Flashcard Supplement to A First Course in Linear Algebra

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Definition SLE System of Linear Equations

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A system of linear equations is a collection of m equations in the variable quantities $x_1, x_2, x_3, \ldots, x_n$ of the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

where the values of a_{ij} , b_i and x_j are from the set of complex numbers, \mathbb{C} .

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Definition SSLE Solution of a System of Linear Equations

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A solution of a system of linear equations in n variables, $x_1, x_2, x_3, \ldots, x_n$ (such as the system given in Definition SLE, is an ordered list of n complex numbers, $s_1, s_2, s_3, \ldots, s_n$ such that if we substitute s_1 for x_1, s_2 for x_2, s_3 for x_3, s_n for x_n , then for every equation of the system the left side will equal the right side, i.e. each equation is true simultaneously.

Definition SSSLE Solution Set of a System of Linear Equations The solution set of a linear system of equations is the set which contains every solution to to system, and nothing more.					
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Definition ESYS Equivalent Systems Two systems of linear equations are equivalent if their solution sets are equal. ©2004 Robert A. Beezer, GFDL License

Definition EO Equation Operations

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Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an equation operation.

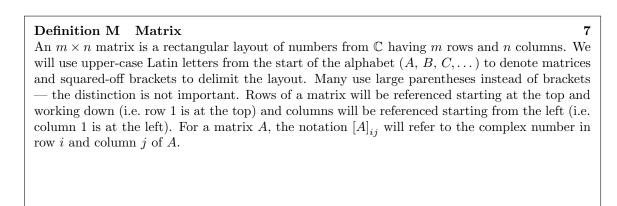
- 1. Swap the locations of two equations in the list of equations.
- 2. Multiply each term of an equation by a nonzero quantity.
- 3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

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Theorem EOPSS Equation Operations Preserve Solution Sets

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If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.



Definition CV Column Vector

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A column vector of size m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a vector. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as \mathbf{u} , \mathbf{v} , \mathbf{w} , \mathbf{x} , \mathbf{y} , \mathbf{z} . Some books like to write vectors with arrows, such as \vec{u} . Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in u. To refer to the entry or component of vector \mathbf{v} in location i of the list, we write $[\mathbf{v}]_i$.

Definition ZCV Zero Column Vector

Q

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The zero vector of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or defined much more compactly, $\left[\mathbf{0}\right]_{i}=0$ for $1\leq i\leq m.$

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Definition CM Coefficient Matrix

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the coefficient matrix is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

Definition VOC Vector of Constants

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the vector of constants is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

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Definition SOLV Solution Vector

For a system of linear equations,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

the solution vector is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

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Definition MRLS Matrix Representation of a Linear System 13 If A is the coefficient matrix of a system of linear equations and $\mathbf b$ is the vector of constants, then we will write $\mathcal{LS}(A, \mathbf b)$ as a shorthand expression for the system of linear equations, which we will refer to as the matrix representation of the linear system.
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Definition AM Augmented Matrix Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants \mathbf{b} . Then the augmented matrix of the system of equations is the $m \times (n+1)$ matrix whose first n columns are the columns of A and whose last column (number $n+1$) is the column vector \mathbf{b} . This matrix will be written as $[A \mid \mathbf{b}]$.

Definition RO Row Operations

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The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a row operation.

- 1. Swap the locations of two rows.
- 2. Multiply each entry of a single row by a nonzero quantity.
- 3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

- 1. $R_i \leftrightarrow R_j$: Swap the location of rows i and j.
- 2. αR_i : Multiply row i by the nonzero scalar α .
- 3. $\alpha R_i + R_j$: Multiply row i by the scalar α and add to row j.

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Definition REM Row-Equivalent Matrices

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Two matrices, A and B, are row-equivalent if one can be obtained from the other by a sequence of row operations.

Theorem REMES Row-Equivalent Matrice	es represent Equival	ent Systems 17
Suppose that A and B are row-equivalent augm	ented matrices. Then	the systems of linear
equations that they represent are equivalent system	ms.	
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Definition RREF Reduced Row-Echelon Form

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A matrix is in reduced row-echelon form if it meets all of the following conditions:

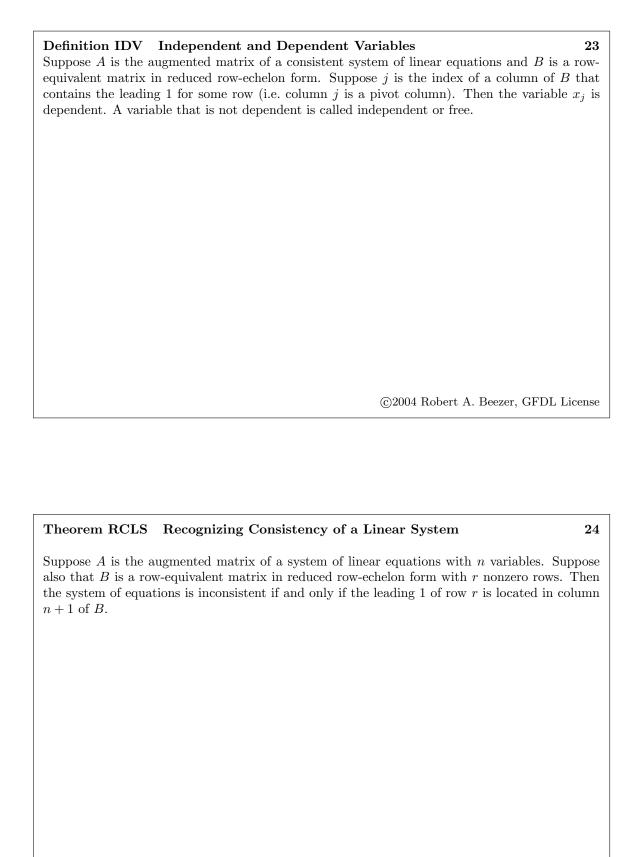
- 1. If there is a row where every entry is zero, then this row lies below any other row that contains a nonzero entry.
- 2. The leftmost nonzero entry of a row is equal to 1.
- 3. The leftmost nonzero entry of a row is the only nonzero entry in its column.
- 4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If s > i, then t > j.

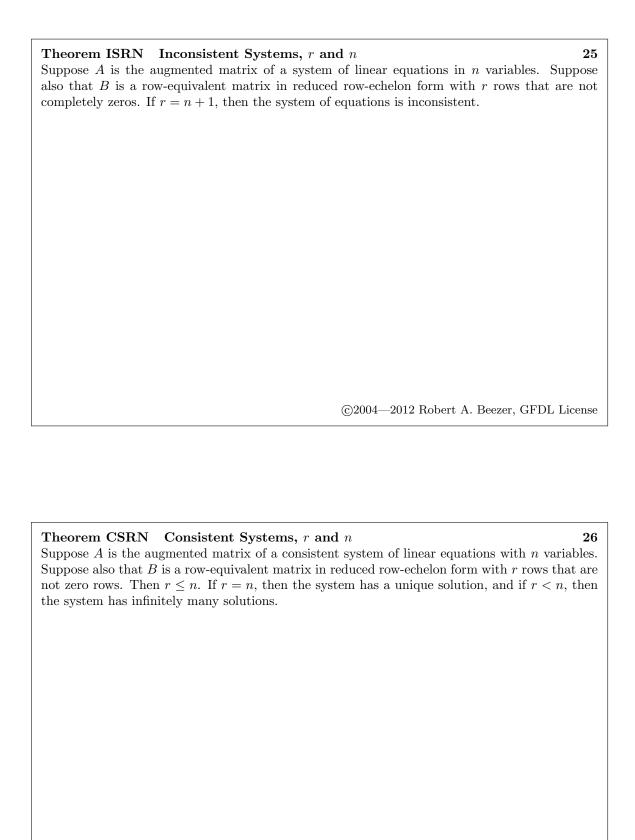
A row of only zero entries will be called a zero row and the leftmost nonzero entry of a nonzero row will be called a leading 1. The number of nonzero rows will be denoted by r.

A column containing a leading 1 will be called a pivot column. The set of column indices for all of the pivot columns will be denoted by $D = \{d_1, d_2, d_3, \ldots, d_r\}$ where $d_1 < d_2 < d_3 < \cdots < d_r$, while the columns that are not pivot columns will be denoted as $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

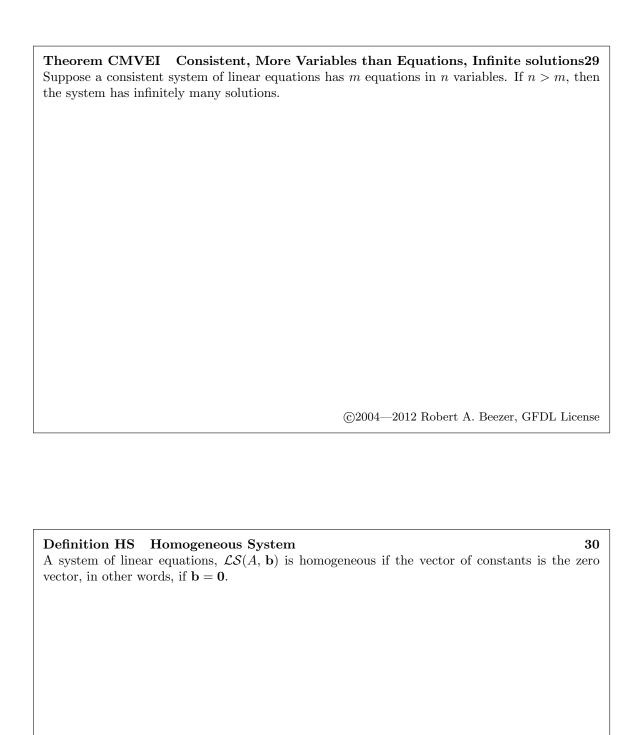
Theorem REMEF Row-Equivalent Matrix in Echelon Form Suppose A is a matrix. Then there is a matrix B so that	19
1. A and B are row-equivalent.	
2. B is in reduced row-echelon form.	
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Theorem RREFU Reduced Row-Echelon Form is Unique Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are row-to A and in reduced row-echelon form. Then $B = C$.	20 equivalent

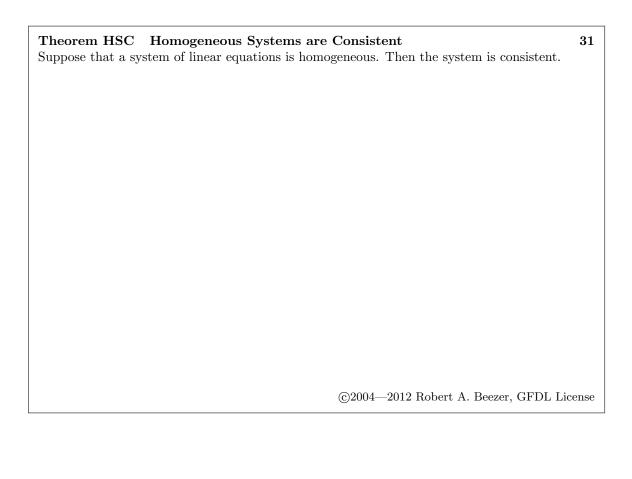
Definition RR Row-Reducing To row-reduce the matrix A means to apply row operations to A and arrive at a row-equival matrix B in reduced row-echelon form.	21 lent
matrix B in reduced row-echelon form.	
©2004 Robert A. Beezer, GFDL Lice	ense
Definition CS Consistent System	22
A system of linear equations is consistent if it has at least one solution. Otherwise, the syst is called inconsistent.	





Theorem FVCS Free Variables for Consistent Systems Suppose A is the augmented matrix of a consistent system of linear equations with n variables Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r rows that not completely zeros. Then the solution set can be described with $n-r$ free variables.	
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Theorem PSSLS Possible Solution Sets for Linear Systems A system of linear equations has no solutions, a unique solution or infinitely many solutions	28

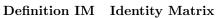




Definition TSHSE Trivial Solution to Homogeneous Systems of Equations 32 Suppose a homogeneous system of linear equations has n variables. The solution $x_1 = 0$, $x_2 = 0$, , $x_n = 0$ (i.e. $\mathbf{x} = \mathbf{0}$) is called the trivial solution.

Theorem HMVEI 33	Homogeneous, More Variables than Equations, Infinite solutions
Suppose that a homoge	eneous system of linear equations has m equations and n variables with m has infinitely many solutions.
70 > 700. Then the Byston	in that infinitely many sortetons.
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	ull Space of a Matrix atrix A , denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions to $\mathcal{LS}(A, 0)$.

Definition SQM Square Matrix A matrix with m rows and n columns is square if $m = n$. In this case, we say the matrix has size n . To emphasize the situation when a matrix is not square, we will call it rectangular.
G
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Definition NM Nonsingular Matrix 36 Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A, 0)$ is $\{0\}$, in other words, the system has only the trivial solution. Then we say that A is a nonsingular matrix. Otherwise we say A is a singular matrix.



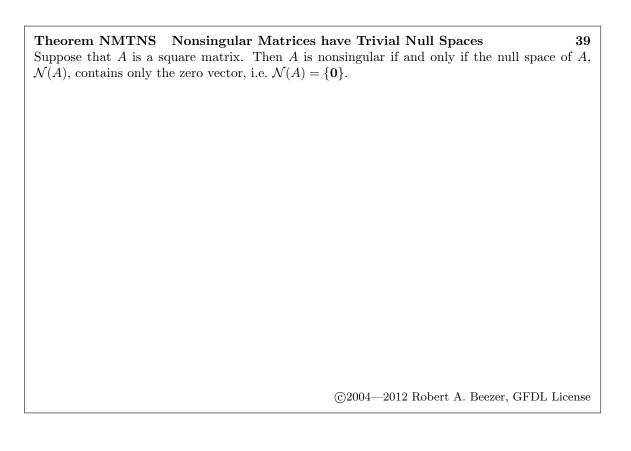
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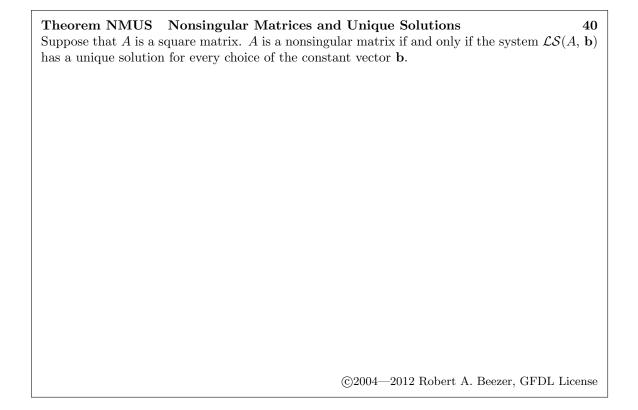
The $m \times m$ identity matrix, I_m , is defined by

$$\left[I_m\right]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad 1 \leq i, j \leq m$$

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Theorem NMRRI Nonsingular Matrices Row Reduce to the Identity matrix 38 Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.





Theorem NME1 Nonsingular Matrix Equivalences, Round 1 Suppose that A is a square matrix. The following are equivalent.	41
1. A is nonsingular.	
2. A row-reduces to the identity matrix.	
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.	
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of	b.
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Definition VSCV Vector Space of Column Vectors	42
The vector space \mathbb{C}^m is the set of all column vectors (Definition CV) of size m with the set of complex numbers, \mathbb{C} .	

Definition CVE Column Vector Equality

43

Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$. Then \mathbf{u} and \mathbf{v} are equal, written $\mathbf{u} = \mathbf{v}$ if

$$[\mathbf{u}]_i = [\mathbf{v}]_i$$

$$1 \leq i \leq m$$

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Definition CVA Column Vector Addition

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Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$. The sum of \mathbf{u} and \mathbf{v} is the vector $\mathbf{u} + \mathbf{v}$ defined by

$$[\mathbf{u} + \mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i$$

$$1 \leq i \leq m$$

Definition CVSM Column Vector Scalar Multiplication

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Suppose $\mathbf{u} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$, then the scalar multiple of \mathbf{u} by α is the vector $\alpha \mathbf{u}$ defined by

$$[\alpha \mathbf{u}]_i = \alpha [\mathbf{u}]_i$$

$$1 \leq i \leq m$$

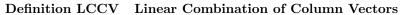
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Theorem VSPCV Vector Space Properties of Column Vectors

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Suppose that \mathbb{C}^m is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- ACC Additive Closure, Column Vectors: If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.
- SCC Scalar Closure, Column Vectors: If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha \mathbf{u} \in \mathbb{C}^m$.
- CC Commutativity, Column Vectors: If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AAC Additive Associativity, Column Vectors: If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- ZC Zero Vector, Column Vectors: There is a vector, $\mathbf{0}$, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.
- AIC Additive Inverses, Column Vectors: If $\mathbf{u} \in \mathbb{C}^m$, then there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMAC Scalar Multiplication Associativity, Column Vectors: If α , $\beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVAC Distributivity across Vector Addition, Column Vectors: If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSAC Distributivity across Scalar Addition, Column Vectors: If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.
- OC One, Column Vectors: If $\mathbf{u} \in \mathbb{C}^m$, then $1\mathbf{u} = \mathbf{u}$.



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Given n vectors \mathbf{u}_1 , \mathbf{u}_2 , \mathbf{u}_3 , ..., \mathbf{u}_n from \mathbb{C}^m and n scalars α_1 , α_2 , α_3 , ..., α_n , their linear combination is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n$$

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Theorem SLSLC Solutions to Linear Systems are Linear Combinations 48 Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$. Then $\mathbf{x} \in \mathbb{C}n$ is a

Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$. Then $\mathbf{x} \in \mathbb{C}n$ is a solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$ if and only if \mathbf{b} equals the linear combination of the columns of A formed with the entries of \mathbf{x} ,

$$\left[\mathbf{x}\right]_{1}\mathbf{A}_{1}+\left[\mathbf{x}\right]_{2}\mathbf{A}_{2}+\left[\mathbf{x}\right]_{3}\mathbf{A}_{3}+\cdots+\left[\mathbf{x}\right]_{n}\mathbf{A}_{n}=\mathbf{b}$$

Theorem VFSLS Vector Form of Solutions to Linear Systems

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Suppose that $[A \mid \mathbf{b}]$ is the augmented matrix for a consistent linear system $\mathcal{LS}(A, \mathbf{b})$ of m equations in n variables. Let B be a row-equivalent $m \times (n+1)$ matrix in reduced row-echelon form. Suppose that B has r nonzero rows, columns without leading 1's with indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$, and columns with leading 1's (pivot columns) having indices $D = \{d_1, d_2, d_3, \ldots, d_r\}$. Define vectors $\mathbf{c}, \mathbf{u}_j, 1 \leq j \leq n-r$ of size n by

$$\begin{aligned} \left[\mathbf{c}\right]_i &= \begin{cases} 0 & \text{if } i \in F \\ \left[B\right]_{k,n+1} & \text{if } i \in D, \, i = d_k \end{cases} \\ \left[\mathbf{u}_j\right]_i &= \begin{cases} 1 & \text{if } i \in F, \, i = f_j \\ 0 & \text{if } i \in F, \, i \neq f_j \\ -\left[B\right]_{k,f_j} & \text{if } i \in D, \, i = d_k \end{cases} . \end{aligned}$$

Then the set of solutions to the system of equations $\mathcal{LS}(A, \mathbf{b})$ is

$$S = \{ \mathbf{c} + \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_{n-r} \mathbf{u}_{n-r} | \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-r} \in \mathbb{C} \}$$

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Theorem PSPHS Particular Solution Plus Homogeneous Solutions

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Suppose that **w** is one solution to the linear system of equations $\mathcal{LS}(A, b)$. Then **y** is a solution to $\mathcal{LS}(A, b)$ if and only if $\mathbf{y} = \mathbf{w} + \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$.

Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$, their span, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p$. Symbolically,

$$\langle S \rangle = \left\{ \left. \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_p \mathbf{u}_p \right| \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$
$$= \left\{ \left. \sum_{i=1}^p \alpha_i \mathbf{u}_i \right| \alpha_i \in \mathbb{C}, \ 1 \le i \le p \right\}$$

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Theorem SSNS Spanning Sets for Null Spaces

 $\mathbf{52}$

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ be the column indices where B has leading 1's (pivot columns) and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the set of column indices where B does not have leading 1's. Construct the n-r vectors \mathbf{z}_j , $1 \le j \le n-r$ of size n as

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F, \ i = f_j \\ 0 & \text{if } i \in F, \ i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D, \ i = d_k \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \, \mathbf{z}_2, \, \mathbf{z}_3, \, \dots, \, \mathbf{z}_{n-r}\} \rangle$$

Definition	RLDCV	\mathbf{R}	elation	of Linear	Dependence	\mathbf{for}	Column	${f Vectors}$	
	_		_						

Given a set of vectors $S = {\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n}$, a true statement of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a relation of linear dependence on S. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \le i \le n$, then we say it is the trivial relation of linear dependence on S.

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Definition LICV Linear Independence of Column Vectors

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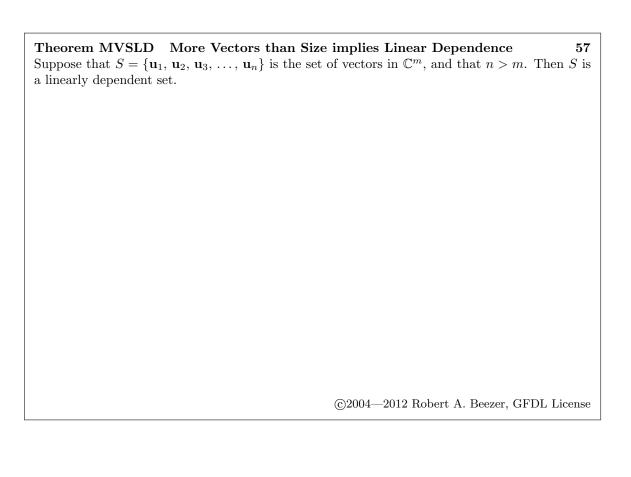
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The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is linearly dependent if there is a relation of linear dependence on S that is not trivial. In the case where the only relation of linear dependence on S is the trivial one, then S is a linearly independent set of vectors.

Theorem LIVRN Linearly Independent Vectors, r and n

 $\mathbf{56}$

Suppose that A is an $m \times n$ matrix and $S = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$ is the set of vectors in \mathbb{C}^m that are the columns of A. Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of non-zero rows in B. Then S is linearly independent if and only if n = r.



Theorem NMLIC Nonsingular Matrices have Linearly Independent Columns 58 Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

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Suppose that A is a square matrix. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of **b**.
- 5. The columns of A form a linearly independent set.

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Theorem BNS Basis for Null Spaces

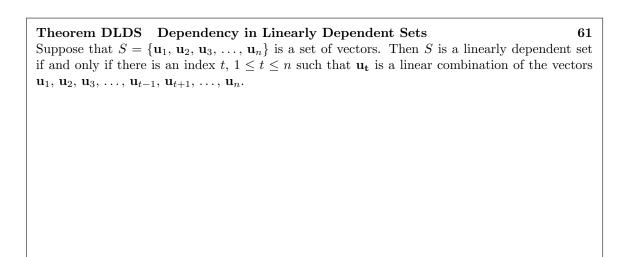
60

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ and $F = \{f_1, f_2, f_3, \ldots, f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have leading 1's. Construct the n-r vectors \mathbf{z}_j , $1 \le j \le n-r$ of size n as

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F, \ i = f_j \\ 0 & \text{if } i \in F, \ i \neq f_j \\ -[B]_{k,f_i} & \text{if } i \in D, \ i = d_k \end{cases}$$

Define the set $S = {\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_{n-r}}$. Then

- 1. $\mathcal{N}(A) = \langle S \rangle$.
- 2. S is a linearly independent set.



Theorem BS Basis of a Span

62

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ is a set of column vectors. Define $W = \langle S \rangle$ and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with $D = \{d_1, d_2, d_3, \dots, d_r\}$ the set of column indices corresponding to the pivot columns of B. Then

- 1. $T = {\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \dots \mathbf{v}_{d_r}}$ is a linearly independent set.
- 2. $W = \langle T \rangle$.

$\ \, \textbf{Definition CCCV} \quad \textbf{Complex Conjugate of a Column Vector} \\$

63

Suppose that **u** is a vector from \mathbb{C}^m . Then the conjugate of the vector, $\overline{\mathbf{u}}$, is defined by

$$[\overline{\mathbf{u}}]_i = \overline{[\mathbf{u}]_i}$$

$$1 \leq i \leq m$$

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Theorem CRVA Conjugation Respects Vector Addition

64

Suppose **x** and **y** are two vectors from \mathbb{C}^m . Then

$$\overline{\mathbf{x} + \mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$$



Suppose **x** is a vector from \mathbb{C}^m , and $\alpha \in \mathbb{C}$ is a scalar. Then

$$\overline{\alpha}\overline{\mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$$

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Definition IP Inner Product

66

65

Given the vectors \mathbf{u} , $\mathbf{v} \in \mathbb{C}^m$ the inner product of \mathbf{u} and \mathbf{v} is the scalar quantity in \mathbb{C} ,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \overline{[\mathbf{u}]_1} [\mathbf{v}]_1 + \overline{[\mathbf{u}]_2} [\mathbf{v}]_2 + \overline{[\mathbf{u}]_3} [\mathbf{v}]_3 + \dots + \overline{[\mathbf{u}]_m} [\mathbf{v}]_m = \sum_{i=1}^m \overline{[\mathbf{u}]_i} [\mathbf{v}]_i$$

Theorem IPVA Inner Product and Vector Addition

Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$. Then

- 1. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
- 2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$

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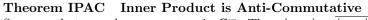
67

68

${\bf Theorem~IPSM~~Inner~Product~and~Scalar~Multiplication}$

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$. Then

- 1. $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \overline{\alpha} \langle \mathbf{u}, \mathbf{v} \rangle$
- 2. $\langle \mathbf{u}, \alpha \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$



Suppose that **u** and **v** are vectors in \mathbb{C}^m . Then $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.

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Definition NV Norm of a Vector

70

69

The norm of the vector ${\bf u}$ is the scalar quantity in ${\mathbb C}$

$$\|\mathbf{u}\| = \sqrt{|[\mathbf{u}]_1|^2 + |[\mathbf{u}]_2|^2 + |[\mathbf{u}]_3|^2 + \dots + |[\mathbf{u}]_m|^2} = \sqrt{\sum_{i=1}^m |[\mathbf{u}]_i|^2}$$

Theorem IPN	Inner Products and Norms					71
Suppose that \mathbf{u} is	a vector in \mathbb{C}^m . Then $\ \mathbf{u}\ ^2 = \langle \mathbf{u} \rangle$	$,\mathbf{u}\rangle.$				
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72

Theorem PIP Positive Inner Products Suppose that \mathbf{u} is a vector in \mathbb{C}^m . Then $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ with equality if and only if $\mathbf{u} = \mathbf{0}$.

Definition OV										73
A pair of vectors $\langle \mathbf{u}, \mathbf{v} \rangle = 0.$, \mathbf{u} and \mathbf{v}	, from \mathbb{C}	\mathbb{C}^m are	orthogonal	if their	inner	product	is zero,	that	is,
, ,										
					©200	4 Robe	rt A. Beez	zer, GFD	L Lice	ense

Definition OSV Orthogonal Set of Vectors

74

Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a set of vectors from \mathbb{C}^m . Then S is an orthogonal set if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0$ whenever $i \neq j$.

Definition SUV Standard Unit Vectors

Let $\mathbf{e}_j \in \mathbb{C}^m$, $1 \leq j \leq m$ denote the column vectors defined by

$$\left[\mathbf{e}_{j}\right]_{i} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Then the set

$$\{\mathbf{e}_1, \, \mathbf{e}_2, \, \mathbf{e}_3, \, \dots, \, \mathbf{e}_m\} = \{\, \mathbf{e}_j | \, 1 \le j \le m\}$$

is the set of standard unit vectors in \mathbb{C}^m .

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Theorem OSLI Orthogonal Sets are Linearly Independent

.

76

75

Suppose that S is an orthogonal set of nonzero vectors. Then S is linearly independent.

Theorem GSP Gram-Schmidt Procedure

77

Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$ is a linearly independent set of vectors in \mathbb{C}^m . Define the vectors $\mathbf{u}_i, 1 \leq i \leq p$ by

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle \mathbf{u}_1, \, \mathbf{v}_i \rangle}{\langle \mathbf{u}_1, \, \mathbf{u}_1 \rangle} \mathbf{u}_1 - \frac{\langle \mathbf{u}_2, \, \mathbf{v}_i \rangle}{\langle \mathbf{u}_2, \, \mathbf{u}_2 \rangle} \mathbf{u}_2 - \frac{\langle \mathbf{u}_3, \, \mathbf{v}_i \rangle}{\langle \mathbf{u}_3, \, \mathbf{u}_3 \rangle} \mathbf{u}_3 - \dots - \frac{\langle \mathbf{u}_{i-1}, \, \mathbf{v}_i \rangle}{\langle \mathbf{u}_{i-1}, \, \mathbf{u}_{i-1} \rangle} \mathbf{u}_{i-1}$$

Then if $T = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_p\}$, then T is an orthogonal set of non-zero vectors, and $\langle T \rangle = \langle S \rangle$.

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Definition ONS OrthoNormal Set

78

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \le i \le n$. Then S is an orthonormal set of vectors.

-	7 9
The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of compl numbers.	.ex
numbers.	
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Definition ME Matrix Equality 80 The $m \times n$ matrices A and B are equal, written A = B provided $[A]_{ij} = [B]_{ij}$ for all $1 \le i \le m$, $1 \le j \le n$.

Definition MA Matrix Addition

81

Given the $m \times n$ matrices A and B, define the sum of A and B as an $m \times n$ matrix, written A + B, according to

$$[A + B]_{ij} = [A]_{ij} + [B]_{ij}$$

$$1 \le i \le m, \ 1 \le j \le n$$

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Definition MSM Matrix Scalar Multiplication

82

Given the $m \times n$ matrix A and the scalar $\alpha \in \mathbb{C}$, the scalar multiple of A is an $m \times n$ matrix, written αA and defined according to

$$[\alpha A]_{ij} = \alpha \, [A]_{ij}$$

$$1 \le i \le m, \ 1 \le j \le n$$

Theorem VSPM Vector Space Properties of Matrices

83

Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- ACM Additive Closure, Matrices: If $A, B \in M_{mn}$, then $A + B \in M_{mn}$.
- SCM Scalar Closure, Matrices: If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.
- CM Commutativity, Matrices: If $A, B \in M_{mn}$, then A + B = B + A.
- AAM Additive Associativity, Matrices: If $A, B, C \in M_{mn}$, then A + (B + C) = (A + B) + C.
- ZM Zero Vector, Matrices: There is a matrix, \mathcal{O} , called the zero matrix, such that $A+\mathcal{O}=A$ for all $A\in M_{mn}$.
- AIM Additive Inverses, Matrices: If $A \in M_{mn}$, then there exists a matrix $-A \in M_{mn}$ so that $A + (-A) = \mathcal{O}$.
- SMAM Scalar Multiplication Associativity, Matrices: If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha(\beta A) = (\alpha \beta)A$.
- DMAM Distributivity across Matrix Addition, Matrices: If $\alpha \in \mathbb{C}$ and $A, B \in M_{mn}$, then $\alpha(A+B) = \alpha A + \alpha B$.
- DSAM Distributivity across Scalar Addition, Matrices: If $\alpha, \beta \in \mathbb{C}$ and $A \in M_{mn}$, then $(\alpha + \beta)A = \alpha A + \beta A$.
- OM One, Matrices: If $A \in M_{mn}$, then 1A = A.

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Definition ZM Zero Matrix

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The $m \times n$ zero matrix is written as $\mathcal{O} = \mathcal{O}_{m \times n}$ and defined by $[\mathcal{O}]_{ij} = 0$, for all $1 \leq i \leq m$, $1 \leq j \leq n$.

Definition TM Transpose of a Matrix

85

86

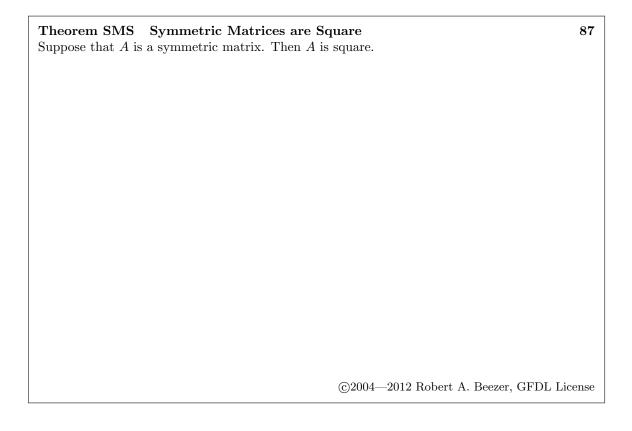
Given an $m \times n$ matrix A, its transpose is the $n \times m$ matrix A^t given by

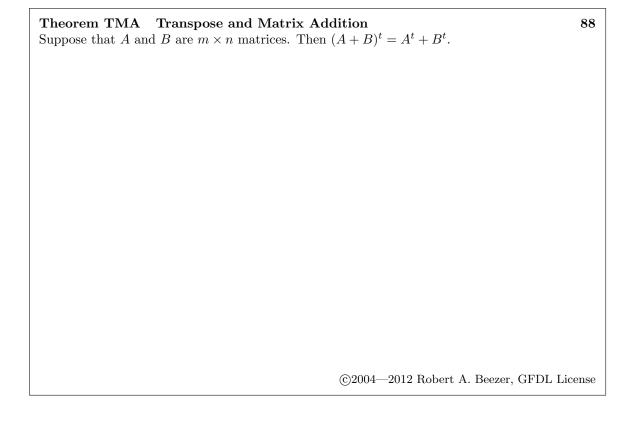
$$\left[A^t\right]_{ij} = [A]_{ji}\,,\quad 1 \leq i \leq n,\, 1 \leq j \leq m.$$

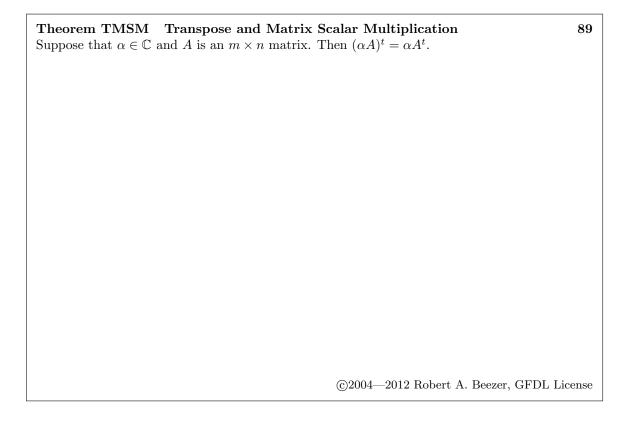
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Definition SYM Symmetric Matrix

The matrix A is symmetric if $A = A^t$.







${\bf Theorem~TT-Transpose~of~a~Transpose}$

90

Suppose that A is an $m \times n$ matrix. Then $(A^t)^t = A$.

Definition CCM Complex Conjugate of a Matrix

91

Suppose A is an $m \times n$ matrix. Then the conjugate of A, written \overline{A} is an $m \times n$ matrix defined by

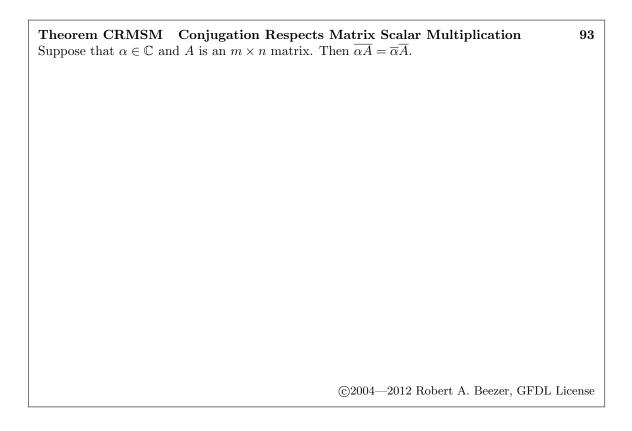
$$\left[\overline{A}\right]_{ij} = \overline{[A]_{ij}}$$

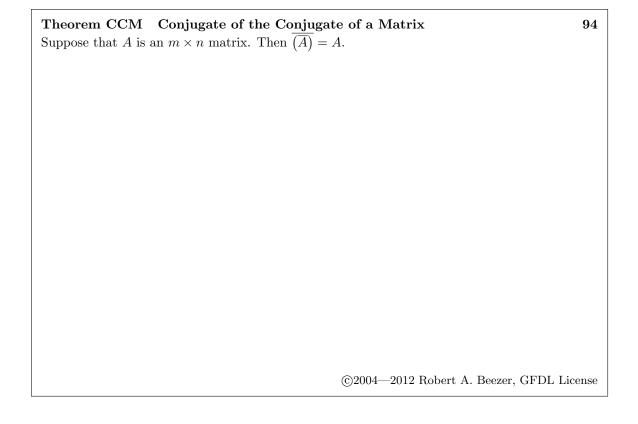
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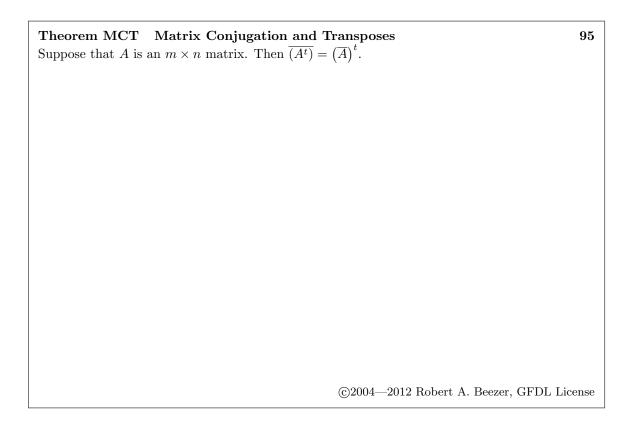
Theorem CRMA Conjugation Respects Matrix Addition

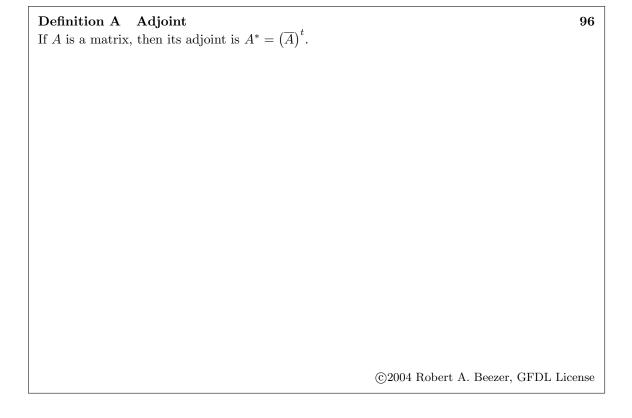
Suppose that A and B are $m \times n$ matrices. Then $\overline{A+B} = \overline{A} + \overline{B}$.

