \left\{ \begin{bmatrix} 4 \\ -10 \\ 7 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 18 \\ -17 \\ 5 \\ 0 \end{bmatrix} \right\}$$

$$\lambda = 4$$

$$\mathcal{E}_K(4) = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Geometric and algebraic multiplicities. (Definition GME [452], Definition AME [452])

$$\gamma_K(-2) = 2$$

$$\alpha_K(-2) = 2$$
\[ \gamma_K(1) = 2 \quad \alpha_K(1) = 2 \]
\[ \gamma_K(4) = 1 \quad \alpha_K(4) = 1 \]

- Diagonalizable? (Definition DZM [487])
  Yes, full eigenspaces, Theorem DFLE [??].

- The diagonalization. (Theorem DC [487])

\[
\begin{bmatrix}
-4 & -3 & -4 & -6 & 7 \\
-7 & -5 & -6 & -8 & 10 \\
1 & -1 & -1 & 1 & -3 \\
1 & 0 & 0 & 1 & -2 \\
2 & 5 & 6 & 4 & 0
\end{bmatrix}
\begin{bmatrix}
10 & 18 & 24 & 24 & -12 \\
12 & -2 & -6 & 0 & -18 \\
-30 & -21 & -23 & -30 & 39 \\
27 & 30 & 36 & 37 & -30 \\
18 & 24 & 30 & 30 & -20
\end{bmatrix}
\begin{bmatrix}
2 & -1 & 4 & -4 & 1 \\
-2 & 2 & -10 & 18 & -1 \\
1 & -2 & 7 & -17 & 0 \\
0 & 1 & 0 & 5 & 1 \\
1 & 0 & 2 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
-2 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 4
\end{bmatrix}
\]
**Archetype L**

**Summary**  Square matrix of size 5. Singular, nullity 2. 2 distinct eigenvalues, each of "high" multiplicity.

1. A matrix:
   
   \[
   \begin{bmatrix}
   -2 & -1 & -2 & -4 & 4 \\
   -6 & -5 & -4 & -4 & 6 \\
   10 & 7 & 7 & 10 & -13 \\
   -7 & -5 & -6 & -9 & 10 \\
   -4 & -3 & -4 & -6 & 6
   \end{bmatrix}
   \]

2. Matrix brought to reduced row-echelon form:
   
   \[
   \begin{bmatrix}
   1 & 0 & 0 & 1 & -2 \\
   0 & 1 & 0 & -2 & 2 \\
   0 & 0 & 1 & 2 & -1 \\
   0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0
   \end{bmatrix}
   \]

3. Analysis of the row-reduced matrix (Notation RREFA 36):
   
   \[ r = 5 \quad D = \{1, 2, 3\} \quad F = \{4, 5\} \]

4. Matrix (coefficient matrix) is nonsingular or singular? (Theorem NMRRI 77) at the same time, examine the size of the set \( F \) above. Notice that this property does not apply to matrices that are not square.

   Singular.

This is the null space of the matrix. The set of vectors used in the span construction is a linearly independent set of column vectors that spans the null space of the matrix (Theorem SSNS 129, Theorem BNS 154). Solve the homogenous system with this matrix as the coefficient matrix and write the solutions in vector form (Theorem VF-SLS 107) to see these vectors arise.
Column space of the matrix, expressed as the span of a set of linearly independent vectors that are also columns of the matrix. These columns have indices that form the set $D$ above. (Theorem BCS [264])

\[
\begin{pmatrix}
-2 \\
-6 \\
-7 \\
-4
\end{pmatrix},
\begin{pmatrix}
-1 \\
-5 \\
7 \\
-3
\end{pmatrix},
\begin{pmatrix}
2 \\
-4 \\
7 \\
-4
\end{pmatrix}
\]

The column space of the matrix, as it arises from the extended echelon form of the matrix. The matrix $L$ is computed as described in Definition EEF [287]. This is followed by the column space described by a set of linearly independent vectors that span the null space of $L$, computed as according to Theorem FS [290] and Theorem BNS [154]. When $r = m$, the matrix $L$ has no rows and the column space is all of $\mathbb{C}^m$.

\[
L = \begin{bmatrix}
1 & 0 & -2 & -6 & 5 \\
0 & 1 & 4 & 10 & -9
\end{bmatrix}
\]

\[
\begin{pmatrix}
-5 \\
9 \\
0 \\
1
\end{pmatrix},
\begin{pmatrix}
6 \\
-10 \\
0 \\
1
\end{pmatrix},
\begin{pmatrix}
2 \\
-4 \\
1 \\
0
\end{pmatrix}
\]

Column space of the matrix, expressed as the span of a set of linearly independent vectors. These vectors are computed by row-reducing the transpose of the matrix into reduced row-echelon form, tossing out the zero rows, and writing the remaining nonzero rows as column vectors. By Theorem CSRST [273] and Theorem BRS [271], and in the style of Example CSROI [273], this yields a linearly independent set of vectors that span the column space.

\[
\begin{pmatrix}
1 \\
0 \\
9
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
5
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

Row space of the matrix, expressed as a span of a set of linearly independent vectors, obtained from the nonzero rows of the equivalent matrix in reduced row-echelon form.
Inverse matrix, if it exists. The inverse is not defined for matrices that are not square, and if the matrix is square, then the matrix must be nonsingular. (Definition MI [232], Theorem NI [251])

Subspace dimensions associated with the matrix. (Definition NOM [385], Definition ROM [385]) Verify Theorem RPNC [387]

Matrix columns: 5 Rank: 3 Nullity: 2

Determinant of the matrix, which is only defined for square matrices. The matrix is nonsingular if and only if the determinant is nonzero (Theorem SMZD [432]). (Product of all eigenvalues?)

Determinant = 0

Eigenvalues, and bases for eigenspaces. (Definition EEM [441], Definition EM [450])

\[
\lambda = -1 \quad \mathcal{E}_L (-1) = \left\{ \begin{bmatrix} -5 \\ 9 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -10 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}
\]

\[
\lambda = 0 \quad \mathcal{E}_L (0) = \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}
\]

Geometric and algebraic multiplicities. (Definition GME [452], Definition AME [452])

\[
\gamma_L (-1) = 3 \quad \alpha_L (-1) = 3
\]
\[ \gamma_L(0) = 2 \quad \alpha_L(0) = 2 \]

- Diagonalizable? [Definition DZM 487]
  Yes, full eigenspaces, [Theorem DMLE ??].

- The diagonalization. [Theorem DC 487]

\[
\begin{bmatrix}
4 & 3 & 4 & 6 & -6 \\
7 & 5 & 6 & 9 & -10 \\
-10 & -7 & -7 & -10 & 13 \\
-4 & -3 & -4 & -6 & 7 \\
-7 & -5 & -6 & -8 & 10
\end{bmatrix}
\begin{bmatrix}
-2 & -1 & -2 & -4 & 4 \\
-6 & -5 & -4 & -4 & 6 \\
10 & 7 & 7 & 10 & -13 \\
-7 & -5 & -6 & -9 & 10 \\
-4 & -3 & -4 & -6 & 6
\end{bmatrix}
\begin{bmatrix}
-5 & 6 & 2 & 2 & -1 \\
9 & -10 & -4 & -2 & 2 \\
0 & 0 & 1 & 1 & -2 \\
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
Archetype M

Summary  Linear transformation with bigger domain than codomain, so it is guaranteed to not be injective. Happens to not be surjective.

A linear transformation: (Definition LT [503])

\[ T: \mathbb{C}^5 \rightarrow \mathbb{C}^3, \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 + 4x_5 \\ 3x_1 + x_2 + 4x_3 - 3x_4 + 7x_5 \\ x_1 - x_2 - 5x_4 + x_5 \end{pmatrix} \]

A basis for the null space of the linear transformation: (Definition KLT [532])

\[ \left\{ \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} \]

Injective: No. (Definition ILT [529])

Since the kernel is nontrivial Theorem KLT [535] tells us that the linear transformation is not injective. Also, since the rank can not exceed 3, we are guaranteed to have a nullity of at least 2, just from checking dimensions of the domain and the codomain. In particular, verify that

\[ T \begin{pmatrix} 1 \\ 2 \\ -1 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 38 \\ 24 \\ -16 \end{pmatrix} \quad T \begin{pmatrix} 0 \\ -3 \\ 0 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 38 \\ 24 \\ -16 \end{pmatrix} \]

This demonstration that \( T \) is not injective is constructed with the observation that

\[ \begin{bmatrix} 0 \\ -3 \\ 0 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} -1 \\ -5 \\ 1 \\ 1 \\ 1 \end{bmatrix} \]
and

\[
z = \begin{bmatrix} -1 \\ -5 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{K}(T)
\]

so the vector \( z \) effectively “does nothing” in the evaluation of \( T \).

- A basis for the range of the linear transformation: [Definition RLT 551]

Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT 556):

\[
\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ -5 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix} \right\}
\]

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTLI 537). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS 271), and perhaps un-coordinatizing. A basis for the range is:

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 5 \end{bmatrix} \right\}
\]

- Surjective: No. [Definition SLT 547]

Notice that the range is not all of \( \mathbb{C}^3 \) since its dimension 2, not 3. In particular, verify that \( \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \notin \mathcal{R}(T) \), by setting the output equal to this vector and seeing that the resulting system of linear equations has no solution, i.e. is inconsistent. So the preimage, \( T^{-1} \left( \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right) \), is empty. This alone is sufficient to see that the linear transformation is not onto.

- Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify [Theorem RPNDD 576].

  Domain dimension: 5  \quad \text{Rank: 2}  \quad \text{Nullity: 3}

- Invertible: No.
Not injective or surjective.

- Matrix representation (Theorem MLTCV 510):

\[ T : \mathbb{C}^5 \rightarrow \mathbb{C}^3, \quad T(x) = Ax, \quad A = \begin{bmatrix} 1 & 2 & 3 & 4 & 4 \\ 3 & 1 & 4 & -3 & 7 \\ 1 & -1 & 0 & -5 & 1 \end{bmatrix} \]
Archetype N

Summary  Linear transformation with domain larger than its codomain, so it is guaranteed to not be injective. Happens to be onto.

- A linear transformation: \[ \text{Definition LT} \ (503) \]
  \[
  T : \mathbb{C}^5 \rightarrow \mathbb{C}^3, \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 + 3x_3 - 4x_4 + 5x_5 \\ x_1 - 2x_2 + 3x_3 - 9x_4 + 3x_5 \\ 3x_1 + 4x_3 - 6x_4 + 5x_5 \end{pmatrix}
  \]

- A basis for the null space of the linear transformation: \[ \text{Definition KLT} \ (532) \]
  \[
  \left\{ \begin{pmatrix} 1 \\ -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}
  \]

- Injective: No. \[ \text{Definition ILT} \ (529) \]
  Since the kernel is nontrivial \[ \text{Theorem KILT} \ (535) \] tells us that the linear transformation is not injective. Also, since the rank can not exceed 3, we are guaranteed to have a nullity of at least 2, just from checking dimensions of the domain and the codomain. In particular, verify that

  \[
  T \begin{pmatrix} -3 \\ 1 \\ -2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 19 \\ 6 \end{pmatrix} \quad T \begin{pmatrix} -4 \\ -4 \\ -2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 19 \\ 6 \end{pmatrix}
  \]

  This demonstration that \( T \) is not injective is constructed with the observation that

  \[
  \begin{pmatrix} -4 \\ -4 \\ -2 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -2 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -5 \\ 0 \\ 2 \\ 3 \end{pmatrix}
  \]
and

\[
\mathbf{z} = \begin{bmatrix}
-1 \\
-5 \\
0 \\
2 \\
3
\end{bmatrix} \in \mathcal{K}(T)
\]

so the vector \( \mathbf{z} \) effectively “does nothing” in the evaluation of \( T \).

- A basis for the range of the linear transformation: (Definition RLT [551])

Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT [556]):

\[
\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ -6 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}
\]

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTLI [537]). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS [271]), and perhaps un-coordinatizing. A basis for the range is:

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}
\]

- Surjective: Yes. (Definition SLT [547])

Notice that the basis for the range above is the standard basis for \( \mathbb{C}^3 \). So the range is all of \( \mathbb{C}^3 \) and thus the linear transformation is surjective.

- Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD [576].

  Domain dimension: 5  
  Rank: 3  
  Nullity: 2

- Invertible: No.

Not surjective, and the relative sizes of the domain and codomain mean the linear transformation cannot be injective. (Theorem ILTIS [571])

- Matrix representation (Theorem MLTCV [510]):
\[ T: \mathbb{C}^5 \rightarrow \mathbb{C}^3, \quad T(x) = Ax, \quad A = \begin{bmatrix} 2 & 1 & 3 & -4 & 5 \\ 1 & -2 & 3 & -9 & 3 \\ 3 & 0 & 4 & -6 & 5 \end{bmatrix} \]
Archetype O

Summary  Linear transformation with a domain smaller than the codomain, so it is guaranteed to not be onto. Happens to not be one-to-one.

A linear transformation: (Definition LT 503)

\[ T: \mathbb{C}^3 \mapsto \mathbb{C}^5, \quad T \left( \begin{array} {c} x_1 \\ x_2 \\ x_3 \end{array} \right) = \begin{bmatrix} -x_1 + x_2 - 3x_3 \\ -x_1 + 2x_2 - 4x_3 \\ x_1 + x_2 + x_3 \\ 2x_1 + 3x_2 + x_3 \\ x_1 + 2x_3 \end{bmatrix} \]

A basis for the null space of the linear transformation: (Definition KLT 532)

\[ \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\} \]

Injective: No. (Definition ILT 529)

Since the kernel is nontrivial Theorem KILT 535 tells us that the linear transformation is not injective. Also, since the rank can not exceed 3, we are guaranteed to have a nullity of at least 2, just from checking dimensions of the domain and the codomain. In particular, verify that

\[ T \left( \begin{array} {c} 5 \\ -1 \\ 3 \end{array} \right) = \begin{bmatrix} -15 \\ -19 \\ 7 \\ 10 \\ 11 \end{bmatrix} \quad T \left( \begin{array} {c} 1 \\ 1 \\ 5 \end{array} \right) = \begin{bmatrix} -15 \\ -19 \\ 7 \\ 10 \\ 11 \end{bmatrix} \]

This demonstration that \( T \) is not injective is constructed with the observation that

\[ \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix} \]

and

\[ z = \begin{bmatrix} -4 \\ 2 \\ 2 \end{bmatrix} \in \mathcal{K}(T) \]
so the vector $z$ effectively “does nothing” in the evaluation of $T$.

- A basis for the range of the linear transformation: (Definition RLT [551])

Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT [556]):

$$
\begin{bmatrix}
-1 & 1 & -3 \\
-1 & 2 & -4 \\
1 & 1 & 1 \\
2 & 3 & 1 \\
1 & 0 & 2
\end{bmatrix}
$$

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTLI [537]). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS [271]), and perhaps un-coordinatizing. A basis for the range is:

$$
\begin{bmatrix}
1 & 0 \\
-3 & 2 \\
-7 & 5 \\
-2 & 1
\end{bmatrix}
$$

- Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD [576].

  Domain dimension: 3  
  Rank: 2  
  Nullity: 1

- Surjective: No. (Definition SLT [547])

The dimension of the range is 2, and the codomain ($\mathbb{C}^5$) has dimension 5. So the transformation is not onto. Notice too that since the domain $\mathbb{C}^3$ has dimension 3, it is impossible for the range to have a dimension greater than 3, and no matter what the actual definition of the function, it cannot possibly be onto.

To be more precise, verify that $\begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \not\in \mathcal{R}(T)$, by setting the output equal to this vector and seeing that the resulting system of linear equations has no solution, i.e. is inconsistent. So the preimage, $T^{-1} \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, is empty. This alone is sufficient to see that
the linear transformation is not onto.

- Invertible: No.

Not injective, and the relative dimensions of the domain and codomain prohibit any possibility of being surjective.