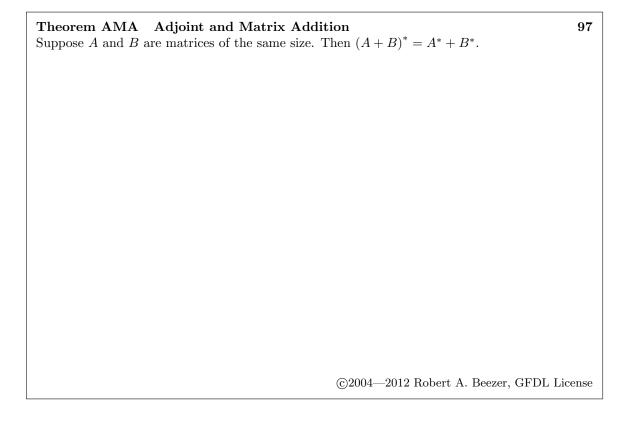
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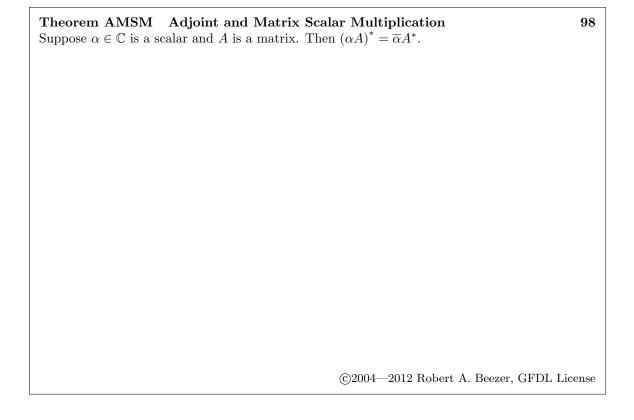


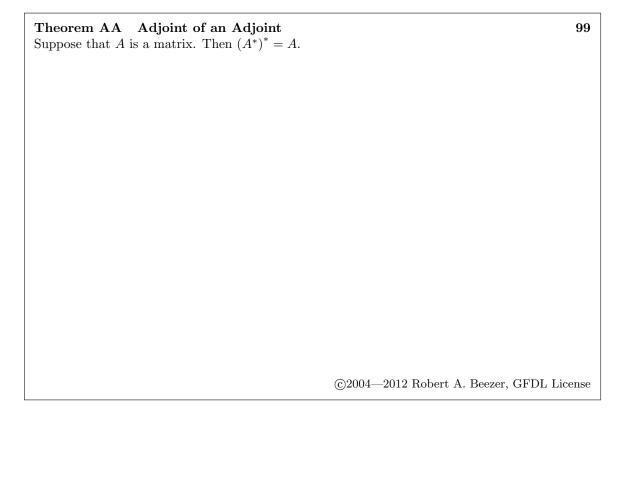










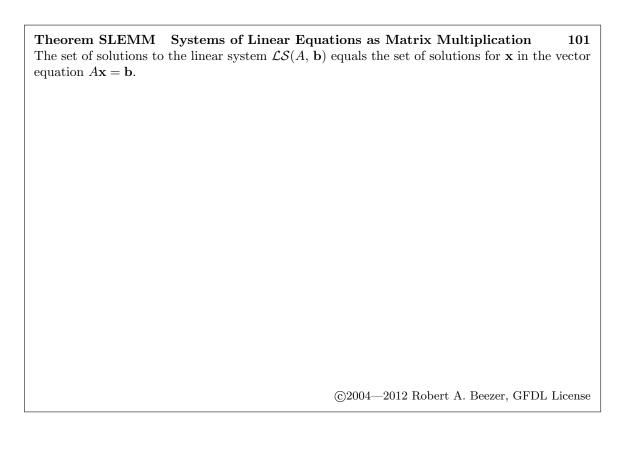


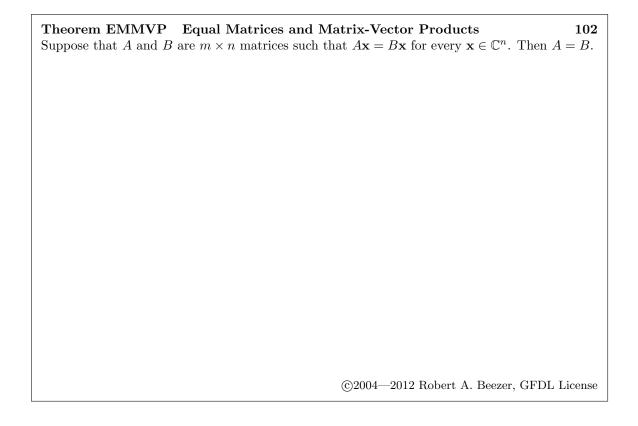
Definition MVP Matrix-Vector Product

100

Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ and \mathbf{u} is a vector of size n. Then the matrix-vector product of A with \mathbf{u} is the linear combination

$$A\mathbf{u} = [\mathbf{u}]_1 \mathbf{A}_1 + [\mathbf{u}]_2 \mathbf{A}_2 + [\mathbf{u}]_3 \mathbf{A}_3 + \dots + [\mathbf{u}]_n \mathbf{A}_n$$





Definition MM Matrix Multiplication

103

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix with columns $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$. Then the matrix product of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

$$AB = A \left[\mathbf{B}_1 | \mathbf{B}_2 | \mathbf{B}_3 | \dots | \mathbf{B}_p \right] = \left[A \mathbf{B}_1 | A \mathbf{B}_2 | A \mathbf{B}_3 | \dots | A \mathbf{B}_p \right].$$

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Theorem EMP Entries of Matrix Products

104

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then for $1 \le i \le m$, $1 \le j \le p$, the individual entries of AB are given by

$$[AB]_{ij} = [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \dots + [A]_{in} [B]_{nj}$$
$$= \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$$

Theorem MMZM Matrix Multiplication and the Zero Matrix

105

Suppose A is an $m \times n$ matrix. Then

- 1. $A\mathcal{O}_{n\times p} = \mathcal{O}_{m\times p}$
- 2. $\mathcal{O}_{p\times m}A = \mathcal{O}_{p\times n}$

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Theorem MMIM Matrix Multiplication and Identity Matrix

106

Suppose A is an $m \times n$ matrix. Then

- 1. $AI_n = A$
- $2. I_m A = A$

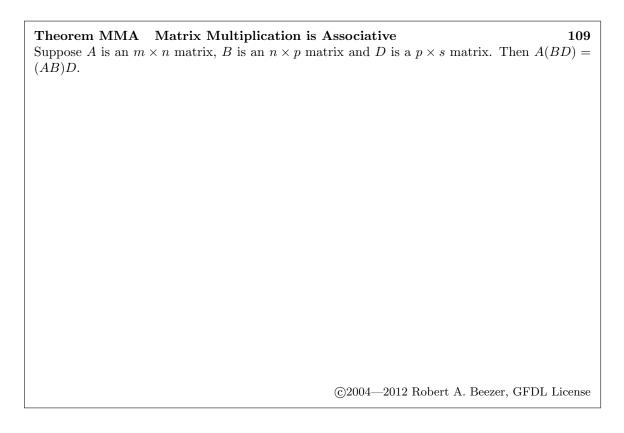
Theorem MMDAA Matrix Multiplication Distributes Across Addition 107 Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then

$$1. \ A(B+C) = AB + AC$$

$$2. \ (B+C)D = BD + CD$$

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Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication 108 Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

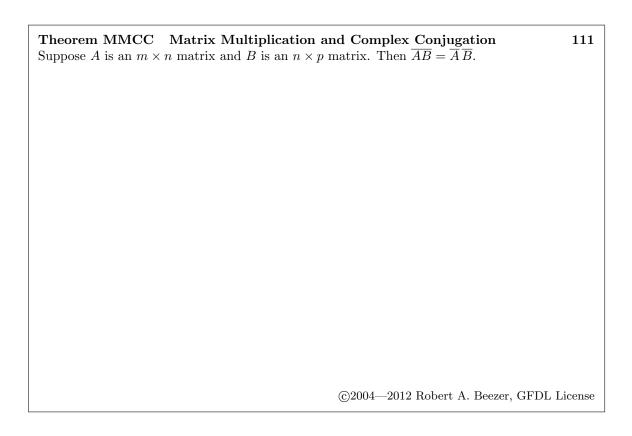


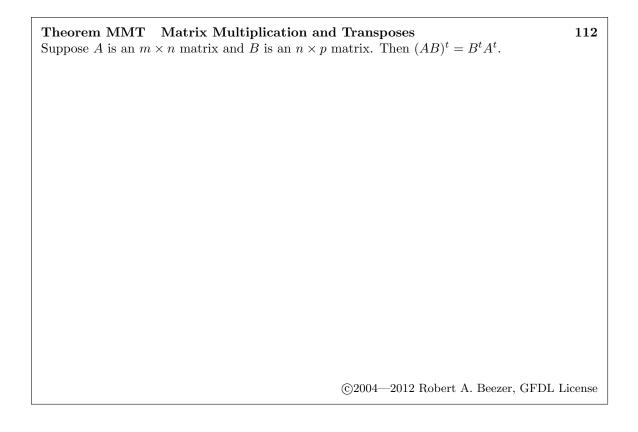
Theorem MMIP Matrix Multiplication and Inner Products

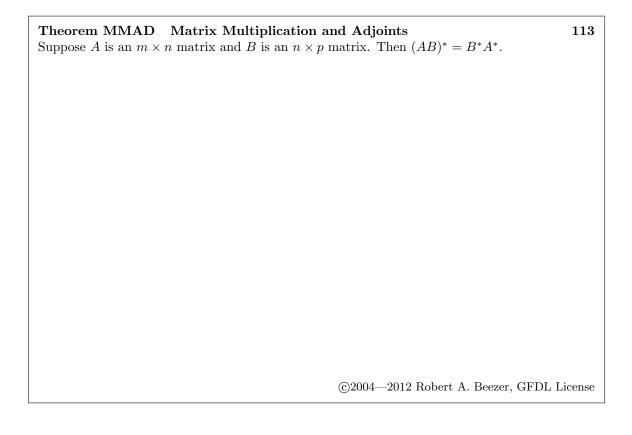
110

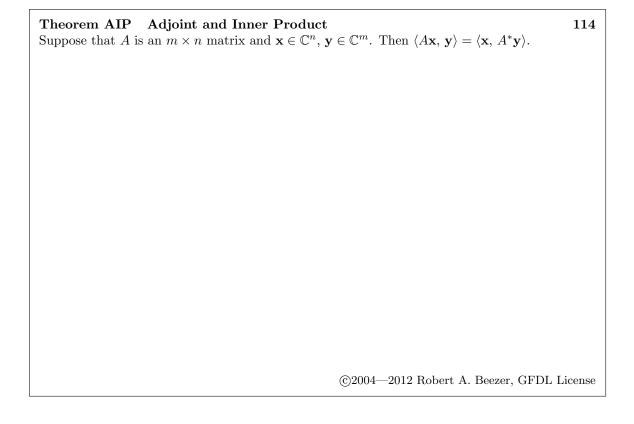
If we consider the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ as $m \times 1$ matrices then

$$\langle \mathbf{u}, \, \mathbf{v} \rangle = \overline{\mathbf{u}}^t \mathbf{v} = \mathbf{u}^* \mathbf{v}$$











Definition MI Matrix Inverse

117

Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is invertible and B is the inverse of A. In this situation, we write $B = A^{-1}$.

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Theorem TTMI Two-by-Two Matrix Inverse $^{\circ}$

118

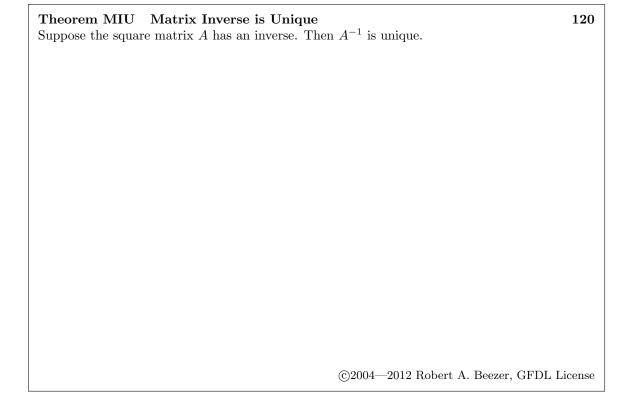
Suppose

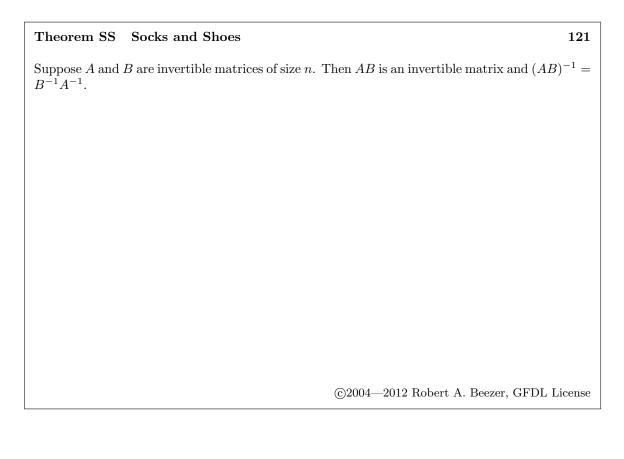
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem CINM Computing the Inverse of a Nonsingular Matrix 119 Suppose A is a nonsingular square matrix of size n . Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A . Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N . Then $AJ = I_n$.
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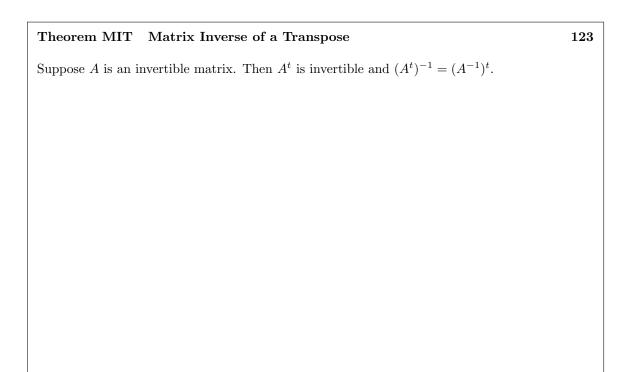




Theorem MIMI Matrix Inverse of a Matrix Inverse

122

Suppose A is an invertible matrix. Then A^{-1} is invertible and $(A^{-1})^{-1} = A$.



Theorem MISM Matrix Inverse of a Scalar Multiple

124

Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ and αA is invertible.

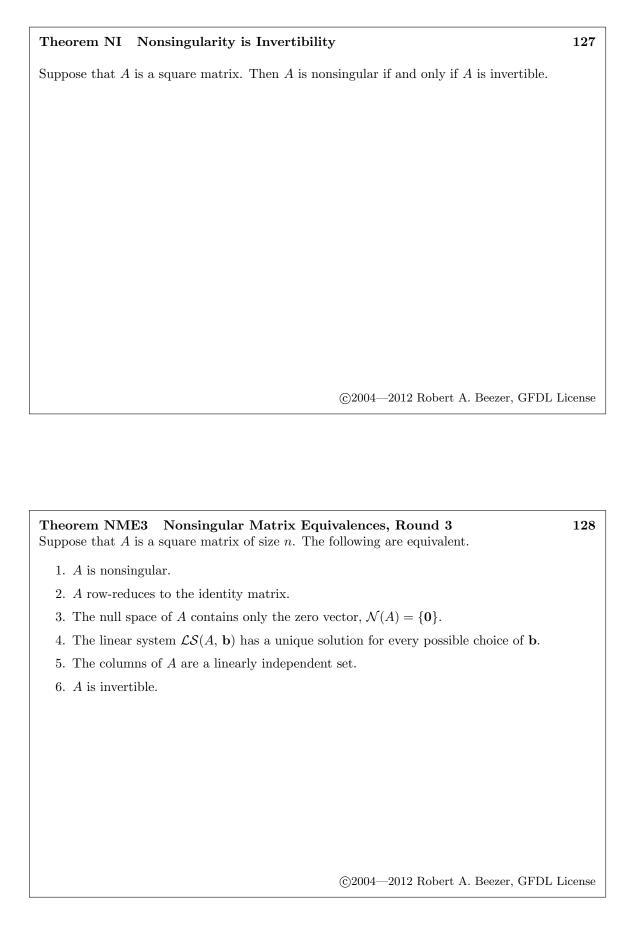
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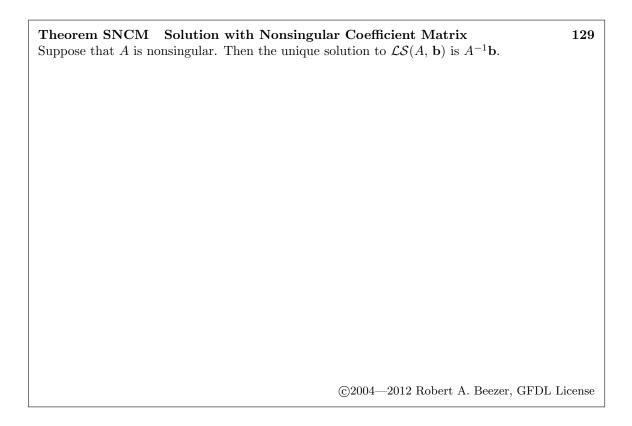
Theorem NPNT Nonsingular Product has Suppose that A and B are square matrices of size if A and B are both nonsingular.	
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Theorem OSIS One-Sided Inverse is Sufficient

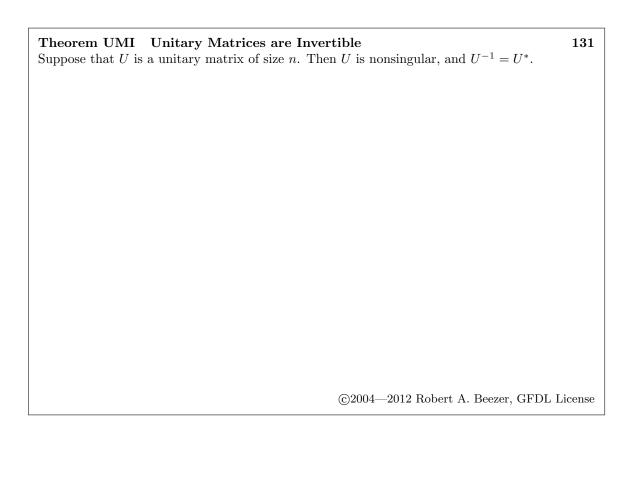
126

Suppose A and B are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.





Definition UM Unitary Matrices 130 Suppose that U is a square matrix of size n such that $U^*U=I_n$. Then we say U is unitary.