- Matrix representation (Theorem MLTCV 510):

\[ T : \mathbb{C}^3 \rightarrow \mathbb{C}^5, \quad T(x) = Ax, \quad A = \begin{bmatrix} -1 & 1 & -3 \\ -1 & 2 & -4 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix} \]


Summary  Linear transformation with a domain smaller that its codomain, so it is guaranteed to not be surjective. Happens to be injective.

A linear transformation: (Definition LT [503])

\[ T: \mathbb{C}^3 \rightarrow \mathbb{C}^5, \quad T \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right) = \begin{bmatrix} -x_1 + x_2 + x_3 \\ -x_1 + 2x_2 + 2x_3 \\ x_1 + x_2 + 3x_3 \\ 2x_1 + 3x_2 + x_3 \\ -2x_1 + x_2 + 3x_3 \end{bmatrix} \]

A basis for the null space of the linear transformation: (Definition KLT [532])

\{ \}  

Injective: Yes. (Definition ILT [529])

Since \( \mathcal{K}(T) = \{0\} \), Theorem KILT [535] tells us that \( T \) is injective.

A basis for the range of the linear transformation: (Definition RLT [551])

Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT [556]):

\[ \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\} \]

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTLI [537]). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS [271]), and perhaps
un-coordinatizing. A basis for the range is:

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
-10 & 0 \\
6 & 7
\end{bmatrix}
, 
\begin{bmatrix}
0 & 0 \\
0 & 1 \\
-3 & -1
\end{bmatrix}
, 
\begin{bmatrix}
0 & 1 \\
1 & 1 \\
1 & 1
\end{bmatrix}
\]

Surjective: No. (Definition SLT 547)

The dimension of the range is 3, and the codomain (\(\mathbb{C}^5\)) has dimension 5. So the transformation is not surjective. Notice too that since the domain \(\mathbb{C}^3\) has dimension 3, it is impossible for the range to have a dimension greater than 3, and no matter what the actual definition of the function, it cannot possibly be surjective in this situation.

To be more precise, verify that

\[
\begin{bmatrix}
2 \\
1 \\
-3 \\
2 \\
6
\end{bmatrix} \not\in \mathcal{R}(T),
\]

by setting the output equal to this vector and seeing that the resulting system of linear equations has no solution, i.e. is inconsistent. So the preimage, \(T^{-1}\left(\begin{bmatrix}
2 \\
1 \\
-3 \\
2 \\
6
\end{bmatrix}\right)\), is empty. This alone is sufficient to see that the linear transformation is not onto.

Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD 576.

Domain dimension: 3
Rank: 3
Nullity: 0

Invertible: No.

The relative dimensions of the domain and codomain prohibit any possibility of being surjective, so apply Theorem ILTIS 571.

Matrix representation (Theorem MLTCV 510):

\[T: \mathbb{C}^3 \mapsto \mathbb{C}^5, \; T(x) = Ax, \; A = \begin{bmatrix}
-1 & 1 & 1 \\
-1 & 2 & 2 \\
1 & 1 & 3 \\
2 & 3 & 1 \\
-2 & 1 & 3
\end{bmatrix}\]
Archetype Q

Summary Linear transformation with equal-sized domain and codomain, so it has the potential to be invertible, but in this case is not. Neither injective nor surjective. Diagonalizable, though.

A linear transformation: (Definition LT [503])

\[
T: \mathbb{C}^5 \mapsto \mathbb{C}^5, \quad T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2x_1 + 3x_2 + 3x_3 - 6x_4 + 3x_5 \\ -16x_1 + 9x_2 + 12x_3 - 28x_4 + 28x_5 \\ -19x_1 + 7x_2 + 14x_3 - 32x_4 + 37x_5 \\ -21x_1 + 9x_2 + 15x_3 - 35x_4 + 39x_5 \\ -9x_1 + 5x_2 + 7x_3 - 16x_4 + 16x_5 \end{pmatrix}
\]

A basis for the null space of the linear transformation: (Definition KLT [532])

\[
\begin{Bmatrix} 3 \\ 4 \\ 1 \\ 3 \\ 3 \end{Bmatrix}
\]

Injective: No. (Definition ILT [529])

Since the kernel is nontrivial Theorem KILT [535] tells us that the linear transformation is not injective. Also, since the rank can not exceed 3, we are guaranteed to have a nullity of at least 2, just from checking dimensions of the domain and the codomain. In particular, verify that

\[
T \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 55 \\ 72 \\ 77 \\ 31 \end{pmatrix} = T \begin{pmatrix} 4 \\ 7 \\ 0 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ 55 \\ 72 \\ 77 \\ 31 \end{pmatrix}
\]

This demonstration that \( T \) is not injective is constructed with the observation that

\[
\begin{pmatrix} 4 \\ 7 \\ 0 \\ 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 1 \\ 3 \\ 3 \end{pmatrix}
\]
and

\[
\mathbf{z} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 3 \\ 3 \end{bmatrix} \in \mathcal{K}(T)
\]

so the vector \( \mathbf{z} \) effectively “does nothing” in the evaluation of \( T \).

\[\square\] A basis for the range of the linear transformation: (Definition RLT [551])

Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT [556]):

\[
\begin{align*}
\begin{bmatrix} -2 \\ -16 \\ -19 \\ -21 \\ -9 \end{bmatrix},
\begin{bmatrix} 3 \\ 9 \\ 7 \\ 9 \\ 5 \end{bmatrix},
\begin{bmatrix} 3 \\ 12 \\ 14 \\ 9 \\ 7 \end{bmatrix},
\begin{bmatrix} -6 \\ -28 \\ -32 \\ -35 \\ -16 \end{bmatrix},
\begin{bmatrix} 3 \\ 28 \\ 37 \\ 39 \\ 16 \end{bmatrix}
\end{align*}
\]

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILT LI [537]). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS [271]), and perhaps un-coordinatizing. A basis for the range is:

\[
\begin{align*}
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},
\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix},
\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix},
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}
\end{align*}
\]

\[\square\] Surjective: No. (Definition SLT [547])

The dimension of the range is 4, and the codomain (\( \mathbb{C}^5 \)) has dimension 5. So \( \mathcal{R}(T) \neq \mathbb{C}^5 \) and by Theorem RSLT [554] the transformation is not surjective.

To be more precise, verify that \( \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \end{bmatrix} \notin \mathcal{R}(T) \), by setting the output equal to this vector and seeing that the resulting system of linear equations has no solution, i.e. is inconsistent. So the preimage, \( T^{-1} \begin{bmatrix} -1 \\ 2 \\ 3 \\ -1 \\ 4 \end{bmatrix} \), is empty. This alone is sufficient to see
that the linear transformation is not onto.

Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD [576].

Domain dimension: 5  
Rank: 4  
Nullity: 1

Invertible: No.

Neither injective nor surjective. Notice that since the domain and codomain have the same dimension, either the transformation is both onto and one-to-one (making it invertible) or else it is both not onto and not one-to-one (as in this case) by Theorem RP-NDD [576].

Matrix representation (Theorem MLTCV [510]):

\[ T : \mathbb{C}^5 \rightarrow \mathbb{C}^5, \quad T(\mathbf{x}) = A\mathbf{x}, \quad A = \begin{bmatrix} -2 & 3 & 3 & -6 & 3 \\ -16 & 9 & 12 & -28 & 28 \\ -19 & 7 & 14 & -32 & 37 \\ -21 & 9 & 15 & -35 & 39 \\ -9 & 5 & 7 & -16 & 16 \end{bmatrix} \]

Eigenvalues and eigenvectors (Definition EELT [637], Theorem EER [650]):

\[ \lambda = -1 \quad E_T(-1) = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 3 \\ 1 \end{bmatrix} \]

\[ \lambda = 0 \quad E_T(0) = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 3 \\ 3 \end{bmatrix} \]

\[ \lambda = 1 \quad E_T(1) = \begin{bmatrix} 5 \\ 3 \\ 0 \\ 2 \\ 2 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \]
Evaluate the linear transformation with each of these eigenvectors as an interesting check.

- A diagonal matrix representation relative to a basis of eigenvectors, $B$.

$$B = \begin{bmatrix}
0 & 3 & 5 & -3 & 1 \\
2 & 4 & 3 & 1 & -1 \\
3 & 1 & 0 & 0 & 2 \\
3 & 3 & 0 & 2 & 0 \\
1 & 3 & 2 & 0 & 0
\end{bmatrix}$$

$$M_{B,B}^T = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}$$
Archetype R

Summary  Linear transformation with equal-sized domain and codomain. Injective, surjective, invertible, diagonalizable, the works.

- A linear transformation: (Definition LT 503)

\[
T: \mathbb{C}^5 \mapsto \mathbb{C}^5, \quad T \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix} = \begin{pmatrix}
-65x_1 + 128x_2 + 10x_3 - 262x_4 + 40x_5 \\
36x_1 - 73x_2 - x_3 + 151x_4 - 16x_5 \\
-44x_1 + 88x_2 + 5x_3 - 180x_4 + 24x_5 \\
34x_1 - 68x_2 - 3x_3 + 140x_4 - 18x_5 \\
12x_1 - 24x_2 - x_3 + 49x_4 - 5x_5
\end{pmatrix}
\]

- A basis for the null space of the linear transformation: (Definition KLT 532)

\[
\{ \} \]

- Injective: Yes. (Definition ILT 529)

Since the kernel is trivial Theorem KLT 535 tells us that the linear transformation is injective.

- A basis for the range of the linear transformation: (Definition RLT 551)

Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT 556):

\[
\left\{ \begin{pmatrix}
-65 \\
36 \\
-44 \\
34 \\
12
\end{pmatrix}, \begin{pmatrix}
128 \\
-73 \\
88 \\
-68 \\
-24
\end{pmatrix}, \begin{pmatrix}
10 \\
-1 \\
5 \\
-3 \\
-1
\end{pmatrix}, \begin{pmatrix}
-262 \\
151 \\
-180 \\
140 \\
49
\end{pmatrix}, \begin{pmatrix}
40 \\
-16 \\
24 \\
-18 \\
-5
\end{pmatrix}\right\}
\]

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTLI 537). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS 271), and perhaps
Archetype R 791

un-coordinatizing. A basis for the range is:

\[
\begin{align*}
\begin{bmatrix}
1 & \ 0 \\
0 & \ 1 \\
0 & \ 0 \\
0 & \ 0 \\
0 & \ 0 \\
\end{bmatrix},
\begin{bmatrix}
0 & \ 1 \\
0 & \ 0 \\
0 & \ 0 \\
0 & \ 0 \\
0 & \ 0 \\
\end{bmatrix},
\begin{bmatrix}
0 & \ 0 \\
1 & \ 0 \\
0 & \ 0 \\
0 & \ 0 \\
0 & \ 0 \\
\end{bmatrix},
\begin{bmatrix}
0 & \ 0 \\
0 & \ 1 \\
0 & \ 0 \\
0 & \ 0 \\
0 & \ 0 \\
\end{bmatrix},
\begin{bmatrix}
0 & \ 0 \\
0 & \ 0 \\
0 & \ 1 \\
0 & \ 0 \\
0 & \ 0 \\
\end{bmatrix}
\end{align*}
\]

\[\square\text{ Surjective: Yes. (Definition SLT 547)}\]

A basis for the range is the standard basis of \(\mathbb{C}^5\), so \(R(T) = \mathbb{C}^5\) and Theorem RSLT 554 tells us \(T\) is surjective. Or, the dimension of the range is 5, and the codomain (\(\mathbb{C}^5\)) has dimension 5. So the transformation is surjective.

\[\square\text{ Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD 576.} \]

\[
\begin{align*}
\text{Domain dimension: } & 5 & \text{Rank: } & 5 & \text{Nullity: } & 0
\end{align*}
\]

\[\square\text{ Invertible: Yes.} \]

Both injective and surjective (Theorem ILTIS 571). Notice that since the domain and codomain have the same dimension, either the transformation is both injective and surjective (making it invertible, as in this case) or else it is both not injective and not surjective.

\[\square\text{ Matrix representation (Theorem MLTCV 510):} \]

\[
T: \mathbb{C}^5 \to \mathbb{C}^5, \quad T(\mathbf{x}) = A\mathbf{x}, \quad A = \\
\begin{bmatrix}
-65 & 128 & 10 & -262 & 40 \\
36 & -73 & -1 & 151 & -16 \\
-44 & 88 & 5 & -180 & 24 \\
34 & -68 & -3 & 140 & -18 \\
12 & -24 & -1 & 49 & -5
\end{bmatrix}
\]

\[\square\text{ The inverse linear transformation (Definition IVLT 567):} \]

\[
T^{-1}: \mathbb{C}^5 \to \mathbb{C}^5, \quad T^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \\
\begin{bmatrix}
-47x_1 + 92x_2 + x_3 - 181x_4 - 14x_5 \\
27x_1 - 55x_2 + \frac{7}{2}x_3 + \frac{221}{4}x_4 + 11x_5 \\
-32x_1 + 64x_2 - x_3 - 126x_4 - 12x_5 \\
25x_1 - 50x_2 + \frac{3}{2}x_3 + \frac{189}{4}x_4 + 9x_5 \\
9x_1 - 18x_2 + \frac{1}{2}x_3 + \frac{77}{2}x_4 + 4x_5
\end{bmatrix}
\]

Verify that \(T(T^{-1}(\mathbf{x})) = \mathbf{x}\) and \(T(T^{-1}(\mathbf{x})) = \mathbf{x}\), and notice that the representations
of the transformation and its inverse are matrix inverses (Theorem IMR 619, Definition MI 232).

Eigenvalues and eigenvectors (Definition EELT 637, Theorem EER 650):

\[ \lambda = -1 \quad \mathcal{E}_T(-1) = \begin{pmatrix} -57 \\ 0 \\ -18 \\ 14 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]

\[ \lambda = 1 \quad \mathcal{E}_T(1) = \begin{pmatrix} -10 \\ -5 \\ -6 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \\ 1 \end{pmatrix} \]

\[ \lambda = 2 \quad \mathcal{E}_T(2) = \begin{pmatrix} -6 \\ 3 \\ -4 \\ 3 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \end{pmatrix} \]

Evaluate the linear transformation with each of these eigenvectors as an interesting check.

A diagonal matrix representation relative to a basis of eigenvectors, \( B \).

\[ B = \begin{pmatrix} -57 & 2 & -10 & 2 & -6 \\ 0 & 1 & -5 & 3 & 3 \\ -18 & 0 & -6 & 1 & -4 \\ 14 & 0 & 0 & 1 & \end{pmatrix} \]

\[ M_{B,B}^T = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \]
Archetype S

Summary Domain is column vectors, codomain is matrices. Domain is dimension 3 and codomain is dimension 4. Not injective, not surjective.

A linear transformation: (Definition LT 503)

\[ T: \mathbb{C}^3 \mapsto M_{22}, \quad T \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a-b & 2a+2b+c \\ 3a+b+c & -2a-6b-2c \end{bmatrix} \]

A basis for the null space of the linear transformation: (Definition KLT 532)

\[ \left\{ \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix} \right\} \]

Injective: No. (Definition ILT 529)

Since the kernel is nontrivial, Theorem KILT 535 tells us that the linear transformation is not injective. Also, since the rank can not exceed 3, we are guaranteed to have a nullity of at least 1, just from checking dimensions of the domain and the codomain. In particular, verify that

\[ T \left( \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 9 \\ 10 & -16 \end{bmatrix} \quad T \left( \begin{bmatrix} 0 \\ -1 \\ 11 \end{bmatrix} \right) = \begin{bmatrix} 1 & 9 \\ 10 & -16 \end{bmatrix} \]

This demonstration that \( T \) is not injective is constructed with the observation that

\[ \begin{bmatrix} 0 \\ -1 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \\ 8 \end{bmatrix} \]

and

\[ \mathbf{z} = \begin{bmatrix} -2 \\ -2 \\ 8 \end{bmatrix} \in \mathcal{K}(T) \]

so the vector \( \mathbf{z} \) effectively “does nothing” in the evaluation of \( T \).