Suppose that A is a square matrix of size n with columns $S = \{\mathbf{A}_1, \, \mathbf{A}_2, \, \mathbf{A}_3, \, \dots, \, \mathbf{A}_n\}$. Then A is a unitary matrix if and only if S is an orthonormal set.

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Theorem CUMOS Columns of Unitary Matrices are Orthonormal Sets

Theorem UMPIP Unitary Matrices Preserve Inner Products

133

Suppose that U is a unitary matrix of size n and **u** and **v** are two vectors from \mathbb{C}^n . Then

$$\langle U\mathbf{u}, U\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

and

$$||U\mathbf{v}|| = ||\mathbf{v}||$$

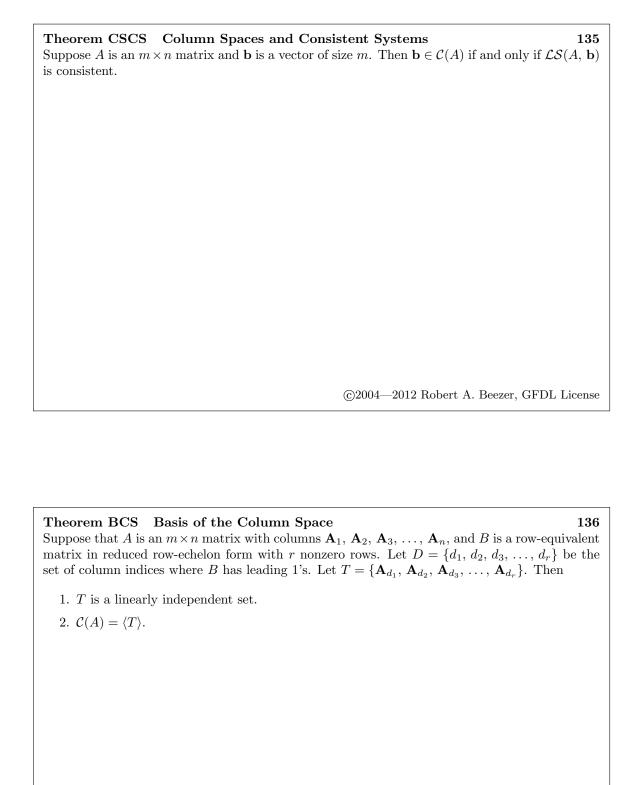
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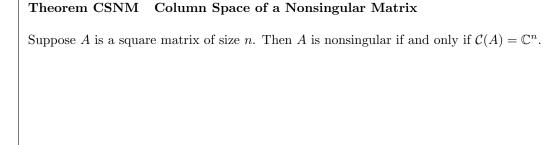
Definition CSM Column Space of a Matrix

134

Suppose that A is an $m \times n$ matrix with columns $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n\}$. Then the column space of A, written $\mathcal{C}(A)$, is the subset of \mathbb{C}^m containing all linear combinations of the columns of A,

$$C(A) = \langle \{ \mathbf{A}_1, \, \mathbf{A}_2, \, \mathbf{A}_3, \, \dots, \, \mathbf{A}_n \} \rangle$$





Theorem NME4 Nonsingular Matrix Equivalences, Round 4

138

137

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.

Definition RSM Row Space of a Matrix	139
Suppose A is an $m \times n$ matrix. Then the row space of A, $\mathcal{R}(A)$, is the column space $\mathcal{R}(A) = \mathcal{C}(A^t)$.	of A^t , i.e.
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Theorem REMRS Row-Equivalent Matrices have equal Row Spaces	140
Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$.	

Theorem BRS Basis for the Row Space

141

Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t . Then

- 1. $\mathcal{R}(A) = \langle S \rangle$.
- 2. S is a linearly independent set.

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Theorem CSRST Column Space, Row Space, Transpose

142

Suppose A is a matrix. Then $C(A) = \mathcal{R}(A^t)$.



143

Suppose A is an $m \times n$ matrix. Then the left null space is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$.

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Definition EEF Extended Echelon Form

144

Suppose A is an $m \times n$ matrix. Extend A on its right side with the addition of an $m \times m$ identity matrix to form an $m \times (n+m)$ matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the extended reduced row-echelon form of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the $m \times n$ matrix formed from the first n columns of N and let J denote the $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the $r \times n$ matrix formed from all of the non-zero rows of B. Let K be the $r \times m$ matrix formed from the first r rows of J, while L will be the $(m-r) \times m$ matrix formed from the bottom m-r rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix} C & K \\ 0 & L \end{bmatrix}$$

Theorem PEEF Properties of Extended Echelon Form

145

Suppose that A is an $m \times n$ matrix and that N is its extended echelon form. Then

- 1. J is nonsingular.
- 2. B = JA.
- 3. If $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$, then $A\mathbf{x} = \mathbf{y}$ if and only if $B\mathbf{x} = J\mathbf{y}$.
- 4. C is in reduced row-echelon form, has no zero rows and has r pivot columns.
- 5. L is in reduced row-echelon form, has no zero rows and has m-r pivot columns.

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Theorem FS Four Subsets

146

Suppose A is an $m \times n$ matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last m-r rows. Then

- 1. The null space of A is the null space of C, $\mathcal{N}(A) = \mathcal{N}(C)$.
- 2. The row space of A is the row space of C, $\mathcal{R}(A) = \mathcal{R}(C)$.
- 3. The column space of A is the null space of L, $C(A) = \mathcal{N}(L)$.
- 4. The left null space of A is the row space of L, $\mathcal{L}(A) = \mathcal{R}(L)$.

Definition VS Vector Space

147

Suppose that V is a set upon which we have defined two operations: (1) vector addition, which combines two elements of V and is denoted by "+", and (2) scalar multiplication, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a vector space over \mathbb{C} if the following ten properties hold.

- AC Additive Closure: If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.
- SC Scalar Closure: If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha \mathbf{u} \in V$.
- C Commutativity: If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AA Additive Associativity: If \mathbf{u} , \mathbf{v} , $\mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- Z Zero Vector: There is a vector, $\mathbf{0}$, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.
- AI Additive Inverses: If $\mathbf{u} \in V$, then there exists a vector $-\mathbf{u} \in V$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMA Scalar Multiplication Associativity: If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha(\beta \mathbf{u}) = (\alpha \beta) \mathbf{u}$.
- DVA Distributivity across Vector Addition: If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in V$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$.
- DSA Distributivity across Scalar Addition: If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in V$, then $(\alpha + \beta)\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{u}$.
- O One: If $\mathbf{u} \in V$, then $1\mathbf{u} = \mathbf{u}$.

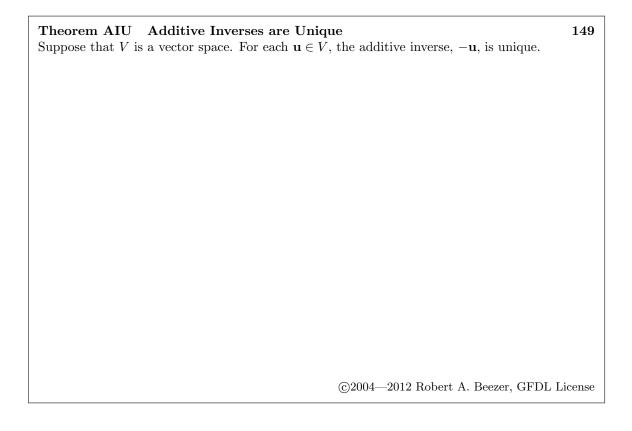
The objects in V are called vectors, no matter what else they might really be, simply by virtue of being elements of a vector space.

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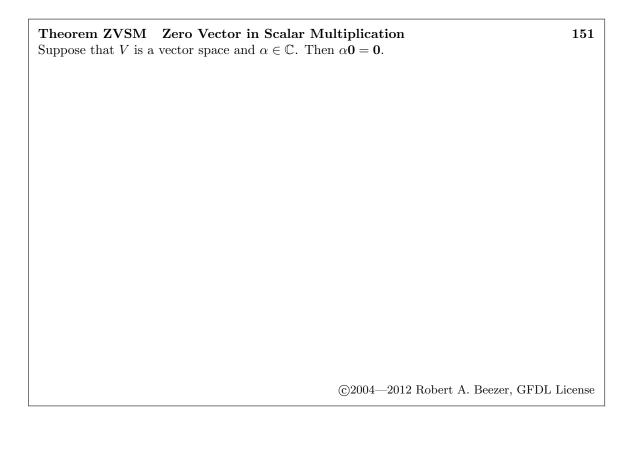
Theorem ZVU Zero Vector is Unique

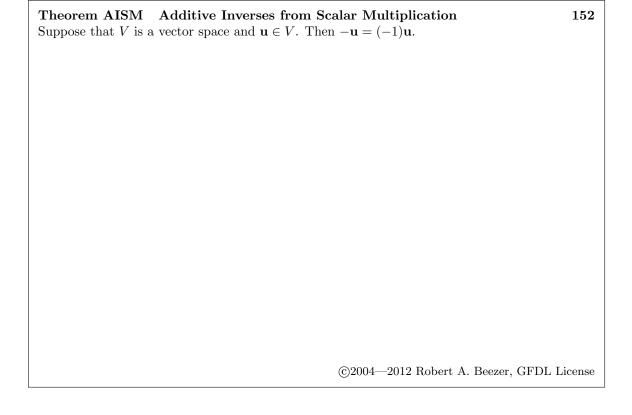
148

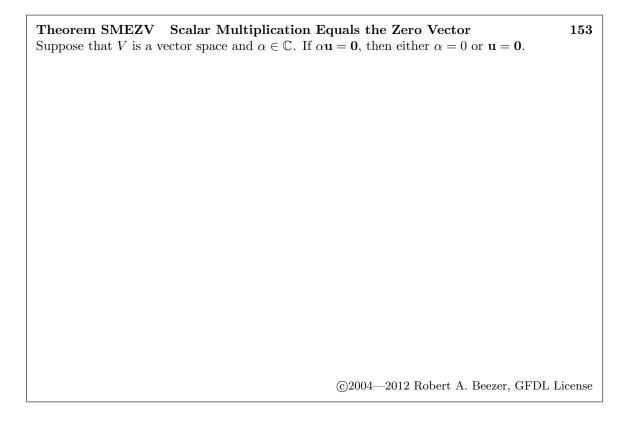
Suppose that V is a vector space. The zero vector, $\mathbf{0}$, is unique.



Theorem ZSSM Zero Scalar in Scalar Multiplication Suppose that V is a vector space and $\mathbf{u} \in V$. Then $0\mathbf{u} = \mathbf{0}$.







Definition S Subspace

154

Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of V, $W \subseteq V$. Then W is a subspace of V.

Theorem TSS Testing Subsets for Subspaces

155

Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

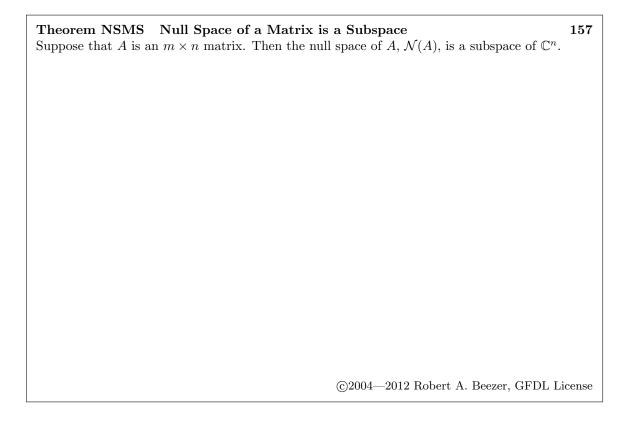
- 1. W is non-empty, $W \neq \emptyset$.
- 2. If $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.
- 3. If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha \mathbf{x} \in W$.

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Definition TS Trivial Subspaces

156

Given the vector space V, the subspaces V and $\{0\}$ are each called a trivial subspace.



Definition LC Linear Combination

158

Suppose that V is a vector space. Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their linear combination is the vector

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n.$$

Definition SS Span of a Set

159

Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$, their span, $\langle S \rangle$, is the set of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t$. Symbolically,

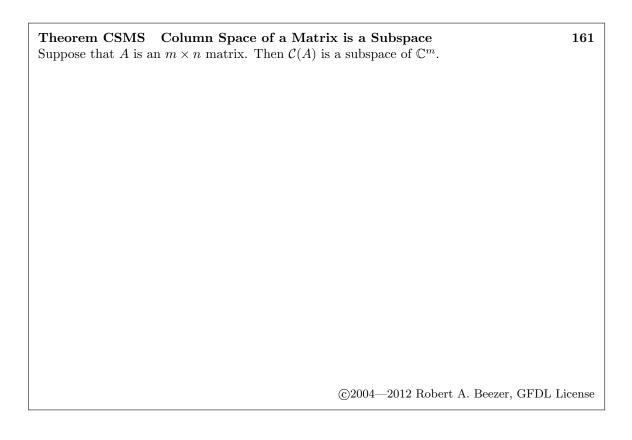
$$\langle S \rangle = \left\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_t \mathbf{u}_t \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$
$$= \left\{ \sum_{i=1}^t \alpha_i \mathbf{u}_i \middle| \alpha_i \in \mathbb{C}, \ 1 \le i \le t \right\}$$

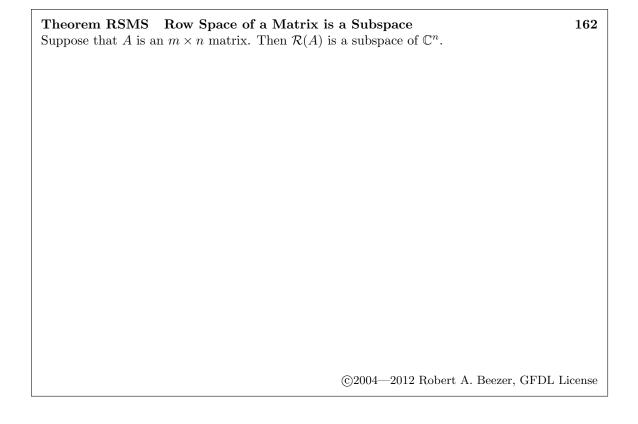
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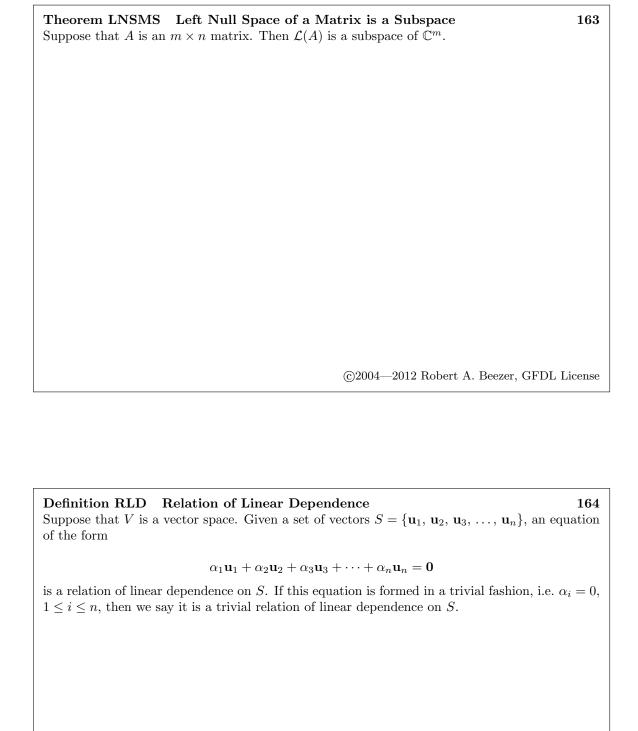
Theorem SSS Span of a Set is a Subspace

160

Suppose V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\} \subseteq V$, their span, $\langle S \rangle$, is a subspace.







Definition LI Linear Independence 165 Suppose that V is a vector space. The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ from V is linearly dependent if there is a relation of linear dependence on S that is not trivial. In the case where the only relation of linear dependence on S is the trivial one, then S is a linearly independent set of vectors.	
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Definition TSVS To Span a Vector Space Suppose V is a vector space. A subset S of V is a spanning set for V if $\langle S \rangle = V$. In this case, we also say S spans V .	

${\bf Theorem~VRRB~~Vector~Representation~Relative~to~a~Basis}$

167

Suppose that V is a vector space and $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$ is a linearly independent set that spans V. Let \mathbf{w} be any vector in V. Then there exist unique scalars $a_1, a_2, a_3, \dots, a_m$ such that

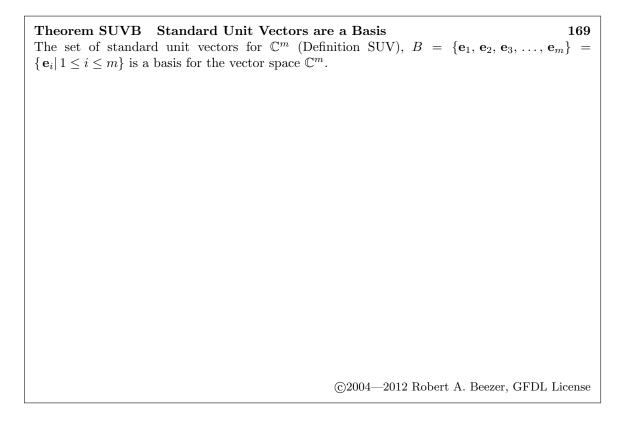
$$\mathbf{w} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + a_3 \mathbf{v}_3 + \dots + a_m \mathbf{v}_m.$$

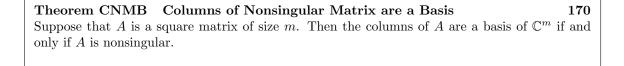
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Definition B Basis

168

Suppose V is a vector space. Then a subset $S \subseteq V$ is a basis of V if it is linearly independent and spans V.





Theorem NME5 Nonsingular Matrix Equivalences, Round 5

171

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .

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Theorem COB Coordinates and Orthonormal Bases

172

Suppose that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_p\}$ is an orthonormal basis of the subspace W of \mathbb{C}^m . For any $\mathbf{w} \in W$,

$$\mathbf{w} = \langle \mathbf{v}_1, \, \mathbf{w} \rangle \, \mathbf{v}_1 + \langle \mathbf{v}_2, \, \mathbf{w} \rangle \, \mathbf{v}_2 + \langle \mathbf{v}_3, \, \mathbf{w} \rangle \, \mathbf{v}_3 + \cdots + \langle \mathbf{v}_p, \, \mathbf{w} \rangle \, \mathbf{v}_p$$



Let A be an $n \times n$ matrix and $B = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_n\}$ be an orthonormal basis of \mathbb{C}^n . Define

$$C = \{A\mathbf{x}_1, A\mathbf{x}_2, A\mathbf{x}_3, \dots, A\mathbf{x}_n\}$$

Then A is a unitary matrix if and only if C is an orthonormal basis of \mathbb{C}^n .

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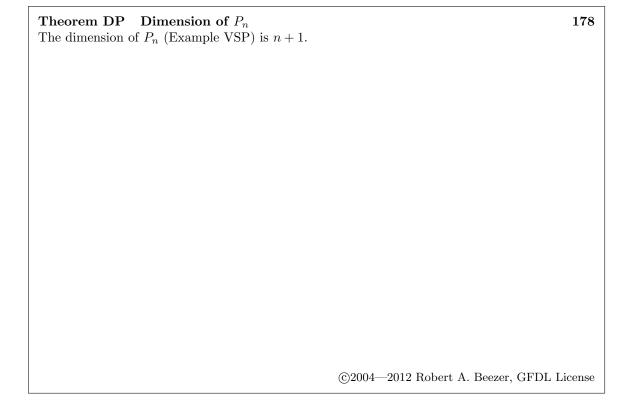
Definition D Dimension

174

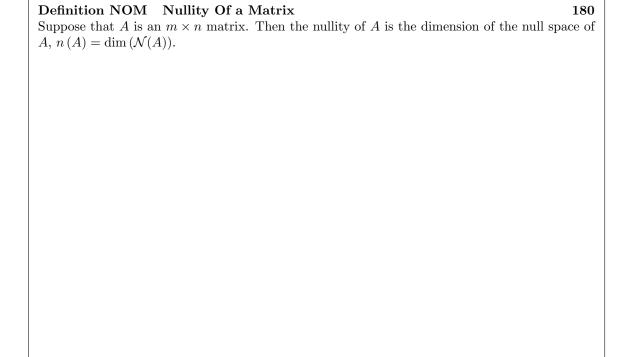
Suppose that V is a vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a basis of V. Then the dimension of V is defined by $\dim(V) = t$. If V has no finite bases, we say V has infinite dimension.

Theorem SSLD Spanning Sets and Linear Dependence	175
Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_t\}$ is a finite set of vectors which spans the	vector space V .
Then any set of $t+1$ or more vectors from V is linearly dependent.	
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©2001 2012 1000010 111 B0010	., GI DI Bicomco
Theorem BIS Bases have Identical Sizes	176
Suppose that V is a vector space with a finite basis B and a second basis C . The	
the same size.	n D and C have
the same size.	

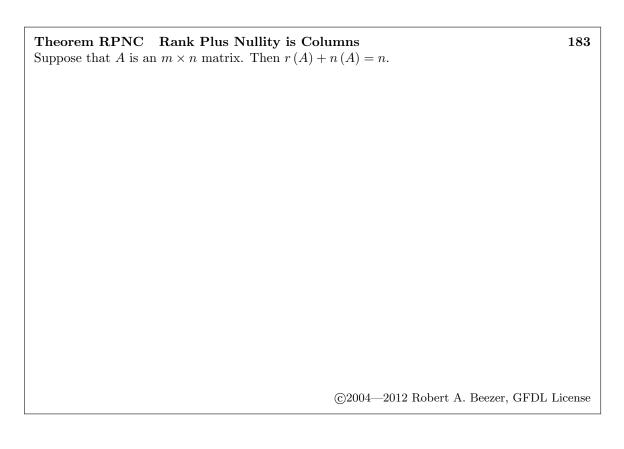
Theorem DCM Dimension of \mathbb{C}^m	177
The dimension of \mathbb{C}^m (Example VSCV) is m .	
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Theorem DM Dimension of M_{mn}	179
The dimension of M_{mn} (Example VSM) is mn .	
	Occasion and D. L. J. D. GDD. I.
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	Rank Of a Matrix in $m \times n$ matrix. Then the rank of A is the dime A is the dime A is the dime A in A is the dime A is the dime A in A is the dime A in A in A in A in A is the dime A in A	181 ension of the column space of
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Theorem CRN C	Computing Rank and Nullity	182
	on $m \times n$ matrix and B is a row-equivalent matrix. Then $r(A) = r$ and $n(A) = n - r$.	atrix in reduced row-echelon



Theorem RNNM Rank and Nullity of a Nonsingular Matrix

184

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. The rank of A is n, r(A) = n.
- 3. The nullity of A is zero, n(A) = 0.

Theorem NME6 Nonsingular Matrix Equivalences, Round 6

185

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- $2.\ A$ row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.

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Theorem ELIS Extending Linearly Independent Sets

186

Suppose V is vector space and S is a linearly independent set of vectors from V. Suppose w is a vector such that $\mathbf{w} \notin \langle S \rangle$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

Theorem G Goldilocks

187

Suppose that V is a vector space of dimension t. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_m\}$ be a set of vectors from V. Then

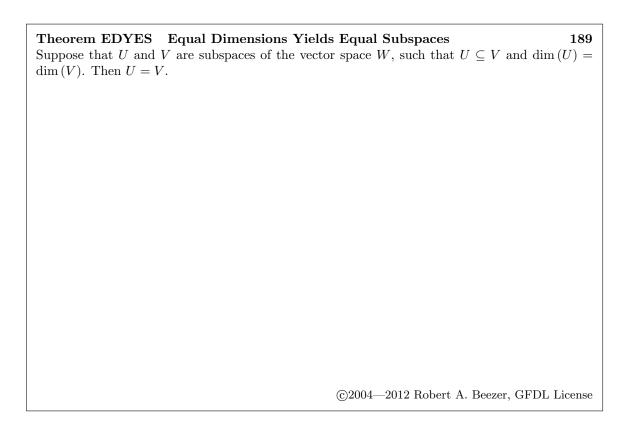
- 1. If m > t, then S is linearly dependent.
- 2. If m < t, then S does not span V.
- 3. If m = t and S is linearly independent, then S spans V.
- 4. If m = t and S spans V, then S is linearly independent.

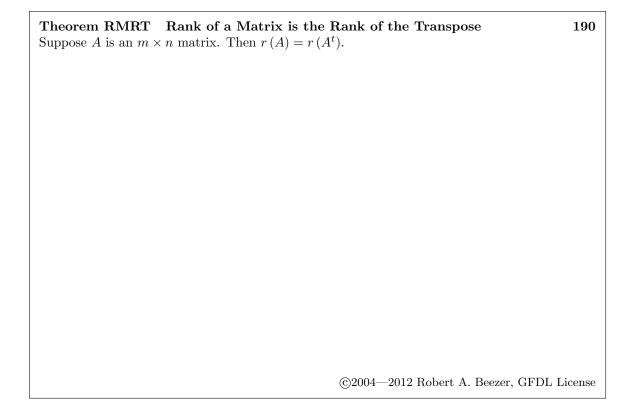
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Theorem PSSD Proper Subspaces have Smaller Dimension

188

Suppose that U and V are subspaces of the vector space W, such that $U \subsetneq V$. Then $\dim(U) < \dim(V)$.





Theorem DFS Dimensions of Four Subspaces

191

Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

- 1. dim $(\mathcal{N}(A)) = n r$
- 2. dim $(\mathcal{C}(A)) = r$
- 3. dim $(\mathcal{R}(A)) = r$
- 4. dim $(\mathcal{L}(A)) = m r$

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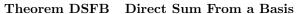
Definition DS Direct Sum

192

Suppose that V is a vector space with two subspaces U and W such that for every $\mathbf{v} \in V$,

- 1. There exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$
- 2. If $\mathbf{v} = \mathbf{u}_1 + \mathbf{w}_1$ and $\mathbf{v} = \mathbf{u}_2 + \mathbf{w}_2$ where $\mathbf{u}_1, \mathbf{u}_2 \in U, \mathbf{w}_1, \mathbf{w}_2 \in W$ then $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{w}_1 = \mathbf{w}_2$.

Then V is the direct sum of U and W and we write $V = U \oplus W$.



193

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ and $m \leq n$. Define

$$U = \langle \{\mathbf{v}_1, \, \mathbf{v}_2, \, \mathbf{v}_3, \, \dots, \, \mathbf{v}_m\} \rangle \qquad W = \langle \{\mathbf{v}_{m+1}, \, \mathbf{v}_{m+2}, \, \mathbf{v}_{m+3}, \, \dots, \, \mathbf{v}_n\} \rangle$$

Then $V = U \oplus W$.

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Theorem DSFOS Direct Sum From One Subspace

194

Suppose that U is a subspace of the vector space V. Then there exists a subspace W of V such that $V = U \oplus W$.

Theorem DSZV Direct Sums and Zero Vectors

195

Suppose U and W are subspaces of the vector space V. Then $V = U \oplus W$ if and only if

- 1. For every $\mathbf{v} \in V$, there exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.
- 2. Whenever $\mathbf{0} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U$, $\mathbf{w} \in W$ then $\mathbf{u} = \mathbf{w} = \mathbf{0}$.

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Theorem DSZI Direct Sums and Zero Intersection

196

Suppose U and W are subspaces of the vector space V. Then $V=U\oplus W$ if and only if

- 1. For every $\mathbf{v} \in V$, there exists vectors $\mathbf{u} \in U$, $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.
- 2. $U \cap W = \{0\}.$

Theorem DSLI Direct Sums and Linear Independence Suppose U and W are subspaces of the vector space V with $V=U$ linearly independent subset of U and S is a linearly independent sublinearly independent subset of V .	
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Theorem DSD Direct Sums and Dimension

198

Suppose U and W are subspaces of the vector space V with $V = U \oplus W$. Then $\dim(V) = \dim(U) + \dim(W)$.

Theorem RDS Repeated Direct Sums

199

Suppose V is a vector space with subspaces U and W with $V = U \oplus W$. Suppose that X and Y are subspaces of W with $W = X \oplus Y$. Then $V = U \oplus X \oplus Y$.

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Definition ELEM Elementary Matrices

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1. For $i \neq j$, $E_{i,j}$ is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

2. For $\alpha \neq 0$, $E_i(\alpha)$ is the square matrix of size n with

$$[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. For $i \neq j$, $E_{i,j}(\alpha)$ is the square matrix of size n with

$$[E_{i,j}(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = i \end{cases}$$

Theorem EMDRO Elementary Matrices Do Row Operations

201

Suppose that A is an $m \times n$ matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO). Then there is an elementary matrix of size m that will convert A to B via matrix multiplication on the left. More precisely,

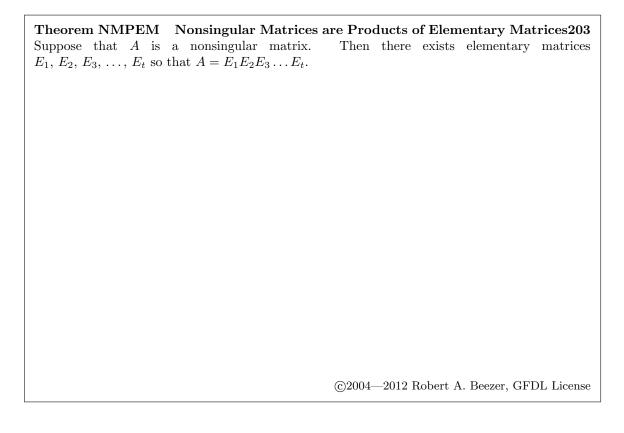
- 1. If the row operation swaps rows i and j, then $B = E_{i,j}A$.
- 2. If the row operation multiplies row i by α , then $B = E_i(\alpha) A$.
- 3. If the row operation multiplies row i by α and adds the result to row j, then $B=E_{i,j}\left(\alpha\right)A$.

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Theorem EMN Elementary Matrices are Nonsingular

202

If E is an elementary matrix, then E is nonsingular.



Definition SM SubMatrix

204

Suppose that A is an $m \times n$ matrix. Then the submatrix A(i|j) is the $(m-1) \times (n-1)$ matrix obtained from A by removing row i and column j.

Definition DM Determinant of a Matrix

205

Suppose A is a square matrix. Then its determinant, $\det(A) = |A|$, is an element of \mathbb{C} defined recursively by:

- 1. If A is a 1×1 matrix, then $det(A) = [A]_{11}$.
- 2. If A is a matrix of size n with $n \geq 2$, then

$$\begin{split} \det{(A)} &= [A]_{11} \det{(A\,(1|1))} - [A]_{12} \det{(A\,(1|2))} + [A]_{13} \det{(A\,(1|3))} - \\ & [A]_{14} \det{(A\,(1|4))} + \dots + (-1)^{n+1} \, [A]_{1n} \det{(A\,(1|n))} \end{split}$$

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206

Theorem DMST Determinant of Matrices of Size Two Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\det{(A)} = ad - bc$.

Theorem DER	Determinant	Expansion	about	Rows

207

Suppose that A is a square matrix of size n. Then

$$\begin{split} \det{(A)} &= (-1)^{i+1} \left[A \right]_{i1} \det{(A\left(i|1\right))} + (-1)^{i+2} \left[A \right]_{i2} \det{(A\left(i|2\right))} \\ &+ (-1)^{i+3} \left[A \right]_{i3} \det{(A\left(i|3\right))} + \dots + (-1)^{i+n} \left[A \right]_{in} \det{(A\left(i|n\right))} \qquad 1 \leq i \leq n \end{split}$$

which is known as expansion about row i.

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Theorem DT Determinant of the Transpose

208

Suppose that A is a square matrix. Then $\det(A^t) = \det(A)$.

Theorem DEC Determinant Expansion about Columns

209

Suppose that A is a square matrix of size n. Then

$$\begin{split} \det{(A)} &= (-1)^{1+j} \left[A \right]_{1j} \det{(A \, (1|j))} + (-1)^{2+j} \left[A \right]_{2j} \det{(A \, (2|j))} \\ &+ (-1)^{3+j} \left[A \right]_{3j} \det{(A \, (3|j))} + \dots + (-1)^{n+j} \left[A \right]_{nj} \det{(A \, (n|j))} \qquad 1 \leq j \leq n \end{split}$$

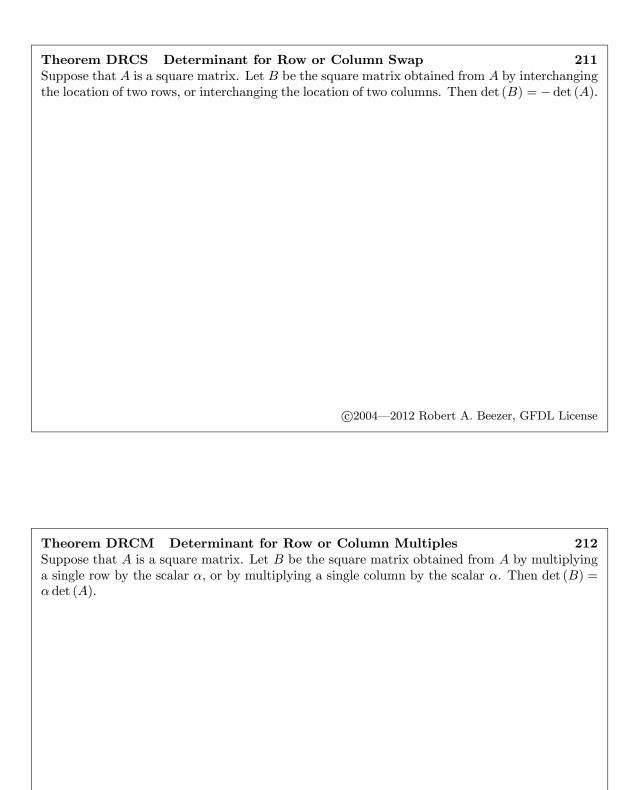
which is known as expansion about column j.

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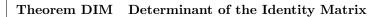
Theorem DZRC Determinant with Zero Row or Column

210

Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then $\det(A) = 0$.







215

For every $n \geq 1$, $\det(I_n) = 1$.

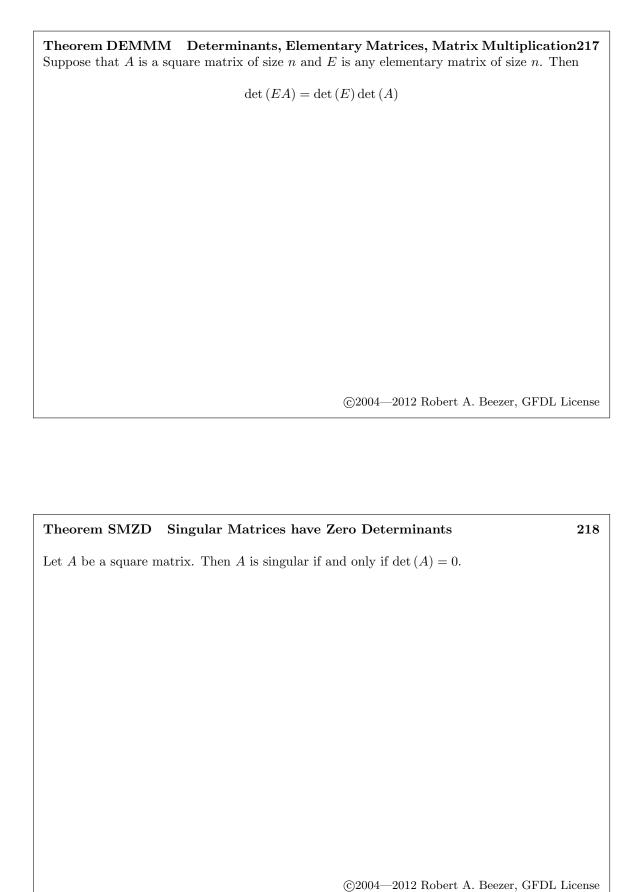
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Theorem DEM Determinants of Elementary Matrices

216

For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

- 1. $\det(E_{i,j}) = -1$
- 2. $\det (E_i(\alpha)) = \alpha$
- 3. $\det (E_{i,j}(\alpha)) = 1$



Theorem NME7 Nonsingular Matrix Equivalences, Round 7

219

Suppose that A is a square matrix of size n. The following are equivalent.

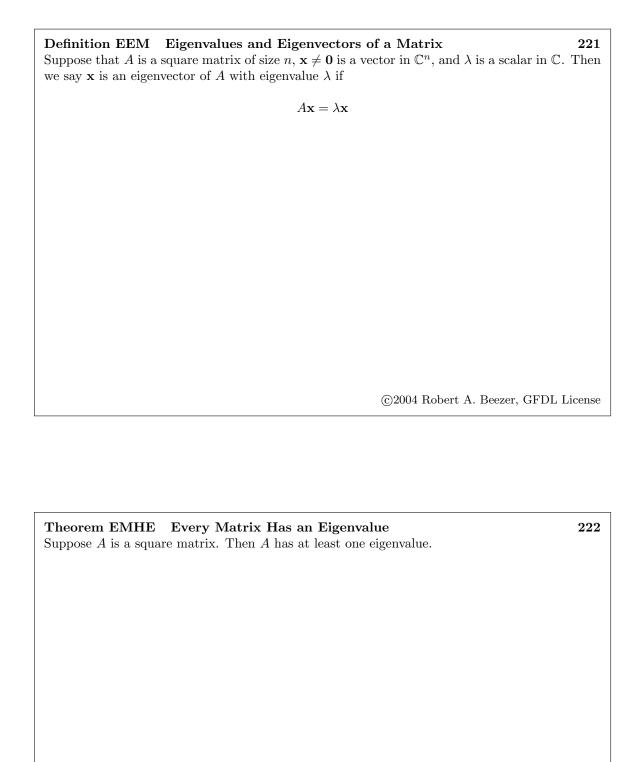
- 1. A is nonsingular.
- $2.\ A$ row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $det(A) \neq 0$.