A basis for the range of the linear transformation: (Definition RLT 551)
Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT 556):

\[
\left\{ \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 1 & -6 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \right\}
\]

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTLI 537). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS 271), and perhaps un-coordinatizing. A basis for the range is:

\[
\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \right\}
\]

\[\square\] Surjective: No. (Definition SLT 547)

The dimension of the range is 2, and the codomain (\(M_{22}\)) has dimension 4. So the transformation is not surjective. Notice too that since the domain \(C^3\) has dimension 3, it is impossible for the range to have a dimension greater than 3, and no matter what the actual definition of the function, it cannot possibly be surjective in this situation.

To be more precise, verify that \(\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \notin \mathcal{R}(T)\), by setting the output of \(T\) equal to this matrix and seeing that the resulting system of linear equations has no solution, i.e. is inconsistent. So the preimage, \(T^{-1}\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right)\), is empty. This alone is sufficient to see that the linear transformation is not onto.

\[\square\] Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD 576.

Domain dimension: 3  
Rank: 2  
Nullity: 1

\[\square\] Invertible: No.

Not injective (Theorem ILTIS 571), and the relative dimensions of the domain and codomain prohibit any possibility of being surjective.

\[\square\] Matrix representation (Definition MR 603):

\[
B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

\[
C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}
\]

Version 0.92
$$M^T_{B,C} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 1 \\ 3 & 1 & 1 \\ -2 & -6 & -2 \end{bmatrix}$$
Archetype T

**Summary**  Domain and codomain are polynomials. Domain has dimension 5, while codomain has dimension 6. Is injective, can’t be surjective.

- A linear transformation: (Definition LT [503])
  \[ T: P_4 \mapsto P_5, \quad T(p(x)) = (x - 2)p(x) \]

- A basis for the null space of the linear transformation: (Definition KLT [532])
  \{ \}

- Injective: Yes. (Definition ILT [529])
  Since the kernel is trivial Theorem KILT [535] tells us that the linear transformation is injective.

- A basis for the range of the linear transformation: (Definition RLT [551])
  Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT [556]):
  \[ \{ x - 2, x^2 - 2x, x^3 - 2x^2, x^4 - 2x^3, x^5 - 2x^4, x^6 - 2x^5 \} \]
  If the linear transformation is injective, then the set above is guaranteed to be linearly independent, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS [271]), and perhaps un-coordinatizing. A basis for the range is:
  \[ \left\{ -\frac{1}{32}x^5 + 1, -\frac{1}{16}x^5 + x, -\frac{1}{8}x^5 + x^2, -\frac{1}{4}x^5 + x^3, -\frac{1}{2}x^5 + x^4 \right\} \]

- Surjective: No. (Definition SLT [547])
  The dimension of the range is 5, and the codomain \( P_5 \) has dimension 6. So the transformation is not surjective. Notice too that since the domain \( P_4 \) has dimension 5, it is impossible for the range to have a dimension greater than 5, and no matter what the actual definition of the function, it cannot possibly be surjective in this situation.
To be more precise, verify that \(1 + x + x^2 + x^3 + x^4 \notin \mathcal{R}(T)\), by setting the output equal to this vector and seeing that the resulting system of linear equations has no solution, i.e. is inconsistent. So the preimage, \(T^{-1}(1 + x + x^2 + x^3 + x^4)\), is nonempty. This alone is sufficient to see that the linear transformation is not onto.

Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD 576.

- Domain dimension: 5
- Rank: 5
- Nullity: 0

Invertible: No.

The relative dimensions of the domain and codomain prohibit any possibility of being surjective, so apply Theorem ILTIS 571.

Matrix representation (Definition MR 603):

\[
B = \{1, x, x^2, x^3, x^4\}
\]
\[
C = \{1, x, x^2, x^3, x^4, x^5\}
\]
\[
M_{B,C}^T = \begin{bmatrix}
-2 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 \\
0 & 1 & -2 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 \\
0 & 0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Archetype U

Summary  Domain is matrices, codomain is column vectors. Domain has dimension 6, while codomain has dimension 4. Can’t be injective, is surjective.

A linear transformation: \( T: M_{23} \rightarrow \mathbb{C}^4, \ T \left( \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \right) = \begin{bmatrix} a + 2b + 12c - 3d + e + 6f \\ 2a - b - c + d - 11f \\ a + b + 7c + 2d + e - 3f \\ a + 2b + 12c + 5e - 5f \end{bmatrix} \)

A basis for the null space of the linear transformation: \( \text{Definition KLT [532]} \)

\( \left\{ \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \)

Injective: No. \( \text{Definition ILT [529]} \)

Since the kernel is nontrivial \[ \text{Theorem KILT [535]} \] tells us that the linear transformation is not injective. Also, since the rank can not exceed 4, we are guaranteed to have a nullity of at least 2, just from checking dimensions of the domain and the codomain. In particular, verify that

\[ T \left( \begin{bmatrix} 1 & 10 & -2 \\ 3 & -1 & 1 \end{bmatrix} \right) = \begin{bmatrix} -7 \\ -14 \\ -1 \\ -13 \end{bmatrix} \quad T \left( \begin{bmatrix} 5 & -3 & -1 \\ 5 & 3 & 3 \end{bmatrix} \right) = \begin{bmatrix} -7 \\ -14 \\ -1 \\ -13 \end{bmatrix} \]

This demonstration that \( T \) is not injective is constructed with the observation that

\[ \begin{bmatrix} 5 & -3 & -1 \\ 5 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 10 & -2 \\ 3 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -13 & 1 \\ 2 & 4 & 2 \end{bmatrix} \]

and

\[ z = \begin{bmatrix} 4 \\ 2 \\ -13 \\ 1 \\ 2 \end{bmatrix} \in \mathcal{K}(T) \]

so the vector \( z \) effectively “does nothing” in the evaluation of \( T \).

A basis for the range of the linear transformation: \( \text{Definition RLT [551]} \)
Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT 556):

\[
\begin{bmatrix}
1 \\
2 \\
1 \\
1
\end{bmatrix}
, \begin{bmatrix}
2 \\
-1 \\
1 \\
2
\end{bmatrix}
, \begin{bmatrix}
12 \\
7 \\
12 \\
0
\end{bmatrix}
, \begin{bmatrix}
-3 \\
1 \\
2 \\
5
\end{bmatrix}
, \begin{bmatrix}
1 \\
0 \\
1 \\
5
\end{bmatrix}
, \begin{bmatrix}
6 \\
-11 \\
-3 \\
-5
\end{bmatrix}
\]

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTLI 537). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS 271), and perhaps un-coordinatizing. A basis for the range is:

\[
\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

\[
\text{Surjective: Yes. (Definition SLT 547)}
\]

A basis for the range is the standard basis of \( \mathbb{C}^4 \), so \( \mathcal{R}(T) = \mathbb{C}^4 \) and Theorem RSLT 554 tells us \( T \) is surjective. Or, the dimension of the range is 4, and the codomain (\( \mathbb{C}^4 \)) has dimension 4. So the transformation is surjective.

\[
\text{Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD 576.}
\]

\[
\text{Domain dimension: 6 Rank: 4 Nullity: 2}
\]

\[
\text{Invertible: No.}
\]

The relative sizes of the domain and codomain mean the linear transformation cannot be injective (Theorem ILTIS 571).

\[
\text{Matrix representation (Definition MR 603):}
\]

\[
B = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}
\]

\[
C = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}
\]
\[ M^T_{B,C} = \begin{bmatrix}
1 & 2 & 12 & -3 & 1 & 6 \\
2 & -1 & -1 & 1 & 0 & -11 \\
1 & 1 & 7 & 2 & 1 & -3 \\
1 & 2 & 12 & 0 & 5 & -5
\end{bmatrix} \]
Archetype V

Summary  Domain is polynomials, codomain is matrices. Domain and codomain both have dimension 4. Injective, surjective, invertible. Square matrix representation, but domain and codomain are unequal, so no eigenvalue information.

- A linear transformation: (Definition LT 503)

\[ T : P_3 \mapsto M_{22}, \quad T (a + bx + cx^2 + dx^3) = \begin{bmatrix} a + b & a - 2c \\ d & b - d \end{bmatrix} \]

- A basis for the null space of the linear transformation: (Definition KLT 532)

\{ \}

- Injective: Yes. (Definition ILT 529)

Since the kernel is trivial, Theorem KILT 535 tells us that the linear transformation is injective.

- A basis for the range of the linear transformation: (Definition RLT 551)

Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT 556):

\[ \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \]

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTLI 537). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS 271), and perhaps un-coordinatizing. A basis for the range is:

\[ \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \]

- Surjective: Yes. (Definition SLT 547)

A basis for the range is the standard basis of \( M_{22} \), so \( \mathcal{R}(T) = M_{22} \) and Theorem RSLT 554.
tells us $T$ is surjective. Or, the dimension of the range is 4, and the codomain ($M_{22}$) has dimension 4. So the transformation is surjective.

Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD [576].

Domain dimension: 4  Rank: 4  Nullity: 0

Invertible: Yes.

Both injective and surjective (Theorem ILTIS [571]). Notice that since the domain and codomain have the same dimension, either the transformation is both injective and surjective (making it invertible, as in this case) or else it is both not injective and not surjective.

Matrix representation (Definition MR [603]):

\[
B = \{1, x, x^2, x^3\}\\
C = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}\\
M_{B,C}^T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}
\]

Since invertible, the inverse linear transformation. (Definition IVLT [567])

\[
T^{-1}: M_{22} \mapsto P_3, \quad T^{-1}\left(\begin{bmatrix} a \\ c \\ d \end{bmatrix}\right) = (a - c - d) + (c + d)x + \frac{1}{2}(a - b - c - d)x^2 + cx^3
\]
Summary  Domain is polynomials, codomain is polynomials. Domain and codomain both have dimension 3. Injective, surjective, invertible, 3 distinct eigenvalues, diagonalizable.

- A linear transformation: (Definition LT 503)

\[ T: P_2 \rightarrow P_2, \quad T(a + bx + cx^2) = (19a + 6b - 4c) + (-24a - 7b + 4c) + (36a + 12b - 9c) \]

- A basis for the null space of the linear transformation: (Definition KLT 532)

\{ \}

- Injective: Yes. (Definition ILT 529)

Since the kernel is trivial Theorem KILT 533 tells us that the linear transformation is injective.

- A basis for the range of the linear transformation: (Definition RLT 551)

Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT 556):

\{ 19 - 24x + 36x^2, 6 - 7x + 12x^2, -4 + 4x - 9x^2 \}

If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTLI 537). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS 271), and perhaps un-coordinatizing. A basis for the range is:

\{ 1, x, x^2 \}

- Surjective: Yes. (Definition SLT 547)

A basis for the range is the standard basis of \( \mathbb{C}^5 \), so \( \mathcal{R}(T) = \mathbb{C}^5 \) and Theorem RSLT 554
tells us $T$ is surjective. Or, the dimension of the range is 5, and the codomain ($\mathbb{C}^5$) has dimension 5. So the transformation is surjective.

Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify [Theorem RPNDD 576].

Domain dimension: 3  
Rank: 3  
Nullity: 0  

Invertible: Yes.  
Both injective and surjective [Theorem ILTIS 571]. Notice that since the domain and codomain have the same dimension, either the transformation is both injective and surjective (making it invertible, as in this case) or else it is both not injective and not surjective.

Matrix representation (Definition MR 603):

$$B = \{1, x, x^2\}$$
$$C = \{1, x, x^2\}$$
$$M_{B,C}^T = \begin{bmatrix} 19 & 6 & -4 \\
-24 & -7 & 4 \\
36 & 12 & -9 \end{bmatrix}$$

Since invertible, the inverse linear transformation. [Definition IVLT 567]

$T^{-1} : P_2 \mapsto P_2$,  
$T^{-1}(a + bx + cx^2) = (-5a - 2b + \frac{4}{3}c) + (24a + 9b - \frac{20}{3}c)x + (12a + 4b - \frac{11}{3}c)x^2$

Eigenvalues and eigenvectors (Definition EELT 637, Theorem EER 650):

$$\lambda = -1 \quad \mathcal{E}_T(-1) = \langle \{2x + 3x^2\} \rangle$$
$$\lambda = 1 \quad \mathcal{E}_T(1) = \langle \{-1 + 3x\} \rangle$$
$$\lambda = 3 \quad \mathcal{E}_T(3) = \langle \{1 - 2x + x^2\} \rangle$$

Evaluate the linear transformation with each of these eigenvectors as an interesting check.

A diagonal matrix representation relative to a basis of eigenvectors, $B$.

$$B = \{2x + 3x^2, -1 + 3x, 1 - 2x + x^2\}$$
\[ M_{B,B}^T = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 3
\end{bmatrix} \]
Summary  Domain and codomain are square matrices. Domain and codomain both have dimension 4. Not injective, not surjective, not invertible, 3 distinct eigenvalues, diagonalizable.

- A linear transformation:  
  \[ T: M_{22} \rightarrow M_{22}, \quad T \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \begin{array}{cc} -2a + 15b + 3c + 27d & 10b + 6c + 18d \\ a - 5b - 9d & -a - 4b - 5c - 8d \end{array} \]

- A basis for the null space of the linear transformation:  
  \[ \left\{ \begin{array}{c} -6 \\ -3 \\ 2 \\ 1 \end{array} \right\} \]

- Injective: No.  
  Since the kernel is nontrivial Theorem KILT tells us that the linear transformation is not injective. In particular, verify that
  \[ T \left( \begin{array}{c} -2 \\ 1 \\ -4 \end{array} \right) = \begin{array}{cc} 115 & 78 \\ -38 & -35 \end{array} \]  
  \[ T \left( \begin{array}{c} 4 \\ -1 \\ 3 \end{array} \right) = \begin{array}{cc} 115 & 78 \\ -38 & -35 \end{array} \]

  This demonstration that \( T \) is not injective is constructed with the observation that
  \[ \begin{array}{cc} 4 & 3 \\ -1 & 3 \end{array} = \begin{array}{cc} -2 & 0 \\ 1 & -4 \end{array} + \begin{array}{cc} 6 & 3 \\ -2 & -1 \end{array} \]

  and
  \[ z = \begin{array}{cc} 6 & 3 \\ -2 & -1 \end{array} \in \mathcal{K}(T) \]

  so the vector \( z \) effectively “does nothing” in the evaluation of \( T \).

- A basis for the range of the linear transformation:  
  \[ \left\{ \begin{array}{c} -2 \\ 1 \\ -1 \end{array} , \begin{array}{c} 15 \\ -5 \\ -4 \end{array} , \begin{array}{c} 3 \\ 0 \\ -5 \end{array} , \begin{array}{c} 27 \\ -9 \\ -8 \end{array} \right\} \]

Evaluate the linear transformation on a standard basis to get a spanning set for the range (Theorem SSRLT):

\[ \left\{ \begin{array}{c} -2 \\ 1 \\ -1 \end{array}, \begin{array}{c} 15 \\ -5 \\ -4 \end{array}, \begin{array}{c} 3 \\ 0 \\ -5 \end{array}, \begin{array}{c} 27 \\ -9 \\ -8 \end{array} \right\} \]
If the linear transformation is injective, then the set above is guaranteed to be linearly independent (Theorem ILTLI \[537\]). This spanning set may be converted to a “nice” basis, by making the vectors the rows of a matrix (perhaps after using a vector representation), row-reducing, and retaining the nonzero rows (Theorem BRS \[271\]), and perhaps un-coordinatizing. A basis for the range is:

\[
\begin{bmatrix}
1 & 0 \\
\frac{-1}{2} & 0 \\
0 & 1 \\
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 1 \\
\frac{1}{4} & 0 \\
0 & 1 \\
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

Surjective: No. (Definition SLT \[547\])

The dimension of the range is 3, and the codomain \((M_{22})\) has dimension 5. So \(R(T) \neq M_{22}\) and by Theorem RSLT \[554\] the transformation is not surjective.