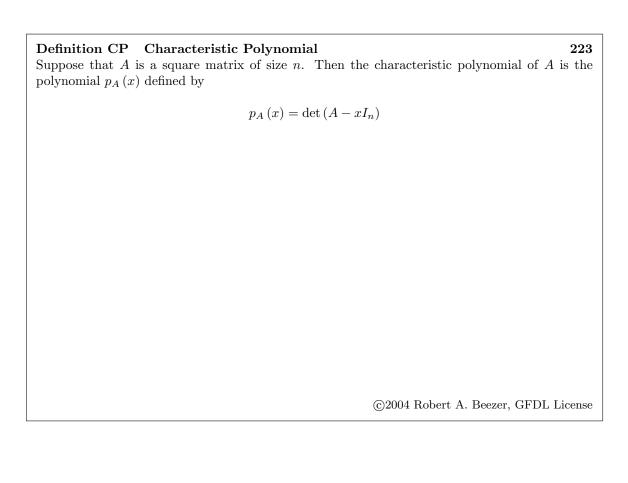
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Theorem DRMM Determinant Respects Matrix Multiplication

220

Suppose that A and B are square matrices of the same size. Then $\det(AB) = \det(A) \det(B)$.

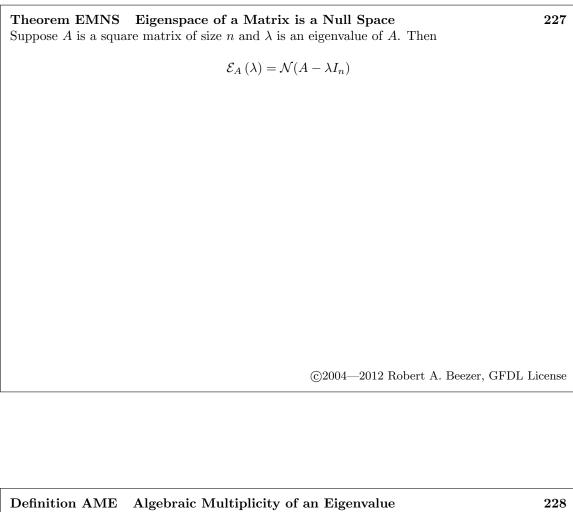




Theorem EMRCP Eigenvalues of a Matrix are Roots of Characteristic Polynomials 224

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$.

Definition EM Eigenspace of a Matrix Suppose that A is a square matrix and λ is an eigenvalue of	
$\mathcal{E}_{A}(\lambda)$, is the set of all the eigenvectors of A for λ , together	with the inclusion of the zero vector.
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Theorem EMS Eigenspace for a Matrix is a Subspace Suppose A is a square matrix of size n and λ is an eigenvalue is a subspace of the vector space \mathbb{C}^n .	



Suppose that A is a square matrix and λ is an eigenvalue of A. Then the algebraic multiplicity of λ , $\alpha_A(\lambda)$, is the highest power of $(x - \lambda)$ that divides the characteristic polynomial, $p_A(x)$.

Definition GME Geometric Multiplicity of an Eigenvalue 229 Suppose that A is a square matrix and λ is an eigenvalue of A . Then the geometric multiplicity of λ , $\gamma_A(\lambda)$, is the dimension of the eigenspace $\mathcal{E}_A(\lambda)$.
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Theorem EDELI Eigenvectors with Distinct Eigenvalues are Linearly Independent
Suppose that A is an $n \times n$ square matrix and $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

orem SMZE Singular Matrices have Zero Eigenvalues 231 ose A is a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of A .

Theorem NME8 Nonsingular Matrix Equivalences, Round 8

232

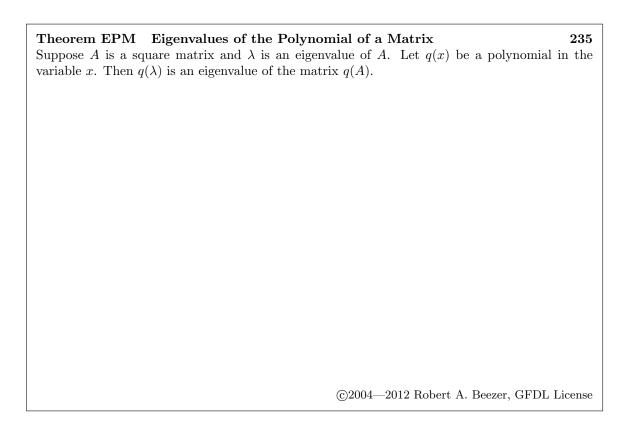
Suppose that A is a square matrix of size n. The following are equivalent.

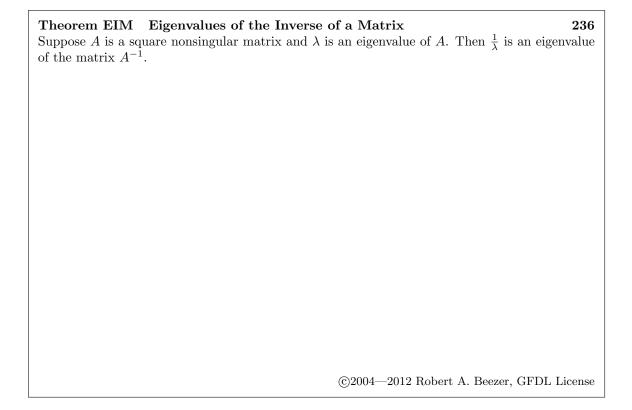
- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{0\}$.
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $det(A) \neq 0$.
- 12. $\lambda = 0$ is not an eigenvalue of A.

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Theorem ESMM Suppose A is a square	_	fultiple of a Matrix ne of A . Then $\alpha\lambda$ is an eigenvalue	233 e of αA .
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Theorem EOMP Eigenvalues Of Matrix Powers 234 Suppose A is a square matrix, λ is an eigenvalue of A, and $s \geq 0$ is an integer. Then λ^s is an eigenvalue of A^s .

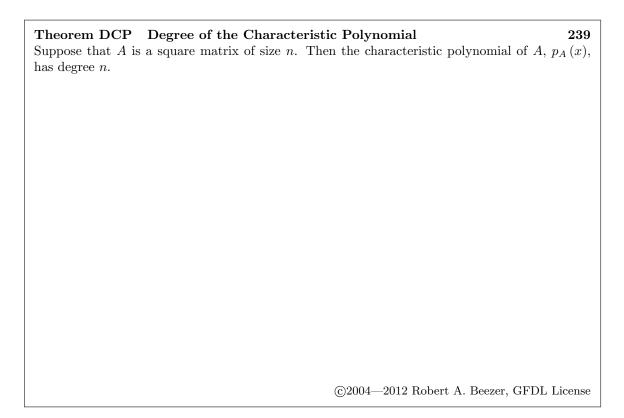




Theorem ETM Eigenvalues of the Transposure A is a square matrix and λ is an eigenvalue.	
Suppose A is a square matrix and λ is an eigenval A^t .	ue of A. Then A is an eigenvalue of the matrix
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Theorem ERMCP Eigenvalues of Real Matrices come in Conjugate Pairs 238

Suppose A is a square matrix with real entries and \mathbf{x} is an eigenvector of A for the eigenvalue λ . Then $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$.



Theorem NEM Number of Eigenvalues of a Matrix

240

Suppose that A is a square matrix of size n with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$. Then

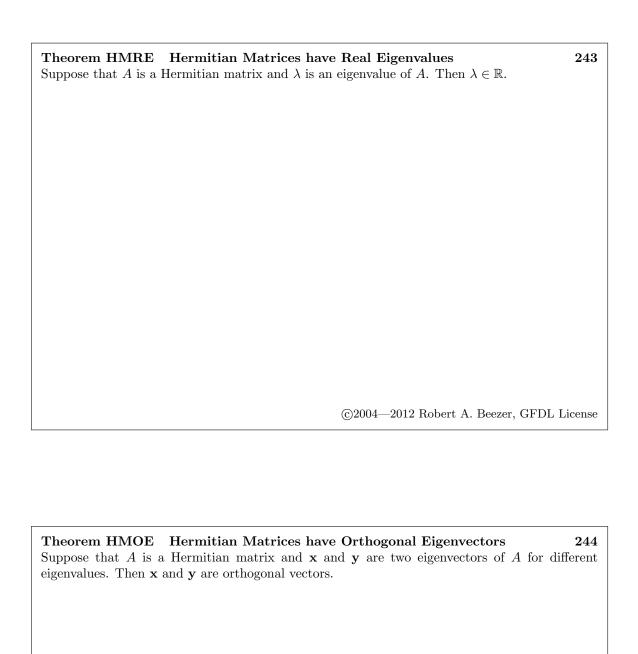
$$\sum_{i=1}^{k} \alpha_A \left(\lambda_i \right) = n$$

Suppose that A is a square matrix of size n and λ is an eigenvalue. Then	
$1 \le \gamma_A(\lambda) \le \alpha_A(\lambda) \le n$	
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Theorem MNEM Maximum Number of Eigenvalues of a Matrix 242	
Theorem MNEM Maximum Number of Eigenvalues of a Matrix 242 Suppose that A is a square matrix of size n . Then A cannot have more than n distinct eigenvalues.	

Theorem ME Multiplicities of an Eigenvalue

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241



Definition SIM Similar Matrices Suppose A and B are two square matrices of size n . Then A and B are similar if there enonsingular matrix of size n , S , such that $A = S^{-1}BS$.	245 xists a
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 Theorem SER Similarity is an Equivalence Relation Suppose A, B and C are square matrices of size n. Then 1. A is similar to A. (Reflexive) 2. If A is similar to B, then B is similar to A. (Symmetric) 3. If A is similar to B and B is similar to C, then A is similar to C. (Transitive) 	246

Theorem SMEE Similar Matrices Suppose A and B are similar matrices. equal, that is, $p_A(x) = p_B(x)$.	have Equal Eigenvalues ${\bf 247}$ Then the characteristic polynomials of A and B are
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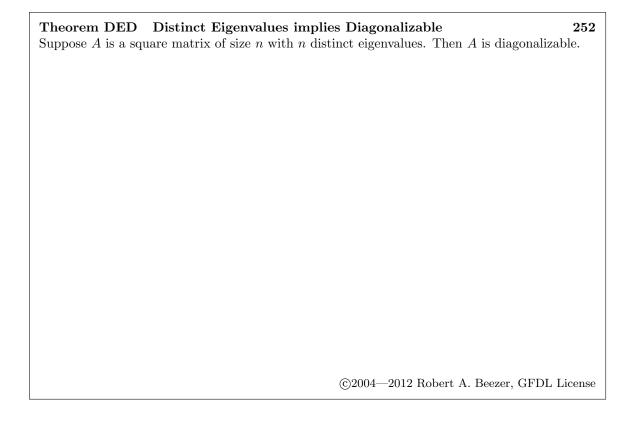
248

Definition DIM Diagonal Matrix Suppose that A is a square matrix. Then A is a diagonal matrix if $[A]_{ij} = 0$ whenever $i \neq j$.

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Theorem DC Diagonalization Characterization Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a

Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a linearly independent set S that contains n eigenvectors of A. ©2004—2012 Robert A. Beezer, GFDL License

Theorem DMFE Diagonalizable Matrices have Full Eigenspaces 251 Suppose A is a square matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for every eigenvalue λ of A .
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Definition LT Linear Transformation

253

A linear transformation, $T \colon U \to V$, is a function that carries elements of the vector space U (called the domain) to the vector space V (called the codomain), and which has two additional properties

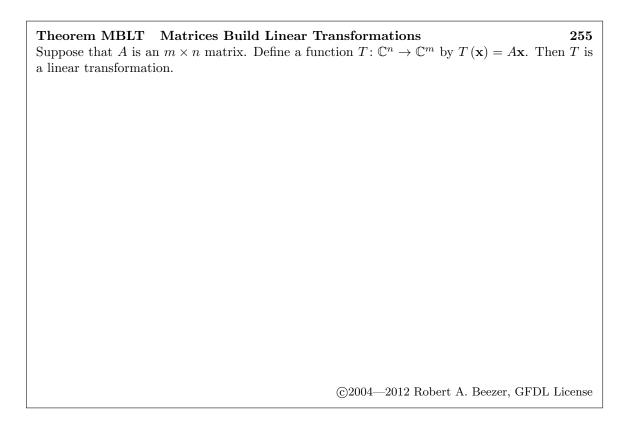
- 1. $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$ for all $\mathbf{u}_1, \mathbf{u}_2 \in U$
- 2. $T(\alpha \mathbf{u}) = \alpha T(\mathbf{u})$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

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${\bf Theorem~LTTZZ~~Linear~Transformations~Take~Zero~to~Zero}$

254

Suppose $T: U \to V$ is a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$.



Theorem MLTCV Matrix of a Linear Transformation, Column Vectors 256

Suppose that $T: \mathbb{C}^n \to \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Theorem LTLC Linear Transformations and Linear Combinations

257

Suppose that $T: U \to V$ is a linear transformation, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$ are vectors from U and $a_1, a_2, a_3, \ldots, a_t$ are scalars from \mathbb{C} . Then

$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \dots + a_t\mathbf{u}_t) = a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \dots + a_tT(\mathbf{u}_t)$$

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Theorem LTDB Linear Transformation Defined on a Basis

258

Suppose $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a basis for the vector space U and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n$ is a list of vectors from the vector space V (which are not necessarily distinct). Then there is a unique linear transformation, $T: U \to V$, such that $T(\mathbf{u}_i) = \mathbf{v}_i, 1 \le i \le n$.

Definition PI Pre-Image

259

Suppose that $T: U \to V$ is a linear transformation. For each \mathbf{v} , define the pre-image of \mathbf{v} to be the subset of U given by

$$T^{-1}(\mathbf{v}) = \{ \mathbf{u} \in U | T(\mathbf{u}) = \mathbf{v} \}$$

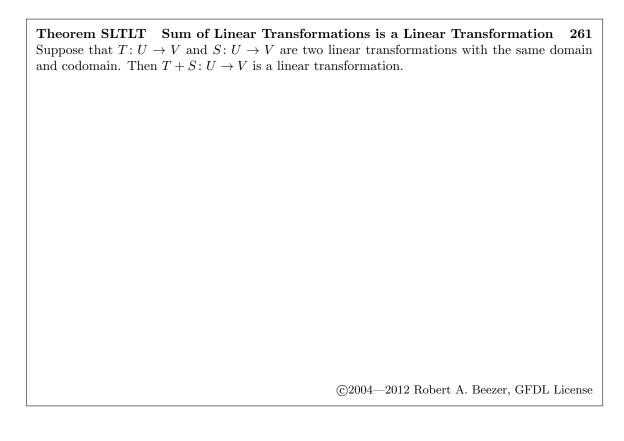
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Definition LTA Linear Transformation Addition

260

Suppose that $T\colon U\to V$ and $S\colon U\to V$ are two linear transformations with the same domain and codomain. Then their sum is the function $T+S\colon U\to V$ whose outputs are defined by

$$(T+S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$



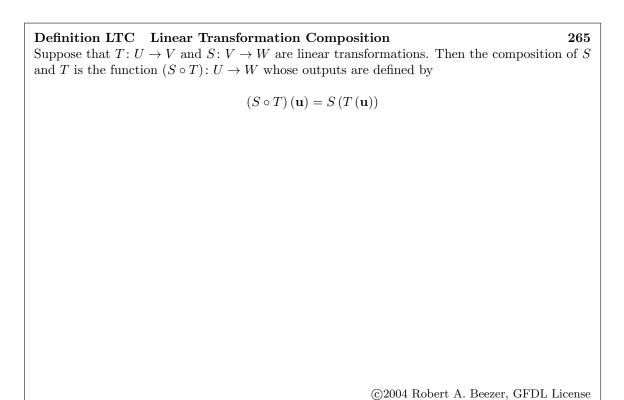
Definition LTSM Linear Transformation Scalar Multiplication

262

Suppose that $T:U\to V$ is a linear transformation and $\alpha\in\mathbb{C}$. Then the scalar multiple is the function $\alpha T\colon U\to V$ whose outputs are defined by

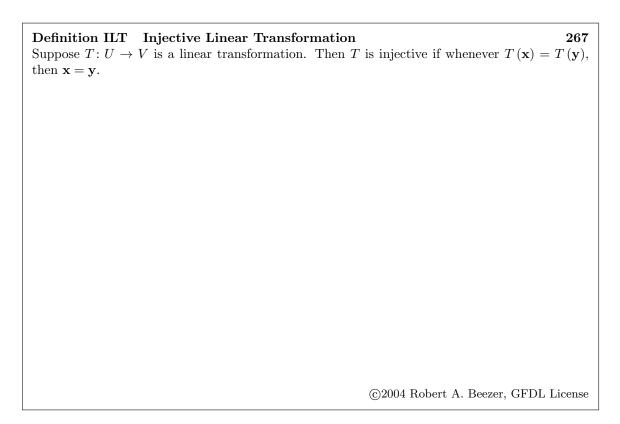
$$(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u})$$

Theorem MLTLT Multiple of a Linear Transformation is a Linear Transformation 263
Suppose that $T: U \to V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then $(\alpha T): U \to V$ is a linear
transformation.
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Theorem VSLT Vector Space of Linear Transformations 264
Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V , $\mathcal{L}T(U,V)$ is a vector space when the operations are those given in Definition LTA and Definition LTSM.



Theorem CLTLT Composition of Linear Transformations is a Linear Transformation 266

Suppose that $T\colon U\to V$ and $S\colon V\to W$ are linear transformations. Then $(S\circ T)\colon U\to W$ is a linear transformation.