To be more precise, verify that \(\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \notin R(T)\), by setting the output of \(T\) equal to this matrix and seeing that the resulting system of linear equations has no solution, i.e. is inconsistent. So the preimage, \(T^{-1}\left(\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}\right)\), is empty. This alone is sufficient to see that the linear transformation is not onto.

Subspace dimensions associated with the linear transformation. Examine parallels with earlier results for matrices. Verify Theorem RPNDD \[576\].

- Domain dimension: 4
- Rank: 3
- Nullity: 1

Invertible: No.

Neither injective nor surjective (Theorem ILTIS \[571\]). Notice that since the domain and codomain have the same dimension, either the transformation is both injective and surjective or else it is both not injective and not surjective (making it not invertible, as in this case).

Matrix representation (Definition MR \[603\]):

\[
B = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1 \\
\end{bmatrix}, \quad
C = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
0 & 1 \\
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 \\
1 & 0 \\
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 \\
0 & 1 \\
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

\[
M_{B,C}^T = \begin{bmatrix}
-2 & 15 & 3 & 27 \\
0 & 10 & 6 & 18 \\
1 & -5 & 0 & -9 \\
-1 & -4 & -5 & -8 \\
\end{bmatrix}
\]

Eigenvalues and eigenvectors (Definition EELT \[637\], Theorem EER \[650\]):
\[ \lambda = 0 \quad \mathcal{E}_T(0) = \left\{ \begin{bmatrix} -6 & -3 \\ 2 & 1 \end{bmatrix} \right\} \]

\[ \lambda = 1 \quad \mathcal{E}_T(1) = \left\{ \begin{bmatrix} -7 & -2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} \right\} \]

\[ \lambda = 3 \quad \mathcal{E}_T(3) = \left\{ \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix} \right\} \]

Evaluate the linear transformation with each of these eigenvectors as an interesting check.

\[ A \text{ diagonal matrix representation relative to a basis of eigenvectors, } B. \]

\[ B = \left\{ \begin{bmatrix} -6 & -3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -7 & -2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix} \right\} \]

\[ M^T_{B,B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \]
Appendix GFDL
GNU Free Documentation License

Version 1.2, November 2002
59 Temple Place, Suite 330, Boston, MA 02111-1307 USA

Everyone is permitted to copy and distribute verbatim copies of this license document,
but changing it is not allowed.

Preamble

The purpose of this License is to make a manual, textbook, or other functional and
useful document “free” in the sense of freedom: to assure everyone the effective freedom
to copy and redistribute it, with or without modifying it, either commercially or non-
commercially. Secondarily, this License preserves for the author and publisher a way to
get credit for their work, while not being considered responsible for modifications made
by others.

This License is a kind of “copyleft”, which means that derivative works of the doc-
ument must themselves be free in the same sense. It complements the GNU General
Public License, which is a copyleft license designed for free software.

We have designed this License in order to use it for manuals for free software, be-
cause free software needs free documentation: a free program should come with manuals
providing the same freedoms that the software does. But this License is not limited to
software manuals; it can be used for any textual work, regardless of subject matter or
whether it is published as a printed book. We recommend this License principally for
works whose purpose is instruction or reference.

1. APPLICABILITY AND DEFINITIONS

This License applies to any manual or other work, in any medium, that contains a
notice placed by the copyright holder saying it can be distributed under the terms of this
License. Such a notice grants a world-wide, royalty-free license, unlimited in duration,
to use that work under the conditions stated herein. The “Document”, below, refers
to any such manual or work. Any member of the public is a licensee, and is addressed
as “you”. You accept the license if you copy, modify or distribute the work in a way
requiring permission under copyright law.

A “Modified Version” of the Document means any work containing the Document
or a portion of it, either copied verbatim, or with modifications and/or translated into
another language.
A **Secondary Section** is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or authors of the Document to the Document’s overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (Thus, if the Document is in part a textbook of mathematics, a Secondary Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.

The **Invariant Sections** are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document is released under this License. If a section does not fit the above definition of Secondary then it is not allowed to be designated as Invariant. The Document may contain zero Invariant Sections. If the Document does not identify any Invariant Sections then there are none.

The **Cover Texts** are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License. A Front-Cover Text may be at most 5 words, and a Back-Cover Text may be at most 25 words.

A **Transparent** copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, that is suitable for revising the document straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings) some widely available drawing editor, and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup, or absence of markup, has been arranged to thwart or discourage subsequent modification by readers is not Transparent. An image format is not Transparent if used for any substantial amount of text. A copy that is not Transparent is called **Opaque**.

Examples of suitable formats for Transparent copies include plain ASCII without markup, Texinfo input format, LaTeX input format, SGML or XML using a publicly available DTD, and standard-conforming simple HTML, PostScript or PDF designed for human modification. Examples of transparent image formats include PNG, XCF and JPG. Opaque formats include proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machine-generated HTML, PostScript or PDF produced by some word processors for output purposes only.

The **Title Page** means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires to appear in the title page. For works in formats which do not have any title page as such, “Title Page” means the text near the most prominent appearance of the work’s title, preceding the beginning of the body of the text.

A section **Entitled XYZ** means a named subunit of the Document whose title either is precisely XYZ or contains XYZ in parentheses following text that translates XYZ in another language. (Here XYZ stands for a specific section name mentioned below, such as “Acknowledgements”, “Dedications”, “Endorsements”, or “History”.) To “Preserve the Title” of such a section when you modify the Document means that it remains a section “Entitled XYZ” according to this definition.

The Document may include Warranty Disclaimers next to the notice which states
that this License applies to the Document. These Warranty Disclaimers are considered
to be included by reference in this License, but only as regards disclaiming warranties: any other implication that these Warranty Disclaimers may have is void and has no effect on the meaning of this License.

2. VERBATIM COPYING

You may copy and distribute the Document in any medium, either commercially or noncommercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section 3.

You may also lend copies, under the same conditions stated above, and you may publicly display copies.

3. COPYING IN QUANTITY

If you publish printed copies (or copies in media that commonly have printed covers) of the Document, numbering more than 100, and the Document's license notice requires Cover Texts, you must enclose the copies in covers that carry, clearly and legibly, all these Cover Texts: Front-Cover Texts on the front cover, and Back-Cover Texts on the back cover. Both covers must also clearly and legibly identify you as the publisher of these copies. The front cover must present the full title with all words of the title equally prominent and visible. You may add other material on the covers in addition. Copying with changes limited to the covers, as long as they preserve the title of the Document and satisfy these conditions, can be treated as verbatim copying in other respects.

If the required texts for either cover are too voluminous to fit legibly, you should put the first ones listed (as many as fit reasonably) on the actual cover, and continue the rest onto adjacent pages.

If you publish or distribute Opaque copies of the Document numbering more than 100, you must either include a machine-readable Transparent copy along with each Opaque copy, or state in or with each Opaque copy a computer-network location from which the general network-using public has access to download using public-standard network protocols a complete Transparent copy of the Document, free of added material. If you use the latter option, you must take reasonably prudent steps, when you begin distribution of Opaque copies in quantity, to ensure that this Transparent copy will remain thus accessible at the stated location until at least one year after the last time you distribute an Opaque copy (directly or through your agents or retailers) of that edition to the public.

It is requested, but not required, that you contact the authors of the Document well before redistributing any large number of copies, to give them a chance to provide you with an updated version of the Document.

4. MODIFICATIONS

You may copy and distribute a Modified Version of the Document under the conditions of sections 2 and 3 above, provided that you release the Modified Version under precisely
this License, with the Modified Version filling the role of the Document, thus licensing
distribution and modification of the Modified Version to whoever possesses a copy of it.
In addition, you must do these things in the Modified Version:

A. Use in the Title Page (and on the covers, if any) a title distinct from that of the
Document, and from those of previous versions (which should, if there were any,
be listed in the History section of the Document). You may use the same title as
a previous version if the original publisher of that version gives permission.

B. List on the Title Page, as authors, one or more persons or entities responsible for
authorship of the modifications in the Modified Version, together with at least five
of the principal authors of the Document (all of its principal authors, if it has fewer
than five), unless they release you from this requirement.

C. State on the Title page the name of the publisher of the Modified Version, as the
publisher.

D. Preserve all the copyright notices of the Document.

E. Add an appropriate copyright notice for your modifications adjacent to the other
copyright notices.

F. Include, immediately after the copyright notices, a license notice giving the public
permission to use the Modified Version under the terms of this License, in the form
shown in the Addendum below.

G. Preserve in that license notice the full lists of Invariant Sections and required Cover
Texts given in the Document’s license notice.

H. Include an unaltered copy of this License.

I. Preserve the section Entitled “History”, Preserve its Title, and add to it an item
stating at least the title, year, new authors, and publisher of the Modified Version as
given on the Title Page. If there is no section Entitled “History” in the Document,
create one stating the title, year, authors, and publisher of the Document as given
on its Title Page, then add an item describing the Modified Version as stated in
the previous sentence.

J. Preserve the network location, if any, given in the Document for public access to
a Transparent copy of the Document, and likewise the network locations given in
the Document for previous versions it was based on. These may be placed in the
“History” section. You may omit a network location for a work that was published
at least four years before the Document itself, or if the original publisher of the
version it refers to gives permission.

K. For any section Entitled “Acknowledgements” or “Dedications”, Preserve the Title
of the section, and preserve in the section all the substance and tone of each of the
contributor acknowledgements and/or dedications given therein.

L. Preserve all the Invariant Sections of the Document, unaltered in their text and
in their titles. Section numbers or the equivalent are not considered part of the
section titles.
M. Delete any section Entitled “Endorsements”. Such a section may not be included in the Modified Version.

N. Do not retitle any existing section to be Entitled “Endorsements” or to conflict in title with any Invariant Section.

O. Preserve any Warranty Disclaimers.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version’s license notice. These titles must be distinct from any other section titles.

You may add a section Entitled “Endorsements”, provided it contains nothing but endorsements of your Modified Version by various parties—for example, statements of peer review or that the text has been approved by an organization as the authoritative definition of a standard.

You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add another; but you may replace the old one, on explicit permission from the previous publisher that added the old one.

The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version.

5. COMBINING DOCUMENTS

You may combine the Document with other documents released under this License, under the terms defined in section 4 above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified, and list them all as Invariant Sections of your combined work in its license notice, and that you preserve all their Warranty Disclaimers.

The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work.

In the combination, you must combine any sections Entitled “History” in the various original documents, forming one section Entitled “History”; likewise combine any sections Entitled “Acknowledgements”, and any sections Entitled “Dedications”. You must delete all sections Entitled “Endorsements”.

6. COLLECTIONS OF DOCUMENTS
You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects.

You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document.

### 7. AGGREGATION WITH INDEPENDENT WORKS

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, is called an “aggregate” if the copyright resulting from the compilation is not used to limit the legal rights of the compilation’s users beyond what the individual works permit. When the Document is included in an aggregate, this License does not apply to the other works in the aggregate which are not themselves derivative works of the Document.

If the Cover Text requirement of section 3 is applicable to these copies of the Document, then if the Document is less than one half of the entire aggregate, the Document’s Cover Texts may be placed on covers that bracket the Document within the aggregate, or the electronic equivalent of covers if the Document is in electronic form. Otherwise they must appear on printed covers that bracket the whole aggregate.

### 8. TRANSLATION

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section 4. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License, and all the license notices in the Document, and any Warranty Disclaimers, provided that you also include the original English version of this License and the original versions of those notices and disclaimers. In case of a disagreement between the translation and the original version of this License or a notice or disclaimer, the original version will prevail.

If a section in the Document is Entitled “Acknowledgements”, “Dedications”, or “History”, the requirement (section 4) to Preserve its Title (section 1) will typically require changing the actual title.

### 9. TERMINATION

You may not copy, modify, sublicense, or distribute the Document except as expressly provided for under this License. Any other attempt to copy, modify, sublicense or distribute the Document is void, and will automatically terminate your rights under this License. However, parties who have received copies, or rights, from you under this License will not have their licenses terminated so long as such parties remain in full compliance.

### 10. FUTURE REVISIONS OF THIS LICENSE
The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See http://www.gnu.org/copyleft/.

Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License “or any later version” applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify a version number of this License, you may choose any version ever published (not as a draft) by the Free Software Foundation.

ADDENDUM: How to use this License for your documents

To use this License in a document you have written, include a copy of the License in the document and put the following copyright and license notices just after the title page:

Copyright ©YEAR YOUR NAME. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled “GNU Free Documentation License”.

If you have Invariant Sections, Front-Cover Texts and Back-Cover Texts, replace the “with...Texts.” line with this:

with the Invariant Sections being LIST THEIR TITLES, with the Front-Cover Texts being LIST, and with the Back-Cover Texts being LIST.

If you have Invariant Sections without Cover Texts, or some other combination of the three, merge those two alternatives to suit the situation.

If your document contains nontrivial examples of program code, we recommend releasing these examples in parallel under your choice of free software license, such as the GNU General Public License, to permit their use in free software.
Part T
Topics
Part A
Applications
Index

A (appendix), 717
A (archetype), 721
A (definition), 255
A (part), 821
A (subsection, section WILA), 4
AA (Property), 310
AAC (Property), 91
AACN (Property), 689
AALC (example), 99
AAM (Property), 199
ABLC (example), 98
ABS (example), 123
AC (Property), 309
ACC (Property), 91
ACCN (Property), 688
ACM (Property), 199
ACN (example), 688
acronym (example), 670
additive associativity
  column vectors
    Property AAC, 91
  complex numbers
    Property AACN, 689
  matrices
    Property AAM, 199
  vectors
    Property AA, 310
additive commutativity
  complex numbers
    Property ACCN, 688
additive inverse
  complex numbers
    Property AICN, 689
from scalar multiplication
  theorem AISM, 318
additive inverses
  column vectors
    Property AIC, 91
  matrices
    Property AIM, 199
unique
  theorem AIU, 317
  vectors
    Property AI, 310
additive closure
  column vectors
    Property ACC, 91
  matrices
    Property ACM, 199
  vectors
    Property AC, 309
adjoint
  definition A, 255
AHSAC (example), 65
AI (Property), 310
AIC (Property), 91
AICN (Property), 689
AIM (Property), 199
AISM (theorem), 318
AIU (theorem), 317
AIVLT (example), 567
ALT (example), 504
ALTMM (example), 607
AM (definition), 32
AM (example), 29
AM (notation), 32
AMA (example), 33
AME (definition), 452
ANILT (example), 568
AOS (example), 189
Archetype A
  column space, 266
  linearly dependent columns, 151
  singular matrix, 76
  solving homogeneous system, 66
  system as linear combination, 99
archetype A
  augmented matrix
    example AAMA, 33
Archetype B
  column space, 267
  inverse
example CMIAB, 238
linearly independent columns, 151
nonsingular matrix, 76
not invertible
example MWIAA, 232
solutions via inverse
example SABMI, 231
solving homogeneous system, 66
system as linear combination, 98
vector equality, 88
archetype B
solutions
example SAB, 39
Archetype C
homogeneous system, 65
Archetype D
column space, original columns, 265
solving homogeneous system, 67
vector form of solutions, 102
Archetype I
column space from row operations, 273
null space, 68
row space, 268
vector form of solutions, 110
Archetype I: casting out vectors, 170
Archetype L
null space span, linearly independent, 155
vector form of solutions, 111
ASC (example), 594
augmented matrix
notation, 32
AVR (example), 354
B (archetype), 726
B (definition), 363
B (section), 363
B (subsection, section B), 363
basis
columns nonsingular matrix
example CABAK, 369
common size
theorem BIS, 383
crazy vector space
example BC, 366
definition B, 363
matrices
example BM, 364
example BSM22, 305
polynomials
example BP, 364
example BPR, 399
example BSP4, 365
example SVP4, 400
subspace of matrices
example BDM22, 399
BC (example), 366
BCS (theorem), 264
BDE (example), 470
BDM22 (example), 399
best cities
money magazine
example MBC, 212
BIS (theorem), 383
BM (example), 364
BNM (subsection, section B), 369
BNS (subsection, section B), 154
BP (example), 364
BPR (example), 399
BRLT (example), 557
BRS (theorem), 271
BS (theorem), 172
BSCV (subsection, section B), 367
BSM22 (example), 365
BSP4 (example), 365
C (archetype), 731
C (definition), 695
C (notation), 695
C (part), 3
C (Property), 310
C (technique, section PT), 703
CABAK (example), 369
CAEHW (example), 446
cancellation
vector addition
theorem VAC, 319
CAV (subsection, section O), 183
CB (section), 637
CB (theorem), 639
CBCV (example), 643
CBM (definition), 638
CBM (subsection, section CB), 638
CBP (example), 640
CC (Property), 91
CCCV (definition), 183
change of basis
  between polynomials
    example CBP, 640
change-of-basis
  between column vectors
    example CBCV, 643
  matrix representation
    theorem MRCB, 644
  similarity
    theorem SCB, 647
    theorem CB, 639
  change-of-basis matrix
    definition CBM, 638
    inverse
      theorem ICBM, 639
characteristic polynomial
  definition CP, 449
  degree
    theorem DCP, 472
size 3 matrix
  example CPMS3, 449
CILT (subsection, section ILT), 539
CIM (subsection, section MISLE), 234
CINM (theorem), 237
CIVLT (theorem), 572
CLI (theorem), 595
CLTLT (theorem), 520
CM (definition), 30
CM (Property), 199
row operations, Archetype I  
example CSROI, 273

subspace 
theorem CSMS, 336

testing membership 
example MCSM, 263

two computations 
example CSTW, 264

column vector addition 
notation, 89

column vector scalar multiplication 
notation, 89

commutativity 
column vectors  
Property CC, 91

matrices  
Property CM, 199

vectors  
Property C, 310

complex m-space 
exmple VSCV, 311

complex arithmetic 
exmple ACN, 688

column vector addition 
notation, 183

column vector scalar multiplication 
notation, 183

conjugate 
column vector 
definition CCCV, 183

matrix 
definition CCM, 203

notation, 203

multiplication 
definition CCRM, 690

notation, 689

scalar multiplication 
definition CRSM, 184

twice 
definition CCT, 690

vector addition 
definition CRVA, 183

conjugate of a vector 
notation, 183

conjugation 
matrix addition 
definition CRMA, 203

matrix scalar multiplication 
definition CRMSM, 204

matrix transpose 
definition MCT, 204

consistent linear system, 54

consistent linear systems 
definition CS, 51

constructive proofs 
technique C, 703

contradiction 
technique CD, 708

contrapositive 
technique CP, 706

converse 
technique CV, 707

coordinates 
orthonormal basis 
definition COB, 371

coordinatization
linear combination of matrices  
  example CM32, 597  
linear independence  
  theorem CLI, 595  
orthonormal basis  
  example CROB3, 372  
  example CROB4, 371  
spanning sets  
  theorem CSS, 595  
coordinatization principle, 596  
coordinatizing  
polynomials  
  example CP2, 596  
COV (example), 170  
COV (subsection, section LDS), 169  
CP (definition), 449  
CP (subsection, section VR), 594  
CP (technique, section PT), 706  
CP2 (example), 596  
CPMS3 (example), 449  
crazy vector space  
  example CVSR, 594  
  properties  
   example PCVS, 318  
CRMA (theorem), 203  
CRMSM (theorem), 204  
CRN (theorem), 386  
CROB3 (example), 372  
CROB4 (example), 371  
CRS (section), 261  
CRS (subsection, section FS), 284  
CRSM (theorem), 184  
CRVA (theorem), 183  
CS (definition), 51  
CS (example), 695  
CSAA (example), 266  
CSAB (example), 267  
CSANS (example), 284  
CSCN (example), 689  
CSCS (theorem), 262  
CSIP (example), 184  
CSLT (subsection, section SLT), 559  
CSLTS (theorem), 559  
CSM (definition), 261  
CSM (notation), 261  
CSMCS (example), 261  
CSMS (theorem), 336  
CSNM (subsection, section CRS), 266  
CSNM (theorem), 267  
CSOCD (example), 265  
CSRN (theorem), 56  
CSROI (example), 273  
CSRST (theorem), 273  
CSS (theorem), 595  
CSSE (subsection, section CRS), 261  
CSSM (theorem), 320  
CSSOC (subsection, section CRS), 263  
CSTW (example), 264  
CTLT (example), 520  
CUMOS (theorem), 253  
CV (definition), 30  
CV (notation), 30  
CV (technique, section PT), 707  
CVA (definition), 89  
CVA (notation), 89  
CVC (notation), 89  
CVE (definition), 88  
CVE (notation), 88  
CVS (example), 314  
CVS (subsection, section VR), 593  
CVSM (definition), 89  
CVSM (example), 90  
CVSM (notation), 89  
CVSM (theorem), 320  
CVSR (example), 594  
D (archetype), 735  
D (chapter), 409  
D (definition), 379  
D (notation), 379  
D (section), 379  
D (subsection, section D), 379  
D (subsection, section SD), 486  
D (technique, section PT), 698  
D33M (example), 415  
DAB (example), 487  
DC (example), 385  
DC (technique, section PT), 712  
DC (theorem), 487  
DCM (theorem), 383  
DCN (Property), 689  
DCP (theorem), 472  
DD (subsection, section DM), 414  
DEC (theorem), 417  
decomposition  
  technique DC, 712
<table>
<thead>
<tr>
<th>Index</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>DED (theorem), definition</td>
<td>492</td>
</tr>
<tr>
<td>A</td>
<td>255</td>
</tr>
<tr>
<td>AM</td>
<td>32</td>
</tr>
<tr>
<td>AME</td>
<td>452</td>
</tr>
<tr>
<td>B</td>
<td>363</td>
</tr>
<tr>
<td>C</td>
<td>695</td>
</tr>
<tr>
<td>CBM</td>
<td>638</td>
</tr>
<tr>
<td>CCCV</td>
<td>183</td>
</tr>
<tr>
<td>CCM</td>
<td>203</td>
</tr>
<tr>
<td>CCN</td>
<td>689</td>
</tr>
<tr>
<td>CM</td>
<td>30</td>
</tr>
<tr>
<td>CNA</td>
<td>688</td>
</tr>
<tr>
<td>CNE</td>
<td>688</td>
</tr>
<tr>
<td>CNM</td>
<td>688</td>
</tr>
<tr>
<td>CP</td>
<td>449</td>
</tr>
<tr>
<td>CS</td>
<td>51</td>
</tr>
<tr>
<td>CSM</td>
<td>201</td>
</tr>
<tr>
<td>CV</td>
<td>30</td>
</tr>
<tr>
<td>CVA</td>
<td>89</td>
</tr>
<tr>
<td>CVE</td>
<td>88</td>
</tr>
<tr>
<td>CVS</td>
<td>89</td>
</tr>
<tr>
<td>D</td>
<td>379</td>
</tr>
<tr>
<td>DIM</td>
<td>486</td>
</tr>
<tr>
<td>DM</td>
<td>414</td>
</tr>
<tr>
<td>DZM</td>
<td>487</td>
</tr>
<tr>
<td>EEF</td>
<td>287</td>
</tr>
<tr>
<td>EELT</td>
<td>637</td>
</tr>
<tr>
<td>EEM</td>
<td>441</td>
</tr>
<tr>
<td>ELE</td>
<td>409</td>
</tr>
<tr>
<td>EM</td>
<td>450</td>
</tr>
<tr>
<td>EO</td>
<td>16</td>
</tr>
<tr>
<td>ES</td>
<td>693</td>
</tr>
<tr>
<td>ESYS</td>
<td>15</td>
</tr>
<tr>
<td>GME</td>
<td>452</td>
</tr>
<tr>
<td>HM</td>
<td>255</td>
</tr>
<tr>
<td>HS</td>
<td>65</td>
</tr>
<tr>
<td>IDLT</td>
<td>567</td>
</tr>
<tr>
<td>IDV</td>
<td>533</td>
</tr>
<tr>
<td>ILT</td>
<td>529</td>
</tr>
<tr>
<td>IM</td>
<td>76</td>
</tr>
<tr>
<td>IP</td>
<td>184</td>
</tr>
<tr>
<td>IVLT</td>
<td>567</td>
</tr>
<tr>
<td>IVS</td>
<td>573</td>
</tr>
<tr>
<td>JB</td>
<td>670</td>
</tr>
<tr>
<td>KLT</td>
<td>532</td>
</tr>
<tr>
<td>LC</td>
<td>331</td>
</tr>
<tr>
<td>LCCV</td>
<td>97</td>
</tr>
<tr>
<td>LI</td>
<td>345</td>
</tr>
<tr>
<td>LICV</td>
<td>145</td>
</tr>
<tr>
<td>LNS</td>
<td>283</td>
</tr>
<tr>
<td>LSMR</td>
<td>32</td>
</tr>
<tr>
<td>LT</td>
<td>503</td>
</tr>
<tr>
<td>LTA</td>
<td>517</td>
</tr>
<tr>
<td>LTC</td>
<td>519</td>
</tr>
<tr>
<td>LTSM</td>
<td>518</td>
</tr>
<tr>
<td>M</td>
<td>29</td>
</tr>
<tr>
<td>MA</td>
<td>198</td>
</tr>
<tr>
<td>MCN</td>
<td>690</td>
</tr>
<tr>
<td>ME</td>
<td>197</td>
</tr>
<tr>
<td>MI</td>
<td>232</td>
</tr>
<tr>
<td>MM</td>
<td>215</td>
</tr>
<tr>
<td>MR</td>
<td>603</td>
</tr>
<tr>
<td>MSM</td>
<td>198</td>
</tr>
<tr>
<td>MVP</td>
<td>211</td>
</tr>
<tr>
<td>NLT</td>
<td>667</td>
</tr>
<tr>
<td>NM</td>
<td>75</td>
</tr>
<tr>
<td>NOLT</td>
<td>575</td>
</tr>
<tr>
<td>NOM</td>
<td>385</td>
</tr>
<tr>
<td>NSM</td>
<td>68</td>
</tr>
<tr>
<td>NV</td>
<td>187</td>
</tr>
<tr>
<td>ONS</td>
<td>193</td>
</tr>
<tr>
<td>OSV</td>
<td>189</td>
</tr>
<tr>
<td>OV</td>
<td>189</td>
</tr>
<tr>
<td>PI</td>
<td>515</td>
</tr>
<tr>
<td>REM</td>
<td>33</td>
</tr>
<tr>
<td>RLD</td>
<td>345</td>
</tr>
<tr>
<td>RLDCCV</td>
<td>145</td>
</tr>
<tr>
<td>RLT</td>
<td>551</td>
</tr>
<tr>
<td>RO</td>
<td>33</td>
</tr>
<tr>
<td>ROLT</td>
<td>575</td>
</tr>
<tr>
<td>ROM</td>
<td>385</td>
</tr>
<tr>
<td>RR</td>
<td>42</td>
</tr>
<tr>
<td>RREF</td>
<td>35</td>
</tr>
<tr>
<td>RSM</td>
<td>268</td>
</tr>
<tr>
<td>S</td>
<td>328</td>
</tr>
<tr>
<td>SC</td>
<td>696</td>
</tr>
<tr>
<td>SE</td>
<td>694</td>
</tr>
<tr>
<td>SET</td>
<td>693</td>
</tr>
<tr>
<td>SI</td>
<td>695</td>
</tr>
<tr>
<td>SIM</td>
<td>483</td>
</tr>
<tr>
<td>SLE</td>
<td>13</td>
</tr>
<tr>
<td>SLT</td>
<td>547</td>
</tr>
<tr>
<td>SM</td>
<td>414</td>
</tr>
<tr>
<td>SQM</td>
<td>75</td>
</tr>
<tr>
<td>SS</td>
<td>332</td>
</tr>
</tbody>
</table>
SSCV, 123
SSET, 693
SU, 695
SUV, 234
SV, 31
SYM, 201
technique D, 698
TM, 200
TS, 330
TSHSE, 66
TSVS, 350
UM, 252
VOC, 31
VR, 587
VS, 309
VSCV, 87
VSM, 197
ZCV, 30
ZM, 200
DEHD (example), 492
DEM (theorem), 431
DEMMM (theorem), 431
DEMS5 (example), 457
DER (theorem), 416
DERC (theorem), 427
determinant
computed two ways
example TCSD, 418
definition DM, 414
equal rows or columns
theorem DER, 427
expansion, columns
theorem DEC, 417
expansion, rows
theorem DER, 416
identity matrix
theorem DIM, 430
matrix multiplication
theorem DRMM, 434
nonsingular matrix, 432
notation, 414
row or column multiple
theorem DRCM, 426
row or column swap
theorem DRCS, 425
size 2 matrix
theorem DMST, 415
size 3 matrix
determinant, upper triangular matrix
example DUTM, 419
determinants
elementary matrices
theorem DEMMM, 431
DFS (subsection, section PD), 402
DFS (theorem), 402
diagonal matrix
definition DIM, 486
diagonalizable
definition DZM, 478
distinct eigenvalues
example DEHD, 492
theorem DED, 492
full eigenspaces
theorem DMFE, 490
not
example NDMS4, 491
diagonalizable matrix
high power
example HPDM, 493
diagonalization
Archetype B
example DAB, 487
criteria
theorem DC, 487
example DMS3, 489
DIM (definition), 486
DIM (theorem), 430
dimension
crazy vector space
example DC, 385
definition D, 379
notation, 379
polynomial subspace
example DSP4, 384
proper subspaces
technique ME, 710

equivalent systems
definition ESYS, 15

ERMCP (theorem), 472
ES (definition), 693
ES (notation), 693

ESEO (subsection, section SSLE), 15
ESLT (subsection, section SLT), 547

ESMM (theorem), 469
ESMS3 (example), 451
ESMS4 (example), 453

ESYS (definition), 15
ETM (theorem), 471

EVS (subsection, section VS), 311

example

AALC, 99
ABLC, 98
ABS, 123
ACN, 688
acronym, 670
AHSAC, 65
AIVLT, 567
ALT, 504
ALTM, 607
AM, 29
AMAA, 33
ANILT, 568
AOS, 189
ASC, 594
AVR, 354
BC, 366
BDE, 470
BDM22, 399
BM, 364
BP, 364
BPR, 399
BRLT, 557
BSM22, 805
BSP4, 365
CABAK, 369
CAEHW, 446
CBCV, 643
CBP, 640
CCM, 203
CELT, 656
CEMS6, 455
CFV, 56
CM32, 597

CMI, 235
CMIAB, 238
CNS1, 69
CNS2, 69
CNSV, 187
COV, 170
CP2, 596
CPMS3, 449
CROB3, 372
CROB4, 371
CS, 695
CSAA, 266
CSAB, 267
CSANS, 284
CSCN, 689
CSIP, 184
CSMCS, 261
CSOCD, 265
CSROI, 273
CSTW, 264
CVS, 314
CVSM, 90
CVSR, 594
D33M, 415
DAB, 487
DC, 385
DEHD, 492
DEM5, 157
DMS3, 489
DRO, 428
DSM22, 383
DSP4, 384
DUTM, 419
EENS, 486
ELTBM, 637
ELTBP, 638
ELTT, 651
EMMS4, 453
EMS3, 450
EMS4, 453
ESMS3, 451
ESMS4, 453
FS1, 294
FS2, 295
FSAG, 296