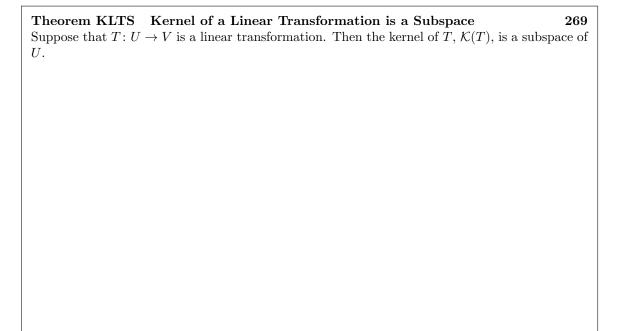


Definition KLT Kernel of a Linear Transformation

268

Suppose $T\colon U\to V$ is a linear transformation. Then the kernel of T is the set

$$\mathcal{K}(T) = \{ \mathbf{u} \in U | T(\mathbf{u}) = \mathbf{0} \}$$

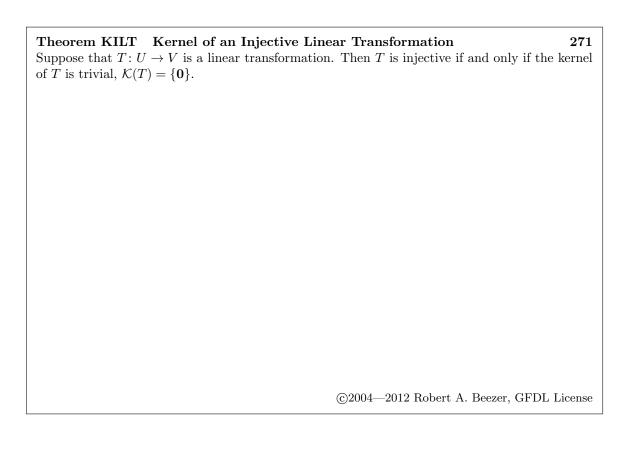


Theorem KPI Kernel and Pre-Image

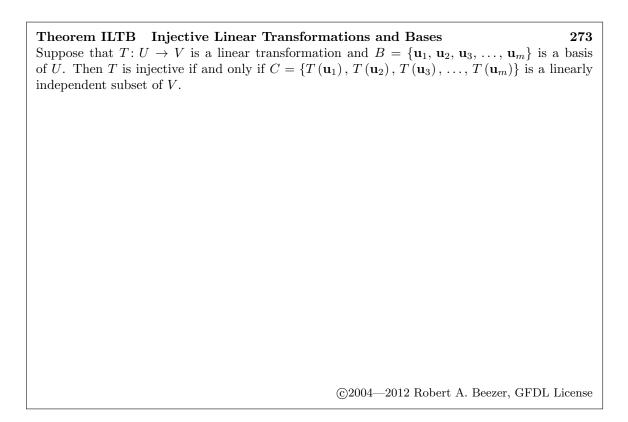
270

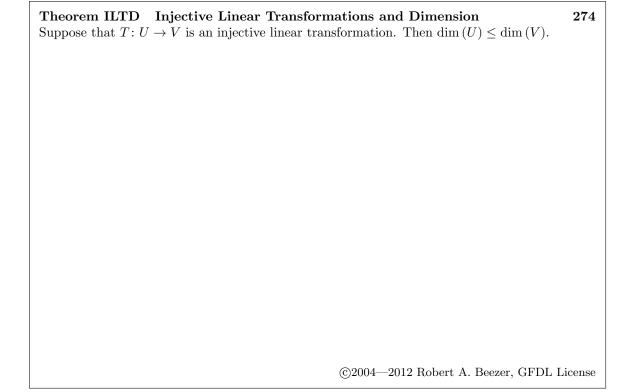
Suppose $T: U \to V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is non-empty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then

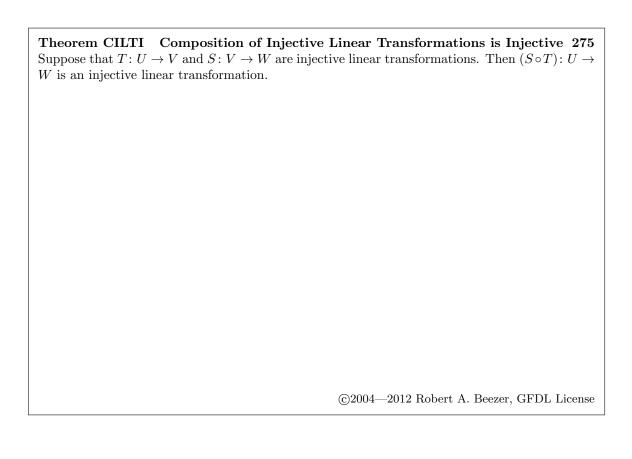
$$T^{-1}(\mathbf{v}) = \{ \mathbf{u} + \mathbf{z} | \mathbf{z} \in \mathcal{K}(T) \} = \mathbf{u} + \mathcal{K}(T)$$

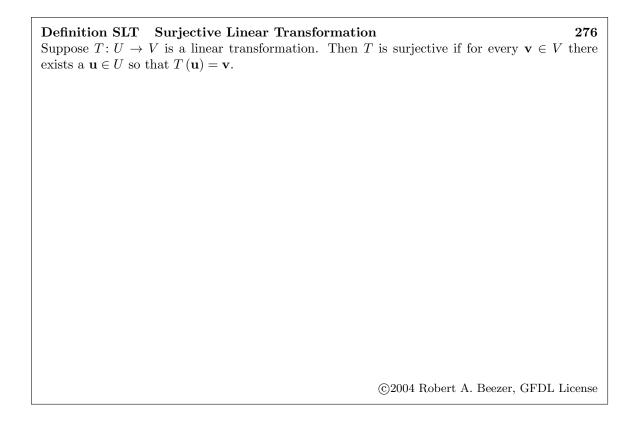


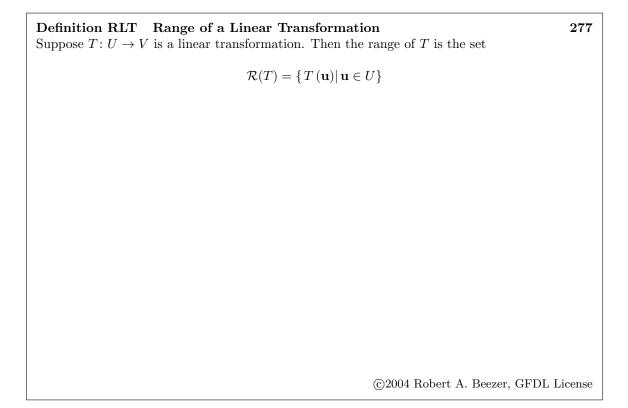
Theorem ILTLI Injective Linear Transformations and Linear Independence 272 Suppose that $T: U \to V$ is an injective linear transformation and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_t\}$ is a linearly independent subset of U. Then $R = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_t)\}$ is a linearly independent subset of V.



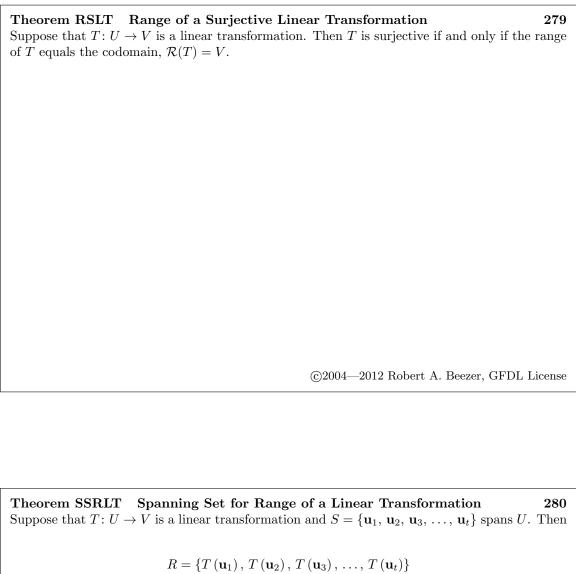




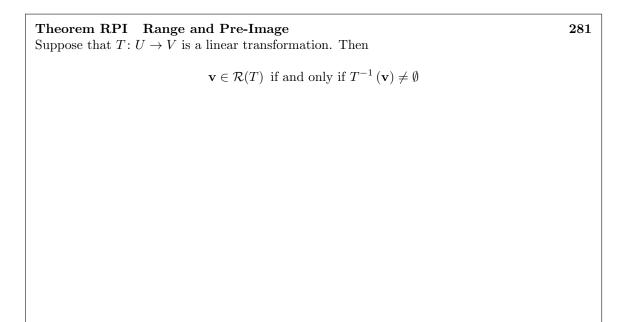




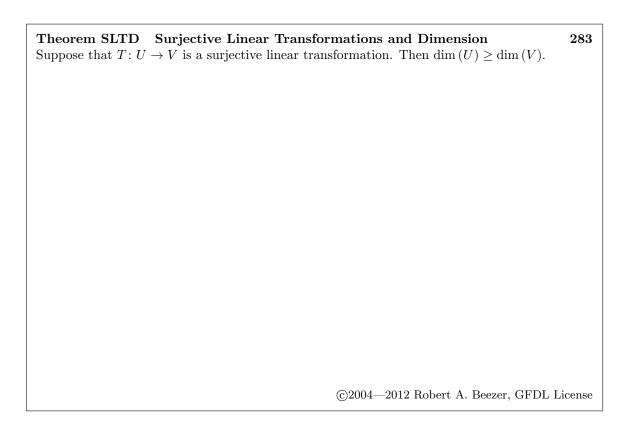
Theorem RLTS Range of a Linear Transformation is a Subspace 278 Suppose that $T: U \to V$ is a linear transformation. Then the range of T, $\mathcal{R}(T)$, is a subspace of V.



spans $\mathcal{R}(T)$.



Theorem SLTB Surjective Linear Transformations and Bases 282 Suppose that $T: U \to V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_m\}$ is a basis of U. Then T is surjective if and only if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \dots, T(\mathbf{u}_m)\}$ is a spanning set for V.



Theorem CSLTS Composition of Surjective Linear Transformations is Surjective 284

Suppose that $T\colon U\to V$ and $S\colon V\to W$ are surjective linear transformations. Then $(S\circ T)\colon U\to W$ is a surjective linear transformation.

Definition IDLT Identity Linear Transformation

285

The identity linear transformation on the vector space W is defined as

$$I_W \colon W \to W, \qquad I_W (\mathbf{w}) = \mathbf{w}$$

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Definition IVLT Invertible Linear Transformations

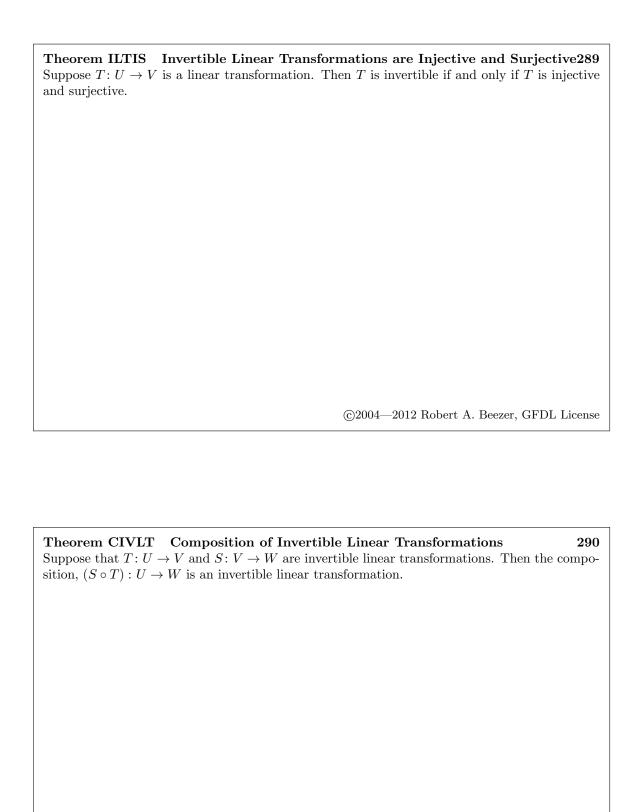
286

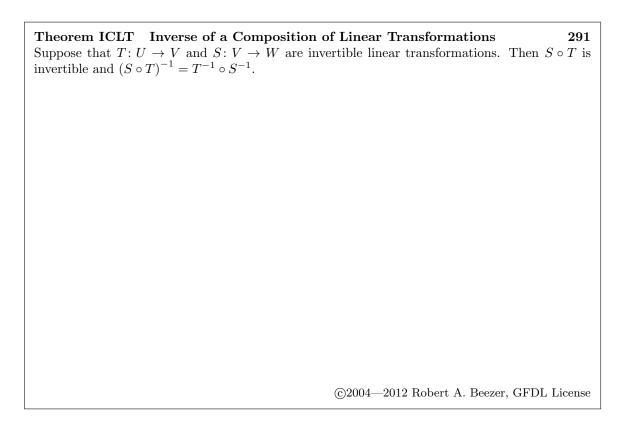
Suppose that $T\colon U\to V$ is a linear transformation. If there is a function $S\colon V\to U$ such that

$$S \circ T = I_U \qquad \qquad T \circ S = I_V$$

then T is invertible. In this case, we call S the inverse of T and write $S = T^{-1}$.



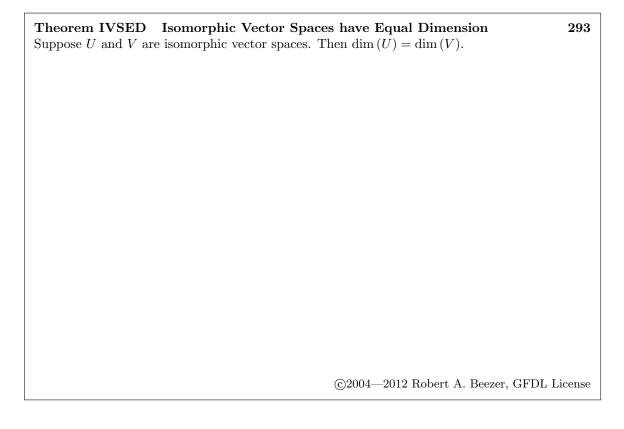




Definition IVS Isomorphic Vector Spaces

292

Two vector spaces U and V are isomorphic if there exists an invertible linear transformation T with domain U and codomain V, $T:U\to V$. In this case, we write $U\cong V$, and the linear transformation T is known as an isomorphism between U and V.

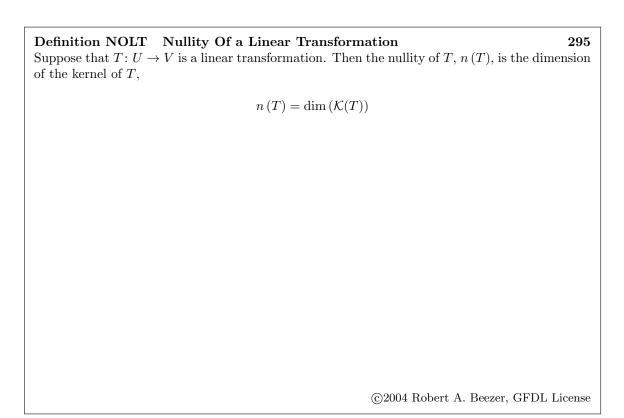


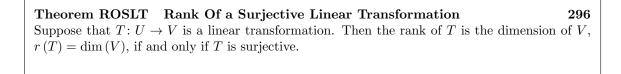
Definition ROLT Rank Of a Linear Transformation

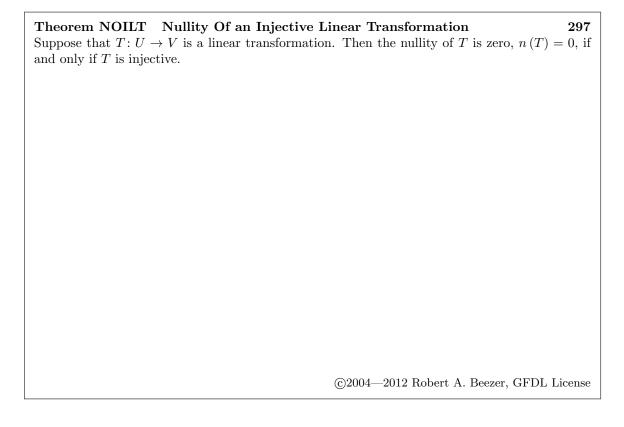
294

Suppose that $T:U\to V$ is a linear transformation. Then the rank of $T,\,r\left(T\right)$, is the dimension of the range of T,

$$r(T) = \dim (\mathcal{R}(T))$$







${\bf Theorem~RPNDD~~Rank~Plus~Nullity~is~Domain~Dimension}$

298

Suppose that $T \colon U \to V$ is a linear transformation. Then

$$r(T) + n(T) = \dim(U)$$

Definition VR Vector Representation

299

Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$. Define a function $\rho_B \colon V \to \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$ define the column vector $\rho_B(\mathbf{w}) \in \mathbb{C}^n$ by

$$\mathbf{w} = \left[\rho_B\left(\mathbf{w}\right)\right]_1 \mathbf{v}_1 + \left[\rho_B\left(\mathbf{w}\right)\right]_2 \mathbf{v}_2 + \left[\rho_B\left(\mathbf{w}\right)\right]_3 \mathbf{v}_3 + \dots + \left[\rho_B\left(\mathbf{w}\right)\right]_n \mathbf{v}_n$$

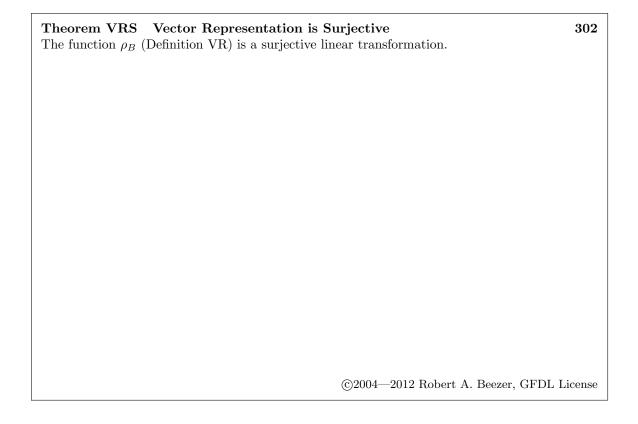
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${\bf Theorem~VRLT~~Vector~Representation~is~a~Linear~Transformation}$

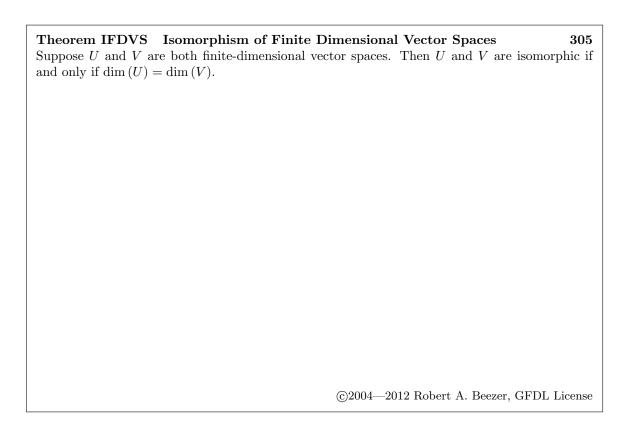
300

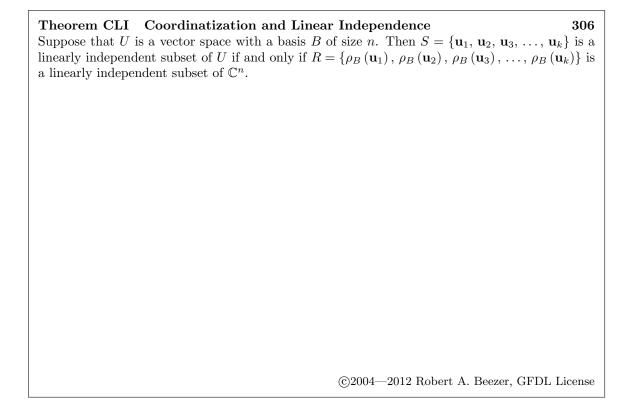
The function ρ_B (Definition VR) is a linear transformation.