Version 0.92
<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>RREF</td>
<td>36</td>
</tr>
<tr>
<td>RREFN</td>
<td>51</td>
</tr>
<tr>
<td>RRTI</td>
<td>402</td>
</tr>
<tr>
<td>RS</td>
<td>368</td>
</tr>
<tr>
<td>RS</td>
<td>268</td>
</tr>
<tr>
<td>RSB</td>
<td>367</td>
</tr>
<tr>
<td>RSC5</td>
<td>168</td>
</tr>
<tr>
<td>RSNS</td>
<td>330</td>
</tr>
<tr>
<td>RSREM</td>
<td>271</td>
</tr>
<tr>
<td>RSSC4</td>
<td>174</td>
</tr>
<tr>
<td>RVMR</td>
<td>618</td>
</tr>
<tr>
<td>S</td>
<td>76</td>
</tr>
<tr>
<td>SAA</td>
<td>39</td>
</tr>
<tr>
<td>SAB</td>
<td>39</td>
</tr>
<tr>
<td>SABMI</td>
<td>231</td>
</tr>
<tr>
<td>SAE</td>
<td>41</td>
</tr>
<tr>
<td>SAN</td>
<td>556</td>
</tr>
<tr>
<td>SAR</td>
<td>548</td>
</tr>
<tr>
<td>SAV</td>
<td>550</td>
</tr>
<tr>
<td>SC</td>
<td>696</td>
</tr>
<tr>
<td>SC3</td>
<td>325</td>
</tr>
<tr>
<td>SCAA</td>
<td>125</td>
</tr>
<tr>
<td>SCAB</td>
<td>127</td>
</tr>
<tr>
<td>SCAD</td>
<td>132</td>
</tr>
<tr>
<td>SEE</td>
<td>442</td>
</tr>
<tr>
<td>SEEF</td>
<td>288</td>
</tr>
<tr>
<td>SETM</td>
<td>693</td>
</tr>
<tr>
<td>SI</td>
<td>696</td>
</tr>
<tr>
<td>SM32</td>
<td>334</td>
</tr>
<tr>
<td>SMLT</td>
<td>519</td>
</tr>
<tr>
<td>SMS3</td>
<td>484</td>
</tr>
<tr>
<td>SMS5</td>
<td>483</td>
</tr>
<tr>
<td>SP4</td>
<td>328</td>
</tr>
<tr>
<td>SPIAS</td>
<td>515</td>
</tr>
<tr>
<td>SRR</td>
<td>77</td>
</tr>
<tr>
<td>SS</td>
<td>414</td>
</tr>
<tr>
<td>SSC</td>
<td>353</td>
</tr>
<tr>
<td>SSET</td>
<td>693</td>
</tr>
<tr>
<td>SSM22</td>
<td>352</td>
</tr>
<tr>
<td>SSSNS</td>
<td>130</td>
</tr>
<tr>
<td>SSP</td>
<td>333</td>
</tr>
<tr>
<td>SSP4</td>
<td>350</td>
</tr>
<tr>
<td>STLT</td>
<td>518</td>
</tr>
<tr>
<td>STNE</td>
<td>13</td>
</tr>
<tr>
<td>SU</td>
<td>495</td>
</tr>
<tr>
<td>SUVOS</td>
<td>189</td>
</tr>
<tr>
<td>SVP4</td>
<td>400</td>
</tr>
<tr>
<td>SYM</td>
<td>201</td>
</tr>
</tbody>
</table>

Version 0.92
INDEX

EXC (subsection, section MR), 625
EXC (subsection, section NM), 83
EXC (subsection, section O), 195
EXC (subsection, section PD), 405
EXC (subsection, section PDM), 437
EXC (subsection, section PEE), 479
EXC (subsection, section RREF), 43
EXC (subsection, section S), 339
EXC (subsection, section SD), 497
EXC (subsection, section SLT), 561
EXC (subsection, section SS), 135
EXC (subsection, section SSLE), 23
EXC (subsection, section TSS), 61
EXC (subsection, section VO), 93
EXC (subsection, section VR), 599
EXC (subsection, section VS), 323
EXC (subsection, section WILA), 9
extended echelon form
submatrices
example SEEF, 288
extended reduced row-echelon form
properties
theorem PEEF, 288
F (archetype), 743
FDV (example), 53
four subsets
example FS1, 294
example FS2, 295
four subspaces
dimension
theorem DFS, 402
FRAN (example), 553
free variables
example CFV, 50
free variables, number
theorem FVCS, 50
free, independent variables
example FDV, 53
FS (section), 283
FS (subsection, section FS), 290
FS (theorem), 290
FS1 (example), 294
FS2 (example), 295
FSAG (example), 296
FTMR (theorem), 606
FVCS (theorem), 56
G (archetype), 748
G (theorem), 398
GFDL (appendix), 809
GME (definition), 452
goldilocks
  theorem G, 398
Gram-Schmidt
column vectors
  theorem GSPCV, 191
three vectors
  example GSTV, 192
gram-schmidt
mathematica, 682
GS (technique, section PT), 702
GSP (subsection, section O), 191
GSP.MMA (computation, section MMA), 682
GSPCV (theorem), 191
GSTV (example), 192
GT (subsection, section PD), 397
H (archetype), 752
hermitian
definition HM, 255
HISAA (example), 66
HISAD (example), 67
HM (definition), 255
HMEM5 (example), 454
HMOE (theorem), 477
HMRE (theorem), 476
HMVEI (theorem), 67
homogeneous system
consistent
  theorem HSC, 65
definition HS, 65
infinitely many solutions
  theorem HMVEI, 67
homogeneous systems
linear independence, 147
homogenous system
Archetype C
  example AHSAC, 65
HPDM (example), 493
HS (definition), 65
HSC (theorem), 65
HSE (section), 65
HUSAB (example), 66
I (archetype), 757
I (technique, section PT), 713
IS (example), 20
isomorphic multiple vector spaces example MIVS, 594
vector spaces example IVSAV, 574
isomorphic vector spaces dimension theorem IVSED, 575
example TIVS, 594
ISRN (theorem), 55
ISSI (example), 52
IV (subsection, section IVLT), 571
IVLT (definition), 567
IVLT (section), 567
IVLT (subsection, section IVLT), 567
IVLT (subsection, section MR), 619
IVS (definition), 573
IVSAV (example), 574
IVSED (theorem), 575
J (archetype), 762
JB (definition), 670
JB (notation), 670
JB4 (example), 670
Jordan block definition JB, 670
notation, 670
size 4 example JB4, 670
K (archetype), 767
kernel injective linear transformation theorem KILT, 535
isomorphic to null space theorem KNSI, 614
linear transformation example NKAO, 532
notation, 532
of a linear transformation definition KLT, 532
pre-image, 535
subspace theorem KLTS, 533
trivial example TKAP, 534
via matrix representation example KVMR, 615
KILT (theorem), 535
KLT (definition), 532
KLT (notation), 532
KLT (subsection, section IILT), 532
KLTS (theorem), 533
KNSI (theorem), 614
KPI (theorem), 535
KPLT (theorem), 673
KPNLT (example), 675
KPNLT (theorem), 675
KVMR (example), 615
L (archetype), 771
L (technique, section PT), 700
LA (subsection, section WILA), 3
LC (definition), 331
LC (section), 97
LC (subsection, section LC), 97
LC (technique, section PT), 716
LCCV (definition), 97
LCM (example), 331
LDCAA (example), 151
LDHS (example), 149
LDP4 (example), 382
LDRN (example), 150
LDS (example), 145
LDS (section), 167
LDSS (subsection, section LDS), 167
left null space as row space, 290
definition LNS, 283
example LNS, 283
notation, 283
subspace theorem LNSMS, 337
lemma technique LC, 716
LI (definition), 345
LI (section), 145
LI (subsection, section LISS), 345
LIC (example), 349
LICAB (example), 151
LICV (definition), 145
LIHS (example), 148
LIM32 (example), 347
linear combination system of equations example ABLC, 98
definition LC, 331
definition LCCV, 97
example TLC, 97
linear transformation, 512
matrices
example LCM, 331
system of equations
example AALC, 99
linear combinations
solutions to linear systems
theorem SLSLC, 100
linear dependence
more vectors than size
theorem MVSLD, 150
linear independence
definition LI, 345
definition LICV, 145
homogeneous systems
theorem LIVHS, 147
injective linear transformation
theorem ILTLI, 537
matrices
example LIM32, 347
orthogonal, 190
r and n
theorem LIVRN, 149
linear solve
mathematica, 680
linear system
consistent
theorem RCLS, 34
matrix representation
definition LSMR, 32
notation, 32
linear systems
notation
example MNSLE, 212
example NSLE, 32
linear transformation
polynomials to polynomials
example LTPP, 507
addition
definition LTA, 517
theorem MLTLT, 518
theorem SLTLT, 517
as matrix multiplication
example ALTMM, 607
basis of range
example BRLT, 557
checking
example ALT, 504
composition
definition LTC, 519
theorem CLTLT, 520
defined by a matrix
example LTM, 508
defined on a basis
example LTDB1, 513
example LTDB2, 513
example LTDB3, 514
theorem LTDB, 512
definition LT, 503
identity
definition IDLT, 567
injection
definition ILT, 529
inverse
theorem IILTLT, 570
inverse of inverse
theorem IILT, 570
invertible
definition IVLT, 567
example AIVLT, 567
invertible, injective and surjective
theorem IILTIS, 571
kernels of powers
theorem KPLT, 673
linear combination
theorem LTLC, 512
matrix of, 510
example MFLT, 509
example MOLT, 511
not
example NLT, 505
not invertible
example ANILT, 568
notation, 503
polynomials to matrices
example LTPM, 506
rank plus nullity
theorem RPNDD, 576
scalar multiple
example SMLT, 519
scalar multiplication
definition LTSM, 518
spanning range
linear transformation inverse via matrix representation
example ILTVR, 620
linear transformations compositions
example CTLT, 520
from matrices
theorem MBLT, 509
linearly dependent
$r < n$
example LDRN, 150
via homogeneous system
example LDHS, 149
linearly dependent columns
Archetype A
example LDCAA, 151
linearly dependent set
example LDS, 145
linear combinations within
theorem DLDS, 167
polynomials
example LDP4, 382
linearly independent

crazy vector space
example LIC, 349
extending sets
theorem ELIS, 397
polynomials
example LIP4, 346
via homogeneous system
example LIHS, 148
linearly independent columns
Archetype B
example LICAB, 151
linearly independent set
example LIS, 147
example LLDS, 150
LINM (subsection, section LI), 151
LINSB (example), 152
LIP4 (example), 346
LIS (example), 147
LISS (section), 345
LISV (subsection, section LI), 145
LIVHS (theorem), 147
LIVRN (theorem), 149
LLDS (example), 150
LNS (definition), 283
LNS (example), 283
LNS (notation), 283
LNS (subsection, section FS), 283
LNSMS (theorem), 337
LS.MMA (computation, section MMA), 680
LSMR (definition), 32
LSMR (notation), 32
LT (chapter), 503
LT (definition), 503
LT (notation), 503
LT (section), 503
LT (subsection, section LT), 503
LTA (definition), 517
LTC (definition), 519
LTDB (theorem), 512
LTDB1 (example), 513
LTDB2 (example), 513
LTDB3 (example), 514
LTLC (subsection, section LT), 512
LTLC (theorem), 512
LTM (example), 508
LTPM (example), 506
LTPP (example), 507
LTSM (definition), 518
LTTZZ (theorem), 307
M (archetype), 775
M (chapter), 197
M (definition), 29
M (notation), 29
MA (definition), 198
MA (example), 198
MA (notation), 198
MACN (Property), 689
mathematica
gram-schmidt (computation), 682
linear solve (computation), 680
matrix entry (computation), 679
matrix inverse (computation), 684
matrix multiplication (computation), 683
null space (computation), 681
row reduce (computation), 679
transpose of a matrix (computation), 683
vector form of solutions (computation), 681
vector linear combinations (computation), 680
mathematical language
  technique L, 700
matrix
  addition
    definition MA, 198
    notation, 198
  augmented
    definition AM, 32
  column space
    definition CSM, 261
  complex conjugate
    example CCM, 203
    definition M, 29
  equality
    definition ME, 197
    notation, 197
    example AM, 29
  identity
    definition IM, 76
  inverse
    definition MI, 232
  nonsingular
    definition NM, 75
    notation, 29
  of a linear transformation
    theorem MLTCV, 510
product
  example PTM, 215
  example PTMEE, 217
  product with vector
    definition MVP, 211
  rectangular, 75
  row space
    definition RSM, 268
  scalar multiplication
    definition MSM, 198
    notation, 198
  singular, 75
  square
    definition SQM, 75
  submatrices
example SS, 414
submatrix
  definition SM, 414
symmetric
  definition SYM, 201
transpose
  definition TM, 200
unitary
  definition UM, 252
  theorem UMI, 253
zero
  definition ZM, 200
matrix addition
  example MA, 198
matrix components
  notation, 29
matrix entry
  mathematica, 679
ti83, 686
ti86, 684
matrix inverse
  Archetype B, 238
  computation
    theorem CINM, 237
  mathematica, 684
  nonsingular matrix
    theorem NI, 251
  of a matrix inverse
    theorem MIMI, 240
  one-sided
    theorem OSIS, 250
  product
    theorem SS, 239
  scalar multiple
    theorem MISM, 240
  size 2 matrices
    theorem TTMI, 234
  transpose
    theorem MIT, 240
  uniqueness
    theorem MIU, 239
matrix multiplication
  associativity
    theorem MMA, 220
complex conjugation
  theorem MMCC, 221
definition MM, 215
MMZM (theorem), 218
MNEM (theorem), 475
MNSLE (example), 212
MO (section), 197
MOLT (example), 511
more variables than equations
  example OSGMD, 57
  theorem CMVEI, 57
MPMR (example), 611
MR (definition), 603
MR (section), 603
MRBE (example), 648
MRCB (theorem), 644
MRCLT (theorem), 610
RCM (example), 645
MRMLT (theorem), 610
MRS (subsection, section CB), 644
MRSLT (theorem), 609
MSCN (example), 690
MSM (definition), 198
MSM (example), 198
MSM (notation), 198
MTV (example), 211
multiplicative associativity
  complex numbers
    Property MACN, 689
multiplicative commutativity
  complex numbers
    Property MCCN, 688
multiplicative inverse
  complex numbers
    Property MICN, 689
MVNSE (subsection, section RREF), 30
MVP (definition), 211
MVP (notation), 211
MVP (subsection, section MM), 211
MVSLD (theorem), 150
MWIAA (example), 232

N (archetype), 778
N (subsection, section O), 187
N (technique, section PT), 705
NDMS4 (example), 491
negation of statements
  technique N, 705
NEM (theorem), 473
NI (theorem), 251
NIAO (example), 536
NIAQ (example), 529
NIAQR (example), 536
NIDAU (example), 538
nilpotent
  linear transformation
    definition NLT, 667
  NKAO (example), 532
  NLT (definition), 667
  NLT (example), 505
  NLT (section), 667
  NLT (subsection, section NLT), 667
  NLTFO (subsection, section LT), 517
  NM (definition), 75
  NM (example), 76
  NM (section), 75
  NM (subsection, section NM), 75
  NM62 (example), 669
  NM64 (example), 668
  NM83 (example), 671
  NME1 (theorem), 81
  NME2 (theorem), 152
  NME3 (theorem), 251
  NME4 (theorem), 267
  NME5 (theorem), 370
  NME6 (theorem), 388
  NME7 (theorem), 433
  NME8 (theorem), 468
  NME9 (theorem), 622
  NMI (subsection, section MINM), 249
  NMLIC (theorem), 151
  NMPM (definition), 413
  NMRI (definition), 77
  NMTNS (theorem), 78
  NMUS (definition), 79
  NOILT (definition), 576
  NOLT (definition), 375
  NOLT (notation), 575
  NOM (definition), 385
  NOM (notation), 385
nonsingular
  columns as basis
    theorem CNMB, 369
nonsingular matrices
  linearly independent columns
    theorem NMLIC, 151
nonsingular matrix
  Archetype B
    example NM, 76
column space, 267
elementary matrices
  theorem NMPEM, 413
equivalences
  theorem NME1, 81
  theorem NME2, 152
  theorem NME3, 251
  theorem NME4, 267
  theorem NME5, 370
  theorem NME6, 388
  theorem NME7, 433
  theorem NME8, 468
  theorem NME9, 622
matrix inverse, 251
null space
  example NSNM, 78
nullity, 387
product of nonsingular matrices
  theorem NPNT, 249
rank
  theorem RNNM, 387
row-reduced
  theorem NMRRI, 77
trivial null space
  theorem NMTNS, 78
unique solutions
  theorem NMUS, 79
nonsingular matrix, row-reduced
  example NSR, 77
norm
  example CNSV, 187
  inner product, 187
notation
  notation, 187
notation for a linear system
  example NSE, 14
NPNT (theorem), 249
NRFO (subsection, section MR), 609
NRREF (example), 36
NS.MMA (computation, section MMA), 681
NSAO (example), 555
INDEX  843

NSAQ (example), 547
NSAQ (example), 555
NSC2A (example), 329
NSC2S (example), 329
NSC2Z (example), 329
NSDAT (example), 558
NSDS (example), 131
NSE (example), 14
NSEAI (example), 68
NSLE (example), 82
NSLIL (example), 155
NSM (definition), 68
NSM (notation), 68
NSM (subsection, section HSE), 68
NSMS (theorem), 330
NSNM (example), 78
NSR (example), 77
NSS (example), 78
NSSLI (subsection, section LI), 152
Null space
  as a span
  example NSDS, 131
null space
  Archetype I
    example NSEAI, 68
  basis
    theorem BNS, 154
  computation
    example CNS1, 69
    example CNS2, 69
  isomorphic to kernel, 614
  linearly independent basis
    example LINSB, 152
  mathematica, 681
  matrix
    definition NSM, 68
  nonsingular matrix, 78
  notation, 68
  singular matrix, 78
  spanning set
    example SSNS, 130
    theorem SSNS, 129
  subspace
    theorem NSMS, 330
null space span, linearly independent
  Archetype L
    example NSLIL, 155
nullity
  computing, 386
  injective linear transformation
    theorem NOILT, 576
  linear transformation
    definition NOLT, 575
    matrix, 386
    definition NOM, 385
    notation, 385, 575
  square matrix, 387
  NV (definition), 187
  NV (notation), 187

O (archetype), 781
O (Property), 310
O (section), 183
OBC (subsection, section B), 370
OC (Property), 91
OCN (Property), 689
OD (subsection, section SD), 494
OLTTR (example), 603
OM (Property), 199
  one
    column vectors
      Property OC, 91
      complex numbers
        Property OCN, 689
      matrices
        Property OM, 199
      vectors
        Property O, 310
    ONFV (example), 194
    ONS (definition), 193
    ONTV (example), 193
  orthogonal
    linear independence
      theorem OSLI, 190
    set
      example AOS, 189
      set of vectors
        definition OSV, 189
      vector pairs
        definition OV, 189
    orthogonal vectors
      example TOV, 189
    orthonormal
      definition ONS, 193
      matrix columns
        example OSMC, 254
orthonormal set
  four vectors
    example ONFV, 194
  three vectors
    example ONTV, 193
OSGMD (example), 57
OSIS (theorem), 250
OSLI (theorem), 190
OSMC (example), 254
OSV (definition), 189
OV (definition), 189
OV (subsection, section O), 188

P (appendix), 687
P (archetype), 784
P (technique, section PT), 715
particular solutions
  example PSHS, 114
PCNA (theorem), 688
PCVS (example), 318
PD (section), 397
PDM (section), 425
PEE (section), 467
PEEF (theorem), 288
PI (definition), 515
PI (subsection, section LT), 515
PI (technique, section PT), 711
PIP (theorem), 188
PM (example), 443
PM (subsection, section EE), 443
PMI (subsection, section MISLE), 238
PMM (subsection, section MM), 218
PMR (subsection, section MR), 614
PNLT (subsection, section NLT), 672

polynomial
  of a matrix
    example PM, 443
polynomial vector space
dimension
  theorem DP, 383
practice
  technique P, 715
pre-image
  definition PI, 515
  kernel
    theorem KPI, 535
pre-images
  example SPIAS, 515

Property
  AA, 310
  AAC, 91
  AACN, 689
  AAM, 199
  AC, 309
  ACC, 91
  ACCN, 688
  ACM, 199
  AI, 310
  AIC, 91
  AICN, 689
  AIM, 199
  C, 310
  CC, 91
  CM, 199
  DCN, 689
  DMAM, 199
  DSA, 310
  DSAC, 91
  DSAM, 199
  DVA, 310
  DVAC, 91
  MACN, 689
  MCCN, 688
  MICN, 689
  O, 310
  OC, 91
  OCN, 689
  OM, 199
  SC, 309
  SCC, 91
  SCM, 199
  SMA, 310
  SMAC, 91
  SMAM, 199
  Z, 310
  ZC, 91
  ZCN, 689
  ZM, 199

PSHS (example), 114
PSHS (subsection, section LC), 113
PSM (subsection, section SD), 484
PSPHS (theorem), 113
PSS (subsection, section SSLE), 14
PSSD (theorem), 400
PSSLS (theorem), 57
PT (section), 697
example RSC5, 168
relation of linear dependence
definition RLD, 345
definition RLDCV, 145
REM (definition), 33
REMEF (theorem), 36
REMES (theorem), 34
REMRS (theorem), 269
RES (example), 175
RLD (definition), 345
RLDCV (definition), 145
RLT (definition), 551
RLT (notation), 551
RLT (subsection, section SLT), 551
RLTS (theorem), 553
RMRT (theorem), 401
RNLT (subsection, section IVLT), 575
RN (example), 386
RN (subsection, section D), 385
RNMM (subsection, section D), 387
RNMM (theorem), 387
RNSM (example), 387
RO (definition), 33
RO (notation), 33
ROLT (definition), 575
ROLT (notation), 575
ROM (definition), 385
ROM (notation), 385
ROSLT (theorem), 575
row operations
definition RO, 33
elementary matrices, 410, 411
notation, 33
row reduce
mathematica, 679
ti83, 686
ti86, 685
row space
Archetype I
element RS, 367
column space
basis
element RS, 367
matrix, 268
notation, 268
row-equivalent matrices
theorem REMRS, 269
row-equivalent matrices
definition REM, 33
element TREM, 34
row space, 269
row spaces
element RSREM, 271
theorem REMES, 34
row-reduce
the verb
definition RR, 42
row-reduced matrices
theorem REMEF, 36
RPI (theorem), 557
RPNC (theorem), 387
RNPD (theorem), 576
RR (definition), 42
RR.MM (computation, section MMA), 679
RR.TI83 (computation, section TI83), 686
RR.TI86 (computation, section TI86), 685
RREF (definition), 35
RREF (example), 36
RREF (section), 29
RREFA (notation), 36
RREFN (example), 51
RREFU (theorem), 116
RRTI (example), 402
RS (example), 368
RSAI (example), 268
RSB (example), 367
RSC5 (example), 168
RSLT (theorem), 554
RSM (definition), 268
RSM (notation), 268
RSM (subsection, section CRS), 268
RSMS (theorem), 337
RSNS (example), 330
RSREM (example), 271
RSSC4 (example), 174
RT (subsection, section PD), 401
RVMR (example), 618
S (archetype), 793
S (definition), 325
S (example), 76
S (section), 325
SAA (example), 39
SAB (example), 39
SABMI (example), 231
SAE (example), 41
SAN (example), 556
SAR (example), 548
SAV (example), 550
SC (definition), 696
SC (example), 696
SC (notation), 696
SC (Property), 309
SC (subsection, section S), 336
SC (subsection, section SET), 695
SC3 (example), 325
SCAA (example), 125
SCAB (example), 127
SCAD (example), 132
scalar closure
  column vectors
    Property SCC, 91
  matrices
    Property SCM, 199
  vectors
    Property SC, 309
scalar multiple
  matrix inverse, 240
scalar multiplication
  canceling scalars
    theorem CSSM, 320
  canceling vectors
    theorem CVSM, 320
  zero scalar
    theorem ZSSM, 317
  zero vector
    theorem ZVSM, 318
  zero vector result
    theorem SMEZV, 319
scalar multiplication associativity
  column vectors
    Property SMAC, 91
  matrices
    Property SMAM, 199
  vectors
    Property SMA, 310
SCB (theorem), 647
SCC (Property), 91
SCM (Property), 199
SD (section), 483
SE (definition), 694
SE (notation), 694
SEE (example), 442
SEEF (example), 288
SER (theorem), 485
set
cardinality
  definition C, 695
  example CS, 695
  notation, 695
complement
  definition SC, 696
  example SC, 696
  notation, 696
definition SET, 693
empty
  definition ES, 693
equality
  definition SE, 694
  notation, 694
intersection
  definition SI, 695
  example SI, 696
  notation, 696
membership
  example SETM, 693
  notation, 693
size, 695
subset, 693
union
  definition SU, 695
  example SU, 695
  notation, 695
SET (definition), 693
SET (section), 693
SETM (example), 693
SETM (notation), 693
shoes, 239
SHS (subsection, section HSE), 65
SI (definition), 695
SI (example), 696
SI (notation), 696
SI (subsection, section IVLT), 573
SIM (definition), 483
similar matrices
  equal eigenvalues
    example EENS, 486
  equal eigenvalues
    theorem SMEE, 485
INDEX  849

SP4 (example), 328
span
  basic
    example ABS, 123
  basis
    theorem BS, 172
definition SS, 332
definition SSCV, 123
improved
  example IAS, 272
notation, 123
reducing
  example RSSC4, 174
reduction
  example RS, 368
removing vectors
  example COV, 170
reworking elements
  example RES, 175
set of polynomials
  example SSP, 333
subspace
  theorem SSS, 332
span of columns
Archetype A
  example SCAA, 125
Archetype B
  example SCAB, 127
Archetype D
  example SCAD, 132
spanning set
  crazy vector space
    example SSC, 353
definition TSVS, 350
matrices
  example SSM22, 352
more vectors
  theorem SSLD, 379
polynomials
  example SSP4, 350
SPIAS (example), 515
SQM (definition), 75
SRR (example), 77
SS (definition), 332
SS (example), 414
SS (section), 123
SS (subsection, section LISS), 350
SS (theorem), 239
SSC (example), 353
SSCV (definition), 123
SSET (definition), 693
SSET (example), 693
SSET (notation), 693
SSLD (theorem), 379
SSLE (section), 13
SSM22 (example), 352
SSNS (example), 130
SSNS (subsection, section SS), 129
SSNS (theorem), 129
SSP (example), 333
SSP4 (example), 350
SSRLT (theorem), 556
SSS (theorem), 332
SSSLT (subsection, section SLT), 556
SSV (notation), 123
SSV (subsection, section SS), 123
starting proofs
  technique GS, 702
STLT (example), 518
STNE (example), 13
SU (definition), 695
SU (example), 695
SU (notation), 695
submatrix
  notation, 414
subset
  definition SSET, 693
notation, 693
subspace
  as null space
    example RSNS, 330
characterized
  example ASC, 594
definition S, 325
in \( P_4 \)
  example SP4, 328
not, additive closure
  example NSC2A, 329
not, scalar closure
  example NSC2S, 329
not, zero vector
  example NSC2Z, 329
testing
  theorem TSS, 327
trivial
  definition TS, 330

Version 0.92
verification
example SC3, 325
example SM32, 334

subspaces
equal dimension
theorem EDYES, 401

surjective
Archetype N
dexample SAN, 556
dexample SAR, 548
not
dexample NSAQ, 547
dexample NSAQR, 555
not, Archetype O
dexample NSAQ, 555
not, by dimension
dexample NSDAT, 558
polynomials to matrices
dexample SAV, 550

surjective linear transformation
bases
theorem SLTB, 557

surjective linear transformations
dimension
theorem SLTD, 558

SUV (definition), 234
SUVB (theorem), 363
SUVOS (example), 189
SV (definition), 31
SVP4 (example), 400
SYM (definition), 201
SYM (example), 201

symmetric matrices
theorem SMS, 201

symmetric matrix
dexample SYM, 201

system of equations
vector equality
dexample VESE, 88

system of linear equations
definition SLE, 13

t (archetype), 796
t (part), 819
t (technique, section PT), 699
TCSD (example), 418
technique
c, 703

cd, 708
cp, 706
cv, 707
d, 698
dc, 712
e, 704
gs, 702
i, 713
l, 700
lc, 716
me, 710
n, 705
p, 715
pl, 711
r, 699
u, 709