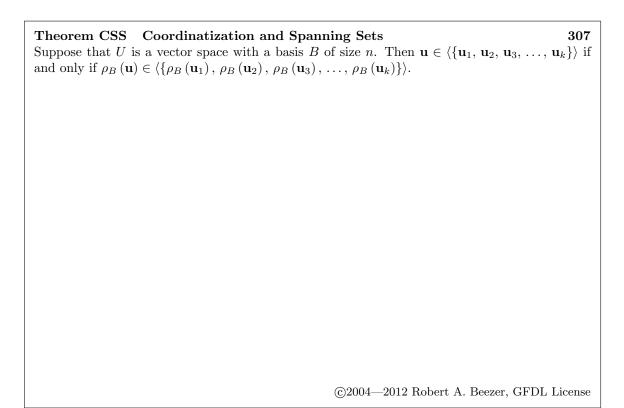
Theorem VRI Vector Representation is Inj The function ρ_B (Definition VR) is an injective line		301
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Definition MR Matrix Representation

308

Suppose that $T: U \to V$ is a linear transformation, $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is a basis for U of size n, and C is a basis for V of size m. Then the matrix representation of T relative to B and C is the $m \times n$ matrix,

$$M_{B,C}^{T} = \left[\rho_{C}\left(T\left(\mathbf{u}_{1}\right)\right) \middle| \rho_{C}\left(T\left(\mathbf{u}_{2}\right)\right) \middle| \rho_{C}\left(T\left(\mathbf{u}_{3}\right)\right) \middle| \dots \middle| \rho_{C}\left(T\left(\mathbf{u}_{n}\right)\right) \right]$$

Theorem FTMR Fundamental Theorem of Matrix Representation

309

Suppose that $T: U \to V$ is a linear transformation, B is a basis for U, C is a basis for V and $M_{B,C}^T$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u} \in U$,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_C^{-1}\left(M_{B,C}^T\left(\rho_B\left(\mathbf{u}\right)\right)\right)$$

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Theorem MRSLT Matrix Representation of a Sum of Linear Transformations 310 Suppose that $T: U \to V$ and $S: U \to V$ are linear transformations, B is a basis of U and C is a basis of V. Then

$$M_{B,C}^{T+S} = M_{B,C}^T + M_{B,C}^S$$

$\begin{array}{ll} {\bf Theorem~MRMLT} & {\bf Matrix~Representation~of~a~Multiple~of~a~Linear~Transformation} \\ {\bf 311} & \\ \end{array}$

Suppose that $T:U\to V$ is a linear transformation, $\alpha\in\mathbb{C},\,B$ is a basis of U and C is a basis of V. Then

$$M_{B,C}^{\alpha T} = \alpha M_{B,C}^T$$

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Theorem MRCLT Matrix Representation of a Composition of Linear Transformations 312

Suppose that $T: U \to V$ and $S: V \to W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M_{B,D}^{S \circ T} = M_{C,D}^S M_{B,C}^T$$

Theorem KNSI Kernel and Null Space Isomorphism

313

Suppose that $T: U \to V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M_{B,C}^T$,

$$\mathcal{K}(T) \cong \mathcal{N}(M_{B,C}^T)$$

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Theorem RCSI Range and Column Space Isomorphism

314

Suppose that $T: U \to V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of $M_{B,C}^T$,

$$\mathcal{R}(T) \cong \mathcal{C}(M_{B,C}^T)$$

Theorem IMR Invertible Matrix Representations

315

Suppose that $T: U \to V$ is a linear transformation, B is a basis for U and C is a basis for V. Then T is an invertible linear transformation if and only if the matrix representation of T relative to B and C, $M_{B,C}^T$ is an invertible matrix. When T is invertible,

$$M_{C,B}^{T^{-1}} = \left(M_{B,C}^T\right)^{-1}$$

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Theorem IMILT Invertible Matrices, Invertible Linear Transformation 316 Suppose that A is a square matrix of size n and $T: \mathbb{C}^n \to \mathbb{C}^n$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$. Then A is invertible matrix if and only if T is an invertible linear transformation.

Theorem NME9 Nonsingular Matrix Equivalences, Round 9

317

Suppose that A is a square matrix of size n. The following are equivalent.

- 1. A is nonsingular.
- 2. A row-reduces to the identity matrix.
- 3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}.$
- 4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
- 5. The columns of A are a linearly independent set.
- 6. A is invertible.
- 7. The column space of A is \mathbb{C}^n , $\mathcal{C}(A) = \mathbb{C}^n$.
- 8. The columns of A are a basis for \mathbb{C}^n .
- 9. The rank of A is n, r(A) = n.
- 10. The nullity of A is zero, n(A) = 0.
- 11. The determinant of A is nonzero, $det(A) \neq 0$.
- 12. $\lambda = 0$ is not an eigenvalue of A.
- 13. The linear transformation $T: \mathbb{C}^n \to \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible.

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Definition EELT Eigenvalue and Eigenvector of a Linear Transformation 318 Suppose that $T: V \to V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if $T(\mathbf{v}) = \lambda \mathbf{v}$.

Definition CBM Change-of-Basis Matrix

319

Suppose that V is a vector space, and $I_V: V \to V$ is the identity linear transformation on V. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$ and C be two bases of V. Then the change-of-basis matrix from B to C is the matrix representation of I_V relative to B and C,

$$C_{B,C} = M_{B,C}^{I_V}$$

$$= \left[\rho_C \left(I_V \left(\mathbf{v}_1 \right) \right) \middle| \rho_C \left(I_V \left(\mathbf{v}_2 \right) \right) \middle| \rho_C \left(I_V \left(\mathbf{v}_3 \right) \right) \middle| \dots \middle| \rho_C \left(I_V \left(\mathbf{v}_n \right) \right) \right]$$

$$= \left[\rho_C \left(\mathbf{v}_1 \right) \middle| \rho_C \left(\mathbf{v}_2 \right) \middle| \rho_C \left(\mathbf{v}_3 \right) \middle| \dots \middle| \rho_C \left(\mathbf{v}_n \right) \right]$$

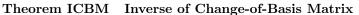
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Theorem CB Change-of-Basis

320

Suppose that \mathbf{v} is a vector in the vector space V and B and C are bases of V. Then

$$\rho_C\left(\mathbf{v}\right) = C_{B,C}\rho_B\left(\mathbf{v}\right)$$



321

Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

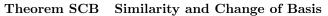
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Theorem MRCB Matrix Representation and Change of Basis

322

Suppose that $T\colon U\to V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

$$M_{B,D}^T = C_{E,D} M_{C,E}^T C_{B,C}$$



323

Suppose that $T: V \to V$ is a linear transformation and B and C are bases of V. Then

$$M_{B,B}^T = C_{B,C}^{-1} M_{C,C}^T C_{B,C}$$

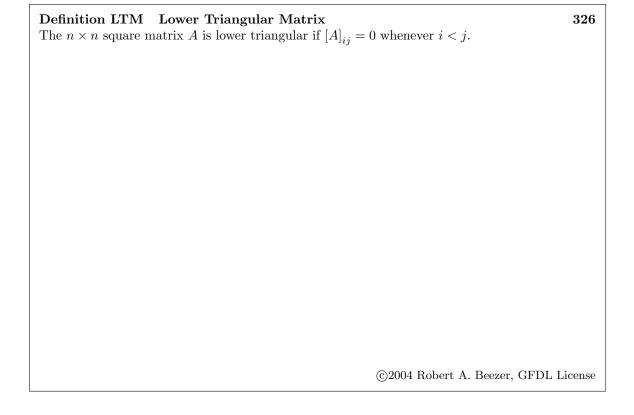
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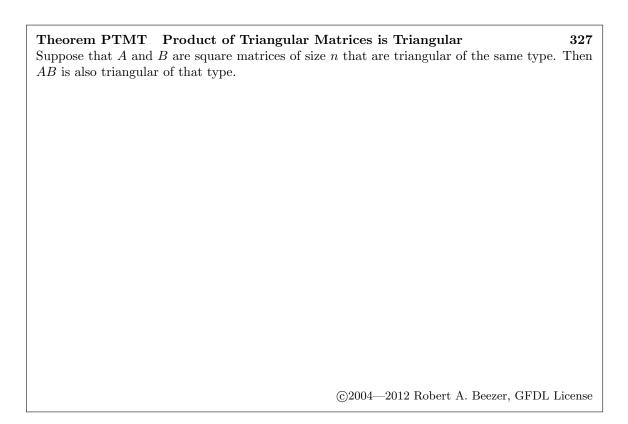
Theorem EER Eigenvalues, Eigenvectors, Representations

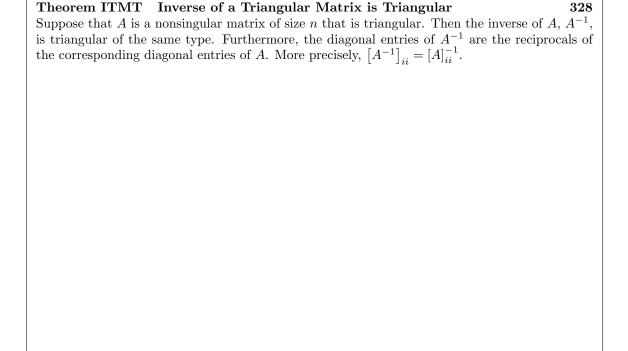
324

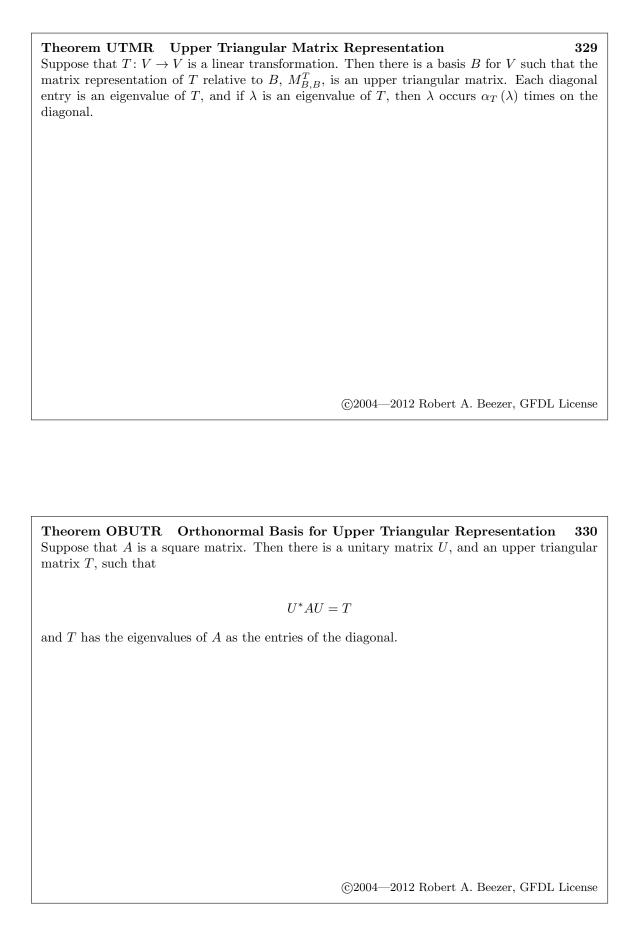
Suppose that $T: V \to V$ is a linear transformation and B is a basis of V. Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M_{B,B}^T$ for the eigenvalue λ .

	Upper Triangular Matrix	325
The $n \times n$ square m	atrix A is upper triangular if $[A]_{ij}$ =	= 0 whenever $i > j$.
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Definition NRML Normal Matrix The square matrix A is normal if $A^*A = AA^*$.	331
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${\bf Theorem~OD~~Orthonormal~Diagonalization}$

332

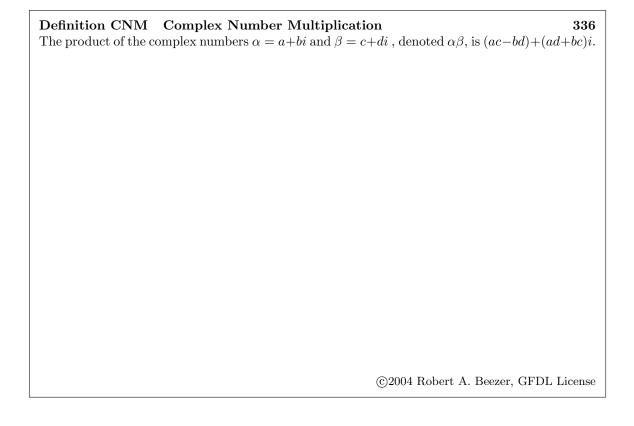
Suppose that A is a square matrix. Then there is a unitary matrix U and a diagonal matrix D, with diagonal entries equal to the eigenvalues of A, such that $U^*AU = D$ if and only if A is a normal matrix.

Theorem OBNM Orthonormal Bases and Normal Matrices 333 Suppose that A is a normal matrix of size n . Then there is an orthonormal basis of \mathbb{C}^n composed of eigenvectors of A .
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Definition CNE Complex Number Equality The complex numbers $\alpha = a + bi$ and $\beta = c + di$ are equal, denoted $\alpha = \beta$, if a = c and b = d.

334

Definition CNA Complex Number Addition	335
The sum of the complex numbers $\alpha = a + bi$ and $\beta = c + bi$	- ai , denoted $\alpha + \beta$, is $(a+c) + (b+a)i$.
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Theorem PCNA Properties of Complex Number Arithmetic

337

The operations of addition and multiplication of complex numbers have the following properties.

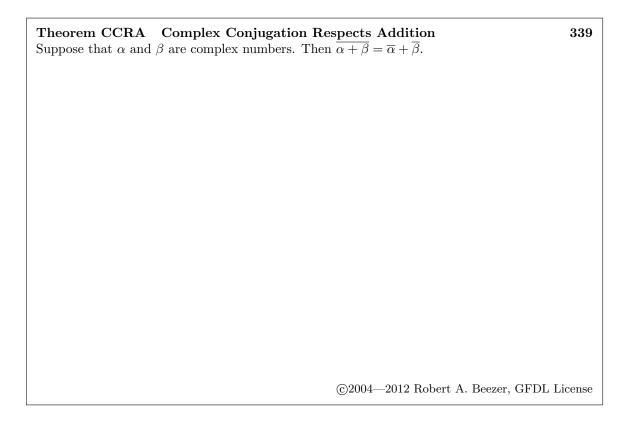
- ACCN Additive Closure, Complex Numbers: If $\alpha, \beta \in \mathbb{C}$, then $\alpha + \beta \in \mathbb{C}$.
- MCCN Multiplicative Closure, Complex Numbers: If $\alpha, \beta \in \mathbb{C}$, then $\alpha\beta \in \mathbb{C}$.
- CACN Commutativity of Addition, Complex Numbers: For any α , $\beta \in \mathbb{C}$, $\alpha + \beta = \beta + \alpha$.
- CMCN Commutativity of Multiplication, Complex Numbers: For any α , $\beta \in \mathbb{C}$, $\alpha\beta = \beta\alpha$.
- AACN Additive Associativity, Complex Numbers: For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
- MACN Multiplicative Associativity, Complex Numbers: For any $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.
- DCN Distributivity, Complex Numbers: For any α , β , $\gamma \in \mathbb{C}$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
- ZCN Zero, Complex Numbers: There is a complex number 0 = 0 + 0i so that for any $\alpha \in \mathbb{C}$, $0 + \alpha = \alpha$.
- OCN One, Complex Numbers: There is a complex number 1 = 1 + 0i so that for any $\alpha \in \mathbb{C}$, $1\alpha = \alpha$.
- AICN Additive Inverse, Complex Numbers: For every $\alpha \in \mathbb{C}$ there exists $-\alpha \in \mathbb{C}$ so that $\alpha + (-\alpha) = 0$.
- MICN Multiplicative Inverse, Complex Numbers: For every $\alpha \in \mathbb{C}$, $\alpha \neq 0$ there exists $\frac{1}{\alpha} \in \mathbb{C}$ so that $\alpha\left(\frac{1}{\alpha}\right) = 1$.

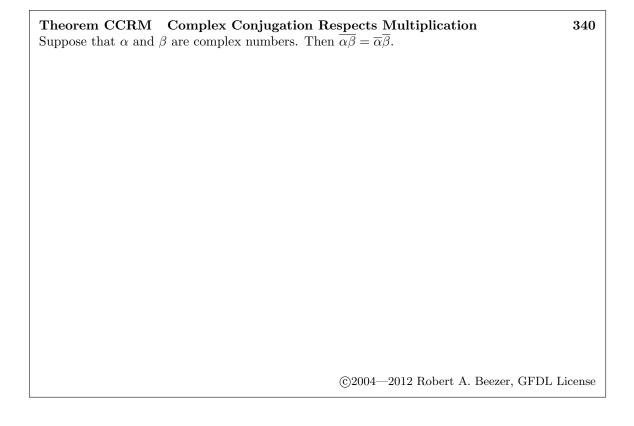
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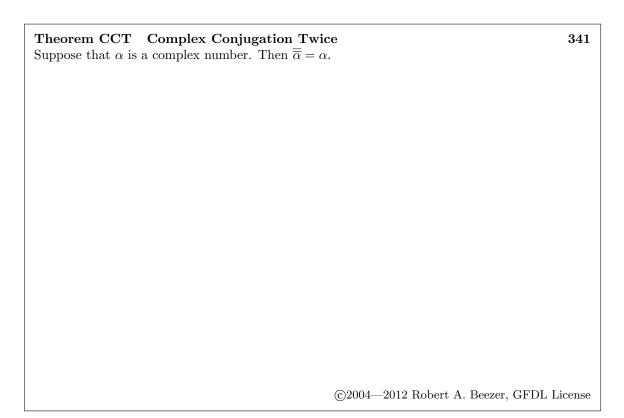
Definition CCN Conjugate of a Complex Number

338

The conjugate of the complex number $\alpha = a + bi \in \mathbb{C}$ is the complex number $\overline{\alpha} = a - bi$.







Definition MCN Modulus of a Complex Number

342

The modulus of the complex number $\alpha = a + bi \in \mathbb{C}$, is the nonnegative real number

$$|\alpha| = \sqrt{\overline{\alpha}\alpha} = \sqrt{a^2 + b^2}.$$

Definition SET Set 343	3
A set is an unordered collection of objects. If S is a set and x is an object that is in the set S	,
we write $x \in S$. If x is not in S, then we write $x \notin S$. We refer to the objects in a set as its	3
elements.	
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Definition SSET Subset

344

If S and T are two sets, then S is a subset of T, written $S \subseteq T$ if whenever $x \in S$ then $x \in T$.

Definition ES Empty Set	345
The empty set is the set with no elements. It is denoted	by \emptyset .
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Definition SE Set Equality 346 Two sets, S and T, are equal, if $S \subseteq T$ and $T \subseteq S$. In this case, we write S = T.

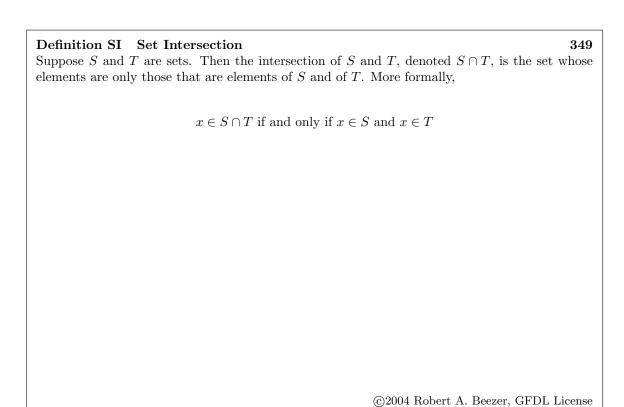
	47
Suppose S is a finite set. Then the number of elements in S is called the cardinality or size of and is denoted $ S $.	S,
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Definition SU Set Union

348

Suppose S and T are sets. Then the union of S and T, denoted $S \cup T$, is the set whose elements are those that are elements of S or of T, or both. More formally,

 $x \in S \cup T$ if and only if $x \in S$ or $x \in T$



Definition SC Set Complement

350

Suppose S is a set that is a subset of a universal set U. Then the complement of S, denoted \overline{S} , is the set whose elements are those that are elements of U and not elements of S. More formally,

 $x \in \overline{S}$ if and only if $x \in U$ and $x \notin S$