theorem
AISM, 318
AIU, 317
BCS, 264
BIS, 383
BNS, 154
BRS, 271
BS, 172
CB, 639
CCRA, 690
CCRM, 690
CCT, 690
CFDVS, 593
CILTI, 539
CINM, 237
CIVLT, 572
CLI, 595
CLTLT, 520
CMVEI, 57
CNMB, 369
COB, 371
CRMA, 203
CRMSM, 204
CRN, 386
CRSM, 184
CRVA, 183
CSCS, 262
CSLTS, 559
CSMS, 336
CSNM, 267
CSRN, 56
CSRST, 273
<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSS</td>
<td>595</td>
</tr>
<tr>
<td>CSSM</td>
<td>320</td>
</tr>
<tr>
<td>CUMOS</td>
<td>253</td>
</tr>
<tr>
<td>CVSM</td>
<td>320</td>
</tr>
<tr>
<td>DC</td>
<td>487</td>
</tr>
<tr>
<td>DCM</td>
<td>383</td>
</tr>
<tr>
<td>DCP</td>
<td>472</td>
</tr>
<tr>
<td>DEC</td>
<td>417</td>
</tr>
<tr>
<td>DED</td>
<td>492</td>
</tr>
<tr>
<td>DEM</td>
<td>431</td>
</tr>
<tr>
<td>DEMMM</td>
<td>431</td>
</tr>
<tr>
<td>DER</td>
<td>416</td>
</tr>
<tr>
<td>DERC</td>
<td>427</td>
</tr>
<tr>
<td>DFS</td>
<td>402</td>
</tr>
<tr>
<td>DIM</td>
<td>430</td>
</tr>
<tr>
<td>DLDS</td>
<td>167</td>
</tr>
<tr>
<td>DM</td>
<td>383</td>
</tr>
<tr>
<td>DMFE</td>
<td>490</td>
</tr>
<tr>
<td>DMST</td>
<td>415</td>
</tr>
<tr>
<td>DNLT</td>
<td>673</td>
</tr>
<tr>
<td>DP</td>
<td>383</td>
</tr>
<tr>
<td>DRCM</td>
<td>426</td>
</tr>
<tr>
<td>DRCMA</td>
<td>427</td>
</tr>
<tr>
<td>DRCS</td>
<td>425</td>
</tr>
<tr>
<td>DRMM</td>
<td>434</td>
</tr>
<tr>
<td>DT</td>
<td>417</td>
</tr>
<tr>
<td>DZRC</td>
<td>425</td>
</tr>
<tr>
<td>EDELI</td>
<td>467</td>
</tr>
<tr>
<td>EDYES</td>
<td>401</td>
</tr>
<tr>
<td>EER</td>
<td>650</td>
</tr>
<tr>
<td>EIM</td>
<td>471</td>
</tr>
<tr>
<td>ELIS</td>
<td>397</td>
</tr>
<tr>
<td>EMDRO</td>
<td>411</td>
</tr>
<tr>
<td>EMHE</td>
<td>445</td>
</tr>
<tr>
<td>EMMVP</td>
<td>214</td>
</tr>
<tr>
<td>EMN</td>
<td>413</td>
</tr>
<tr>
<td>EMNS</td>
<td>451</td>
</tr>
<tr>
<td>EMP</td>
<td>216</td>
</tr>
<tr>
<td>EMRCP</td>
<td>449</td>
</tr>
<tr>
<td>EMS</td>
<td>450</td>
</tr>
<tr>
<td>ENLT</td>
<td>672</td>
</tr>
<tr>
<td>EOMP</td>
<td>469</td>
</tr>
<tr>
<td>EOPSS</td>
<td>16</td>
</tr>
<tr>
<td>EPM</td>
<td>470</td>
</tr>
<tr>
<td>ERMCP</td>
<td>472</td>
</tr>
<tr>
<td>ESMM</td>
<td>469</td>
</tr>
<tr>
<td>ETM</td>
<td>471</td>
</tr>
<tr>
<td>FS</td>
<td>290</td>
</tr>
<tr>
<td>FTMR</td>
<td>606</td>
</tr>
<tr>
<td>FVCS</td>
<td>56</td>
</tr>
<tr>
<td>G</td>
<td>398</td>
</tr>
<tr>
<td>GSPCV</td>
<td>191</td>
</tr>
<tr>
<td>HMOE</td>
<td>477</td>
</tr>
<tr>
<td>HMRE</td>
<td>476</td>
</tr>
<tr>
<td>HMVEI</td>
<td>67</td>
</tr>
<tr>
<td>HSC</td>
<td>65</td>
</tr>
<tr>
<td>ICBM</td>
<td>639</td>
</tr>
<tr>
<td>ICVT</td>
<td>572</td>
</tr>
<tr>
<td>IFDVS</td>
<td>594</td>
</tr>
<tr>
<td>IILT</td>
<td>570</td>
</tr>
<tr>
<td>ILTB</td>
<td>537</td>
</tr>
<tr>
<td>ILTD</td>
<td>538</td>
</tr>
<tr>
<td>ILTIS</td>
<td>571</td>
</tr>
<tr>
<td>ILTLA</td>
<td>537</td>
</tr>
<tr>
<td>ILTLT</td>
<td>570</td>
</tr>
<tr>
<td>IMILN</td>
<td>521</td>
</tr>
<tr>
<td>IMR</td>
<td>619</td>
</tr>
<tr>
<td>IPAC</td>
<td>186</td>
</tr>
<tr>
<td>IPN</td>
<td>187</td>
</tr>
<tr>
<td>IPSM</td>
<td>186</td>
</tr>
<tr>
<td>IPVA</td>
<td>185</td>
</tr>
<tr>
<td>ISRM</td>
<td>55</td>
</tr>
<tr>
<td>IVSED</td>
<td>575</td>
</tr>
<tr>
<td>KILT</td>
<td>535</td>
</tr>
<tr>
<td>KLTS</td>
<td>533</td>
</tr>
<tr>
<td>KNSI</td>
<td>614</td>
</tr>
<tr>
<td>KPI</td>
<td>535</td>
</tr>
<tr>
<td>KPLT</td>
<td>535</td>
</tr>
<tr>
<td>KPNLT</td>
<td>673</td>
</tr>
<tr>
<td>LIVHS</td>
<td>675</td>
</tr>
<tr>
<td>LIVRN</td>
<td>149</td>
</tr>
<tr>
<td>LNSMS</td>
<td>337</td>
</tr>
<tr>
<td>LTDB</td>
<td>512</td>
</tr>
<tr>
<td>LTLC</td>
<td>512</td>
</tr>
<tr>
<td>LTTZ</td>
<td>507</td>
</tr>
<tr>
<td>MBLT</td>
<td>509</td>
</tr>
<tr>
<td>MCT</td>
<td>204</td>
</tr>
<tr>
<td>ME</td>
<td>474</td>
</tr>
<tr>
<td>MIMI</td>
<td>240</td>
</tr>
<tr>
<td>MISM</td>
<td>240</td>
</tr>
<tr>
<td>MIT</td>
<td>240</td>
</tr>
<tr>
<td>MIU</td>
<td>239</td>
</tr>
<tr>
<td>MLTCV</td>
<td>510</td>
</tr>
<tr>
<td>MLTLT</td>
<td>518</td>
</tr>
<tr>
<td>MMA</td>
<td>220</td>
</tr>
<tr>
<td>MMCC</td>
<td>221</td>
</tr>
<tr>
<td>Term</td>
<td>Page</td>
</tr>
<tr>
<td>----------</td>
<td>------</td>
</tr>
<tr>
<td>MMDAA</td>
<td>219</td>
</tr>
<tr>
<td>MMIM</td>
<td>218</td>
</tr>
<tr>
<td>MMIP</td>
<td>220</td>
</tr>
<tr>
<td>MMSMM</td>
<td>219</td>
</tr>
<tr>
<td>MMT</td>
<td>221</td>
</tr>
<tr>
<td>MMZM</td>
<td>218</td>
</tr>
<tr>
<td>MNEM</td>
<td>475</td>
</tr>
<tr>
<td>MRCB</td>
<td>644</td>
</tr>
<tr>
<td>MRCLT</td>
<td>610</td>
</tr>
<tr>
<td>MRMLT</td>
<td>610</td>
</tr>
<tr>
<td>MRSLT</td>
<td>609</td>
</tr>
<tr>
<td>MVSLD</td>
<td>150</td>
</tr>
<tr>
<td>NEM</td>
<td>373</td>
</tr>
<tr>
<td>NI</td>
<td>251</td>
</tr>
<tr>
<td>NME1</td>
<td>81</td>
</tr>
<tr>
<td>NME2</td>
<td>152</td>
</tr>
<tr>
<td>NME3</td>
<td>251</td>
</tr>
<tr>
<td>NME4</td>
<td>267</td>
</tr>
<tr>
<td>NME5</td>
<td>370</td>
</tr>
<tr>
<td>NME6</td>
<td>388</td>
</tr>
<tr>
<td>NME7</td>
<td>433</td>
</tr>
<tr>
<td>NME8</td>
<td>408</td>
</tr>
<tr>
<td>NME9</td>
<td>622</td>
</tr>
<tr>
<td>NMLIC</td>
<td>151</td>
</tr>
<tr>
<td>NMPREM</td>
<td>413</td>
</tr>
<tr>
<td>NMRR1</td>
<td>77</td>
</tr>
<tr>
<td>NMTNS</td>
<td>78</td>
</tr>
<tr>
<td>NMUS</td>
<td>79</td>
</tr>
<tr>
<td>NOILT</td>
<td>576</td>
</tr>
<tr>
<td>NPNT</td>
<td>249</td>
</tr>
<tr>
<td>NSMS</td>
<td>330</td>
</tr>
<tr>
<td>OSIS</td>
<td>250</td>
</tr>
<tr>
<td>OSLL</td>
<td>190</td>
</tr>
<tr>
<td>PCNA</td>
<td>688</td>
</tr>
<tr>
<td>PEEF</td>
<td>288</td>
</tr>
<tr>
<td>PIP</td>
<td>188</td>
</tr>
<tr>
<td>PSPHS</td>
<td>113</td>
</tr>
<tr>
<td>PSSD</td>
<td>400</td>
</tr>
<tr>
<td>PSSLS</td>
<td>57</td>
</tr>
<tr>
<td>RCLS</td>
<td>54</td>
</tr>
<tr>
<td>RCSI</td>
<td>617</td>
</tr>
<tr>
<td>REMEF</td>
<td>36</td>
</tr>
<tr>
<td>REMES</td>
<td>34</td>
</tr>
<tr>
<td>REMRS</td>
<td>269</td>
</tr>
<tr>
<td>RLTTS</td>
<td>553</td>
</tr>
<tr>
<td>RMRT</td>
<td>401</td>
</tr>
<tr>
<td>RNNM</td>
<td>387</td>
</tr>
<tr>
<td>ROSLT</td>
<td>375</td>
</tr>
<tr>
<td>RPI</td>
<td>557</td>
</tr>
<tr>
<td>RPNC</td>
<td>387</td>
</tr>
<tr>
<td>RPNDD</td>
<td>576</td>
</tr>
<tr>
<td>RREFU</td>
<td>116</td>
</tr>
<tr>
<td>RSLT</td>
<td>555</td>
</tr>
<tr>
<td>RSMS</td>
<td>337</td>
</tr>
<tr>
<td>SCB</td>
<td>647</td>
</tr>
<tr>
<td>SER</td>
<td>485</td>
</tr>
<tr>
<td>SLEMM</td>
<td>212</td>
</tr>
<tr>
<td>SLSLC</td>
<td>100</td>
</tr>
<tr>
<td>SLTB</td>
<td>557</td>
</tr>
<tr>
<td>SLTD</td>
<td>558</td>
</tr>
<tr>
<td>SLEMM</td>
<td>212</td>
</tr>
<tr>
<td>SMEE</td>
<td>485</td>
</tr>
<tr>
<td>SMEZV</td>
<td>319</td>
</tr>
<tr>
<td>SMS</td>
<td>201</td>
</tr>
<tr>
<td>SMZD</td>
<td>432</td>
</tr>
<tr>
<td>SMZE</td>
<td>408</td>
</tr>
<tr>
<td>SNCM</td>
<td>252</td>
</tr>
<tr>
<td>SSS</td>
<td>239</td>
</tr>
<tr>
<td>SSLD</td>
<td>379</td>
</tr>
<tr>
<td>SSNS</td>
<td>129</td>
</tr>
<tr>
<td>SSRLT</td>
<td>556</td>
</tr>
<tr>
<td>SSS</td>
<td>332</td>
</tr>
<tr>
<td>TMA</td>
<td>202</td>
</tr>
<tr>
<td>TMSM</td>
<td>202</td>
</tr>
<tr>
<td>TSS</td>
<td>327</td>
</tr>
<tr>
<td>TT</td>
<td>202</td>
</tr>
<tr>
<td>TTTMI</td>
<td>234</td>
</tr>
<tr>
<td>UMI</td>
<td>253</td>
</tr>
<tr>
<td>UMPHIP</td>
<td>254</td>
</tr>
<tr>
<td>VAC</td>
<td>319</td>
</tr>
<tr>
<td>VFSLS</td>
<td>107</td>
</tr>
<tr>
<td>VRI</td>
<td>592</td>
</tr>
<tr>
<td>VRILT</td>
<td>593</td>
</tr>
<tr>
<td>VRLT</td>
<td>587</td>
</tr>
<tr>
<td>VRBB</td>
<td>355</td>
</tr>
<tr>
<td>VRS</td>
<td>592</td>
</tr>
<tr>
<td>VSLT</td>
<td>519</td>
</tr>
<tr>
<td>VSPCV</td>
<td>91</td>
</tr>
<tr>
<td>VSPM</td>
<td>199</td>
</tr>
<tr>
<td>ZSSM</td>
<td>317</td>
</tr>
<tr>
<td>ZVSM</td>
<td>318</td>
</tr>
<tr>
<td>ZVU</td>
<td>317</td>
</tr>
<tr>
<td>ti83</td>
<td></td>
</tr>
<tr>
<td>matrix entry (computation)</td>
<td>686</td>
</tr>
</tbody>
</table>
row reduce (computation), 686
vector linear combinations (computation), 686
TI83 (section), 686
ti86
matrix entry (computation), 684
row reduce (computation), 685
transpose of a matrix (computation), 685
vector linear combinations (computation), 686
TI86 (section), 684
TIVS (example), 594
TKAP (example), 534
TLC (example), 97
TM (definition), 200
TM (example), 201
TM (notation), 201
TM.MMA (computation, section MMA), 683
TM.TI86 (computation, section TI86), 685
TMA (theorem), 202
TMP (example), 4
TMSM (theorem), 202
TOV (example), 189
trail mix
example TMP, 4
transpose
matrix scalar multiplication
theorem TMSM, 202
example TM, 201
matrix addition
theorem TMA, 202
matrix inverse, 240
notation, 201
scalar multiplication, 202
transpose of a matrix
mathematica, 683
ti86, 685
transpose of a transpose
theorem TT, 202
TREM (example), 34
trivial solution
system of equations
definition TSHSE, 66
TS (definition), 330
TS (subsection, section S), 326
TSHSE (definition), 66
TSM (subsection, section MO), 200
TSS (section), 51
TSS (subsection, section S), 331
TSS (theorem), 327
TSVS (definition), 350
TT (theorem), 202
TTMI (theorem), 234
TTS (example), 14
typical systems, 2 × 2
typical systems, example TTS, 14
U (archetype), 798
U (technique, section PT), 709
UM (definition), 252
UM (subsection, section MINM), 252
UM3 (example), 252
UMI (theorem), 253
UMPI (theorem), 254
unique solution, 3 × 3
unique solution, example US, 19
unique solution, example USR, 34
uniqueness
technique U, 709
unit vectors
basis
theorem SUVB, 363
definition SUV, 234
orthogonal
text example SUVOS, 189
unitary
permutation matrix
example UPM, 253
size 3
example UM3, 252
unitary matrices
columns
theorem CUMOS, 253
unitary matrix
inner product
theorem UMPIP, 254
UPM (example), 253
URREF (subsection, section LC), 116
US (example), 19
USR (example), 34
V (archetype), 801
V (chapter), 87
VA (example), 89
VAC (theorem), 319
VEASM (subsection, section VO), 88
vector addition
definition CVA, 89
vector component
notation, 30
vector form of solutions
Archetype D
example VFSAD, 102
Archetype I
example VFSAI, 110
Archetype L
example VFSAL, 111
example VFS, 104
mathematica, 681
theorem VFSLS, 107
vector linear combinations
mathematica, 680
ti83, 686
ti86, 685
vector representation
example AVR, 354
example VRC4, 589
injective
theorem VRI, 592
invertible
theorem VRILT, 593
linear transformation
definition VR, 587
theorem VRLT, 587
surjective
theorem VRS, 592
vector scalar multiplication
example CVSM, 90
vector space
characterization
theorem CFDVS, 593
column vectors
definition VSCV, 87
definition VS, 309
infinite dimension
equation VSFUD, 385
linear transformations
theorem VSLT, 519
vector space of column vectors
notation, 87
vector space of functions
example VSF, 313
vector space of infinite sequences
example VSIS, 313
vector space of matrices
definition VSM, 197
example VSM, 311
notation, 197
vector space of polynomials
example VSP, 312
vector space properties
column vectors
theorem VSPCV, 91
matrices
theorem VSPM, 199
vector space, crazy
example CVS, 314
vector space, singleton
example VSS, 314
vector spaces
isomorphic
definition IVS, 573
theorem IFDVS, 594
VESE (example), 88
VFS (example), 104
VFSAD (example), 102
VFSAI (example), 110
VFSAL (example), 111
VFSLS (theorem), 107
VFSS (subsection, section LC), 102
VFSS.MMA (computation, section MMA), 681
VLC.MMA (computation, section MMA), 680
VLC.TI83 (computation, section TI83), 686
VLC.TI86 (computation, section TI86), 685
VO (section), 87
VOC (definition), 31
VR (definition), 587
VR (section), 587
VR (subsection, section LISS), 354
VRC4 (example), 589
VRI (theorem), 592
VRLLT (theorem), 593
VRLT (theorem), 587
VRP2 (example), 591
VRRB (theorem), 355
VRS (theorem), 592
VS (chapter), 309
VS (definition), 309
VS (section), 309
VS (subsection, section VS), 309
VSCV (definition), 87
VSCV (example), 311
VSCV (notation), 87
VSF (example), 313
VSIS (example), 313
VSLT (theorem), 319
VSM (definition), 197
VSM (example), 311
VSM (notation), 197
VSP (example), 312
VSP (subsection, section MO), 199
VSP (subsection, section VO), 90
VSP (subsection, section VS), 316
VSPCV (theorem), 91
VSPM (theorem), 199
VSPUD (example), 385
VSS (example), 314

W (archetype), 803
WILA (section), 3

X (archetype), 806

Z (Property), 310
ZC (Property), 91
ZCN (Property), 689
ZCV (definition), 30

zero complex numbers
Property ZCN, 689
zero column vector
definition ZCV, 30
notation, 30
zero matrix
notation, 200
zero vector
column vectors
Property ZC, 91
matrices
Property ZM, 199
unique
theorem ZVU, 317
vectors
Property Z, 310
ZM (definition), 200
ZM (notation), 200
ZM (Property), 199
ZNDAB (example), 433
ZSSM (theorem), 317
ZVSM (theorem), 318
ZVU (theorem), 317